CHAPTER IX

ADDITIONAL TYPES OF ORDINARY EQUATIONS

100. Equations of the first order and higher degree. The *degree* of a differential equation is defined as the degree of the derivative of highest order which enters in the equation. In the case of the equation $\Psi(x, y, y') = 0$ of the first order, the degree will be the degree of the equation in y'. From the idea of the lineal element (§ 85) it appears that if the degree of Ψ in y' is n, there will be n lineal elements through each point (x, y). Hence it is seen that there are n curves, which are compounded of these elements, passing through each point. It may be pointed out that equations such as $y' = x\sqrt{1+y^2}$, which are apparently of the first degree in y', are really of higher degree if the multiple value of the functions, such as $\sqrt{1+y^2}$, which enter in the equation, is taken into consideration; the equation above is replaceable by $y'^2 = x^2 + x^2y^2$, which is of the second degree and without any multiple valued function.*

First suppose that *the differential equation*

$$\Psi(x, y, y') = [y' - \psi_1(x, y)] \times [y' - \psi_2(x, y)] \cdots = 0 \tag{1}$$

may be solved for y'. It then becomes equivalent to the set

$$y' - \psi_1(x, y) = 0, \qquad y' - \psi_2(x, y) = 0, \cdots \tag{1'}$$

of equations each of the first order, and each of these may be treated by the methods of Chap. VIII. Thus a set of integrals †

$$F_1(x, y, C) = 0, \qquad F_2(x, y, C) = 0, \cdots \tag{2}$$

may be obtained, and the product of these separate integrals

$$F(x, y, C) = F_1(x, y, C) \cdot F_2(x, y, C) \cdots = 0 \tag{2'}$$

is the complete solution of the original equation. Geometrically speaking, each integral $F_i(x, y, C) = 0$ represents a family of curves and the product represents all the families simultaneously.

* It is therefore apparent that the idea of degree as applied in practice is somewhat indefinite.

† The same constant C or any desired function of C may be used in the different solutions because C is an arbitrary constant and no specialization is introduced by its repeated use in this way.

As an example consider $y'^2 + 2y'y\cot x = y^2$. Solve.

$$y'^2 + 2y'y\cot x + y^2\cot^2 x = y^2(1 + \cot^2 x) = y^2\csc^2 x,$$

and
$$(y' + y\cot x - y\csc x)(y' + y\cot x + y\csc x) = 0.$$

These equations both come under the type of variables separable. Integrate.

$$\frac{dy}{y} = \frac{1-\cos x}{\sin x}dx = -\frac{d\cos x}{1+\cos x}, \qquad y(1+\cos x) = C,$$

and
$$\frac{dy}{y} = -\frac{1+\cos x}{\sin x}dx = \frac{d\cos x}{1-\cos x}, \qquad y(1-\cos x) = C.$$

Hence
$$[y(1+\cos x) + C][y(1-\cos x) + C] = 0$$

is the solution. It may be put in a different form by multiplying out. Then

$$y^2\sin^2 x + 2Cy + C^2 = 0.$$

If the equation cannot be solved for y' or if the equations resulting from the solution cannot be integrated, this first method fails. In that case *it may be possible to solve for y or for x* and treat the equation by differentiation. Let $y' = p$. Then if

$$y = f(x, p), \qquad \frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx}. \tag{3}$$

The equation thus found by differentiation is a differential equation of the first order in dp/dx and it may be solved by the methods of Chap. VIII to find $F(p, x, C) = 0$. The two equations

$$y = f(x, p) \quad \text{and} \quad F(p, x, C) = 0 \tag{3'}$$

may be regarded as defining x and y parametrically in terms of p, or p may be eliminated between them to determine the solution in the form $\Omega(x, y, C) = 0$ if this is more convenient. If the given differential equation had been solved for x, then

$$x = f(y, p) \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}. \tag{4}$$

The resulting equation on the right is an equation of the first order in dp/dy and may be treated in the same way.

As an example take $xp^2 - 2yp + ax = 0$ and solve for y. Then

$$2y = xp + \frac{ax}{p}, \qquad 2\frac{dy}{dx} = 2p = p + x\frac{dp}{dx} - \frac{ax}{p^2}\frac{dp}{dx} + \frac{a}{p},$$

or
$$\frac{x}{p}\left[p - \frac{a}{p}\right]\frac{dp}{dx} + \left(\frac{a}{p} - p\right) = 0, \quad \text{or} \quad xdp - pdx = 0.$$

The solution of this equation is $x = Cp$. The solution of the given equation is

$$2y = xp + \frac{ax}{p}, \qquad x = Cp$$

when expressed parametrically in terms of p. If p be eliminated, then

$$2y = \frac{x^2}{C} + aC \qquad \text{parabolas.}$$

As another example take $p^2y + 2px = y$ and solve for x. Then

$$2x = y\left(\frac{1}{p} - p\right), \qquad 2\frac{dx}{dy} = \frac{2}{p} = \frac{1}{p} - p + y\left(-\frac{1}{p^2} - 1\right)\frac{dp}{dy},$$

or
$$\frac{1}{p} + p + y\left(\frac{1}{p^2} + 1\right)\frac{dp}{dy} = 0, \quad \text{or} \quad ydp + pdy = 0.$$

The solution of this is $py = C$ and the solution of the given equation is

$$2x = y\left(\frac{1}{p} - p\right), \qquad py = C, \quad \text{or} \quad y^2 = 2Cx + C^2.$$

Two special types of equation may be mentioned in addition, although their method of solution is a mere corollary of the methods already given in general. They are the equation *homogeneous* in (x, y) and *Clairaut's* equation. The general form of the homogeneous equation is $\Psi(p, y/x) = 0$. This equation may be solved as

$$p = \psi\left(\frac{y}{x}\right) \qquad \text{or as} \quad \frac{y}{x} = f(p), \qquad y = xf(p); \tag{5}$$

and in the first case is treated by the methods of Chap. VIII, and in the second by the methods of this article. Which method is chosen rests with the solver. The Clairaut type of equation is

$$y = px + f(p) \tag{6}$$

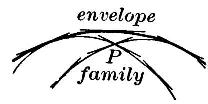
and comes directly under the methods of this article. It is especially noteworthy, however, that on differentiating with respect to x the resulting equation is

$$[x + f'(p)]\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{dx} = 0. \tag{6'}$$

Hence the solution for p is $p = C$, and thus $y = Cx + f(C)$ is the solution for the Clairaut equation and represents a family of straight lines. The rule is merely to substitute C in place of p. This type occurs very frequently in geometric applications either directly or in a disguised form requiring a preliminary change of variable.

101. To this point the only solution of the differential equation $\Psi(x, y, p) = 0$ which has been considered is the *general solution* $F(x, y, C) = 0$ containing an arbitrary constant. If a special value, say 2, is given to C, the solution $F(x, y, 2) = 0$ is called a *particular solution*. It may happen that the arbitrary constant C enters into the expression $F(x, y, C) = 0$ in such a way that when C becomes positively infinite (or negatively infinite) the curve $F(x, y, C) = 0$ approaches a definite limiting position which is a solution of the differential equation; such solutions are called *infinite solutions*. In addition to these types of solution which naturally group themselves in connection with the general solution, there is often a solution of a different kind which is known as the *singular solution.* There are several different definitions for the singular solution. That which will be adopted here is: *A singular solution is the envelope of the family of curves defined by the general solution.*

The consideration of the lineal elements (§ 85) will show how it is that the envelope (§ 65) of the family of particular solutions which constitute the general solution is itself a solution of the equation. For consider the figure, which represents the particular solutions broken up into their lineal elements. Note that the envelope is made up of those lineal elements, one taken from each particular solution, which are at the points of contact of the envelope with the curves of the family. It is seen that the envelope is a curve all of whose lineal elements satisfy the equation $\Psi(x, y, p) = 0$ for the reason that they lie upon solutions of the equation. Now any curve whose lineal elements satisfy the equation is by definition a solution of the equation; and so the envelope must be a solution. It might conceivably happen that the family $F(x, y, C) = 0$ was so constituted as to envelope one of its own curves. In that case that curve would be both a particular and a singular solution.



If the general solution $F(x, y, C) = 0$ of a given differential equation is known, the singular solution may be found according to the rule for finding envelopes (§ 65) by eliminating C from

$$F(x, y, C) = 0 \quad \text{and} \quad \frac{\partial}{\partial C} F(x, y, C) = 0. \tag{7}$$

It should be borne in mind that in the eliminant of these two equations there may occur some factors which do not represent envelopes and which must be discarded from the singular solution. If only the singular solution is desired and the general solution is not known, this method is inconvenient. In the case of Clairaut's equation, however, where the solution is known, it gives the result immediately as that obtained by eliminating C from the two equations

$$y = Cx + f(C) \quad \text{and} \quad 0 = x + f'(C). \tag{8}$$

It may be noted that as $p = C$, the second of the equations is merely the factor $x + f'(p) = 0$ discarded from (6′). The singular solution may therefore be found by eliminating p between the given Clairaut equation and the discarded factor $x + f'(p) = 0$.

A reëxamination of the figure will suggest a means of finding the singular solution without integrating the given equation. For it is seen that when two neighboring curves of the family intersect in a point P

near the envelope, then through this point there are two lineal elements which satisfy the differential equation. These two lineal elements have nearly the same direction, and indeed the nearer the two neighboring curves are to each other the nearer will their intersection lie to the envelope and the nearer will the two lineal elements approach coincidence with each other and with the element upon the envelope at the point of contact. Hence for all points (x, y) on the envelope the equation $\Psi(x, y, p) = 0$ of the lineal elements must have *double roots for p*. Now if an equation has double roots, the derivative of the equation must have a root. Hence the requirement that the two equations

$$\psi(x, y, p) = 0 \quad \text{and} \quad \frac{\partial}{\partial p}\psi(x, y, p) = 0 \qquad (9)$$

have a common solution for p will insure that the first has a double root for p; and the points (x, y) which satisfy these equations simultaneously must surely include all the points of the envelope. The rule for finding the singular solution is therefore: *Eliminate p from the given differential equation and its derivative with respect to p*, that is, from (9). The result should be tested.

If the equation $xp^2 - 2yp + ax = 0$ treated above be tried for a singular solution, the elimination of p is required between the two equations

$$xp^2 - 2yp + ax = 0 \quad \text{and} \quad xp - y = 0.$$

The result is $y^2 = ax^2$, which gives a pair of lines through the origin. The substitution of $y = \pm\sqrt{a}x$ and $p = \pm\sqrt{a}$ in the given equation shows at once that $y^2 = ax^2$ satisfies the equation. Thus $y^2 = ax^2$ is a singular solution. The same result is found by finding the envelope of the general solution given above. It is clear that in this case the singular solution is not a particular solution, as the particular solutions are parabolas.

