Advanced Studies in Pure Mathematics 72, 2017
Geometry, Dynamics, and Foliations 2013
pp. 145–180

Circle diffeomorphisms, rigidity of symmetric conjugation and affine foliation of the universal Teichmüller space

Katsuhiko Matsuzaki

Abstract.

The little Teichmüller space of symmetric homeomorphisms of the circle defines a Banach foliated structure of the universal Teichmüller space. First we consider rigidity of Möbius representations given by symmetric conjugation and failure of the fixed point property for isometric group action on the little Teichmüller space. This space includes the Teichmüller space of circle diffeomorphisms with Hölder continuous derivatives. Then we characterize these diffeomorphisms by Beltrami coefficients of quasiconformal extensions and Schwarzian derivatives of their Bers embeddings. This is used for proving certain rigidity of representations by symmetric conjugation in the group of circle diffeomorphisms. We also consider Teichmüller spaces of integrable symmetric homeomorphisms, which induce another Banach foliated structure and the generalized Weil–Petersson metric on the universal Teichmüller space. As an application, we investigate the fixed point property for isometric group action on these spaces and give a condition for a group of circle diffeomorphisms with Hölder continuous derivatives to be conjugate to a Möbius group in the same class.

§1. Introduction

In this article, we will explain Teichmüller spaces of circle diffeomorphisms with Hölder continuous derivatives and certain rigidity for the representation of a Möbius group in the group of such diffeomorphisms.

Received June 30, 2014.
Revised September 29, 2014.
2010 *Mathematics Subject Classification.* Primary 30F60, 58B20, 58C30; Secondary 37E10, 58D05.
Key words and phrases. quasisymmetric, quasiconformal, Beltrami coefficients, Schwarzian derivative, conformally natural extension, Weil–Petersson metric, isometric action, fixed point property, uniformly convex.
This work was supported by JSPS KAKENHI 24654035.

The Teichmüller space of a Riemann surface is the deformation space of complex structures on it and it can be regarded as a subspace of the universal Teichmüller space that is fixed pointwise by the Fuchsian group Γ uniformizing the Riemann surface. The universal Teichmüller space $T = \text{Möb}(\mathbb{S})\backslash\text{QS}$ is given by the group QS of quasisymmetric self-homeomorphisms of the unit circle $\mathbb{S}$ modulo the group $\text{Möb}(\mathbb{S})$ of Möbius transformations. Quasisymmetric self-homeomorphisms are the boundary extension of quasiconformal homeomorphisms of the unit disk $\mathbb{D}$ and they equip necessary regularity for the deformation space to be parametrized complex analytically. Also, T is provided with the Teichmüller metric defined by dilatations of these mappings.

A family of circle homeomorphisms of higher regularity can be also considered as a subspace of the universal Teichmüller space. In this article, we deal with symmetric homeomorphisms Sym, circle diffeomorphisms $\text{Diff}^{1+\alpha}(\mathbb{S})$ with α-Hölder continuous derivatives, and p-integrable symmetric homeomorphisms Sym^p. A parameter space of C^∞ diffeomorphisms was already studied by Nag and Verjovsky [36] among others in connection with the Teichmüller theory but the class of diffeomorphisms with Hölder continuous derivatives fits well to the framework of complex analytic theory of Teichmüller spaces. One of the purposes of this article is to show the feature of the Teichmüller space $T_0^\alpha = \text{Möb}(\mathbb{S})\backslash\text{Diff}^{1+\alpha}(\mathbb{S})$.

The group QS of quasisymmetric homeomorphisms of $\mathbb{S}$ is the automorphism group of the universal Teichmüller space T. This space possesses a property that if a subgroup $G < \text{QS}$ acts on T with a bounded orbit then G has a fixed point in it (Markovic [29]). In general, we say that a complete metric space X has the fixed point property if every subgroup of its automorphism group with a bounded orbit always has a fixed point in X. The fixed point property of T is the basis of considering the generalization of the Nielsen realization problem and the conjugation problem to Möbius groups. The arguments in this article are going along questions whether a subspace of T has the fixed point property or whether we can endow another metric to the subspace so that it has the fixed point property.

The universal Teichmüller space T is embedded into the Banach space $B(\mathbb{D}^*)$ of hyperbolically bounded holomorphic functions on the exterior $\mathbb{D}^*$ of the unit disk in the Riemann sphere $\widehat{\mathbb{C}}$. Through this Bers embedding $\beta\colon T \to B(\mathbb{D}^*)$, the action of $\text{Möb}(\mathbb{S}) \subset \text{QS}$ on T is realized as the group of isometric linear transformations of $B(\mathbb{D}^*)$. This linearization of Teichmüller space produces powerful methods in our arguments.

A basic subspace is the little Teichmüller space $T_0 = \text{Möb}(\mathbb{S})\backslash\text{Sym}$ of symmetric homeomorphisms. This was studied by Gardiner and Sullivan [21] and now it is of importance in the theory of asymptotic Teichmüller spaces of Riemann surfaces. In our arguments, T_0 offers a fundamental structure at the following points:

(1) The little subspace T_0 is embedded into the little subspace $B_0(\mathbb{D}^*)$ of $B(\mathbb{D}^*)$ under the Bers embedding β. Moreover, the affine foliation of $\beta(T)$ by $B_0(\mathbb{D}^*)$ corresponds to the coset decomposition of T by T_0 under β. In particular, the quotient space $AT = T/T_0$ has a complex structure modeled on the quotient Banach space $B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$.

(2) The little subspace T_0 does not have the fixed point property. More strongly, quasisymmetric conjugations of a non-rigid Fuchsian group $\Gamma < \text{Möb}(\mathbb{S})$ into Sym defines an infinite dimensional representation space in AT even if the Teichmüller space $T(\Gamma)$ for Γ is finite dimensional.

(3) There is certain rigidity for the conjugation of a Möbius group Γ by a symmetric homeomorphism. More precisely, every symmetric conjugation of Γ both into $\text{Möb}(\mathbb{S})$ and into $\text{Diff}^{1+\alpha}(\mathbb{S})$ is trivial, namely, it is actually an inner automorphism.

The Teichmüller space $T^p = \text{Möb}(\mathbb{S})\backslash\text{Sym}^p$ of p-integrable symmetric homeomorphisms is contained in T_0. The corresponding subspace $A^p(\mathbb{D}^*) \subset B_0(\mathbb{D}^*)$ under the Bers embedding is the Banach space of hyperbolically p-integrable holomorphic functions on $\mathbb{D}^*$. Similarly to the case of T_0 as in (1), the affine foliation of $T \cong \beta(T)$ by $A^p(\mathbb{D}^*)$ is compatible with the coset decomposition T/T^p under the Bers embedding β.

In the case where $p = 2$, Cui [11] introduced the Weil–Petersson metric in T^2 by the inner product of the Hilbert space $A^2(\mathbb{D}^*)$. Later, Takhtajan and Teo [41] extended it to each leaf of the affine foliation of T and investigated its curvature properties. In particular, we see that T^2 has the fixed point property with respect to the Weil–Petersson metric. In the general case $p \geq 2$, the generalized Weil–Petersson metric can be similarly defined in each leaf of the affine foliation of T by $A^p(\mathbb{D}^*)$. However, regarding its fixed point property, we only propose a problem on uniform convexity of this metric on T^p.

In the case where $\alpha > 1/2$, we have $\text{Diff}^{1+\alpha}(\mathbb{S}) < \text{Sym}^p$ and hence T_0^α is contained in T^2, which has the fixed point property. Combining this with such a rigidity that any symmetric conjugation of a non-elementary Möbius group Γ into $\text{Diff}^{1+\alpha}(\mathbb{S})$ is trivial, we obtain a condition for a subgroup of $\text{Diff}^{1+\alpha}(\mathbb{S})$ to be isomorphic to some Möbius group under an inner automorphism in terms of certain quantities related to the metric on T^2.

This article is based on the author's talks in the conference "Rigidity School" held at the University of Tokyo on January 6–10, 2014.

Acknowledgements. The author would like to thank the referee for his/her careful reading of the manuscript and valuable suggestions, in particular, for Sections 3.2 and 3.4.

§2. The universal Teichmüller space

We begin with defining the universal Teichmüller space T, which can be represented by using the following concepts:

- Quasisymmetric self-homeomorphisms of $\mathbb{S}$;
- Quasiconformal self-homeomorphisms of $\mathbb{D}$;
- Complex projective structures (Schwarzian derivatives) on $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$.

These enable us to provide metric and complex structure for T. We can consult the monographs by Lehto [27] and Nag [35] for necessary arguments in this section. Then we consider the group of automorphisms of T and the fixed point property for its subgroups.

2.1. Quasisymmetric homeomorphisms

There are several equivalent definitions for quasisymmetry of a self-homeomorphism of $\mathbb{S}$; one of them is the following.

Definition 2.1. *An orientation-preserving homeomorphism $g\colon \mathbb{S} \to \mathbb{S}$ is called* quasisymmetric *if there is a constant $M \geq 1$ such that*

$$[g(z_1), g(z_2), g(z_3), g(z_4)]_* \leq M$$

for every positively ordered quadruple (z_1, z_2, z_3, z_4) on $\mathbb{S} \subset \mathbb{C}$ with $[z_1, z_2, z_3, z_4]_* = 1$. Here, for the usual cross ratio

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3},$$

the alternative cross ratio as above is defined by

$$[z_1, z_2, z_3, z_4]_* = \frac{[z_2, z_3, z_4, z_1]}{[z_1, z_2, z_3, z_4]}.$$

This takes its value in $(0, \infty)$ for a positively ordered quadruple of distinct points on $\mathbb{S}$.

Now we define the following subsets (actually subgroups) of the group $\mathrm{Homeo}(\mathbb{S})$ of orientation-preserving self-homeomorphisms of $\mathbb{S}$:

$$\mathrm{QS} = \{g \in \mathrm{Homeo}(\mathbb{S}) \mid g\colon \text{ quasisymmetric}\};$$
$$\mathrm{M\ddot{o}b}(\mathbb{S}) = \{g \in \mathrm{Homeo}(\mathbb{S}) \mid g\colon \text{ Möbius transformation}\} \cong \mathrm{PSL}(2, \mathbb{R}).$$

Clearly $\mathrm{M\ddot{o}b}(\mathbb{S}) < \mathrm{QS}$ because $g \in \mathrm{M\ddot{o}b}(\mathbb{S})$ implies $M(g) = 1$, where $M(g)$ is the optimal constant $M \geq 1$ that satisfies the condition in the above definition for $g \in \mathrm{QS}$.

Definition 2.2. *The* universal Teichmüller space T *is defined by the set of cosets*

$$T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS}.$$

2.2. Quasiconformal homeomorphisms

A counterpart of quasisymmetry is quasiconformality. This also has several equivalent definitions but for our purpose we adopt the following.

Definition 2.3. *An orientation-preserving homeomorphism $f : D \to \widehat{\mathbb{C}}$ of a domain $D \subset \widehat{\mathbb{C}}$ is called* quasiconformal *if it has partial derivative in the distribution sense and if there is a constant $k < 1$ such that the* complex dilatation

$$\mu_f(z) = \frac{\bar{\partial} f(z)}{\partial f(z)}$$

satisfies $|\mu_f(z)| \leq k$ (a.e. $z \in D$). The maximal dilatation *of f is defined by*

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

The complex dilatation determines the quasiconformal map. This fact can be stated as the existence and the uniqueness of a solution for the *Beltrami equation*: given $\mu \in L^\infty(D)$ with $\|\mu\|_\infty < 1$, which is called a *Beltrami coefficient* on $D \subset \widehat{\mathbb{C}}$, find a quasiconformal homeomorphism f of D such that $\bar{\partial} f(z) = \mu(z) \partial f(z)$ (a.e. $z \in D$). The solution is described as the following measurable Riemann mapping theorem due to Ahlfors and Bers [2].

Theorem 2.4. *The Beltrami equation $\bar{\partial} f(z) = \mu(z) \partial f(z)$ on D can be solved. The solution is unique up to post-composition of a conformal homeomorphism of the image. Under a certain normalization, the solution $f(z)$ depends holomorphically on μ for each fixed $z \in D$.*

As before, we define the following subsets (actually subgroups) of the group $\mathrm{Homeo}(D)$ of orientation-preserving self-homeomorphisms of $D \subset \widehat{\mathbb{C}}$:

$$\mathrm{QC}(D) = \{f \in \mathrm{Homeo}(D) \mid f\colon \text{ quasiconformal}\};$$
$$\mathrm{Conf}(D) = \{f \in \mathrm{Homeo}(D) \mid f\colon \text{ conformal}\}.$$

Clearly $\mathrm{Conf}(D) < \mathrm{QC}(D)$ because $f \in \mathrm{Conf}(D)$ implies $\mu_f(z) \equiv 0$.