If the elimination had been carried on by Sylvester's method, then

$$\begin{vmatrix} 0 & x & -y \\ x & -2y & a \\ x & -y & 0 \end{vmatrix} = -x(y^2 - ax^2) = 0;$$

and the eliminant is the product of two factors $x = 0$ and $y^2 - ax^2 = 0$, of which the second is that just found and the first is the y-axis. As the slope of the y-axis is infinite, the substitution in the equation is hardly legitimate, and the equation can hardly be said to be satisfied. The occurrence of these extraneous factors in the eliminant is the real reason for the necessity of testing the result to see if it actually represents a singular solution. These extraneous factors may represent a great variety of conditions. Thus in the case of the equation $p^2 + 2yp\cot x = y^2$ previously treated, the elimination gives $y^2\csc^2 x = 0$, and as $\csc x$ cannot vanish, the result reduces to $y^2 = 0$, or the x-axis. As the slope along the x-axis is 0 and y is 0, the equation is clearly satisfied. Yet the line $y = 0$ is *not* the envelope of the general solution; for the curves of the family touch the line only at the points $n\pi$. It is a particular solution and corresponds to $C = 0$. There is no singular solution.

Many authors use a great deal of time and space discussing just what may and what may not occur among the extraneous loci and how many times it may occur. The result is a considerable number of statements which in their details are either grossly incomplete or glaringly false or both (cf. §§ 65–67). The rules here given for finding singular solutions should not be regarded in any other light than as leading to some expressions which are to be examined, the best way one can, to find out whether or not they are singular solutions. One curve which may appear in the elimination of p and which deserves a note is the *tac-locus* or locus of points of tangency of the particular solutions with each other. Thus in the system of circles $(x - C)^2 + y^2 = r^2$ there may be found two which are tangent to each other at any assigned point of the x-axis. This tangency represents two coincident lineal elements and hence may be expected to occur in the elimination of p between the differential equation of the family and its derivative with respect to p; but not in the eliminant from (7).

EXERCISES

1. Integrate the following equations by solving for $p = y'$:

(α) $p^2 - 6p + 5 = 0$, (β) $p^3 - (2x + y^2)p^2 + (x^2 - y^2 + 2xy^2)p - (x^2 - y^2)y^2 = 0$,
(γ) $xp^2 - 2yp - x = 0$, (δ) $p^3(x + 2y) + 3p^2(x + y) + p(y + 2x) = 0$,
(ϵ) $y^2 + p^2 = 1$, (ζ) $p^2 - ax^3 = 0$, (η) $p = (a - x)\sqrt{1 + p^2}$.

2. Integrate the following equations by solving for y or x:

(α) $4xp^2 + 2xp - y = 0$, (β) $y = -xp + x^4p^2$, (γ) $p + 2xy - x^2 - y^2 = 0$,
(δ) $2px - y + \log p = 0$, (ϵ) $x - yp = ap^2$, (ζ) $y = x + a\tan^{-1}p$,
(η) $x = y + a\log p$, (θ) $x + py(2p^2 + 3) = 0$, (ι) $a^2yp^2 - 2xp + y = 0$,
(κ) $p^3 - 4xyp + 8y^2 = 0$, (λ) $x = p + \log p$, (μ) $p^2(x^2 + 2ax) = a^2$.

3. Integrate these equations [substitutions suggested in (ι) and (κ)]:

(α) $xy^2(p^2 + 2) = 2py^3 + x^3$, ($\beta$) $(nx + py)^2 = (1 + p^2)(y^2 + nx^2)$,
(γ) $y^2 + xyp - x^2p^2 = 0$, (δ) $y = yp^2 + 2px$,
(ϵ) $y = px + \sin^{-1}p$, (ζ) $y = p(x - b) + a/p$,
(η) $y = px + p(1 - p^2)$, (θ) $y^2 - 2pxy - 1 = p^2(1 - x^2)$,
(ι) $4e^{2y}p^2 + 2xp - 1 = 0$, $z = e^{2y}$, (κ) $y = 2px + y^2p^3$, $y^2 = z$,
(λ) $4e^{2y}p^2 + 2e^{2x}p - e^x = 0$, ($\mu$) $x^2(y - px) = yp^2$.

4. Treat these equations by the p method (9) to find the singular solutions. Also solve and treat by the C method (7). Sketch the family of solutions and examine the significance of the extraneous factors as well as that of the factor which gives the singular solution:

(α) $p^2y + p(x - y) - x = 0$, (β) $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$,
(γ) $4xp^2 = (3x - a)^2$, (δ) $yp^2x(x - a)(x - b) = [3x^2 - 2x(a + b) + ab]^2$,
(ϵ) $p^2 + xp - y = 0$, (ζ) $8a(1 + p)^3 = 27(x + y)(1 - p)^3$,
(η) $x^3p^2 + x^2yp + a^3 = 0$, (θ) $y(3 - 4y)^2p^2 = 4(1 - y)$.

5. Examine sundry of the equations of Exs. 1, 2, 3, for singular solutions.

6. Show that the solution of $y = x\phi(p) + f(p)$ is given parametrically by the given equation and the solution of the linear equation:

$$\frac{dx}{dp} + x\frac{\phi'(p)}{\phi(p) - p} = \frac{f'(p)}{p - \phi(p)}.$$ Solve (α) $y = mxp + n(1 + p^3)^{\frac{1}{2}}$,

(β) $y = x\left(p + a\sqrt{1 + p^2}\right)$, ($\gamma$) $x = yp + ap^2$, (δ) $y = (1 + p)x + p^2$.

7. As any straight line is $y = mx + b$, any family of lines may be represented as $y = mx + f(m)$ or by the Clairaut equation $y = px + f(p)$. Show that the orthogonal trajectories of any family of lines leads to an equation of the type of Ex. 6. The same is true of the trajectories at any constant angle. Express the equations of the following systems of lines in the Clairaut form, write the equations of the orthogonal trajectories, and integrate:

(α) tangents to $x^2 + y^2 = 1$, (β) tangents to $y^2 = 2ax$,
(γ) tangents to $y^2 = x^3$, (δ) normals to $y^2 = 2ax$,
(ϵ) normals to $y^2 = x^3$, (ζ) normals to $b^2x^2 + a^2y^2 = a^2b^2$.

8. The *evolute* of a given curve is the locus of the center of curvature of the curve, or, what amounts to the same thing, it is the envelope of the normals of the given curve. If the Clairaut equation of the normals is known, the evolute may be obtained as its singular solution. Thus find the evolutes of

(α) $y^2 = 4ax$, (β) $2xy = a^2$, (γ) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$,
(δ) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, ($\epsilon$) $y^2 = \dfrac{x^3}{2a - x}$, ($\zeta$) $y = \frac{1}{2}(e^x + e^{-x})$.

9. The *involutes* of a given curve are the curves which cut the tangents of the given curve orthogonally, or, what amounts to the same thing, they are the curves which have the given curve as the locus of their centers of curvature. Find the involutes of

(α) $x^2 + y^2 = a^2$, (β) $y^2 = 2mx$, (γ) $y = a\cosh(x/a)$.

10. As any curve is the envelope of its tangents, it follows that when the curve is described by a property of its tangents the curve may be regarded as the singular solution of the Clairaut equation of its tangent lines. Determine thus what curves have these properties:

(α) length of the tangent intercepted between the axes is l,
(β) sum of the intercepts of the tangent on the axes is c,
(γ) area between the tangent and axes is the constant k^2,
(δ) product of perpendiculars from two fixed points to tangent is k^2,
(ϵ) product of ordinates from two points of x-axis to tangent is k^2.

11. From the relation $\dfrac{dF}{dn} = \mu\sqrt{M^2 + N^2}$ of Proposition 3, p. 212, show that as the curve $F = C$ is moving tangentially to itself along its envelope, the singular solution of $Mdx + Ndy = 0$ may be expected to be found in the equation $1/\mu = 0$; also the infinite solutions. Discuss the equation $1/\mu = 0$ in the following cases:

(α) $\sqrt{1 - y^2}\,dx = \sqrt{1 - x^2}\,dy$, ($\beta$) $xdx + ydy = \sqrt{x^2 + y^2 - a^2}\,dy$.

102. Equations of higher order. In the treatment of special problems (§ 82) it was seen that the substitutions

$$\frac{dy}{dx} = p, \qquad \frac{d^2y}{dx^2} = \frac{dp}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} = p\frac{dp}{dy} \tag{10}$$

rendered the differential equations integrable by reducing them to integrable equations of the first order. These substitutions or others like them are useful in treating certain cases of the differential equation

$\Psi(x, y, y', y'', \cdots, y^{(n)}) = 0$ of the nth order, namely, when one of the variables and perhaps some of the derivatives of lowest order do not occur in the equation.

In case
$$\Psi\left(x, \frac{d^i y}{dx^i}, \frac{d^{i+1}y}{dx^{i+1}}, \cdots, \frac{d^n y}{dx^n}\right) = 0, \tag{11}$$
y and the first $i-1$ derivatives being absent, substitute
$$\frac{d^i y}{dx^i} = q \quad \text{so that} \quad \Psi\left(x, q, \frac{dq}{dx}, \cdots, \frac{d^{n-i}q}{dx^{n-i}}\right) = 0. \tag{11'}$$
The original equation is therefore replaced by one of lower order. If the integral of this be $F(x, q) = 0$, which will of course contain $n-i$ arbitrary constants, the solution for q gives
$$q = f(x) \quad \text{and} \quad y = \int\cdots\int f(x)\,(dx)^i. \tag{12}$$
The solution has therefore been accomplished. If it were more convenient to solve $F(x, q) = 0$ for $x = \phi(q)$, the integration would be
$$y = \int\cdots\int q\,(dx)^i = \int\cdots\int q\,[\phi'(q)\,dq]^i; \tag{12'}$$
and this equation with $x = \phi(q)$ would give a parametric expression for the integral of the differential equation.

In case
$$\Psi\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^n y}{dx^n}\right) = 0, \tag{13}$$
x being absent, substitute p and regard p as a function of y. Then
$$\begin{aligned} &\frac{dy}{dx} = p, \quad \frac{d^2y}{dx^2} = p\frac{dp}{dy}, \quad \frac{d^3y}{dx^3} = p\frac{d}{dy}\left(p\frac{dp}{dy}\right), \cdots \\ \text{and} \quad &\Psi_1\left(y, p, \frac{dp}{dy}, \cdots, \frac{d^{n-1}p}{dy^{n-1}}\right) = 0. \end{aligned} \tag{13'}$$
In this way the order of the equation is lowered by unity. If this equation can be integrated as $F(y, p) = 0$, the last step in the solution may be obtained either directly or parametrically as
$$p = f(y), \qquad \int \frac{dy}{f(y)} = x \tag{14}$$
or
$$y = \phi(p), \qquad x = \int \frac{dy}{p} = \int \frac{\phi'(p)\,dp}{p}. \tag{14'}$$
It is no particular simplification in this case to have some of the lower derivatives of y absent from $\Psi = 0$, because in general the lower derivatives of p will none the less be introduced by the substitution that is made.

As an example consider $\left(x\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}\right)^2=\left(\frac{d^3y}{dx^3}\right)^2+1$,

which is
$$\left(x\frac{dq}{dx}-q\right)^2=\left(\frac{dq}{dx}\right)^2+1 \quad \text{if} \quad q=\frac{d^2y}{dx^2}.$$

Then
$$q=x\frac{dq}{dx}\pm\sqrt{\left(\frac{dq}{dx}\right)^2+1} \quad \text{and} \quad q=C_1x\pm\sqrt{C_1^2+1};$$

for the equation is a Clairaut type. Hence, finally,

$$y=\int\int\left[C_1x\pm\sqrt{C_1^2+1}\right](dx)^2=\tfrac{1}{6}C_1x^3\pm\tfrac{1}{2}x^2\sqrt{C_1^2+1}+C_2x+C_3.$$

As another example consider $y''-y'^2=y^2\log y$. This becomes

$$p\frac{dp}{dy}-p^2=y^2\log y \quad \text{or} \quad \frac{d(p^2)}{dy}-2p^2=2y^2\log y.$$

The equation is linear in p^2 and has the integrating factor e^{-2y}.

$$\frac{1}{2}p^2e^{-2y}=\int y^2e^{-2y}\log y\,dy, \qquad \frac{1}{\sqrt{2}}p=\left[e^{2y}\int y^2e^{-2y}\log y\,dy\right]^{\frac{1}{2}},$$

and
$$\int\frac{dy}{\left[e^{2y}\int y^2e^{-2y}\log y\,dy\right]^{\frac{1}{2}}}=\sqrt{2}\,x.$$

The integration is therefore reduced to quadratures and becomes a problem in ordinary integration.

If an equation is *homogeneous with respect to y and its derivatives,* that is, if the equation is multiplied by a power of k when y is replaced by ky, the order of the equation may be lowered by the substitution $y=e^z$ and by taking z' as the new variable. If the equation is *homogeneous with respect to x and dx,* that is, if the equation is multiplied by a power of k when x is replaced by kx, the order of the equation may be reduced by the substitution $x=e^t$. The work may be simplified (Ex. 9, p. 152) by the use of

$$D_x^ny=e^{-nt}D_t(D_t-1)\cdots(D_t-n+1)y. \tag{15}$$

If the equation is *homogeneous with respect to x and y and the differentials dx, dy, d^2y, $\cdots$,* the order may be lowered by the substitution $x=e^t$, $y=e^tz$, where it may be recalled that

$$\begin{aligned}D_x^ny&=e^{-nt}D_t(D_t-1)\cdots(D_t-n+1)y\\&=e^{-(n-1)t}(D_t+1)D_t\cdots(D_t-n+2)z.\end{aligned} \tag{15'}$$

Finally, if the equation is *homogeneous with respect to x considered of dimensions* 1, *and y considered of dimensions m,* that is, if the equation is multiplied by a power of k when kx replaces x and k^my replaces y, the substitution $x=e^t$, $y=e^{mt}z$ will lower the degree of the equation. It may be recalled that

$$D_x^ny=e^{(m-n)t}(D_t+m)(D_t+m-1)\cdots(D_t+m-n+1)z. \tag{15''}$$

Consider $xyy'' - xy'^2 = yy' + bxy'^2/\sqrt{a^2 - x^2}$. If in this equation y be replaced by ky so that y' and y'' are also replaced by ky' and ky'', it appears that the equation is merely multiplied by k^2 and is therefore homogeneous of the first sort mentioned. Substitute

$$y = e^z, \qquad y' = e^z z', \qquad y'' = e^z(z'' + z'^2).$$

Then e^{2z} will cancel from the whole equation, leaving merely

$$xz'' = z' + bxz'^2/\sqrt{a^2 - x^2} \quad \text{or} \quad \frac{xdz'}{z'^2} - \frac{1}{z'}dx = \frac{bxdx}{\sqrt{a^2 - x^2}}.$$

The equation in the first form is Bernoulli; in the second form, exact. Then

$$\frac{x}{z'} = b\sqrt{a^2 - x^2} + C \quad \text{and} \quad dz = \frac{xdx}{b\sqrt{a^2 - x^2} + C}.$$

The variables are separated for the last integration which will determine $z = \log y$ as a function of x.

Again consider $x^4 \dfrac{d^2y}{dx^2} = (x^3 + 2xy)\dfrac{dy}{dx} - 4y^2$. If x be replaced by kx and y by k^2y so that y' is replaced by ky' and y'' remains unchanged, the equation is multiplied by k^4 and hence comes under the fourth type mentioned above. Substitute

$$x = e^t, \qquad y = e^{2t}z, \qquad D_xy = e^t(D_t + 2)z, \qquad D_x^2y = (D_t + 2)(D_t + 1)z.$$

Then e^{4t} will cancel and leave $z'' + 2(1 - z)z' = 0$, if accents denote differentiation with respect to t. This equation lacks the independent variable t and is reduced by the substitution $z'' = z'dz'/dz$. Then

$$\frac{dz'}{dz} + 2(1 - z) = 0, \qquad z' = \frac{dz}{dt} = (1 - z)^2 + C, \qquad \frac{dz}{(1 - z^2) + C} = dt.$$

There remains only to perform the quadrature and replace z and t by x and y.

103. If the equation may be obtained by differentiation, as

$$\Psi\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = \frac{d\Omega}{dx} = \frac{\partial\Omega}{\partial x} + \frac{\partial\Omega}{\partial y}y' + \cdots + \frac{\partial\Omega}{\partial y^{(n-1)}}y^{(n)}, \quad (16)$$

it is called an *exact equation*, and $\Omega(x, y, y', \cdots, y^{(n-1)}) = C$ is an integral of $\Psi = 0$. Thus in case the equation is exact, the order may be lowered by unity. It may be noted that unless the degree of the nth derivative is 1 the equation cannot be exact. Consider

$$\Psi(x, y, y', \cdots, y^{(n)}) = \phi_1 y^{(n)} + \phi_2,$$

where the coefficient of $y^{(n)}$ is collected into ϕ_1. Now integrate ϕ_1, partially regarding only $y^{(n-1)}$ as variable so that

$$\int \phi_1 dy^{(n-1)} = \Omega_1, \qquad \frac{d}{dx}\Omega_1 = \frac{\partial\Omega_1}{\partial x} + \cdots + \frac{\partial\Omega_1}{\partial y^{(n-2)}}y^{(n-1)} + \phi_1 y^{(n)}.$$

Then
$$\Psi - \frac{d\Omega_1}{dx} = \phi_3\left[\frac{d^{n-k}y}{dx^{n-k}}\right]^m + \phi_4.$$

That is, the expression $\Psi - \Omega_1'$ does not contain $y^{(n)}$ and may contain no derivative of order higher than $n - k$, and may be collected as indicated. Now if Ψ was an exact derivative, so must $\Psi - \Omega_1'$ be. Hence if $m \neq 1$, the conclusion is that Ψ was not exact. If $m = 1$, the process of integration may be continued to obtain Ω_2 by integrating partially with respect to $y^{(n-k-1)}$. And so on until it is shown that Ψ is not exact or until Ψ is seen to be the derivative of an expression $\Omega_1 + \Omega_2 + \cdots = C$.

As an example consider $\Psi = x^2y''' + xy'' + (2xy - 1)y' + y^2 = 0$. Then

$$\Omega_1 = \int x^2 dy'' = x^2y'', \qquad \Psi - \Omega_1' = -xy'' + (2xy-1)y' + y^2,$$

$$\Omega_2 = \int -x dy' = -xy', \qquad \Psi - \Omega_1' - \Omega_2' = 2xyy' + y^2 = (xy^2)'.$$

As the expression of the first order is an exact derivative, the result is

$$\Psi - \Omega_1' - \Omega_2' - (xy^2)' = 0; \quad \text{and} \quad \Psi_1 = x^2y'' - xy' + xy^2 - C_1 = 0$$

is the new equation. The method may be tried again.

$$\Omega_1 = \int x^2 dy' = x^2y', \qquad \Psi_1 - \Omega_1' = -3xy' + xy^2 - C_1.$$

This is not an exact derivative and the equation $\Psi_1 = 0$ is not exact. Moreover the equation $\Psi_1 = 0$ contains both x and y and is not homogeneous of any type except when $C_1 = 0$. It therefore appears as though the further integration of the equation $\Psi = 0$ were impossible.

The method is applied with especial ease to the case of

$$X_0 \frac{d^ny}{dx^n} + X_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + X_{n-1}\frac{dy}{dx} + X_n y - R(x) = 0, \tag{17}$$

where the coefficients are functions of x alone. This is known as the *linear equation*, the integration of which has been treated only when the order is 1 or when the coefficients are constants. The application of successive integration by parts gives

$$\Omega_1 = X_0 y^{(n-1)}, \quad \Omega_2 = (X_1 - X_0')y^{(n-2)}, \quad \Omega_3 = (X_2 - X_1' + X_0'')y^{(n-3)}, \cdots;$$

and after n such integrations there is left merely

$$(X_n - X_{n-1}' + \cdots + (-1)^{n-1}X_1 + (-1)^n X_0)y - R,$$

which is a derivative only when it is a function of x. Hence

$$X_n - X_{n-1}' + \cdots + (-1)^{n-1}X_1 + (-1)^n X_0 \equiv 0 \tag{18}$$

is the condition that the linear equation shall be exact, and

$$X_0 y^{(n-1)} + (X_1 - X_0')y^{(n-2)} + (X_2 - X_1' + X_0'')y^{(n-3)} + \cdots = \int R dx \tag{19}$$

is the first solution in case it is exact.