For $D = \mathbb{D}$, we denote the space of Beltrami coefficients on $\mathbb{D}$ by

$$\mathrm{Bel}(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1\},$$

which is the open unit ball of the Banach space $L^\infty(\mathbb{D})$. Then Theorem 2.4 implies that

$$\mathrm{Bel}(\mathbb{D}) \cong \mathrm{Conf}(\mathbb{D})\backslash\mathrm{QC}(\mathbb{D}) = \mathrm{Möb}(\mathbb{D})\backslash\mathrm{QC}(\mathbb{D}),$$

where $\mathrm{Möb}(\mathbb{D})$ is the group of Möbius transformations preserving $\mathbb{D}$.

2.3. Quasiconformal extension

We investigate the relation between QS and $\mathrm{QC}(\mathbb{D})$. A basic fact is that every $f \in \mathrm{QC}(\mathbb{D})$ extends continuously to a self-homeomorphism of $\mathbb{S} = \partial\mathbb{D}$. This defines the boundary extension map (homomorphism)

$$q\colon \mathrm{QC}(\mathbb{D}) \to \mathrm{Homeo}(\mathbb{S}).$$

Then the boundary extension is actually a quasisymmetric automorphism of $\mathbb{S}$. Conversely, every $g \in \mathrm{QS}$ extends continuously to a quasiconformal automorphism of $\mathbb{D}$. Namely, there is a section

$$e\colon \mathrm{QS} \to \mathrm{QC}(\mathbb{D})$$

for q. In particular, $\mathrm{Im}\, q = \mathrm{QS}$. These results as well as the following continuity were shown by Beurling and Ahlfors [7].

Proposition 2.5. *In the above circumstances, $M(q(f))$ for $f \in \mathrm{QC}(\mathbb{D})$ is estimated in terms of $K(f)$ and $M(q(f)) \to 1$ as $K(f) \to 1$; $K(e(g))$ for $g \in \mathrm{QS}$ is estimated in terms of $M(g)$ and $K(e(g)) \to 1$ as $M(g) \to 1$.*

The *Beurling–Ahlfors extension* $e(g) \in \mathrm{QC}(\mathbb{D})$ is given by an explicit formula in the upper half-plane model $\mathbb{H}$ and its boundary $\mathbb{R}$. There is certain advantage of using this quasiconformal extension due to its explicit representation. We will discuss this later in Section 4.

2.4. Conformally natural extension

Douady and Earle [12] introduced later the *conformally natural extension* $e_{\mathrm{DE}}\colon \mathrm{QS} \to \mathrm{QC}(\mathbb{D})$ having similar properties to the Beurling–Ahlfors extension. Moreover, this is compatible with the composition of elements of $\mathrm{Möb}(\mathbb{S})$ and $\mathrm{Möb}(\mathbb{D})$.

The complex average of a probability measure m on $\mathbb{S}$ viewed at $0 \in \mathbb{D}$ is defined by

$$\xi_m(0) = \int_{\mathbb{S}} \zeta \, dm(\zeta).$$

By the homogeneity of $\mathbb{D}$ under $\mathrm{Möb}(\mathbb{D})$, the average of m viewed at an arbitrary point $w \in \mathbb{D}$ is given by

$$\xi_m(w) = (1 - |w|^2) \int_{\mathbb{S}} \frac{\zeta - w}{1 - \bar{w}\zeta} \, dm(\zeta).$$

If m has no point mass, it is known that there is a unique point $w \in \mathbb{D}$ such that $\xi_m(w) = 0$, which is called the *barycenter* of m.

For a probability measure m on $\mathbb{S}$, the family of conformal measures $\{m_z\}_{z\in\mathbb{D}}$ is given by the relation

$$\frac{dm_z}{dm}(\zeta) = \frac{1 - |z|^2}{|\zeta - z|^2}.$$

The correspondence of the barycenter $w(z) \in \mathbb{D}$ of m_z to each $z \in \mathbb{D}$ defines the barycentric map $w_m\colon \mathbb{D} \to \mathbb{D}$.

Consider the push-forward of the normalized Lebesgue measure on $\mathbb{S}$ by $g\colon \mathbb{S} \to \mathbb{S}$; $dm_g = g_*(d\theta/2\pi)$. Then the conformally natural extension $e_{\mathrm{DE}}(g)$ is defined by w_{m_g}. The results by Douady and Earle [12] are as follows.

Theorem 2.6. *If $g \in \mathrm{QS}$, then $e_{\mathrm{DE}}(g) \in \mathrm{QC}(\mathbb{D})$ with $q(e_{\mathrm{DE}}(g)) = g$. This also satisfies $K(e_{\mathrm{DE}}(g)) \to 1$ as $M(g) \to 1$. Moreover, $e_{\mathrm{DE}}(g)$ is a bi-Lipschitz diffeomorphism with respect to the hyperbolic metric on $\mathbb{D}$. For any $g \in \mathrm{QS}$ and $h_1, h_2 \in \mathrm{Möb}(\mathbb{S}) \cong \mathrm{Möb}(\mathbb{D})$, it satisfies the conformal naturality*

$$e_{\mathrm{DE}}(h_1 \circ g \circ h_2) = h_1 \circ e_{\mathrm{DE}}(g) \circ h_2.$$

2.5. The Teichmüller projection

The boundary extension $q\colon \mathrm{QC}(\mathbb{D}) \to \mathrm{QS}$ modulo $\mathrm{Möb}(\mathbb{D}) \cong \mathrm{Möb}(\mathbb{S})$ defines a map

$$\pi\colon \mathrm{Bel}(\mathbb{D}) = \mathrm{Möb}(\mathbb{D})\backslash\mathrm{QC}(\mathbb{D}) \longrightarrow T = \mathrm{Möb}(\mathbb{S})\backslash\mathrm{QS},$$

which is called the *Teichmüller projection.* See the following diagram.

$$\begin{tikzcd}
\mathrm{QC}(\mathbb{D}) \arrow[r, "q"] \arrow[d, "\mathrm{M\ddot{o}b}(\mathbb{D})\backslash"'] & \mathrm{QS} \arrow[d, "\mathrm{M\ddot{o}b}(\mathbb{S})\backslash"'] \\
\mathrm{Bel}(\mathbb{D}) \arrow[r, "\pi"'] & T
\end{tikzcd}$$

We provide $T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{QS}$ with the topology induced by the quasisymmetry constant $M(g)$ for $g \in \mathrm{QS}$. The open ball $\mathrm{Bel}(\mathbb{D}) \subset L^\infty(\mathbb{D})$ is equipped with the norm topology. Then topological properties of the Teichmüller projection can be described as follows. The continuity of π is a consequence of Proposition 2.5.

Proposition 2.7. *The Teichmüller projection* $\pi\colon \mathrm{Bel}(\mathbb{D}) \to T$ *is a continuous open map. In particular, the topology on* T *coincides with the quotient topology of* $\mathrm{Bel}(\mathbb{D})$ *by* π.

The conformally natural extension $e_{\mathrm{DE}}\colon \mathrm{QS} \to \mathrm{QC}(\mathbb{D})$ descends to a section $s_{\mathrm{DE}}\colon T \to \mathrm{Bel}(\mathbb{D})$ for the Teichmüller projection π. Since e_{DE} is continuous, so is s_{DE}. Then we see that T is contractible since so is $\mathrm{Bel}(\mathbb{D})$.

2.6. The Teichmüller metric

The *Teichmüller distance* on T is the quotient distance of that on $\mathrm{Bel}(\mathbb{D})$ defined as follows. The unit ball $\mathrm{Bel}(\mathbb{D})$ of $L^\infty(\mathbb{D})$ has a hyperbolic distance: for any $\mu, \nu \in \mathrm{Bel}(\mathbb{D})$, it is defined by

$$d_{\mathrm{Bel}}(\mu,\nu) = \log \frac{1 + \left\| \frac{\mu-\nu}{1-\bar{\nu}\mu} \right\|_\infty}{1 - \left\| \frac{\mu-\nu}{1-\bar{\nu}\mu} \right\|_\infty}.$$

Its infinitesimal form is a Finsler metric on $\mathrm{Bel}(\mathbb{D})$ with the density $2(1-\|\nu\|_\infty^2)^{-1}$ at $\nu \in \mathrm{Bel}(\mathbb{D})$.

To induce the infinitesimal metric on T, a differentiable structure on T by which π is differentiable is required. In fact, a complex structure modeled on a certain Banach space is provided for T by using the parametrization of complex projective structures on $\mathbb{D}^*$ through Schwarzian derivatives. This will be explained below.

Definition 2.8. *The* Teichmüller metric *on* T *is the quotient metric of the hyperbolic metric on* $\mathrm{Bel}(\mathbb{D})$ *by the Teichmüller projection* $\pi\colon \mathrm{Bel}(\mathbb{D}) \to T$. *The Teichmüller distance induced by this metric is denoted by* d_T.

It is known that the metric space (T, d_T) is complete. We also see later the Finsler structure of the Teichmüller metric.

2.7. Bounded projective structures

A complex projective structure on a two dimensional manifold is a $(\mathrm{PSL}(2,\mathbb{C}),\widehat{\mathbb{C}})$-structure in the language of (G,X)-structures. A projective structure on $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$ is realized by the developing map $f\colon \mathbb{D}^* \to \widehat{\mathbb{C}}$, which is a holomorphic local homeomorphism.

The Schwarzian derivative of the developing map

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

measures the difference from the standard projective structure on $\mathbb{D}^* \subset \widehat{\mathbb{C}}$ and actually it determines the projective structure uniquely.

In the representation of the Teichmüller space, only *bounded projective structures* are involved, for which the developing map $f\colon \mathbb{D}^* \to \widehat{\mathbb{C}}$ satisfies

$$\|S_f\|_\infty = \sup_{z\in\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z)|S_f(z)| < \infty,$$

where $\rho_{\mathbb{D}^*}(z) = 2/(|z|^2-1)$ is the hyperbolic density on $\mathbb{D}^*$. The space of bounded projective structures on $\mathbb{D}^*$ is identified with the Banach space of holomorphic functions

$$B(\mathbb{D}^*) = \left\{\varphi \in \mathrm{Hol}(\mathbb{D}^*) \;\middle|\; \|\varphi\|_\infty = \sup_{z\in\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| < \infty\right\}.$$

2.8. The Bers embedding

For each $\mu \in \mathrm{Bel}(\mathbb{D})$, a projective structure on $\mathbb{D}^*$ is defined as follows: Extend μ to $\mathrm{Bel}(\widehat{\mathbb{C}})$ by setting $\mu(z) \equiv 0$ on $\mathbb{D}^*$. As the unique solution of the Beltrami equation on $\widehat{\mathbb{C}}$ up to post-composition of $\mathrm{Möb}(\widehat{\mathbb{C}}) \cong \mathrm{PSL}(2,\mathbb{C})$, one has a quasiconformal automorphism $f_\mu\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ conformal on $\mathbb{D}^*$. Then $f_\mu|_{\mathbb{D}^*}$ gives the developing map of a projective structure.

The *Bers projection* $\Phi\colon \mathrm{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)$ is defined by $\mu \mapsto S_{f_\mu|_{\mathbb{D}^*}}$; the Nehari–Kraus theorem implies that $\|\Phi(\mu)\|_\infty = \|S_{f_\mu|_{\mathbb{D}^*}}\|_\infty \le 3/2$ for every $\mu \in \mathrm{Bel}(\mathbb{D})$, which shows that this Schwarzian derivative corresponds to a bounded projective structure. Bers showed that $\Phi \circ \pi^{-1}$ is well-defined and injective; this map $\beta = \Phi \circ \pi^{-1}\colon T \to B(\mathbb{D}^*)$ is called the *Bers embedding*. See the diagram below. Moreover, with the aid of the holomorphic dependence of the solution of Beltrami equation (Theorem 2.4) and the boundedness of the norm, we see that the Bers projection Φ is holomorphic.