As an example take $y''' + y''\cos x - 2y'\sin x - y\cos x = \sin 2x$. The test

$$X_3 - X_2' + X_1'' - X_0''' = -\cos x + 2\cos x - \cos x = 0$$

is satisfied. The integral is therefore $y'' + y'\cos x - y\sin x = -\frac{1}{2}\cos 2x + C_1$. This equation still satisfies the test for exactness. Hence it may be integrated again with the result $y' + y\cos x = -\frac{1}{4}\sin 2x + C_1x + C_2$. This belongs to the linear type. The final result is therefore

$$y = e^{-\sin x}\int e^{\sin x}(C_1x + C_2)\,dx + C_3e^{-\sin x} + \tfrac{1}{2}(1 - \sin x).$$

EXERCISES

1. Integrate these equations or at least reduce them to quadratures:

(α) $2xy'''y'' = y''^2 - a^2$, (β) $(1 + x^2)y'' + 1 + y'^2 = 0$,
(γ) $y^{\text{iv}} + a^2y'' = 0$, ($\delta$) $y^{\text{v}} - m^2y''' = e^{ax}$, ($\epsilon$) $x^2y^{\text{iv}} + a^2y'' = 0$,
(ζ) $a^2y''y' = x$, (η) $xy'' + y' = 0$, (θ) $y'''y'' = 4$,
(ι) $(1 - x^2)y'' - xy' = 2$, (κ) $y^{\text{iv}} = \sqrt{y'''}$, ($\lambda$) $y'' = f(y)$,
(μ) $2(2a - y)y'' = 1 + y'^2$, (ν) $yy'' - y'^2 - y^2y' = 0$,
(o) $yy'' + y'^2 + 1 = 0$, (π) $2y'' = e^y$, (ρ) $y^3y'' = a$.

2. Carry the integration as far as possible in these cases:

(α) $x^2y'' = (mx^2y'^2 + ny^2)^{\frac{1}{2}}$, ($\beta$) $mx^3y'' = (y - xy')^2$,
(γ) $x^4y'' = (y - xy')^3$, (δ) $x^4y'' - x^3y' - x^2y'^2 + 4y^2 = 0$,
(ϵ) $x^{-2}y'' + x^{-4}y = \frac{1}{4}y'^2$, ($\zeta$) $ayy'' + by'^2 = yy'(c^2 + x^2)^{-\frac{1}{2}}$.

3. Carry the integration as far as possible in these cases:

(α) $(y^2 + x)y''' + 6yy'y'' + y'' + 2y'^2 = 0$, ($\beta$) $y'y'' - yx^2y' = xy^2$,
(γ) $x^3yy''' + 3x^3y'y'' + 9x^2yy'' + 9x^2y'^2 + 18xyy' + 3y^2 = 0$,
(δ) $y + 3xy' + 2yy'^3 + (x^2 + 2y^2y')y'' = 0$,
(ϵ) $(2x^3y' + x^2y)y'' + 4x^2y'^2 + 2xyy' = 0$.

4. Treat these linear equations:

(α) $xy'' + 2y = 2x$, (β) $(x^2 - 1)y'' + 4xy' + 2y = 2x$,
(γ) $y'' - y'\cot x + y\csc^2 x = \cos x$, ($\delta$) $(x^2 - x)y'' + (3x - 2)y' + y = 0$,
(ϵ) $(x - x^3)y''' + (1 - 5x^2)y'' - 2xy' + 2y = 6x$,
(ζ) $(x^3 + x^2 - 3x + 1)y''' + (9x^2 + 6x - 9)y'' + (18x + 6)y' + 6y = x^3$,
(η) $(x + 2)^2y''' + (x + 2)y'' + y' = 1$, ($\theta$) $x^2y'' + 3xy' + y = x$,
(ι) $(x^3 - x)y''' + (8x^2 - 3)y'' + 14xy' + 4y = 0$.

5. Note that Ex. 4 (θ) comes under the third homogeneous type, and that Ex. 4 (η) may be brought under that type by multiplying by $(x + 2)$. Test sundry of Exs. 1, 2, 3 for exactness. Show that any linear equation in which the coefficients are polynomials of degree less than the order of the derivatives of which they are the coefficients, is surely exact.

6. Sometimes, when the condition that an equation be exact is not satisfied, it is possible to find an integrating factor for the equation so that after multiplication by the factor the equation becomes exact. For linear equations try x^m. Integrate

(α) $x^5y'' + (2x^4 - x)y' - (2x^3 - 1)y = 0$, ($\beta$) $(x^2 - x^4)y'' - x^3y' - 2y = 0$.

7. Show that the equation $y'' + Py' + Qy'^2 = 0$ may be reduced to quadratures 1° when P and Q are both functions of y, or 2° when both are functions of x, or 3° when P is a function of x and Q is a function of y (integrating factor $1/y'$). In each case find the general expression for y in terms of quadratures. Integrate $y'' + 2y'\cot x + 2y'^2\tan y = 0$.

8. Find and discuss the curves for which the radius of curvature is proportional to the radius r of the curve.

9. If the radius of curvature R is expressed as a function $R = R(s)$ of the arc s measured from some point, the equation $R = R(s)$ or $s = s(R)$ is called the *intrinsic equation* of the curve. To find the relation between x and y the second equation may be differentiated as $ds = s'(R)\,dR$, and this equation of the third order may be solved. Show that if the origin be taken on the curve at the point $s = 0$ and if the x-axis be tangent to the curve, the equations

$$x = \int_0^s \cos\left[\int_0^s \frac{ds}{R}\right] ds, \qquad y = \int_0^s \sin\left[\int_0^s \frac{ds}{R}\right] ds$$

express the curve parametrically. Find the curves whose intrinsic equations are

$$(\alpha)\ R = a, \qquad (\beta)\ aR = s^2 + a^2, \qquad (\gamma)\ R^2 + s^2 = 16\,a^2.$$

10. Given $F = y^{(n)} + X_1 y^{(n-1)} + X_2 y^{(n-2)} + \cdots + X_{n-1} y' + X_n y = 0$. Show that if μ, a function of x alone, is an integrating factor of the equation, then

$$\Phi = \mu^{(n)} - (X_1\mu)^{(n-1)} + (X_2\mu)^{(n-2)} - \cdots + (-1)^{n-1}(X_{n-1}\mu)' + (-1)^n X_n \mu = 0$$

is the equation satisfied by μ. Collect the coefficient of μ to show that the condition that the given equation be exact is the condition that this coefficient vanish. The equation $\Phi = 0$ is called the *adjoint* of the given equation $F = 0$. Any integral μ of the adjoint equation is an integrating factor of the original equation. Moreover note that

$$\int \mu F dx = \mu y^{(n-1)} + (\mu X_1 - \mu')\, y^{(n-2)} + \cdots + (-1)^n \int y \Phi dx,$$

or $$d[\mu F - (-1)^n y \Phi] = d\,[\mu y^{(n-1)} + (\mu X_1 - \mu')\, y^{(n-2)} + \cdots] = d\Omega.$$

Hence if μF is an exact differential, so is $y\Phi$. In other words, any solution y of the original equation is an integrating factor for the adjoint equation.

104. Linear differential equations. The equations

$$\begin{aligned} X_0 D^n y + X_1 D^{n-1} y + \cdots + X_{n-1} D y + X_n y &= R(x), \\ X_0 D^n y + X_1 D^{n-1} y + \cdots + X_{n-1} D y + X_n y &= 0 \end{aligned} \tag{20}$$

are linear differential equations of the nth order; the first is called the *complete equation* and the second the *reduced equation*. If $y_1, y_2, y_3, \cdots$ are any solutions of the reduced equation, and $C_1, C_2, C_3, \cdots$ are any constants, then $y = C_1 y_1 + C_2 y_2 + C_3 y_3 + \cdots$ is also a solution of the reduced equation. This follows at once from the linearity of the reduced equation and is proved by direct substitution. Furthermore if I is any solution of the complete equation, then $y + I$ is also a solution of the complete equation (cf. § 96).

As the equations (20) are of the nth order, they will determine $y^{(n)}$ and, by differentiation, all higher derivatives in terms of the values of $x, y, y', \cdots, y^{(n-1)}$. Hence if the values of the n quantities $y_0, y_0', \cdots, y_0^{(n-1)}$ which correspond to the value $x = x_0$ be given, all the higher derivatives are determined (§§ 87–88). Hence there are n and no more than n arbitrary conditions that may be imposed as initial conditions. A solution of the equations (20) which contains n distinct arbitrary constants is called the general solution. By distinct is meant that the constants can actually be determined to suit the n initial conditions.

If $y_1, y_2, \cdots, y_n$ are n solutions of the reduced equation, and

$$\begin{aligned} y &= C_1y_1 + C_2y_2 + \cdots + C_ny_n, \\ y' &= C_1y_1' + C_2y_2' + \cdots + C_ny_n', \\ &\cdots\cdots\cdots\cdots \\ y^{(n-1)} &= C_1y_1^{(n-1)} + C_2y_2^{(n-1)} + \cdots + C_ny_n^{(n-1)}, \end{aligned} \tag{21}$$

then y is a solution and $y', \cdots, y^{(n-1)}$ are its first $n-1$ derivatives. If x_0 be substituted on the right and the assumed corresponding initial values $y_0, y_0', \cdots, y_0^{(n-1)}$ be substituted on the left, the above n equations become linear equations in the n unknowns $C_1, C_2, \cdots, C_n$; and if they are to be soluble for the C's, the condition

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0 \tag{22}$$

must hold for every value of $x = x_0$. Conversely if the condition does hold, the equations will be soluble for the C's.

The determinant $W(y_1, y_2, \cdots, y_n)$ is called the *Wronskian* of the n functions $y_1, y_2, \cdots, y_n$. The result may be stated as: If n functions $y_1, y_2, \cdots, y_n$ which are solutions of the reduced equation, and of which the Wronskian does not vanish, can be found, the general solution of the reduced equation can be written down. In general no solution of the equation can be found, whether by a definite process or by inspection; but in the rare instances in which the n solutions can be seen by inspection the problem of the solution of the reduced equation is completed. Frequently one solution may be found by inspection, and it is therefore important to see how much this contributes toward effecting the solution.

If y_1 is a solution of the reduced equation, make the substitution $y = y_1z$. The derivatives of y may be obtained by Leibniz's Theorem (§ 8). As the formula is linear in the derivatives of z, it follows that the result of the substitution will leave the equation linear in the new variable z. Moreover, to collect the coefficient of z itself, it is necessary to take only the first term $y_1^{(k)}z$ in the expansions for the derivative $y^{(k)}$. Hence

$$(X_0y_1^{(n)} + X_1y_1^{(n-1)} + \cdots + X_{n-1}y_1' + X_ny_1)\,z = 0$$

is the coefficient of z and vanishes by the assumption that y_1 is a solution of the reduced equation. Then the equation for z is

$$P_0z^{(n)} + P_1z^{(n-1)} + \cdots + P_{n-2}z'' + P_{n-1}z' = 0; \tag{23}$$

and if z' be taken as the variable, the equation is of the order $n-1$. It therefore appears that *the knowledge of a solution y_1 reduces the order of the equation by one.*

Now if $y_2, y_3, \cdots, y_p$ were other solutions, the derived ratios

$$z_1' = \left(\frac{y_2}{y_1}\right)', \qquad z_2' = \left(\frac{y_3}{y_1}\right)', \qquad \cdots, \qquad z_{p-1}' = \left(\frac{y_p}{y_1}\right)' \qquad (23')$$

would be solutions of the equation in z'; for by substitution,

$$y = y_1 z_1 = y_2, \qquad y = y_1 z_2 = y_3, \qquad \cdots, \qquad y = y_1 z_{p-1} = y_p$$

are all solutions of the equation in y. Moreover, if there were a linear relation $C_1 z_1' + C_2 z_2' + \cdots + C_{p-1} z_{p-1}' = 0$ connecting the solutions z_i', an integration would give a linear relation

$$C_1 y_2 + C_2 y_3 + \cdots + C_{p-1} y_n + C_p y_1 = 0$$

connecting the p solutions y_i. Hence if there is no linear relation (of which the coefficients are not all zero) connecting the p solutions y_i of the original equation, there can be none connecting the $p-1$ solutions z_i' of the transformed equation. Hence *a knowledge of p solutions of the original reduced equation gives a new reduced equation of which $p-1$ solutions are known.* And the process of substitution may be continued to reduce the order further until the order $n-p$ is reached.

As an example consider the equation of the third order

$$(1-x)y''' + (x^2-1)y'' - x^2 y + xy = 0.$$

Here a simple trial shows that x and e^x are two solutions. Substitute

$$y = e^x z, \qquad y' = e^x(z+z'), \qquad y'' = e^x(2z'+z''), \qquad y''' = e^x(3z''+z''').$$

Then $\qquad (1-x)z''' + (x^2-3x+2)z'' + (x^2-3x+1)z' = 0$

is of the second order in z'. A known solution is the derived ratio $(x/e^x)'$.

$$z' = (xe^{-x})' = e^{-x}(1-x). \quad \text{Let } z' = e^{-x}(1-x)w.$$

From this, z'' and z''' may be found and the equation takes the form

$$(1-x)w'' + (1+x)(x-2)w' = 0 \quad \text{or} \quad \frac{dw'}{w'} = xdx - \frac{2}{x-1}dx.$$

This is a linear equation of the first order and may be solved.

$$\log w' = \tfrac{1}{2}x^2 - 2\log(x-1) + C \quad \text{or} \quad w' = C_1 e^{\frac{1}{2}x^2}(x-1)^{-2}.$$

Hence $\qquad w = C_1 \int e^{\frac{1}{2}x^2}(x-1)^{-2}dx + C_2,$

$$z' = \left(\frac{x}{e^x}\right)' w = C_1\left(\frac{x}{e^x}\right)' \int e^{\frac{1}{2}x^2}(x-1)^{-2}dx + C_2\left(\frac{x}{e^x}\right)',$$

$$z = C_1 \int \left(\frac{x}{e^x}\right)' \int e^{\frac{1}{2}x^2}(x-1)^{-2}(dx)^2 + C_2\frac{x}{e^x} + C_3,$$

$$y = e^x z = C_1 e^x \int \left(\frac{x}{e^x}\right)' \int e^{\frac{1}{2}x^2}(x-1)^{-2}(dx)^2 + C_2 x + C_3 e^x.$$

The value for y is thus obtained in terms of quadratures. It may be shown that in case the equation is of the nth degree with p known solutions, the final result will call for $p(n-p)$ quadratures.

105. If the general solution $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ of the reduced equation has been found (called the *complementary function* for the complete equation), the general solution of the complete equation may always be obtained in terms of quadratures by the important and far-reaching *method of the variation of constants* due to Lagrange. The question is: Cannot functions of x be found so that the expression

$$y = C_1(x)\,y_1 + C_2(x)\,y_2 + \cdots + C_n(x)\,y_n \tag{24}$$

shall be the solution of the complete equation? As there are n of these functions to be determined, it should be possible to impose $n-1$ conditions upon them and still find the functions.

Differentiate y on the supposition that the C's are variable.

$$y' = C_1y_1' + C_2y_2' + \cdots + C_ny_n' + y_1C_1' + y_2C_2' + \cdots + y_nC_n'.$$

As one of the conditions on the C's suppose that

$$y_1C_1' + y_2C_2' + \cdots + y_nC_n' = 0.$$

Differentiate again and impose the new condition

$$y_1'C_1' + y_2'C_2' + \cdots + y_n'C_n' = 0,$$

so that $$y'' = C_1y_1'' + C_2y_2'' + \cdots + C_ny_n''.$$

The differentiation may be continued to the $(n-1)$st condition

$$y_1^{(n-2)}C_1' + y_2^{(n-2)}C_2' + \cdots + y_n^{(n-2)}C_n' = 0,$$

and $$y^{(n-1)} = C_1y_1^{(n-1)} + C_2y_2^{(n-1)} + \cdots + C_ny_n^{(n-1)}.$$

Then $$y^{(n)} = C_1y_1^{(n)} + C_2y_2^{(n)} + \cdots + C_ny_n^{(n)} + y_1^{(n-1)}C_1' + y_2^{(n-1)}C_2' + \cdots + y_n^{(n-1)}C_n'.$$

Now if the expressions thus found for $y, y', y'', \cdots, y^{(n-1)}, y^{(n)}$ be substituted in the complete equation, and it be remembered that $y_1, y_2, \cdots, y_n$ are solutions of the reduced equation and hence give 0 when substituted in the left-hand side of the equation, the result is

$$y_1^{(n-1)}C_1' + y_2^{(n-1)}C_2' + \cdots + y_n^{(n-1)}C_n' = R.$$

Hence, in all, there are n linear equations

$$\begin{aligned}
y_1C_1' &+ y_2C_2' + \cdots + y_nC_n' = 0,\\
y_1'C_1' &+ y_2'C_2' + \cdots + y_n'C_n' = 0,\\
&\cdots\cdots\cdots\cdots\\
y_1^{(n-2)}C_1' &+ y_2^{(n-2)}C_2' + \cdots + y_n^{(n-2)}C_n' = 0,\\
y_1^{(n-1)}C_1' &+ y_2^{(n-1)}C_2' + \cdots + y_n^{(n-1)}C_n' = R.
\end{aligned} \tag{25}$$

connecting the derivatives of the C's; and these may actually be solved for those derivatives which will then be expressed in terms of x. The C's may then be found by quadrature.

As an example consider the equation with constant coefficients

$$(D^3 + D)y = \sec x \quad \text{with} \quad y = C_1 + C_2 \cos x + C_3 \sin x$$

as the solution of the reduced equation. Here the solutions y_1, y_2, y_3 may be taken as 1, $\cos x$, $\sin x$ respectively. The conditions on the derivatives of the C's become by direct substitution in (25)

$$C_1' + \cos x C_2' + \sin x C_3' = 0, \quad -\sin x C_2' + \cos x C_3' = 0, \quad -\cos x C_2' - \sin x C_3' = \sec x.$$

Hence $\qquad C_1' = \sec x, \qquad C_2' = -1, \qquad C_3' = -\tan x$

and $\quad C_1 = \log\tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi) + c_1, \qquad C_2 = -x + c_2, \qquad C_3 = \log\cos x + c_3.$

Hence $\quad y = c_1 + \log\tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi) + (c_2 - x)\cos x + (c_3 + \log\cos x)\sin x$

is the general solution of the complete equation. This result could not be obtained by any of the real short methods of §§ 96–97. It could be obtained by the general method of § 95, but with little if any advantage over the method of variation of constants here given. The present method is equally available for equations with variable coefficients.

106. *Linear equations of the second order* are especially frequent in practical problems. In a number of cases the solution may be found. Thus 1° when the coefficients are constant or may be made constant by a change of variable as in Ex. 7, p. 222, the general solution of the reduced equation may be written down at once. The solution of the complete equation may then be found by obtaining a particular integral I by the methods of §§ 95–97 or by the application of the method of variation of constants. And 2° when the equation is exact, the solution may be had by integrating the linear equation (19) of § 103 of the first order by the ordinary methods. And 3° when one solution of the reduced equation is known (§ 104), the reduced equation may be completely solved and the complete equation may then be solved by the method of variation of constants, or the complete equation may be solved directly by Ex. 6 below.

Otherwise, write the differential equation in the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = R. \tag{26}$$

The substitution $y = uz$ gives the new equation

$$\frac{d^2z}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dz}{dx} + \frac{1}{u}(u'' + Pu' + Qu)z = \frac{R}{u}. \tag{26'}$$

If u be determined so that the coefficient of z' vanishes, then

$$u = e^{-\frac{1}{2}\int P dx} \quad \text{and} \quad \frac{d^2z}{dx^2} + \left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right)z = Re^{\frac{1}{2}\int P dx}. \tag{27}$$

Now 4° if $Q - \frac{1}{2}P' - \frac{1}{4}P^2$ is constant, the new reduced equation in (27) may be integrated; and 5° if it is k/x^2, the equation may also be integrated by the method of Ex. 7, p. 222. The integral of the complete equation may then be found. (In other cases this method may be useful in that the equation is reduced to a simpler form where solutions of the reduced equation are more evident.)

Again, suppose that the independent variable is changed to z. Then

$$\frac{d^2y}{dz^2} + \frac{z'' + Pz'}{z'^2}\frac{dy}{dz} + \frac{Q}{z'^2}y = \frac{R}{z'^2}. \qquad (28)$$

Now 6° if $z'^2 = \pm Q$ will make $z'' + Pz' = kz'^2$, so that the coefficient of dy/dz becomes a constant k, the equation is integrable. (Trying if $z'^2 = \pm Qz^2$ will make $z'' + Pz' = kz'^2/z$ is needless because nothing in addition to 6° is thereby obtained. It may happen that if z be determined so as to make $z'' + Pz' = 0$, the equation will be so far simplified that a solution of the reduced equation becomes evident.)