For $\varphi \in B(\mathbb{D}^*)$ with $\|\varphi\|_\infty < 1/2$, the correspondence

$$\mu(z) = -2\rho_{\mathbb{D}^*}^{-2}(z^*)(zz^*)^2\varphi(z^*) \in \mathrm{Bel}(\mathbb{D}) \qquad (z^* = 1/\bar{z})$$

defines a holomorphic local section for the Bers projection $\Phi\colon \mathrm{Bel}(\mathbb{D}) \to \beta(T)$. This is called the *Ahlfors–Weill local section* ([3]). Based on this, Bers [6] gave a holomorphic local section at every point $\varphi \in \beta(T)$. In particular, we see that Φ is a holomorphic split submersion and that β is a homeomorphism onto the image. This provides a complex structure for T modeled on the complex Banach space $B(\mathbb{D}^*)$.

$$\begin{tikzcd} & \mathrm{Bel}(\mathbb{D}) \arrow[dl, "\pi"] \arrow[dr, "\Phi"] & \\ T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS} \arrow[rr, "\beta"'] & & \beta(T) \subset B(\mathbb{D}^*) \end{tikzcd}$$

2.9. Metrics associated with the complex structure

It is known that the Kobayashi metric on the complex manifold T coincides with the Teichmüller metric (Royden and Gardiner among others). See the monograph by Gardiner and Lakic [20, Chapter 7] for a modern proof of this fact. In particular, every biholomorphic automorphism of T is also isometric.

The Teichmüller metric is also given by a certain norm of the tangent space $B(\mathbb{D}^*)$ at every point of $T \cong \beta(T) \subset B(\mathbb{D}^*)$. It is enough to present this at the base point because of the homogeneity of T. Each tangent vector $\varphi \in B(\mathbb{D}^*)$ defines a bounded linear functional on the Banach space $A^1(\mathbb{D}^*)$ of integrable holomorphic functions on $\mathbb{D}^*$:

$$\ell_\varphi\colon A^1(\mathbb{D}^*) \to \mathbb{C}; \quad \psi \mapsto \int_{\mathbb{D}^*} \psi(z)\overline{\varphi(z)}\rho_{\mathbb{D}^*}^{-2}(z)\, dx\, dy.$$

Then the operator norm of ℓ_φ defines a Finsler metric on T, which coincides with the Teichmüller metric up to a constant multiple.

§3. Isometric group action and the fixed point property

We consider a group G of isometric automorphisms of the universal Teichmüller space T and find a condition under which G has a fixed point in T. We first see that the group of automorphisms of T coincides with QS and then formulate the above problem for a subgroup $G < \mathrm{QS}$.

3.1. The automorphism group of the universal Teichmüller space

The group QS of quasisymmetric automorphisms acts on the universal Teichmüller space $T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS}$ canonically:

$$\mathrm{QS} \times T \to T, \quad (g, [f]) \mapsto [f \circ g^{-1}] =: g_*[f].$$

This action is biholomorphic and isometric with respect to d_T. The group QS plays the role of mapping class group for T.

Consider the representation of QS in the group $\mathrm{Aut}(T)$ of biholomorphic automorphisms:

$$\iota\colon \mathrm{QS} \to \mathrm{Aut}(T).$$

Note that a biholomorphic automorphism of T is isometric because the Kobayashi metric coincides with the Teichmüller metric. The representation ι is known to be a bijective isomorphism. The injectivity is clear and the surjectivity can be proved by results of Earle and Gardiner [13] and Markovic [28].

The group QS of quasisymmetric automorphisms also acts on the Bers embedding $\beta(T) \subset B(\mathbb{D}^*)$ of T as biholomorphic automorphisms. The isotropy subgroup $\mathrm{Stab}([\mathrm{id}]) = \mathrm{Möb}(\mathbb{S})$ acts on $\beta(T)$ as a linear transformation of $B(\mathbb{D}^*)$ because it preserves the tangent space $B(\mathbb{D}^*)$. Actually, it is of the following explicit form: for every $\varphi \in \beta(T)$ ($\varphi = \beta([f])$ with $[f] \in T$) and for every $g \in \mathrm{Möb}(\mathbb{S}) \cong \mathrm{Möb}(\mathbb{D}^*)$, the image $g_*\varphi = \beta([f \circ g^{-1}])$ is given by

$$(g_*\varphi)(z) = \varphi(g^{-1}(z))(g^{-1})'(z)^2 \quad (z \in \mathbb{D}^*).$$

Then g_* is a linear isometric automorphism of $B(\mathbb{D}^*)$.

If $G \subset \mathrm{QS}$ has a fixed point $[f] \in T$, then $G \subset \mathrm{Stab}([f])$ is conjugate to a subgroup of $\mathrm{Stab}([\mathrm{id}]) = \mathrm{Möb}(\mathbb{S})$. Hence G is conjugate to the linear isometric action on $\beta(T)$.

3.2. Fixed point problem on Teichmüller space

We formulate the following problem for a general metric space (X, d) and its isometric automorphism group $\mathrm{Aut}(X, d)$:

Problem 3.1. If a subgroup $G < \mathrm{Aut}(X, d)$ has a bounded orbit in X, does G have a fixed point in X?

We consider this problem for (T, d_T) and $G < \mathrm{QS} = \mathrm{Aut}(T)$. As a special case, we first review the *Nielsen realization problem*.

Let Σ_g be a closed surface of genus $g \geq 2$. We choose a Fuchsian representation $\Gamma < \mathrm{Möb}(\mathbb{S})$ of the fundamental group $\pi_1(\Sigma_g)$. We introduce the following subset (not a subgroup) of QS:

$$\mathrm{QS}(\Gamma) = \{f \in \mathrm{QS} \mid f\Gamma f^{-1} < \mathrm{Möb}(\mathbb{S})\}.$$

Then the Teichmüller space of Σ_g is given as the $\mathrm{Möb}(\mathbb{S})$-representation space of Γ:

$$T(\Gamma) = \mathrm{Möb}(\mathbb{S}) \backslash \mathrm{QS}(\Gamma).$$

The mapping class group of Σ_g coincides with the quotient by Γ of the normalizer

$$N_{\mathrm{QS}}(\Gamma) = \{g \in \mathrm{QS} \mid g^{-1}\Gamma g = \Gamma\}.$$

The group $N_{\mathrm{QS}}(\Gamma)$ acts on $T(\Gamma)$ by $g_*[f] = [f \circ g^{-1}]$ for $g \in N_{\mathrm{QS}}(\Gamma)$ and $[f] \in T(\Gamma)$, with $\Gamma \lhd N_{\mathrm{QS}}(\Gamma)$ acting trivially. Therefore the mapping class group $N_{\mathrm{QS}}(\Gamma)/\Gamma$ acts on the Teichmüller space $T(\Gamma)$.

The fixed point problem in this case asks whether a finite subgroup G/Γ of $N_{\mathrm{QS}}(\Gamma)/\Gamma$, where $\Gamma \lhd G < N_{\mathrm{QS}}(\Gamma)$, has a fixed point in $T(\Gamma)$ or not. The answer is yes by Kerckhoff [25]; there is $[f] \in T(\Gamma)$ such that $[g]_*[f] = [f \circ g^{-1}] = [f]$ for every $[g] \in G/\Gamma$. This is equivalent to the condition that $fGf^{-1} < \mathrm{M\ddot{o}b}(\mathbb{S})$. Then the canonical extension $\mathrm{M\ddot{o}b}(\mathbb{S}) \to \mathrm{M\ddot{o}b}(\mathbb{D})$ gives the extension $\widetilde{H} < \mathrm{M\ddot{o}b}(\mathbb{D})$ of $H = fGf^{-1}$. Take the conformally natural extension $\tilde{f} = e_{\mathrm{DE}}(f) \in \mathrm{QC}(\mathbb{D})$ and set $\widetilde{G} = \tilde{f}^{-1}\widetilde{H}\tilde{f} < \mathrm{QC}(\mathbb{D})$. Then an isomorphism $G \to \widetilde{G}$ is given by $f^{-1}hf \mapsto \tilde{f}^{-1}h\tilde{f}$ for $h \in H \cong \widetilde{H}$. By the conformal naturality, the restriction of the isomorphism $G \to \widetilde{G}$ to the subgroup $\Gamma \lhd G$ is the identity under the identification $\mathrm{M\ddot{o}b}(\mathbb{S}) \cong \mathrm{M\ddot{o}b}(\mathbb{D})$. Indeed, for every $\gamma \in \Gamma$ written as $\gamma = f^{-1}hf$ for some $h \in H$, the conformally natural extension yields $\gamma = \tilde{f}^{-1}h\tilde{f}$. Since $\Gamma < \mathrm{M\ddot{o}b}(\mathbb{S})$ is normal in G, so is $\Gamma < \mathrm{M\ddot{o}b}(\mathbb{D})$ in $\widetilde{G}$. Hence the group $\widetilde{G}$ descends to a group of self-homeomorphisms of Σ_g, which means that the group G/Γ of mapping classes is realized as a group of self-homeomorphisms of Σ_g.

3.3. Uniformly quasisymmetric group

The fixed point problem in the general case where $G < \mathrm{QS}$ acts on T with a bounded orbit was solved affirmatively by Markovic [29]. We say that a subgroup $G < \mathrm{QS}$ is *uniformly quasisymmetric* if $M(g)$ are uniformly bounded for all $g \in G$. This condition is equivalent to the condition that the action of G on T has a bounded orbit.

Theorem 3.2. *If $G < \mathrm{QS}$ is uniformly quasisymmetric, then G is conjugate to a subgroup of $\mathrm{M\ddot{o}b}(\mathbb{S})$ by some $f \in \mathrm{QS}$.*

The conclusion of the theorem is equivalent to the existence of a fixed point $[f]$ in T for G. The proof of [29] uses the fact that a convergence group is Fuchsian due to Tukia [45], Gabai [19] and Casson and Jungreis [10]. Namely, if $G < \mathrm{QS}$ is uniformly quasisymmetric then G has a convergence property, from which it is proved that G is homeomorphic conjugate to a subgroup of $\mathrm{M\ddot{o}b}(\mathbb{S})$. Then complicated arguments using the conformally natural extension make the homeomorphism to be quasisymmetric.

The existence of a fixed point of G in T implies the realization of G in $\mathrm{QC}(\mathbb{D})$ as before also in this general case. Conversely, a uniformly quasiconformal realization of G as a subgroup $\widetilde{G} < \mathrm{QC}(\mathbb{D})$ ensures the existence of a fixed point in T, which was shown by Sullivan [40] and Tukia [44]. We will sketch this argument below. A subgroup $\widetilde{G} < \mathrm{QC}(\mathbb{D})$ is called *uniformly quasiconformal* if $K(\tilde{g})$ are uniformly bounded for all $\tilde{g} \in \widetilde{G}$. Note that $e_{\mathrm{DE}} \colon \mathrm{QS} \to \mathrm{QC}(\mathbb{D})$ is not a homomorphism; this does not yield a realization of $G < \mathrm{QS}$.

Proposition 3.3. *A uniformly quasiconformal group* $\widetilde{G} < \mathrm{QC}(\mathbb{D})$ *is quasiconformally conjugate to a subgroup of* $\mathrm{M\ddot{o}b}(\mathbb{D})$, *which is equivalent to the existence of a fixed point in T for $G = q(\widetilde{G}) < \mathrm{QS}$.*

Indeed, for almost every $z \in \mathbb{D}$, consider the unique circumcenter $\nu(z)$ of the bounded set $\{\mu_{\tilde{g}}(z)\}_{\tilde{g}\in\widetilde{G}}$ with respect to the hyperbolic metric on $\mathbb{D}$. By the uniform boundedness of $K(\tilde{g})$ for all $\tilde{g} \in \widetilde{G}$, we see that $\|\nu\|_\infty < 1$. Then one can show that $\pi(\nu) \in T$ is a fixed point of $G = q(\widetilde{G})$.

3.4. The fixed point property of metric spaces

In this article, we say that a metric space (X, d) has the *fixed point property* if every subgroup $G < \mathrm{Aut}(X, d)$ with a bounded orbit has a fixed point in X. We will consider a sufficient condition for a complete metric space to have the fixed point property.

Definition 3.4. *A complete metric space (X, d) possesses* uniformly normal structure *if there is a constant $c \in (0, 1)$ such that every nonempty subset $A \subset X$ that is the intersection of closed metric balls satisfies*

$$\mathrm{rad}(A) \le c\, \mathrm{diam}(A).$$

Here, $\mathrm{rad}(A)$ *denotes the infimum of radii of closed metric balls with center in A that contain A.*

It is known that $L^\infty(\Omega)$ for a measurable space Ω has uniformly normal structure for $c = 1/2$. See Khamsi and Kirk [26, p. 204]. The following claim can be also found in this book or in [32]. Here, we introduce a direct proof pointed out by the referee. Actually, this is essentially due to Khamsi [50].

Proposition 3.5. *Uniformly normal structure implies the fixed point property.*

Proof. Assume that $G < \mathrm{Aut}(X, d)$ has a bounded orbit $G(x)$. Let A_0 be the intersection of all closed metric balls that contain $G(x)$.