Consider the example $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{a^2}{x^4}y = 0$. Here no solution is apparent. Hence compute $Q - \frac{1}{2}P' - \frac{1}{4}P^2$. This is a^2/x^4 and is neither constant nor proportional to $1/x^2$. Hence the methods 4° and 5° will not work. From $z'^2 = Q = a^2/x^4$ or $z' = a/x^2$, it appears that $z'' + Pz' = 0$, and 6° works; the new equation is

$$\frac{d^2y}{dz^2} + y = 0 \quad \text{with} \quad z = -\frac{a}{x}.$$

The solution is therefore seen immediately to be

$$y = C_1 \cos z - C_2 \sin z \quad \text{or} \quad y = C_1 \cos(a/x) + C_2 \sin(a/x).$$

If there had been a right-hand member in the original equation, the solution could have been found by the method of variation of constants, or by some of the short methods for finding a particular solution if R had been of the proper form.

EXERCISES

1. If a relation $C_1y_1 + C_2y_2 + \cdots + C_ny_n = 0$, with constant coefficients not all 0, exists between n functions $y_1, y_2, \cdots, y_n$ of x for all values of x, the functions are by definition said to be *linearly dependent;* if no such relation exists, they are said to be *linearly independent.* Show that the nonvanishing of the Wronskian is a criterion for linear independence.

2. If the general solution $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ is the same for

$$X_0y^{(n)} + X_1y^{(n-1)} + \cdots + X_ny = 0 \quad \text{and} \quad P_0y^{(n)} + P_1y^{(n-1)} + \cdots + P_ny = 0,$$

two linear equations of the nth order, show that y satisfies the equation

$$(X_1P_0 - X_0P_1)y^{(n-1)} + \cdots + (X_nP_0 - X_0P_n)y = 0$$

of the $(n-1)$st order; and hence infer, from the fact that y contains n arbitrary constants corresponding to n arbitrary initial conditions, the important theorem: If two linear equations of the nth order have the same general solution, the corresponding coefficients are proportional.

3. If $y_1, y_2, \cdots, y_n$ are n independent solutions of an equation of the nth order, show that the equation may be taken in the form $W(y_1, y_2, \cdots, y_n, y) = 0$.

4. Show that if, in any reduced equation, $X_{n-1}' + xX_n = 0$ identically, then x is a solution. Find the condition that x^m be a solution; also that e^{mx} be a solution.

5. Find by inspection one or more independent solutions and integrate:

(α) $(1+x^2)y'' - 2xy' + 2y = 0$, ($\beta$) $xy'' + (1-x)y' - y = 0$,
(γ) $(ax - bx^2)y'' - ay' + 2by = 0$, ($\delta$) $\frac{1}{2}y'' + xy' - (x+2)y = 0$,
(ϵ) $\left(\log x + \frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{x}\right)y''' + \left(\log x + \frac{1}{x^4} + \frac{1}{x^3} - \frac{1}{x^2}\right)y'' + \left(\frac{1}{x^2} - \frac{1}{x}\right)(y' - xy) = 0$,
(ζ) $y^{\text{iv}} - xy''' + xy' - y = 0$, ($\eta$) $(4x^2 - x + 1)y''' + 8x^2y'' - 4xy' - 8y = 0$.

6. If y_1 is a known solution of the equation $y'' + Py' + Qy = R$ of the second order, show that the general solution may be written as

$$y = C_1y_1 + C_2y_1\int e^{-\int Pdx}\frac{dx}{y_1^2} + y_1\int\frac{1}{y_1^2}e^{-\int Pdx}\int y_1e^{\int Pdx}R\,(dx)^2.$$

7. Integrate:

(α) $xy'' - (2x+1)y' + (x+1)y = x^2 - x - 1$,
(β) $y'' - x^2y' + xy = x$, (γ) $xy'' + (1-x)y' - y = e^x$,
(δ) $y'' - xy' + (x-1)y = R$, (ϵ) $y''\sin^2 x + y'\sin x\cos x - y = x - \sin x$.

8. After writing down the integral of the reduced equation by inspection, apply the method of the variation of constants to these equations:

(α) $(D^2+1)y = \tan x$, (β) $(D^2+1)y = \sec^2 x$, (γ) $(D-1)^2y = e^x(1-x)^{-2}$,
(δ) $(1-x)y'' + xy' - y = (1-x)^2$, ($\epsilon$) $(1 - 2x + x^2)(y''' - 1) - x^2y'' + 2xy' - y = 1$.

9. Integrate the following equations of the second order:

(α) $4x^2y'' + 4x^3y' + (x^2+1)^2y = 0$, ($\beta$) $y'' - 2y'\tan x - (a^2+1)y = 0$,
(γ) $xy'' + 2y' - xy = 2e^x$, (δ) $y''\sin x + 2y'\cos x + 3y\sin x = e^x$,
(ϵ) $y'' + y'\tan x + y\cos^2 x = 0$, ($\zeta$) $(1-x^2)y'' - xy' + 4y = 0$,
(η) $y'' + (2e^x - 1)y' + e^{2x}y = e^{4x}$, ($\theta$) $x^6y'' + 3x^5y' + y = x^{-2}$.

10. Show that if $X_0y'' + X_1y' + X_2y = R$ may be written in factors as

$$(X_0D^2 + X_1D + X_2)y = (p_1D + q_1)(p_2D + q_2)y = R,$$

where the factors are not commutative inasmuch as the differentiation in one factor is applied to the variable coefficients of the succeeding factor as well as to D, then the solution is obtainable in terms of quadratures. Show that

$$q_1p_2 + p_1p_2' + p_1q_2 = X_1 \quad\text{and}\quad q_1q_2 + p_1q_2' = X_2.$$

In this manner integrate the following equations, choosing p_1 and p_2 as factors of X_0 and determining q_1 and q_2 by inspection or by assuming them in some form and applying the method of undetermined coefficients:

(α) $xy'' + (1-x)y' - y = e^x$, (β) $3x^2y'' + (2 - 6x^2)y' - 4 = 0$,
(γ) $3x^2y'' + (2 + 6x - 6x^2)y' - 4y = 0$, ($\delta$) $(x^2-1)y'' - (3x+1)y' - x(x-1)y = 0$,
(ϵ) $axy'' + (3a + bx)y' + 3by = 0$, ($\zeta$) $xy'' - 2x(1+x)y' + 2(1+x)y = x^3$.

11. Integrate these equations in any manner:

(α) $y'' - \frac{1}{\sqrt{x}}y' + \frac{x + \sqrt{x} - 8}{4x^2}y = 0$, ($\beta$) $y'' - \frac{2}{x}y' + \left(a^2 + \frac{2}{x^2}\right)y = 0$,

(γ) $y'' + y' \tan x + y \cos^2 x = 0$, ($\delta$) $y'' - 2\left(n - \frac{a}{x}\right) y' + \left(n^2 - 2\frac{na}{x}\right) y = e^{nx}$,

(ϵ) $(1 - x^2) y'' - xy' - c^2 y = 0$, ($\zeta$) $(a^2 - x^2) y'' - 8xy' - 12y = 0$,

(η) $y'' + \frac{1}{x^2 \log x} y = e^x\left(\frac{2}{x} + \log x\right)$, ($\theta$) $y'' - \frac{9 - 4x}{3 - x} y' + \frac{6 - 3x}{3 - x} y = 0$,

(ι) $y'' + 2x^{-1} y' - n^2 y = 0$, ($\kappa$) $y'' - 4xy' + (4x^2 - 3) y = e^{x^2}$,

(λ) $y'' + 2ny' \cot nx + (m^2 - n^2) y = 0$, ($\mu$) $y'' + 2(x^{-1} + Bx^{-2}) y' + Ax^{-4} y = 0$.

12. If y_1 and y_2 are solutions of $y'' + Py' + R = 0$, show by eliminating Q and integrating that

$$y_1 y_2' - y_2 y_1' = Ce^{-\int P dx}.$$

What if $C = 0$? If $C \neq 0$, note that y_1 and y_1' cannot vanish together; and if $y_1(a) = y_1(b) = 0$, use the relation $(y_2 y_1')_a : (y_2 y_1')_b = k > 0$ to show that as y_{1a}' and y_{1b}' have opposite signs, y_{2a} and y_{2b} have opposite signs and hence $y_2(\xi) = 0$ where $a < \xi < b$. Hence the theorem: Between any two roots of a solution of an equation of the second order there is one root of every solution independent of the given solution. What conditions of continuity for y and y' are tacitly assumed here?

107. The cylinder functions. Suppose that $C_n(x)$ is a function of x which is different for different values of n and which satisfies the two equations

$$C_{n-1}(x) - C_{n+1}(x) = 2\frac{d}{dx} C_n(x), \quad C_{n-1}(x) + C_{n+1}(x) = \frac{2n}{x} C_n(x). \tag{29}$$

Such a function is called a *cylinder function* and the index n is called the *order* of the function and may have any real value. The two equations are supposed to hold for all values of n and for all values of x. They do not completely determine the functions but from them follow the chief rules of operation with the functions. For instance, by addition and subtraction,

$$C_n'(x) = C_{n-1}(x) - \frac{n}{x} C_n(x) = \frac{n}{x} C_n(x) - C_{n+1}(x). \tag{30}$$

Other relations which are easily deduced are

$$D_x[x^n C_n(ax)] = ax^n C_{n-1}(ax), \quad D_x[x^{-n} C_n(ax)] = -ax^{-n} C_{n+1}(x), \tag{31}$$

$$D_x\left[x^{\frac{n}{2}} C_n(\sqrt{ax})\right] = \tfrac{1}{2}\sqrt{a}x^{\frac{n-1}{2}} C_{n-1}(\sqrt{ax}), \tag{32}$$

$$C_0'(x) = -C_1(x), \qquad C_{-n}(x) = (-1)^n C_n(x), \qquad n \text{ integral}, \tag{33}$$

$$C_n(x) K_n'(x) - C_n'(x) K_n(x) = C_{n+1}(x) K_n(x) - C_n(x) K_{n+1}(x) = \frac{A}{x}, \tag{34}$$

where C and K denote any two cylinder functions.

The proof of these relations is simple, but will be given to show the use of (29). In the first case differentiate directly and substitute from (29).

$$D_x[x^n C_n(ax)] = x^n\left[aD_{ax} C_n(ax) + \frac{n}{x} C_n(ax)\right]$$

$$= x^n\left[aC_{n-1}(ax) - a\frac{n}{ax} C_n(ax) + \frac{n}{x} C_n(ax)\right].$$

The second of (31) is proved similarly. For (32), differentiate.

$$D_x[x^{\frac{n}{2}}C_n(\sqrt{\alpha x})] = \frac{1}{2}nx^{\frac{n}{2}-1}C_n(\sqrt{\alpha x}) + x^{\frac{n}{2}}\frac{1}{2}\sqrt{\frac{\alpha}{x}}D_{\sqrt{\alpha x}}C_n(\sqrt{\alpha x})$$

$$= \frac{1}{2}\sqrt{\alpha x}^{\frac{n-1}{2}}\left[\frac{n}{\sqrt{\alpha x}}C_n(\sqrt{\alpha x}) + C_{n-1}(\sqrt{\alpha x}) - \frac{n}{\sqrt{\alpha x}}C_n(\sqrt{\alpha x})\right].$$

Next (33) is obtained 1° by substituting 0 for n in both equations (29).

$$C_{-1}(x) - C_1(x) = 2\,C_0'(x), \quad C_{-1}(x) + C_1(x) = 0, \quad \text{hence} \quad C_0'(x) = -\,C_1(x);$$

and 2° by substituting successive values for n in the second of (29) written in the form $xC_{n-1} + xC_{n+1} = 2\,nC_n$. Then

$$xC_{-1} + xC_1 = 0, \qquad xC_{-2} + xC_0 = -\,2\,C_{-1}, \qquad xC_0 + xC_2 = 2\,C_1,$$
$$xC_{-3} + xC_{-1} = -\,4\,C_{-2}, \qquad xC_1 + xC_3 = 4\,C_2,$$
$$xC_{-4} + xC_{-2} = -\,6\,C_3, \qquad xC_2 + xC_4 = 6\,C_3,$$

and so on. The first gives $C_{-1} = -\,C_1$. Subtract the next two and use $C_{-1} + C_1 = 0$. Then $C_{-2} - C_2 = 0$ or $C_{-2} = (-1)^2C_2$. Add the next two and use the relations already found. Then $C_{-3} + C_3 = 0$ or $C_{-3} = (-1)^3C_3$. Subtract the next two, and so on. For the last of the relations, a very important one, note first that the two expressions become equivalent by virtue of (29); for

$$C_nK_n' - C_n'K_n = \frac{n}{x}C_nK_n - C_nK_{n+1} - \frac{n}{x}C_nK_n + C_{n+1}K_n.$$

Now $$\frac{d}{dx}[x(C_{n+1}K_n - C_nK_{n+1})] = C_{n+1}K_n - C_nK_{n+1} + xK_n\left(C_n - \frac{n+1}{x}C_{n+1}\right)$$
$$+\,xC_{n+1}\left(\frac{n}{x}K_n - K_{n+1}\right) - xK_{n+1}\left(\frac{n}{x}C_n - C_{n+1}\right)$$
$$-\,xC_n\left(K_n - \frac{n+1}{x}K_{n+1}\right) = 0.$$

Hence $x(C_{n+1}K_n - C_nK_{n+1}) = \text{const.} = A$, and the relation is proved.

The cylinder functions of a given order n satisfy a linear differential equation of the second order. This may be obtained by differentiating the first of (29) and combining with (30).

$$2\,C_n'' = C_{n-1}' - C_{n+1}' = \frac{n-1}{x}C_{n-1} - 2\,C_n + \frac{n+1}{x}C_{n+1}$$
$$= \frac{n}{x}(C_{n-1} + C_{n+1}) - \frac{1}{x}(C_{n-1} - C_{n+1}) - 2\,C_n.$$

Hence $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0, \qquad y = C_n(x). \tag{35}$$

This equation is known as *Bessel's equation;* the functions $C_n(x)$, which have been called cylinder functions, are often called *Bessel's functions.* From the equation it follows that any three functions of the same order n are connected by a linear relation and there are only two independent functions of any given order.

By a change of the independent variable, the Bessel equation may take on several other forms. The easiest way to find them is to operate directly with the relations (31), (32). Thus

$$D_x[x^{-n}C_n(x)] = -x^{-n}C_{n+1} = -x\cdot x^{-n-1}C_{n+1},$$
$$D_x^2[x^{-n}C_n(x)] = -x^{-n-1}C_{n+1} + x\cdot x^{-n-1}C_{n+2}$$
$$= -x^{-n-1}C_{n+1} + 2(n+1)x^{-n-1}C_{n+1} - x^{-n}C_n.$$

Hence $$\frac{d^2y}{dx^2} + \frac{(1+2n)}{x}\frac{dy}{dx} + y = 0, \qquad y = x^{-n}C_n(x). \tag{36}$$

Again $$\frac{d^2y}{dx^2} + \frac{(1-2n)}{x}\frac{dy}{dx} + y = 0, \qquad y = x^{n}C_n(x). \tag{37}$$

Also $$xy'' + (1+n)y' + y = 0, \qquad y = x^{-\frac{n}{2}}C_n(2\sqrt{x}). \tag{38}$$

And $$xy'' + (1-n)y' + y = 0, \qquad y = x^{\frac{n}{2}}C_n(2\sqrt{x}). \tag{39}$$

In all these differential equations it is well to restrict x to positive values inasmuch as, if n is not specialized, the powers of x, as x^n, x^{-n}, $x^{\frac{n}{2}}$, $x^{-\frac{n}{2}}$, are not always real.

108. The fact that n occurs only squared in (35) shows that both $C_n(x)$ and $C_{-n}(x)$ are solutions, so that if these functions are independent, the complete solution is $y = aC_n + bC_{-n}$. In like manner the equations (36), (37) form a pair which differ only in the sign of n. Hence if H_n and H_{-n} denote particular integrals of the first and second respectively, the complete integrals are respectively

$$y = aH_n + bH_{-n}x^{-2n} \quad \text{and} \quad y = aH_{-n} + bH_nx^{2n};$$

and similarly the respective integrals of (38), (39) are

$$y = aI_n + bI_{-n}x^{-n} \quad \text{and} \quad y = aI_{-n} + bI_nx^{n},$$

where I_n and I_{-n} denote particular integrals of these two equations. It should be noted that these forms are the complete solutions only when the two integrals are independent. Note that

$$I_n(x) = x^{-\frac{1}{2}n}C_n(2\sqrt{x}), \qquad C_n(x) = (\tfrac{1}{2}x)^nI_n(\tfrac{1}{4}x^2). \tag{40}$$

As it has been seen that $C_n = (-1)^nC_{-n}$ when n is integral, it follows that in this case the above forms do not give the complete solution.

A particular solution of (38) may readily be obtained in series by the method of undetermined coefficients (§ 88). It is

$$I_n(x) = \sum_0^\infty a_ix^i, \qquad a_i = \frac{(-1)^i}{i!\,(n+1)(n+2)\cdots(n+i)}, \tag{41}$$

as is derived below. It should be noted that I_{-n} formed by changing the sign of n is meaningless when n is an integer, for the reason that from a certain point on, the coefficients a_i have zeros in the denominator. The determination of a series for the second independent solution when n is integral will be omitted. The solutions of (35), (36) corresponding to $I_n(x)$ are, by (40) and (41),

$$J_n(x) = \frac{x^n}{2^n} \sum_0^\infty \frac{(-1)^i x^{2i}}{2^{2i} i!\,(n+i)!} = \frac{x^n}{2^n n!} I_n(\tfrac{1}{4}x^2), \tag{42}$$

$$x^{-n} J_n(x) = \frac{1}{2^n n!} I_n(\tfrac{1}{4}x^2), \tag{42'}$$

where the factor $n!$ has been introduced in the denominator merely to conform to usage.* The chief cylinder function $C_n(x)$ is $J_n(x)$ and it always carries the name of Bessel.

To derive the series for $I_n(x)$ write

$$\begin{array}{r|l}
1 & I_n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} + \cdots, \\
(1+n) & I_n' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (k-1)a_{k-1} x^{k-2} + \cdots, \\
x & I_n'' = \phantom{a_1 +{}} 2a_2 + 3\cdot 2a_3 x + \cdots + (k-1)(k-2)a_{k-1} x^{k-3} + \cdots, \\
\hline
0 & = [a_0 + a_1(n+1)] + x[a_1 + a_2 2(n+2)] + x^2[a_2 + a_3 3(n+3)] \\
 & \quad + \cdots + x^{k-1}[a_{k-1} + a_k k(n+k)] + \cdots.
\end{array}$$

Hence $a_0 + a_1(n+1) = 0, \quad a_1 + a_2 2(n+2) = 0, \cdots, \quad a_{k-1} + a_k k(n+k) = 0,$

or
$$a_1 = -\frac{a_0}{n+1}, \qquad a_2 = \frac{-a_1}{2(n+2)} = \frac{a_0}{2!\,(n+1)(n+2)}, \cdots,$$

$$a_k = \frac{(-1)^k a_0}{k!\,(n+1)\cdots(n+k)}.$$

If now the choice $a_0 = 1$ is made, the series for $I_n(x)$ is as given in (41).

The famous differential equation of the first order

$$xy' - ay + by^2 = cx^n, \tag{43}$$

known as *Riccati's equation*, may be integrated in terms of cylinder functions. Note that if $n = 0$ or $c = 0$, the variables are separable; and if $b = 0$, the equation is linear. As these cases are immediately integrable, assume $bcn \neq 0$. By a suitable change of variable, the equation takes the form

$$\xi \frac{d^2\eta}{d\xi^2} + \left(1 - \frac{a}{n}\right)\frac{d\eta}{d\xi} - bc\eta = 0, \qquad \xi = \frac{1}{n^2}x^n, \qquad y = \frac{n}{b}\frac{d\eta}{d\xi}\frac{\xi}{\eta}. \tag{43'}$$

A comparison of this with (39) shows that the solution is

$$\eta = AI_{-\frac{a}{n}}(-bc\xi) + BI_{\frac{a}{n}}(-bc\xi)\cdot(-bc\xi)^{\frac{a}{n}},$$

which in terms of Bessel functions J becomes, by (40),

$$\eta = \xi^{\frac{a}{2n}}\left[AJ_{\frac{a}{n}}(2\sqrt{-bc\xi}) + BJ_{-\frac{a}{n}}(2\sqrt{-bc\xi})\right].$$

* If n is not integral, both $n!$ and $(n+i)!$ must be replaced (§ 147) by $\Gamma(n+1)$ and $\Gamma(n+i+1)$.