Clearly A_0 is non-empty and invariant under G. Set $\tilde{c} = (1+c)/2 < 1$ for the constant $c \in (0,1)$ concerning the uniformly normal structure of (X,d). Then $\mathrm{rad}(A_0) < \tilde{c}\,\mathrm{diam}(A_0)$. For $r_0 = \tilde{c}\,\mathrm{diam}(A_0)$, consider a subset

$$A_1 = \{x \in A_0 \mid B(x,r_0) \supset A_0\} = \bigcap_{y \in A_0} B(y,r_0) \cap A_0,$$

where $B(x,r)$ denotes the closed metric ball of center x and radius r. This is non-empty by the first equality with $\mathrm{rad}(A_0) < r_0$ and is given as the intersection of closed metric balls by the second equality. We also have $\mathrm{diam}(A_1) \leq r_0$ because any two points $x, y \in A_1$ satisfy $y \in A_0$ and $x \in B(y,r_0)$. Moreover, G-invariance of A_1 follows from that of A_0.

Similarly, we define A_2 from A_1 by using $r_1 = \tilde{c}\,\mathrm{diam}(A_1)$, and so forth. Now $A_0, A_1, A_2, \ldots$ form a decreasing sequence of G-invariant closed subsets such that

$$\mathrm{diam}(A_n) \leq \tilde{c}^n\,\mathrm{diam}(A_0) \to 0 \quad (n \to \infty).$$

By the completeness of X, the intersection is a singleton, which is invariant by G. Q.E.D.

It is natural to ask the following question whose affirmative solution implies the fixed point theorem for the universal Teichmüller space (Theorem 3.2).

Problem 3.6. Does the universal Teichmüller space (T, d_T) with the Teichmüller distance have uniformly normal structure?

§4. The little Teichmüller space and the symmetric representation space

In this section, we consider the little subspace T_0 of the universal Teichmüller space T. This is analogous to the relationship between ℓ^∞ and its subspace $\ell_0^\infty = c_0$, where $\{\xi(n)\}_{n \in \mathbb{N}} \in \ell^\infty$ belongs to ℓ_0^∞ if $\xi(n) \to 0$ $(n \to \infty)$. The representation of the little subspace T_0 is given by the following concepts:

- Symmetric self-homeomorphisms of $\mathbb{S}$;
- Asymptotically conformal self-homeomorphisms of $\mathbb{D}$ (Beltrami coefficients on $\mathbb{D}$ vanishing at the boundary $\mathbb{S}$);
- Complex projective structures (Schwarzian derivatives) on $\mathbb{D}^*$ vanishing at the boundary $\mathbb{S}$.

We start with defining the spaces of these elements.

4.1. The spaces for the little Teichmüller space

We have the triangle diagram among $\mathrm{Bel}(\mathbb{D}) = \mathrm{M\ddot{o}b}(\mathbb{D})\backslash \mathrm{QC}(\mathbb{D})$, $T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS}$ and $B(\mathbb{D}^*)$ in Section 2. We consider the corresponding diagram for the following subspaces.

(1) We say that a measurable function $\mu \in L^\infty(\mathbb{D})$ *vanishes at the boundary* if $\mathrm{ess.sup}_{|z|\geq 1-t}|\mu(z)| \to 0$ as $t \to 0$. The closed subspace of $L^\infty(\mathbb{D})$ consisting of these elements is denoted by $L_0^\infty(\mathbb{D})$. Then we set

$$\mathrm{Bel}_0(\mathbb{D}) = \mathrm{Bel}(\mathbb{D}) \cap L_0^\infty(\mathbb{D}),$$

which is the space of all Beltrami coefficients on $\mathbb{D}$ vanishing at the boundary. Moreover, if a quasiconformal homeomorphism f of $\mathbb{D}$ has complex dilatation μ_f vanishing at the boundary, then we say that f is *asymptotically conformal*.

(2) A quasisymmetric homeomorphism $g \in \mathrm{QS}$ is called *symmetric* if

$$\left|\frac{g(e^{i(x+t)}) - g(e^{ix})}{g(e^{ix}) - g(e^{i(x-t)})}\right| \to 1$$

as $t \to 0$ uniformly for all $x \in \mathbb{R}$. For example, an orientation-preserving diffeomorphism of $\mathbb{S}$ satisfies this condition. The subset of QS consisting of these elements is denoted by Sym. The *little subspace* of the universal Teichmüller space $T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS}$ is defined by

$$T_0 = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{Sym}.$$

It can be shown that Sym is a subgroup of QS containing $\mathrm{M\ddot{o}b}(\mathbb{S})$. Actually Sym is a topological subgroup whereas QS is not a topological group. In fact, the left translation and the inverse operation are not continuous with respect to the M-topology defined by the quasisymmetry constant M on QS. Gardiner and Sullivan [21] introduced Sym as the characteristic topological subgroup of QS consisting of all elements $g \in \mathrm{QS}$ such that the adjoint map $\mathrm{QS} \to \mathrm{QS}$ given by conjugation of g is continuous at the identity.

(3) We can also define vanishing at the boundary for functions on $\mathbb{D}^*$. We define $B_0(\mathbb{D}^*)$ to be the set of all elements $\varphi \in B(\mathbb{D}^*)$ such that the function $\rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)|$ on $\mathbb{D}^*$ vanishes at $\mathbb{S}$. Then $B_0(\mathbb{D}^*)$ is a closed (Banach) subspace of $B(\mathbb{D}^*)$.

Noticing the earlier works by Becker and Pommerenke [5] and by Fehlmann [18], Gardiner and Sullivan [21] proved that the same triangle diagram holds true by restricting the maps π, Φ and β to the above subspaces.

$$
\begin{tikzcd}
 & \mathrm{Bel}_0(\mathbb{D}) \arrow[dl, "\pi"'] \arrow[dr, "\Phi"] & \\
T_0 = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash\mathrm{Sym} \arrow[rr, "\beta"'] & & \beta(T) \cap B_0(\mathbb{D}^*)
\end{tikzcd}
$$

This implies that T_0 has a complex structure modeled on the Banach space $B_0(\mathbb{D}^*)$. Moreover, $g_* \in \mathrm{Aut}(T)$ preserves T_0 for every $g \in \mathrm{Sym}$ and hence it is a biholomorphic automorphism of T_0.

4.2. Foliation of the universal Teichmüller space

Obviously, the action of QS on T is compatible with the coset decomposition $T/T_0 = \mathrm{Sym}\backslash\mathrm{QS}$. Moreover, we see that the image of each coset in T/T_0 under the Bers embedding β is precisely the intersection of $\beta(T)$ with the corresponding coset in $B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$. A proof using the conformally natural extension can be found in Earle, Markovic and Saric [17].

Proposition 4.1. *For every $g \in \mathrm{QS}$ and for any $\nu \in \mathrm{Bel}(\mathbb{D})$ with $\pi(\nu) = [g^{-1}]$, the Bers embedding of the image of T_0 under $g_* \in \mathrm{Aut}(T)$ is*

$$\beta(g_*(T_0)) = \beta(T) \cap \{\beta([g^{-1}]) + B_0(\mathbb{D}^*)\} = \beta(T) \cap \{\Phi(\nu) + B_0(\mathbb{D}^*)\}.$$

We call such a foliation of $\beta(T) \subset B(\mathbb{D}^*)$ induced by $B_0(\mathbb{D}^*)$ the *affine foliation*. Proposition 4.1 implies that this corresponds to the coset decomposition T/T_0 under β. Hence the action of QS on $\beta(T)$ preserves the affine foliation by $B_0(\mathbb{D}^*)$.

4.3. The asymptotic Teichmüller space

Take the quotient of the Bers embedding $\beta\colon T \to \beta(T) \subset B(\mathbb{D}^*)$ by T_0 and $B_0(\mathbb{D}^*)$ respectively. Then Proposition 4.1 implies that the following map is well-defined and injective:

$$\hat{\beta}\colon T/T_0 \to \beta(T)/B_0(\mathbb{D}^*) \subset B(\mathbb{D}^*)/B_0(\mathbb{D}^*).$$

Under the quotient topology, it can be shown that $\hat{\beta}$ is a homeomorphism onto the image.

Gardiner and Sullivan [21] defined $AT = T/T_0$ as the space of symmetric structures on $\mathbb{S}$. Nowadays it is called the *asymptotic Teichmüller space*. The asymptotic Teichmüller space AT has the complex structure modeled on the quotient Banach space $B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$. Also it is equipped with the quotient Teichmüller metric. These results were generalized to asymptotic Teichmüller spaces of Riemann surfaces by Earle, Gardiner and Lakic [15], [16].

Since QS acts on T preserving the affine foliation of its Bers embedding by $B_0(\mathbb{D}^*)$, it also acts on AT as biholomorphic and isometric automorphisms; this gives a homomorphism

$$\iota_{\mathrm{AT}} \colon \mathrm{QS} \to \mathrm{Aut}(AT).$$

Moreover, it was proved that ι_{AT} is injective by Earle, Gardiner and Lakic [14].

Problem 4.2. We state problems concerning the asymptotic Teichmüller space AT. Note that the corresponding statements are all valid for the universal Teichmüller space T.

(1) The Kobayashi metric on AT coincides with the quotient Teichmüller metric?
(2) $\iota_{\mathrm{AT}} \colon \mathrm{QS} \to \mathrm{Aut}(AT)$ is surjective?
(3) If $G \subset \mathrm{QS}$ has a bounded orbit then G has a fixed point in AT?

4.4. Failure of the fixed point property

We are interested in the little subspace T_0 regarding its failure of the fixed point property. First we show this fact for a prototype in a discrete model and then apply it to T_0.

We consider the Banach space $\ell^\infty(\mathbb{Z})$ of all bounded bilateral sequences and the shift operator

$$\sigma \colon \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z}); \quad \{\xi(n)\}_{n\in\mathbb{Z}} \mapsto \{\xi(n+1)\}_{n\in\mathbb{Z}}.$$

Then σ is a linear isometric automorphism of $\ell^\infty(\mathbb{Z})$ and its fixed point set $\mathrm{Fix}(\sigma)$ consists of all constant sequences $\{\xi(n) \equiv c\}_{c\in\mathbb{R}}$. Set the closed subspace consisting of the elements that vanish at the infinity;

$$\ell^\infty_0(\mathbb{Z}) = c_0(\mathbb{Z}) = \{\xi \in \ell^\infty(\mathbb{Z}) \mid \xi(n) \to 0 \ (n \to \pm\infty)\}.$$

Take $\xi_0 = (\ldots, 0, 0, 1, 1, \ldots)$ and the affine subspace $\xi_0 + \ell^\infty_0(\mathbb{Z})$ isomorphic to $\ell^\infty_0(\mathbb{Z})$. This is invariant under σ and σ acts on it with a bounded orbit. However, there is no fixed point on it.

This construction can be generalized to any countable discrete group Γ. Namely, we consider the Banach space $\ell^\infty(\Gamma)$ of all bounded functions $\xi \colon \Gamma \to \mathbb{R}$ and define the action of Γ on $\ell^\infty(\Gamma)$ by

$$(\gamma_*\xi)(g) = \xi(\gamma^{-1}g)$$

for every $\gamma \in \Gamma$ and for every $\xi \in \ell^\infty(\Gamma)$. This gives the left regular representation of Γ. The closed subspace of the elements vanishing at the infinity is

$$\ell_0^\infty(\Gamma) = \{\xi \in \ell^\infty(\Gamma) \mid \xi(g) \to 0 \ (g \to \infty)\},$$

where $g \to \infty$ means that the sequence exits from every finite subset of Γ. We can find some $\xi_0 \in \ell^\infty(\Gamma)$ such that the affine subspace $\xi_0 + \ell_0^\infty(\Gamma) \cong \ell_0^\infty(\Gamma)$ is invariant under Γ, with a bounded orbit, but without a fixed point.

Now using the discrete model as above, we exhibit an example of a subgroup of QS that acts isometrically on T_0 with a bounded orbit but without a fixed point. Take a Fuchsian group $\Gamma < \mathrm{Möb}(\mathbb{S}) = \mathrm{Möb}(\mathbb{D})$ such that $T(\Gamma) \neq \{[\mathrm{id}]\}$ (non-rigid), where $T(\Gamma) = \mathrm{Möb}(\mathbb{S})\backslash \mathrm{QS}(\Gamma)$ is the Teichmüller space for Γ.

Choose a small neighborhood W of some $z_0 \in \mathbb{D}$ such that $\{W_\gamma\}_{\gamma \in \Gamma}$ are mutually disjoint for $W_\gamma = \gamma(W)$ $(\gamma \in \Gamma)$. Then there is some $\mu_0 \in \mathrm{Bel}(\mathbb{D})$ with support on $\bigsqcup_{\gamma \in \Gamma} W_\gamma$ such that $\pi(\mu_0) \in T(\Gamma) - \{[\mathrm{id}]\}$. For $\xi_0 \in \ell^\infty(\Gamma)$ as above and assuming $\xi_0(\gamma) \in [0,1]$, we consider a Beltrami coefficient

$$\mu(z) = \sum_{\gamma \in \Gamma} \xi_0(\gamma) 1_{W_\gamma}(z) \mu_0(z),$$

where 1_W is the characteristic function of W. Then we have the following result, which was given in [30].

Proposition 4.3. *Under the circumstances as above, suppose that* $\xi_0 \in \ell^\infty(\Gamma)$ *satisfies*

- ($\xi_0 + \ell_0^\infty(\Gamma)$ is invariant) $\gamma_* \xi_0 - \xi_0 \in \ell_0^\infty(\Gamma)$ *for every* $\gamma \in \Gamma$;
- (stronger than fixed point free) $((\gamma_i)_* \xi_0)(g) \to 1$ $(i \to \infty)$ *for some sequence* $\{\gamma_i\} \subset \Gamma$ *and* $((\gamma_j)_* \xi_0)(g) \to 0$ $(j \to \infty)$ *for some sequence* $\{\gamma_j\} \subset \Gamma$, *pointwise for all* $g \in \Gamma$.

Then there exists some $\mu \in \mathrm{Bel}(\mathbb{D})$ *such that the subspace*

$$\beta(T) \cap \{\Phi(\mu) + B_0(\mathbb{D}^*)\}$$

isomorphic to T_0 *is invariant under* Γ, *with a bounded orbit, but without a fixed point.*

We represent the point $\pi(\mu)$ for μ as in this proposition by $[f]$ for $f \in \mathrm{QS}$. Then $f \Gamma f^{-1} < \mathrm{Sym}$ and it acts isometrically on T_0 with a bounded orbit but without a fixed point. This implies that T_0 does not hold the fixed point property.

Remark 4.4. For each $\gamma \in \Gamma$, the decay order of $\gamma_* \xi_0 - \xi_0 \in \ell_0^\infty(\Gamma)$ at infinity affects the regularity of the conjugate $f \gamma f^{-1} \in \mathrm{Sym}$ for $f \in \mathrm{QS}$ with $[f] = \pi(\mu)$. For example, when Γ is cocompact, we can make each element of $f \Gamma f^{-1}$ absolutely continuous with L^p $(0 < p < \infty)$ derivative.

4.5. Symmetric representation space

For a Fuchsian group $\Gamma < \mathrm{Möb}(\mathbb{S})$, we define the following subspace of QS:

$$\begin{aligned}\mathrm{QS}_{\mathrm{Sym}}(\Gamma) &= \{f \in \mathrm{QS} \mid f\Gamma f^{-1} < \mathrm{Sym}\} \\ &\supset \{f \in \mathrm{QS} \mid f\Gamma f^{-1} < \mathrm{Möb}(\mathbb{S})\} = \mathrm{QS}(\Gamma).\end{aligned}$$

Then $\mathrm{Möb}(\mathbb{S}) \backslash \mathrm{QS}_{\mathrm{Sym}}(\Gamma)$ contains $T(\Gamma) = \mathrm{Möb}(\mathbb{S}) \backslash \mathrm{QS}(\Gamma)$ in T.

Let $a\colon T \to AT = T/T_0$ denote the canonical projection. The *symmetric representation space* of $\Gamma < \mathrm{Möb}(\mathbb{S})$ is defined by

$$AT(\Gamma) = \mathrm{Sym} \backslash \mathrm{QS}_{\mathrm{Sym}}(\Gamma),$$

which contains $aT(\Gamma) = \mathrm{Sym} \backslash \mathrm{QS}(\Gamma)$. Under the quotient Bers embedding $\hat{\beta}\colon AT \to B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$, the symmetric representation space $AT(\Gamma)$ is mapped onto a bounded domain in the Banach subspace of $B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$ that is the fixed point locus of Γ. In this manner, $AT(\Gamma)$ is provided with the complex structure.

In [51], we see the following results concerning these spaces.

Theorem 4.5. *For any non-rigid infinite Fuchsian group Γ, the symmetric representation space $AT(\Gamma)$ is an infinite-dimensional complex Banach manifold. The Teichmüller space $T(\Gamma)$ is biholomorphically embedded into $AT(\Gamma)$ by the projection a, and $aT(\Gamma)$ is strictly contained in $AT(\Gamma)$.*

Proof. We only show the injectivity of $a|_{T(\Gamma)}$. The rest of the proof appears in [51]. We have only to consider the projection a in the fiber T_0 over the origin by replacing the base point. We utilize the Bers embedding $\beta(T_0) \subset B_0(\mathbb{D}^*)$. Suppose that $\varphi \in \beta(T_0)$ is fixed by Γ, that is, $\varphi \in \beta(T(\Gamma))$. Then, for every $\gamma \in \Gamma$, we have

$$\rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| = \rho_{\mathbb{D}^*}^{-2}(z)|(\gamma_*\varphi)(z)| = \rho_{\mathbb{D}^*}^{-2}(\gamma^{-1}(z))|\varphi(\gamma^{-1}(z))|$$

for $z \in \mathbb{D}^*$. The last term tends to 0 if we choose a sequence in Γ such that $|\gamma^{-1}(z)| \to 1$. This is because $\varphi \in B_0(\mathbb{D}^*)$ (that is, $\rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)|$ vanishes at the boundary). Hence $\varphi = 0$. Q.E.D.

Expressing the last statement ($aT(\Gamma) \subsetneqq AT(\Gamma)$) of the above theorem, we say that the symmetric representation of Γ is not rigid if Γ is not rigid. This raises the following question naturally.

Problem 4.6. Is the symmetric representation of a rigid Fuchsian group rigid?

§5. Teichmüller spaces of circle diffeomorphisms

We have investigated the little Teichmüller space T_0, which is the space of symmetric homeomorphisms of $\mathbb{S}$. In this section, we consider a smaller subspace than T_0, which is the space of self-homeomorphisms of $\mathbb{S}$ with higher regularity, i.e. diffeomorphisms with Hölder continuous derivatives. The contents of this section is as follows:

(1) Characterization of a circle diffeomorphism with Hölder continuous derivative of exponent α in terms of quasiconformal Teichmüller theory;
(2) Teichmüller spaces T_0^α defined by the group of such circle diffeomorphisms $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ and their structure;
(3) The affine foliation of T associated with T_0^α;
(4) $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$-representation space and its rigidity.

5.1. The spaces related to $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$

We deal with an orientation-preserving diffeomorphism $g\colon \mathbb{S} \to \mathbb{S}$ having Hölder continuous derivative of exponent $\alpha \in (0,1)$, that is, the lift $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ of g under the universal cover $\mathbb{R} \to \mathbb{S}$ ($x \mapsto e^{ix}$) satisfies

$$|\tilde{g}'(x) - \tilde{g}'(y)| \leq c|x-y|^\alpha$$

for some constant $c > 0$ uniformly for any $x, y \in \mathbb{R}$. Let $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ denote the group of all these diffeomorphisms. Then we have the following inclusion relation:

$$\mathrm{Diff}^{1+\alpha}(\mathbb{S}) < \mathrm{Sym} < \mathrm{QS}\,.$$

We can provide a stronger topology for $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ than the M-topology on QS, which is the right uniform $C^{1+\alpha}$-topology. We see that $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ is a topological group with respect to this topology ([33]).

The decay order of Beltrami coefficients defines a subspace of $\mathrm{Bel}(\mathbb{D})$. We have seen that a symmetric homeomorphism in Sym corresponds to a Beltrami coefficient μ with $|\mu(z)| = o(1)$ a.e. ($|z| \to 1$). For a diffeomorphism in $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$, we consider a Beltrami coefficient μ with $|\mu(z)| = O((1-|z|)^\alpha)$ a.e. ($|z| \to 1$). More precisely, setting a norm

$$\|\mu\|_{\infty,\alpha} = \mathrm{ess.sup}_{z\in\mathbb{D}}\, \rho_{\mathbb{D}}^\alpha(z)|\mu(z)|$$

for $\mu \in \mathrm{Bel}(\mathbb{D})$ where $\rho_{\mathbb{D}}(z) = 2/(1-|z|^2)$ is the hyperbolic density, we deal with

$$\mathrm{Bel}_0^{\alpha}(\mathbb{D}) = \{\mu \in \mathrm{Bel}(\mathbb{D}) \mid \|\mu\|_{\infty,\alpha} < \infty\}.$$

Note that this is not closed in $\mathrm{Bel}(\mathbb{D})$ but the inclusion map $\mathrm{Bel}_0^{\alpha}(\mathbb{D}) \to \mathrm{Bel}(\mathbb{D})$ is continuous.

The decay order of Schwarzian derivatives defines a subspace of $B(\mathbb{D}^*)$. We have seen that a symmetric homeomorphism in Sym corresponds to a Schwarzian derivative $\varphi = S_{f_\mu|_{\mathbb{D}^*}}$ with $\rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| = o(1)$ ($|z| \to 1$). For a diffeomorphism in $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$, we consider a Schwarzian derivative φ with $\rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| = O((|z|-1)^{\alpha})$ ($|z| \to 1$). More precisely, setting a norm

$$\|\varphi\|_{\infty,\alpha} = \sup_{z \in \mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2+\alpha}(z)|\varphi(z)|$$

for $\varphi \in B(\mathbb{D}^*)$, we deal with a Banach space

$$B_0^{\alpha}(\mathbb{D}^*) = \{\varphi \in B(\mathbb{D}^*) \mid \|\varphi\|_{\infty,\alpha} < \infty\}.$$

Note that this is not closed in $B(\mathbb{D}^*)$ but the inclusion map $B_0^{\alpha}(\mathbb{D}^*) \to B(\mathbb{D}^*)$ is continuous.

5.2. Characterization of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$

We summarize the relationship between the spaces defined above, which is presented and proved in [31] and [33].

Theorem 5.1. *The following conditions are equivalent for $g \in \mathrm{QS}$:*

(1) $g \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$;

(2′) $s_{\mathrm{DE}}([g]) \in \mathrm{Bel}_0^{\alpha}(\mathbb{D})$ (*the complex dilatation of* $e_{\mathrm{DE}}(g) \in \mathrm{QC}(\mathbb{D})$ *belongs to* $\mathrm{Bel}_0^{\alpha}(\mathbb{D})$);

(2) *there is some* $\mu \in \mathrm{Bel}_0^{\alpha}(\mathbb{D})$ *such that* $\pi(\mu) = [g]$;

(3) $\beta([g]) \in B_0^{\alpha}(\mathbb{D}^*)$.

Here we give some commentaries on previously known results. Implications which are not previously known are only concerning (2′).

(1) $\Rightarrow$ (2) is essentially due to Carleson [9]. We easily see that if $g \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ then the lift $\tilde{g} \colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\frac{\tilde{g}(x+t) - \tilde{g}(x)}{\tilde{g}(x) - \tilde{g}(x-t)} = 1 + O(t^{\alpha}) \quad (t \to 0).$$

Set

$$\alpha(x,y) = \int_0^1 \tilde{g}(x+ty)\,dt; \qquad \beta(x,y) = \int_0^1 \tilde{g}(x-ty)\,dt$$

and define the *Beurling–Ahlfors extension* of $\tilde{g}$ by

$$F(z) = \frac{1}{2}\{\alpha(x,y) + \beta(x,y)\} + i\{\alpha(x,y) - \beta(x,y)\}$$

for $z = x + iy \in \mathbb{H}$. This is a quasiconformal self-homeomorphism of $\mathbb{H}$ and the decay order of its complex dilatation $\mu_F(z)$ as $y \to 0$ was estimated in [9]. By projecting down this quasiconformal homeomorphism to $\mathbb{D} - \{0\}$ by the holomorphic universal cover $\mathbb{H} \to \mathbb{D} - \{0\}$ and extending to 0, we have a quasiconformal extension of g whose complex dilatation μ satisfies $|\mu(z)| = O((1 - |z|)^\alpha)$ $(|z| \to 1)$.

(1) $\Rightarrow$ (3) is due to Tam and Wan [42]. They used the harmonic quasiconformal extension $\mathbb{D} \to \mathbb{D}$ of g. Here "harmonic" means a harmonic map with respect to the hyperbolic metric on $\mathbb{D}$. (3) $\Rightarrow$ (2) is the "boundary version" of the Ahlfors–Weill local section due to Becker and Pommerenke [5]. The proof used a Löwner chains in which the associated conformal homeomorphism of $\mathbb{D}^*$ is contained. The converse implication (2) $\Rightarrow$ (1) had been a problem, which was settled by Dyn'kin [49] and Anderson, Cantón and Fernández [48].