The value of y may be found by substitution and use of (29).

$$y=\sqrt{-\frac{c}{b}}x^{\frac{n}{2}}\frac{J_{\frac{a}{n}-1}(2x^{\frac{n}{2}}\sqrt{-bc}/n)-AJ_{1-\frac{a}{n}}(2x^{\frac{n}{2}}\sqrt{-bc}/n)}{J_{\frac{a}{n}}(2x^{\frac{n}{2}}\sqrt{-bc}/n)+AJ_{-\frac{a}{n}}(2x^{\frac{n}{2}}\sqrt{-bc}/n)}, \tag{44}$$

where A denotes the one arbitrary constant of integration.

It is noteworthy that the cylinder functions are sometimes expressible in terms of trigonometric functions. For when $n=\frac{1}{2}$ the equation (35) has the integrals

$$y=A\sin x+B\cos x \quad \text{and} \quad y=x^{\frac{1}{2}}[AC_{\frac{1}{2}}(x)+BC_{-\frac{1}{2}}(x)].$$

Hence it is permissible to write the relations

$$x^{\frac{1}{2}}C_{\frac{1}{2}}(x)=\sin x, \qquad x^{\frac{1}{2}}C_{-\frac{1}{2}}(x)=\cos x, \tag{45}$$

where C is a suitably chosen cylinder function of order $\frac{1}{2}$. From these equations by application of (29) the cylinder functions of order $p+\frac{1}{2}$, where p is any integer, may be found.

Now if Riccati's equation is such that b and c have opposite signs and a/n is of the form $p+\frac{1}{2}$, the integral (44) can be expressed in terms of trigonometric functions by using the values of the functions $C_{p+\frac{1}{2}}$ just found in place of the J's. Moreover if b and c have the same sign, the trigonometric solution will still hold formally and may be converted into exponential or hyperbolic form. Thus Riccati's equation is integrable in terms of the elementary functions when $a/n=p+\frac{1}{2}$ no matter what the sign of bc is.

EXERCISES

1. Prove the following relations:

(α) $4C_n''=C_{n-2}-2C_n+C_{n+2}$, ($\beta$) $xC_n=2(n+1)C_{n+1}-xC_{n+2}$,
(γ) $2^3C_n'''=C_{n-3}-3C_{n-1}+3C_{n+1}-C_{n+3}$, generalize,
(δ) $xC_n=2(n+1)C_{n+1}-2(n+3)C_{n+3}+2(n+5)C_{n+5}-xC_{n+6}$.

2. Study the functions defined by the pair of relations

$$F_{n-1}(x)+F_{n+1}(x)=2\frac{d}{dx}F_n(x), \qquad F_{n-1}(x)-F_{n+1}(x)=\frac{2}{x}F_n(x)$$

especially to find results analogous to (30)–(35).

3. Use Ex. 12, p. 247, to obtain (34) and the corresponding relation in Ex. 2.

4. Show that the solution of (38) is $y=AI_n\int\frac{dx}{x^{n+1}I_n^2}+BI_n$.

5. Write out five terms in the expansions of $I_0, I_1, I_{-\frac{1}{2}}, J_0, J_1$.

6. Show from the expansion (42) that $\frac{1}{2}!\sqrt{\frac{2}{x}}J_{\frac{1}{2}}(x)=\frac{1}{x}\sin x$.

7. From (45), (29) obtain the following:

$$x^{\frac{1}{2}}C_{\frac{3}{2}}(x)=\frac{\sin x}{x}-\cos x, \qquad x^{\frac{1}{2}}C_{\frac{5}{2}}(x)=\left(\frac{3}{x^2}-1\right)\sin x-\frac{3}{x}\cos x,$$

$$x^{\frac{1}{2}}C_{-\frac{3}{2}}(x)=-\sin x-\frac{\cos x}{x}, \qquad x^{\frac{1}{2}}C_{-\frac{5}{2}}(x)=\frac{3}{x}\sin x+\left(\frac{3}{x^2}-1\right)\cos x.$$

8. Prove by integration by parts: $\int \frac{J_2 dx}{x^3} dx = \frac{J_3}{x^3} + 6\frac{J_4}{x^4} + 6 \cdot 8 \int \frac{J_5 dx}{x^5}$.

9. Suppose $C_n(x)$ and $K_n(x)$ so chosen that $A = 1$ in (34). Show that

$$y = AC_n(x) + BK_n(x) + L\left[K_n(x)\int \frac{C_n(x)}{x^3}dx - C_n(x)\int \frac{K_n(x)}{x^3}dx\right]$$

is the integral of the differential equation $x^2y'' + xy' + (x^2 - n^2)y = Lx^{-2}$.

10. Note that the solution of Riccati's equation has the form

$$y = \frac{f(x) + Ag(x)}{F(x) + AG(x)}, \quad \text{and show that} \quad \frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x)$$

will be the form of the equation which has such an expression for its integral.

11. Integrate these equations in terms of cylinder functions and reduce the results whenever possible by means of Ex. 7:

(α) $xy' - 5y + y^2 + x^2 = 0$, (β) $xy' - 3y + y^2 = x^2$,
(γ) $y'' + ye^{2x} = 0$, (δ) $x^2y'' + nxy' + (b + cx^{2m})y = 0$.

12. Identify the functions of Ex. 2 with the cylinder functions of ix.

13. Let $(x^2 - 1)P'_n = (n+1)(P_{n+1} - xP_n)$, $P'_{n+1} = xP'_n + (n+1)P_n$ (46)

be taken as defining the *Legendre functions* $P_n(x)$ of order n. Prove

(α) $(x^2 - 1)P'_n = n(xP_n - P_{n-1})$, (β) $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$,
(γ) $(2n+1)P_n = P'_{n+1} - P'_{n-1}$, (δ) $(1 - x^2)P''_n - 2xP'_n + n(n+1)P_n = 0$.

14. Show that $P_nQ'_n - P'_nQ_n = \frac{A}{x^2 - 1}$ and $P_nQ_{n+1} - P_{n+1}Q_n = \frac{A}{n+1}$,

where P and Q are any two Legendre functions. Express the general solution of the differential equation of Ex. 13 (δ) analogously to Ex. 4.

15. Let $u = x^2 - 1$ and let D denote differentiation by x. Show

$$D^{n+1}u^{n+1} = D^{n+1}(uu^n) = uD^{n+1}u^n + 2(n+1)xD^nu^n + n(n+1)D^{n-1}u^n,$$
$$D^{n+1}u^{n+1} = D^nDu^{n+1} = 2(n+1)D^n(xu^n) = 2(n+1)xD^nu^n + 2n(n+1)D^{n-1}u^n.$$

Hence show that the derivative of the second equation and the eliminant of $D^{n-1}u^n$ between the two equations give two equations which reduce to (46) if

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2 - 1)^n. \qquad \begin{cases}\text{When } n \text{ is integral these are} \\ \textit{Legendre's polynomials.}\end{cases}$$

16. Determine the solutions of Ex. 13 (δ) in series for the initial conditions

(α) $P_n(0) = 1$, $P'_n(0) = 0$, (β) $P_n(0) = 0$, $P'_n(0) = 1$.

17. Take $P_0 = 1$ and $P_1 = x$. Show that these are solutions of (46) and compute P_2, P_3, P_4 from Ex. 13 (β). If $x = \cos\theta$, show

$P_2 = \frac{3}{4}\cos 2\theta + \frac{1}{4}$, $P_3 = \frac{5}{8}\cos 3\theta + \frac{3}{8}\cos\theta$, $P_4 = \frac{35}{64}\cos 4\theta + \frac{20}{64}\cos 2\theta + \frac{9}{64}$.

18. Write Ex. 13 (δ) as $\frac{d}{dx}[(1 - x^2)P'_n] + n(n+1)P_n = 0$ and show

$$[m(m+1) - n(n+1)]\int_{-1}^{+1} P_nP_m dx = \int_{-1}^{+1}\left[P_m\frac{d(1 - x^2)P'_n}{dx} - P_n\frac{d(1 - x^2)P'_m}{dx}\right]dx.$$

Integrate by parts, assume the functions and their derivatives are finite, and show

$$\int_{-1}^{+1} P_n P_m dx = 0, \quad \text{if} \quad n \neq m.$$

19. By successive integration by parts and by reduction formulas show

$$\int_{-1}^{+1} P_n^2 dx = \frac{1}{2^{2n}(n\,!)^2}\int_{-1}^{+1}\frac{d^n(x^2-1)^n}{dx^n}\cdot\frac{d^n(x^2-1)^n}{dx^n}dx = \frac{(-1)^n}{2^n\cdot n\,!}\int_{-1}^{+1}(x^2-1)^n dx$$

and $$\int_{-1}^{+1} P_n^2 dx = \frac{2}{2n+1}, \qquad n \text{ integral}.$$

20. Show $$\int_{-1}^{+1} x^m P_n dx = \int_{-1}^{+1} x^m \frac{d^n(x^2-1)^n}{dx^n} = 0, \qquad \text{if } m < n.$$

Determine the value of the integral when $m = n$. Cannot the results of Exs. 18, 19 for m and n integral be obtained simply from these results?

21. Consider (38) and its solution $I_0 = 1 - x + \frac{x^2}{2\,!^2} - \frac{x^3}{3\,!^2} + \frac{x^4}{4\,!^2} - \cdots$ when $n = 0$. Assume a solution of the form $y = I_0 v + w$ so that

$$x\frac{d^2w}{dx^2} + \frac{dw}{dx} + w + 2x\frac{dI_0}{dx}\frac{dv}{dx} = 0, \quad \text{if} \quad x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 0,$$

is the equation for w if v satisfies the equation $xv'' + v' = 0$. Show

$$v = A + B\log x, \qquad xw'' + w' + w = 2B - \frac{2Bx}{2\,!} + \frac{2Bx^2}{2\,!\,3\,!} - \frac{2Bx^3}{3\,!\,4\,!} + \cdots.$$

By assuming $w = a_1 x + a_2 x^2 + \cdots$, determine the a's and hence obtain

$$w = 2B\left[x - \frac{x^2}{2\,!^2}\left(1 + \frac{1}{2}\right) + \frac{x^3}{3\,!^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) - \frac{x^4}{4\,!^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \cdots\right];$$

and $(A + B\log x)I_0 + w$ is then the complete solution containing two constants. As AI_0 is one solution, $B\log x \cdot I_0 + w$ is another. From this second solution for $n = 0$, the second solution for any integral value of n may be obtained by differentiation; the work, however, is long and the result is somewhat complicated.