5.3. The Teichmüller space of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$

Based on the characterization of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$, we define its Teichmüller space by

$$T_0^\alpha = \mathrm{Möb}(\mathbb{S})\backslash \mathrm{Diff}^{1+\alpha}(\mathbb{S}).$$

Theorem 5.1 implies that the triangle diagram is also valid for $\mathrm{Bel}_0^\alpha(\mathbb{D})$, T_0^α and $B_0^\alpha(\mathbb{D}^*)$ with the involved maps restricted to these spaces.

$$\begin{tikzcd}
& \mathrm{Bel}_0^\alpha(\mathbb{D}) \arrow[dl, "\pi"'] \arrow[dr, "\Phi"] & \\
T_0^\alpha = \mathrm{Möb}(\mathbb{S})\backslash \mathrm{Diff}^{1+\alpha}(\mathbb{S}) \arrow[rr, "\beta"'] & & \beta(T) \cap B_0^\alpha(\mathbb{D}^*)
\end{tikzcd}$$

Moreover, if we equip these spaces with the canonical topology, we see that the involved maps are all continuous. Holomorphy also follows from the continuity in this situation. The precise statements are as follows, which are obtained in [33].

Theorem 5.2. *Providing the norm topology* $\|\cdot\|_{\infty,\alpha}$ *for* $\mathrm{Bel}_0^\alpha(\mathbb{D})$ *and* $\beta(T) \cap B_0^\alpha(\mathbb{D}^*)$, *and the induced topology for* T_0^α *from* $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$, *one has the following*:

(1) *the Teichmüller projection* $\pi\colon \mathrm{Bel}_0^\alpha(\mathbb{D}) \to T_0^\alpha$ *is continuous and open*;
(2) *the Bers embedding* $\beta\colon T_0^\alpha \to \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ *is a homeomorphism*;
(3) *the Bers projection* $\Phi\colon \mathrm{Bel}_0^\alpha(\mathbb{D}) \to \beta(T) \cap B_0^\alpha(\mathbb{D}^*)$ *is holomorphic with holomorphic local section at every point.*

This shows that the Teichmüller space T_0^α has a complex structure modeled on the Banach space $B_0^\alpha(\mathbb{D}^*)$. In addition, $g_* \in \mathrm{Aut}(T)$ is a biholomorphic automorphism of T_0^α for every $g \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$. Note that the continuity of the conformally natural section s_{DE} can be also proved ([52]).

5.4. Affine foliation of T and T_0 by $B_0^{>\alpha}(\mathbb{D}^*)$

We have seen in Section 4 that the Bers embedding $\beta\colon T \to \beta(T) \subset B(\mathbb{D}^*)$ is compatible with the coset decomposition T/T_0 and $B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$:

$$\beta(g_*(T_0)) = \beta(T) \cap \{\beta([g^{-1}]) + B_0(\mathbb{D}^*)\} \quad (\forall g \in \mathrm{QS}).$$

We called $\{\beta(T) \cap (\psi + B_0(\mathbb{D}^*))\}_\psi$ the affine foliation of $\beta(T)$ by $B_0(\mathbb{D}^*)$.

We want to obtain a similar fact for $B_0^\alpha(\mathbb{D}^*)$ but certain modification is necessary. Fix $\alpha \in [0, 1)$. Set the following spaces by taking the union of increasing subspaces:

$$\mathrm{Diff}^{>1+\alpha}(\mathbb{S}) = \bigcup_{\varepsilon>0} \mathrm{Diff}^{1+\alpha+\varepsilon}(\mathbb{S});$$
$$T_0^{>\alpha} = \bigcup_{\varepsilon>0} T_0^{\alpha+\varepsilon}; \quad B_0^{>\alpha}(\mathbb{D}^*) = \bigcup_{\varepsilon>0} B_0^{\alpha+\varepsilon}(\mathbb{D}^*).$$

Then the corresponding results are as follows ([34]). The first one is concerning the affine foliation of $\beta(T_0)$.

Proposition 5.3. *The Bers embedding $\beta\colon T_0 \to \beta(T_0) \subset B_0(\mathbb{D}^*)$ is compatible with the coset decomposition $T_0/T_0^{>\alpha}$ and $B_0(\mathbb{D}^*)/B_0^{>\alpha}(\mathbb{D}^*)$:*

$$\beta(g_*(T_0^{>\alpha})) = \beta(T_0) \cap \{\beta([g^{-1}]) + B_0^{>\alpha}(\mathbb{D}^*)\}$$

for every $\alpha \in [0, 1)$ and for every $g \in \mathrm{Sym}$.

Proposition 5.4. *The Bers embedding $\beta\colon T \to \beta(T) \subset B(\mathbb{D}^*)$ is compatible with the coset decomposition $T/T_0^{>0}$ and $B(\mathbb{D}^*)/B_0^{>0}(\mathbb{D}^*)$:*

$$\beta(g_*(T_0^{>0})) \subset \beta(T) \cap \{\beta([g^{-1}]) + B_0^{>0}(\mathbb{D}^*)\}$$

for every $g \in \mathrm{QS}$.

Remark 5.5. In the previous case (Proposition 4.1), the inclusion relation in Proposition 5.4 was equality. Also, $\beta(T) \cap \{\psi + B_0(\mathbb{D}^*)\}$ is connected for every $\psi \in B(\mathbb{D}^*)$. However, in the present case, we have just proved these results. Accordingly, as $\hat{\beta}\colon T/T_0 \to B(\mathbb{D}^*)/B_0(\mathbb{D}^*)$ was injective, we also know that this is true for the quotient Bers embedding

$$\hat{\beta}_0^{>0}\colon T/T_0^{>0} \to B(\mathbb{D}^*)/B_0^{>0}(\mathbb{D}^*)$$

in the present case.

5.5. $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$**-representation space**

For a subgroup $\Gamma < \text{Möb}(\mathbb{S})$, we define the following subspaces as before:

$$\begin{aligned}\mathrm{QS}_{\mathrm{Diff}^{1+\alpha}}(\Gamma) &= \{f \in \mathrm{QS} \mid f\Gamma f^{-1} < \mathrm{Diff}^{1+\alpha}(\mathbb{S})\} \\ &\subset \{f \in \mathrm{QS} \mid f\Gamma f^{-1} < \mathrm{Sym}\} = \mathrm{QS}_{\mathrm{Sym}}(\Gamma).\end{aligned}$$

Then by taking the quotient of Sym, we have

$$\mathrm{Sym}\backslash\mathrm{QS}_{\mathrm{Diff}^{1+\alpha}}(\Gamma) \subset \mathrm{Sym}\backslash\mathrm{QS}_{\mathrm{Sym}}(\Gamma) = AT(\Gamma)$$

in the asymptotic Teichmüller space AT.

However, the $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$-representation space of Γ should be defined as

$$DT^{\alpha}(\Gamma) = \mathrm{Diff}^{1+\alpha}(\mathbb{S})\backslash\mathrm{QS}_{\mathrm{Diff}^{1+\alpha}}(\Gamma).$$

The next theorem says that, concerning the equivalence class of id, they are identical. It implies a rigidity of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$-representation obtained by the conjugation of Sym. A complete proof appears in [34].

Theorem 5.6. *Let* $\Gamma < \text{Möb}(\mathbb{S})$ *be an infinite non-abelian group. Then*

$$\mathrm{Sym} \cap \mathrm{QS}_{\mathrm{Diff}^{1+\alpha}}(\Gamma) = \mathrm{Diff}^{1+\alpha}(\mathbb{S}) \cap \mathrm{QS}_{\mathrm{Diff}^{1+\alpha}}(\Gamma).$$

In other words, if $f \in \mathrm{Sym}$ *satisfies the condition* $f\Gamma f^{-1} < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$, *then* $f \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$.

Proof. We only sketch an outline of the proof here. Choose a hyperbolic element $h \in \Gamma$, which exists by the assumption on Γ. Set $\varphi = \beta([f]) \in B_0(\mathbb{D}^*)$ for $f \in \mathrm{Sym}$. By Propositions 5.3 or 5.4, the condition $f\Gamma f^{-1} < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ implies that $h_*\varphi - \varphi \in B_0^{\alpha'}(\mathbb{D}^*)$ for some $\alpha' > 0$. We denote this element by ψ and apply the action of h^n to ψ:

$$\psi = h_*\varphi - \varphi, \quad h_*\psi = h_*^2\varphi - h_*\varphi, \quad \ldots, \quad h_*^n\psi = h_*^{n+1}\varphi - h_*^n\varphi.$$

By summing up all these equations, we have

$$\sum_{i=0}^{n} h_*^i\psi = h_*^{n+1}\varphi - \varphi.$$

Since $\varphi \in B_0(\mathbb{D}^*)$, a similar equation as in the proof of Theorem 4.5 can be used to show that $\lim_{n\to\infty}(h_*^{n+1}\varphi)(z) = 0$ pointwise. These are also true for negative power n. Hence

$$\varphi = -\sum_{i=0}^{\infty} h_*^i\psi = \sum_{i=1}^{\infty} h_*^{-i}\psi,$$

and φ is represented by the infinite sum in terms of $\psi \in B_0^{\alpha'}(\mathbb{D}^*)$. Using the assumption that h is hyperbolic, we can prove that $\varphi \in B_0^{\alpha'}(\mathbb{D}^*)$.

Having this conclusion, we repeat again the same argument from the beginning. Now φ has been promoted from $B_0(\mathbb{D}^*)$ to $B_0^{\alpha'}(\mathbb{D}^*)$. Once we have a definite decay order for φ, we can show that the condition $f\Gamma f^{-1} < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ exactly implies that

$$\psi = h_*\varphi - \varphi \in B_0^{\alpha}(\mathbb{D}^*).$$

Then the same argument concludes that $\varphi \in B_0^{\alpha}(\mathbb{D}^*)$. This implies that $f \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ by Theorem 5.1. Q.E.D.

5.6. A problem on rigidity

A generalization of Theorem 5.6 gives us a question on certain rigidity of the diffeomorphic representation of a Fuchsian group. We first set up our representation spaces and then formulate the rigidity. For simplicity, we deal with the case of $\mathrm{Diff}^{>1}(\mathbb{S})$. To embed the representation space of Γ into $AT(\Gamma)$, we consider

$$DT^{>0}(\Gamma) = \mathrm{Diff}^{>1}(\mathbb{S})\backslash \mathrm{QS}_{\mathrm{Diff}^{>1}}(\Gamma).$$

By Proposition 5.4 and Remark 5.5, we have the quotient Bers embedding

$$\hat{\beta}_0^{>0} \colon T/T_0^{>0} \to B(\mathbb{D}^*)/B_0^{>0}(\mathbb{D}^*)$$

though the injectivity is going to be proved. If $\hat{\beta}_0^{>0}$ is injective, we can embed $DT^{>0}(\Gamma)$ into $AT(\Gamma)$ in a similar argument to the proof of Theorem 5.6. Here, we just identify $DT^{>0}(\Gamma)$ with its image under the projection. Then

$$aT(\Gamma) \subset DT^{>0}(\Gamma) \subset AT(\Gamma)$$

and the inclusion of the first term in the third term is strict (Theorem 4.5). We regard the $\mathrm{Diff}^{>1}(\mathbb{S})$-representation of a Fuchsian group Γ rigid if $aT(\Gamma) = DT^{>0}(\Gamma)$.

Problem 5.7. Determine whether $aT(\Gamma) = DT^{>0}(\Gamma)$ or not for an infinite non-abelian Fuchsian group Γ.

Assume that Γ is a cocompact Fuchsian group. The Ghys rigidity theorem [22] implies that the diffeomorphic representation of Γ with higher regularity is rigid. So the class $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ is suitable for asking the problem on rigidity. On the other hand, the argument for Proposition 4.3 shows that there is a non-trivial conjugation of Γ by $f \in \mathrm{QS}$ such that each element of $f\Gamma f^{-1}$ is symmetric and absolutely continuous with L^p $(0 < p < \infty)$ derivative, which was mentioned in Remark 4.4.

A theorem due to Ghys and Tsuboi [23] combined with Theorem 5.6 implies the following:

Corollary 5.8. *If* $G < \mathrm{Diff}^r(\mathbb{S})$ $(r \geq 2)$ *is conjugate to an infinite non-abelian subgroup of* $\mathrm{M\ddot{o}b}(\mathbb{S})$ *having a dense orbit in* $\mathbb{S}$ *by a symmetric homeomorphism of* $\mathbb{S}$, *then it is actually conjugate in* $\mathrm{Diff}^r(\mathbb{S})$.

By Theorem 3.2, the quasisymmetric conjugate to a Möbius group is equivalent to being a uniformly quasisymmetric group. Hence the assumption in the above corollary implies that G is uniformly quasisymmetric and actually our arguments on rigidity are only applicable to subgroups of QS that are uniformly quasisymmetric.

§6. Integrable Teichmüller spaces and the Weil–Petersson metric

In this section, we introduce the integrable Teichmüller space defined by integrable Beltrami coefficients with respect to the hyperbolic metric. This Teichmüller space has a complex structure modeled on the Banach space $A^p(\mathbb{D}^*)$ consisting of hyperbolically integrable elements of $B(\mathbb{D}^*)$. Also, another affine foliation on the universal Teichmüller space $T \cong \beta(T)$ is induced by $A^p(\mathbb{D}^*)$. This gives a (leaf-wise) Banach manifold structure on T and the generalized Weil–Petersson metric associated with this Banach structure. As an application, we consider a conjugation problem for a group of circle diffeomorphisms with Hölder continuous derivatives.

6.1. The integrable subspaces

We define p-integrable norms for $\mathrm{Bel}(\mathbb{D})$ and $B(\mathbb{D}^*)$ respectively and subspaces consisting of p-integrable elements for $p \geq 1$:

$$\mathrm{Ael}^p(\mathbb{D}) = \left\{ \mu \in \mathrm{Bel}(\mathbb{D}) \;\middle|\; \|\mu\|_p^p = \int_{\mathbb{D}} |\mu(z)|^p \rho_{\mathbb{D}}^2(z)\, dx\, dy < \infty \right\};$$

$$A^p(\mathbb{D}^*) = \left\{ \varphi \in B(\mathbb{D}^*) \;\middle|\; \|\varphi\|_p^p = \int_{\mathbb{D}^*} |\varphi(z)|^p \rho_{\mathbb{D}^*}^{2-2p}(z)\, dx\, dy < \infty \right\}.$$

Here $A^p(\mathbb{D}^*)$ is a Banach space with norm $\|\cdot\|_p$ and the inclusion map $A^p(\mathbb{D}^*) \to B(\mathbb{D}^*)$ is continuous but it is not closed in $B(\mathbb{D}^*)$.

We mention the inclusion relations of these spaces with the previous ones. There are no inclusion relations in the level of Beltrami coefficients, that is, between $\mathrm{Ael}^p(\mathbb{D})$ and $\mathrm{Bel}_0(\mathbb{D}) \supset \mathrm{Bel}_0^\alpha(\mathbb{D})$. There are inclusion relations in the level of Schwarzian derivatives as follows:

(1) $A^p(\mathbb{D}^*) \subset A^{p'}(\mathbb{D}^*) \subset B_0(\mathbb{D}^*)$ for $1 \leq p < p' < \infty$;
(2) $B_0^\alpha(\mathbb{D}^*) \subset A^p(\mathbb{D}^*)$ for $\alpha p > 1$.

6.2. Integrable symmetric homeomorphisms

Using p-integrable Beltrami coefficients, we will define conversely p-integrable symmetric homeomorphisms as follows: $g \in \mathrm{QS}$ is *p-integrable symmetric* if there exists $\mu \in \mathrm{Ael}^p(\mathbb{D})$ such that $\pi(\mu) = [g]$. The set of all p-integrable symmetric homeomorphisms is denoted by Sym^p. We will see later that this is contained in Sym.

Assume that $p \geq 2$. It was essentially proved by Cui [11] (see also Tang [43]) that $g \in \mathrm{Sym}^p$ if and only if the complex dilatation of the conformally natural extension $e_{\mathrm{DE}}(g)$ belongs to $\mathrm{Ael}^p(\mathbb{D})$. Moreover Sym^p is a subgroup of QS.

Remark 6.1. In the above definition of Sym^p, we rely on the quasiconformal extension of $g \in \mathrm{QS}$. However, there are several studies about conditions on the mapping of $\mathbb{S}$ itself which is related to Sym^p. Among them, Shen [39] recently proves the following characterization of Sym^2: a quasisymmetric homeomorphism g belongs to Sym^2 if and only if g is absolutely continuous and $\log g'$ is in the Sobolev class $H^{1/2}$. Here, an integrable function h on $\mathbb{S}$ belongs to $H^{1/2}$ if and only if

$$\int_{\mathbb{S}\times\mathbb{S}} \frac{|h(x)-h(y)|^2}{\sin^2((x-y)/2)}\, dx\, dy < \infty.$$

6.3. The p-integrable Teichmüller space

We define the *p-integrable Teichmüller space* by

$$T^p = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{Sym}^p$$

for $p \geq 2$. Cui [11] proved in the case $p = 2$ and later extended by Guo [24] that the triangle diagram also holds true for $\mathrm{Ael}^p(\mathbb{D})$, $A^p(\mathbb{D}^*)$ and T^p with the involved maps restricted to these spaces.

$$\begin{tikzcd}
 & \mathrm{Ael}^p(\mathbb{D}) \arrow[dl, "\pi"] \arrow[dr, "\Phi"] & \\
T^p = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{Sym}^p \arrow[rr, "\beta"'] & & \beta(T) \cap A^p(\mathbb{D}^*)
\end{tikzcd}$$

Since there are inclusions for the spaces of Schwarzian derivatives and since the Bers embedding β is injective, this diagram also implies the following:

(1) $\mathrm{Sym}^p < \mathrm{Sym}^{p'} < \mathrm{Sym}$ and $T^p \subset T^{p'} \subset T_0$ for $2 \leq p < p' < \infty$;
(2) $\mathrm{Diff}^{1+\alpha}(\mathbb{S}) < \mathrm{Sym}^p$ and $T_0^\alpha \subset T^p$ for $\alpha p > 1$.

We provide $\mathrm{Ael}^p(\mathbb{D})$ with the stronger topology induced by both norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$. The topology of T^p is the quotient topology by the Teichmüller projection π. Then the continuity of the mappings in the diagram also follows, which was first given by Cui [11] for $p = 2$ and generalized later by Tang [43]. See Takhtajan and Teo [41] and Yanagishita [47] for additional proofs.

Theorem 6.2. *The Bers embedding $\beta\colon T^p \to \beta(T) \cap A^p(\mathbb{D}^*)$ is a homeomorphism and the Bers projection $\Phi\colon \mathrm{Ael}^p(\mathbb{D}) \to \beta(T) \cap A^p(\mathbb{D}^*)$ is holomorphic with holomorphic local section at every point. Moreover, the conformally natural section $s_{\mathrm{DE}}\colon T^p \to \mathrm{Ael}^p(\mathbb{D})$ is continuous.*

In particular, we see that T^p has a complex structure modeled on the Banach space $A^p(\mathbb{D}^*)$, and $g_* \in \mathrm{Aut}(T)$ is a biholomorphic automorphism of T^p for every $g \in \mathrm{Sym}^p$. The continuity of the section s_{DE} implies that T^p is contractible.

6.4. Affine foliation of T by $A^p(\mathbb{D}^*)$

As before, we will show that the Bers embedding

$$\beta\colon T \to \beta(T) \subset B(\mathbb{D}^*)$$

is compatible with the coset decomposition $T/T^p = \mathrm{Sym}^p\backslash\mathrm{QS}$ and $B(\mathbb{D}^*)/A^p(\mathbb{D}^*)$. Then this gives the affine foliation of $T \cong \beta(T)$ by $A^p(\mathbb{D}^*)$. The compatibility in the case of $p = 2$ was proved by Takhtajan and Teo [41]. The general case is remarked in [34].

Theorem 6.3. *The Bers embedding β is compatible with the coset decompositions T/T^p and $B(\mathbb{D}^*)/A^p(\mathbb{D}^*)$. More precisely,*

$$\beta(g_*(T^p)) \subset \beta(T) \cap \{\beta([g^{-1}]) + A^p(\mathbb{D}^*)\}$$

for every $g \in \mathrm{QS}$.

Remark 6.4. In the same situation as $T_0^{>0}$, the above inclusion is recently proved to be equality.

6.5. The p-Weil–Petersson metric

The integrable Teichmüller space T^2 is equipped with the Weil–Petersson metric. This was first introduced by Cui [11]. Then the universal Teichmüller space T has the leaf-wise Weil–Petersson metric under the affine foliation, which was investigated by Takhtajan and Teo [41]. The generalization for $p > 2$ can be done similarly ([34]).

Definition 6.5. *The* p-Weil–Petersson metric *on* $T^p \cong \beta(T) \cap A^p(\mathbb{D}^*)$ *at the origin is defined by* $2\|\varphi\|_{p*}$ *for a tangent vector* $\varphi \in A^p(\mathbb{D}^*)$, *where* $\|\varphi\|_{p*}$ *is the operator norm of a bounded linear functional*

$$\ell_\varphi \colon A^q(\mathbb{D}^*) \to \mathbb{C}; \quad \psi \mapsto \int_{\mathbb{D}^*} \psi(z)\overline{\varphi(z)}\rho_{\mathbb{D}^*}^{-2}(z)\,dx\,dy$$

for $1/p + 1/q = 1$. *At an arbitrary point of* T^p, *it is defined after sending the point to the origin by a biholomorphic automorphism* $g_* \colon T^p \to T^p$ *induced by* $g \in \mathrm{Sym}^p$.

The p-Weil–Petersson metric is a Finsler metric on T^p invariant under the action of Sym^p. The p-Weil–Petersson distance induced by this metric is denoted by d_{WP}^p.

6.6. Properties of the Weil–Petersson metric

A remarkable property of the Weil–Petersson metric on T^p is that it is also complete as the Teichmüller metric is.

Proposition 6.6. (T^p, d_{WP}^p) *is a complete metric space.*

Remark 6.7. For the Teichmüller space $T(\Gamma)$ of a cocompact Fuchsian group Γ, the Weil–Petersson metric was originally defined. This is not a complete metric as proved by Wolpert [46]; the Weil–Petersson length of a path in $T(\Gamma)$ along pinching deformation of a non-trivial simple closed curve is finite.

The completeness follows from the next lemma (Cui [11] for $p = 2$ and [34] for the general case).

Lemma 6.8. *The distance* d_{WP}^p *is comparable to the norm* $\|\cdot\|_p$ *on a small neighborhood* U *of the origin in* $\beta(T) \cap A^p(\mathbb{D}^*)$, *which is contained in the domain of the Ahlfors–Weill local section. Namely, there is a constant* $c \geq 1$ *such that*

$$\frac{1}{c}\|\varphi_1 - \varphi_2\|_p \leq d_{\mathrm{WP}}^p(\beta^{-1}(\varphi_1), \beta^{-1}(\varphi_2)) \leq c\|\varphi_1 - \varphi_2\|_p$$

for any $\varphi_1, \varphi_2 \in U$.

On the other hand, the Weil–Petersson distance can be estimated by the norm of Beltrami coefficients ([34]).

Lemma 6.9. *For every* $\mu \in \mathrm{Ael}^p(\mathbb{D})$, *one has*

$$d_{\mathrm{WP}}^p(\pi(0), \pi(\mu)) \leq C\|\mu\|_p,$$

where $C > 0$ *is a constant depending only on* $\|\mu\|_\infty$.

6.7. Comparison with the Teichmüller metric

The universal Teichmüller space T is equipped with the Teichmüller metric. This induces a path metric on $T^p \subset T$ by giving the length of each tangent vector on T^p. We denote this distance on T^p also by d_T. It has been proved by Yanagishita [47] that the Kobayashi distance on the complex manifold T^p coincides with the restriction of the Teichmüller distance on T to T^p. Then we see that they are equal to d_T.

In a usual way, the Weil–Petersson distance can be compared with the Teichmüller distance as follows.

Proposition 6.10. *The Teichmüller distance* (*metric*) *and the Weil–Petersson distance* (*metric*) *on T^p satisfy*

$$d_T \leq c_{q1} d^p_{\mathrm{WP}}.$$

Here p and q are related by $1/p + 1/q = 1$ and c_{q1} is the operator norm of the inclusion map $A^1(\mathbb{D}^*) \to A^q(\mathbb{D}^*)$:

$$c_{q1} = \sup_{\psi \in A^1(\mathbb{D}^*) - \{0\}} \frac{\|\psi\|_q}{\|\psi\|_1} < \infty.$$

Proof. Take $\varphi \in A^p(\mathbb{D}^*) \subset B(\mathbb{D}^*)$ as a tangent vector at the origin of T^p. The p-Weil–Petersson metric is given by the double of the operator norm

$$\|\varphi\|_{p*} = \sup_{\psi \in A^q(\mathbb{D}^*) - \{0\}} \frac{\int_{\mathbb{D}^*} \psi(z)\overline{\varphi(z)}\rho_{\mathbb{D}^*}^{-2}(z)\, dx\, dy}{\|\psi\|_q}.$$

On the other hand, the Teichmüller metric is given by the double of the operator norm

$$\|\varphi\|_{\infty*} = \sup_{\psi \in A^1(\mathbb{D}^*) - \{0\}} \frac{\int_{\mathbb{D}^*} \psi(z)\overline{\varphi(z)}\rho_{\mathbb{D}^*}^{-2}(z)\, dx\, dy}{\|\psi\|_1}.$$

Hence we have $\|\varphi\|_* \leq c_{q1}\|\varphi\|_p$. Integration over a path yields the required inequality for the distances. Q.E.D.

6.8. Curvature of the Weil–Petersson metric

In the case of $p = 2$, the Weil–Petersson metric on T^2 is the Hermitian metric induced by the inner product of the Hilbert space $A^2(\mathbb{D}^*)$. As in the case of finite dimensional Teichmüller spaces due to Ahlfors [1], the Weil–Petersson metric satisfies the following, which was proved by Takhtajan and Teo [41].

Theorem 6.11. *The Weil–Petersson metric on T^2 is Kähler and its Ricci, sectional, and holomorphic sectional curvatures are all negative.*

Combined with Proposition 6.6, this implies that T^2 has a negatively curved property authorized for a general metric space.

Corollary 6.12. (T^2, d^2_{WP}) *is a complete* CAT(0) *space.*

It is well known that a complete CAT(0) space has the fixed point property. See Bridson and Haefliger [8, p. 179].

6.9. Conjugation of a group of circle diffeomorphisms

As an application of the Weil–Petersson metric on T^2, we consider the conjugation problem for a subgroup G of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$. It asks a condition on G under which G is conjugate to a subgroup of $\mathrm{M\ddot{o}b}(\mathbb{S})$ by an element of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$. In other words, G is the image of a trivial $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$-representation of some Möbius group.

Our strategy is as follows. Finding a conjugating element of G in $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ is equivalent to finding a fixed point of G in T_0^α. When T_0^α is contained in T^2, we can utilize the fixed point property of the complete CAT(0) space (T^2, d^2_{WP}). If we formulate a condition for G to act isometrically on (T^2, d^2_{WP}) with a bounded orbit, then it guarantees that G has a fixed point in T^2. Then our rigidity theorem (Theorem 5.6) can be applied to see that the fixed point is actually in T_0^α. The following result is proved in [34].

Theorem 6.13. *Assume $\alpha > 1/2$. For an infinite non-abelian group $G < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$, there exists $f \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ such that $f^{-1}Gf < \mathrm{M\ddot{o}b}(\mathbb{S})$ if and only if $\|s_{\mathrm{DE}}([g])\|_2$ are uniformly bounded and $\|s_{\mathrm{DE}}([g])\|_\infty$ are uniformly less than 1 for all $g \in G$.*

Remark 6.14. The second condition (on $\|s_{\mathrm{DE}}([g])\|_\infty$) is equivalent for G to be uniformly quasisymmetric. Solely by this condition, Theorem 3.2 asserts that there is some $f \in \mathrm{QS}$ such that $f^{-1}Gf < \mathrm{M\ddot{o}b}(\mathbb{S})$.

Proof. "Only if" part is easy. Note that $\mathrm{Diff}^{1+\alpha}(\mathbb{S}) < \mathrm{Sym}^2$ for $\alpha > 1/2$. If there is $f \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ such that $\gamma = f^{-1}gf$ belongs to $\mathrm{M\ddot{o}b}(\mathbb{S})$ for every $g \in G$, then $g = f\gamma f^{-1} \in \mathrm{Sym}^2$. From this, $\|s_{\mathrm{DE}}([g])\|_2$ and $\|s_{\mathrm{DE}}([g])\|_\infty$ are estimated uniformly.

"If" part is essential. Since $G < \mathrm{Diff}^{1+\alpha}(\mathbb{S}) < \mathrm{Sym}^2$, G acts on T^2 isometrically with respect to d^2_{WP}. By Lemma 6.9, we see that G has a bounded orbit in (T^2, d^2_{WP}) from the uniform boundedness of both $\|s_{\mathrm{DE}}([g])\|_2$ and $\|s_{\mathrm{DE}}([g])\|_\infty$. Corollary 6.12 says that (T^2, d^2_{WP}) is a complete CAT(0) space, which ensures the fixed point property. Hence

G has a fixed point $[f] \in T^2$ for $f \in \mathrm{Sym}^2 < \mathrm{Sym}$. Then $\Gamma = f^{-1}Gf$ is a subgroup of $\mathrm{Möb}(\mathbb{S})$.

The last condition is conversely expressed as $f\Gamma f^{-1} < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$. Then the rigidity theorem for $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$-representation (Theorem 5.6) implies that $f \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$. Q.E.D.

A related result has been obtained by Navas [38] by using a unitary representation of $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$ in $L^2(\mathbb{S} \times \mathbb{S}, m)$ for the Liouville measure m.

6.10. An extension of the conjugation problem to the general case

We will try to generalize the statement of Theorem 6.13 to an arbitrary $\alpha \in (0,1)$. However, we need an extra assumption as in the following theorem. We will discuss an idea for removing it later.

Theorem 6.15. *For each $\alpha \in (0,1)$, there is a constant $c > 0$ depending only on p with $p > 1/\alpha$ such that if an infinite non-abelian group $G < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ satisfies*

$$\inf_{\pi(\mu)=[g]} \left(\int_{\mathbb{D}} \left(\frac{|\mu(z)|^2}{1-|\mu(z)|^2} \right)^{p/2} \rho_{\mathbb{D}}^2(z)\, dx\, dy \right)^{1/p} \leq c$$

for every $g \in G$ then there is $f \in \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ such that $f^{-1}Gf < \mathrm{Möb}(\mathbb{S})$.

Remark 6.16. The condition that G is uniformly quasisymmetric follows from the above assumption. Indeed, the assumption implies that $\|\beta([g])\|_p$ are sufficiently small for all $g \in G$. Then, in a small neighborhood of the origin, the Teichmüller distance can be estimated by the p-norm.

Proof. An outline is as follows. We rely on Theorem 3.2 to see that G is quasisymmetric conjugate to a Möbius group. Then the assumption implies that G acts on $A^p(\mathbb{D}^*)$ for some p isometrically with a bounded orbit. In this case, we use the fixed point property of $A^p(\mathbb{D}^*)$ to have a fixed point of G in it. To guarantee that the fixed point is actually in the image $\beta(T)$ of the Bers embedding, we need the assumption that the norms $\|\beta([g])\|_p$ are sufficiently small. Theorem 5.6 can conclude that the fixed point is in $B_0^\alpha(\mathbb{D}^*)$. Q.E.D.

We mention an application of the above theorem. For $[g] \in T_0^\alpha = \mathrm{Möb}(\mathbb{S})\backslash\mathrm{Diff}^{1+\alpha}(\mathbb{S})$, define the infimum of the α-Hölder constant in the Teichmüller class by

$$c_\alpha([g]) = \inf_{g \in [g]} \sup_{x,y \in \mathbb{R}} \frac{|\tilde{g}'(x) - \tilde{g}'(y)|}{|x-y|^\alpha}.$$

The following result is obtained in [31].

Corollary 6.17. *For each* $\alpha \in (0,1)$, *there is a constant* $c_0 > 0$ *depending only on* α *such that if an infinite non-abelian group* $G < \mathrm{Diff}^{1+\alpha}(\mathbb{S})$ *satisfies* $c_\alpha([g]) \le c_0$ *for every* $g \in G$ *then* $G < \text{Möb}(\mathbb{S})$.

6.11. Uniform convexity

For the proof of Theorem 6.15, we have used the fixed point property of $A^p(\mathbb{D}^*)$. This comes from uniform convexity of the Banach space $A^p(\mathbb{D}^*)$. To extend Theorem 6.15 to the statement similar to Theorem 6.13, we have only to show that (T^p, d^p_{WP}) possesses the fixed point property for any $p \ge 2$.

Definition 6.18. *A complete metric space* (X,d) *is* p-uniformly convex $(p \ge 1)$ *with constant* $c > 0$ *if for any* $x, y \in X$ *there is* $m \in X$ *such that every* $z \in X$ *satisfies*

$$d(z,m)^p \le \frac{1}{2}\{d(z,x)^p + d(z,y)^p\} - c\, d(x,y)^p.$$

Remark 6.19. For $p = 2$ and $c = 1/4$, the above condition is equivalent to that (X,d) is a CAT(0) space. See Ballmann [4, Proposition I.5.1].

Uniform convexity implies the fixed point property. See Naor and Silberman [37]. Also we see that uniform convexity implies uniformly normal structure ([32]). For the generalization of Theorem 6.13, the following question might be interesting.

Problem 6.20. Is (T^p, d^p_{WP}) p-uniformly convex?

References

[1] L. Ahlfors, Curvature properties of Teichmüller's space, *J. Analyse Math.* **9** (1961), 161–176.

[2] L. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, *Ann. of Math.* (2) **72** (1960), 385–404.

[3] L. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.* **13** (1962), 975–978.

[4] W. Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar **25**, Birkhäuser, Basel, 1995.

[5] J. Becker and C. Pommerenke, Über die quasikonforme Fortsetzung schlichter Funktionen, *Math. Z.* **161** (1978), 69–80.

[6] L. Bers, A non-standard integral equation with applications to quasiconformal mappings, *Acta Math.* **116** (1966), 113–134.

[7] A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* **96** (1956), 125–142.
[8] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften **319**, Springer, Berlin, 1999.
[9] L. Carleson, On mappings, conformal at the boundary, *J. Analyse Math.* **19** (1967), 1–13.
[10] A. Casson and D. Jungreis, Convergence groups and Seifert fibered 3-manifolds, *Invent. Math.* **118** (1994), 441–456.
[11] G. Cui, Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, *Sci. China Ser. A* **43** (2000), 267–279.
[12] A. Douady and C. J. Earle, Conformally natural extension of homeomorphisms of the circle, *Acta Math.* **157** (1986), 23–48.
[13] C. J. Earle and F. P. Gardiner, Geometric isomorphisms between infinite-dimensional Teichmüller spaces, *Trans. Amer. Math. Soc.* **348** (1996), 1163–1190.
[14] C. J. Earle, F. P. Gardiner and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence*, Inst. Hautes Études Sci. (1995), unpublished manuscript.
[15] C. J. Earle, F. P. Gardiner and N. Lakic, Asymptotic Teichmüller space, I, The complex structure, in *In the tradition of Ahlfors and Bers* (*Stony Brook, NY, 1998*), Contemp. Math. **256**, Amer. Math. Soc., Providence, RI, 2000, 17–38.
[16] C. J. Earle, F. P. Gardiner and N. Lakic, Asymptotic Teichmüller space, II, The metric structure, in *In the tradition of Ahlfors and Bers, III*, Contemp. Math. **355**, Amer. Math. Soc., Providence, RI, 2004, 187–219.
[17] C. J. Earle, V. Markovic and D. Saric, Barycentric extension and the Bers embedding for asymptotic Teichmüller space, in *Complex manifolds and hyperbolic geometry* (*Guanajuato, 2001*), Contemp. Math. **311**, Amer. Math. Soc., Providence, RI, 2002, 87–105.
[18] R. Fehlmann, Über extremale quasikonforme Abbildungen, *Comment. Math. Helv.* **56** (1981), 558–580.
[19] D. Gabai, Convergence groups are Fuchsian groups, *Ann. of Math.* (2) **136** (1992), 447–510.
[20] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory*, Mathematical Surveys and Monographs **76**, Amer. Math. Soc., Providence, RI, 2000.
[21] F. P. Gardiner and D. P. Sullivan, Symmetric structures on a closed curve, *Amer. J. Math.* **114** (1992), 683–736.
[22] É. Ghys, Rigidité différentiable des groupes fuchsiens, *Inst. Hautes Études Sci. Publ. Math.* **78** (1993), 163–185.
[23] É. Ghys and T. Tsuboi, Différentiabilité des conjugaisons entre systèmes dynamiques de dimension 1, *Ann. Inst. Fourier* (*Grenoble*) **38** (1988), 215–244.
[24] H. Guo, Integrable Teichmüller spaces, *Sci. China Ser. A* **43** (2000), 47–58.

[25] S. P. Kerckhoff, The Nielsen realization problem, *Ann. of Math.* (2) **117** (1983), 235–265.

[26] M. A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001.

[27] O. Lehto, *Univalent functions and Teichmüller spaces*, Graduate Texts in Mathematics **109**, Springer, New York, 1987.

[28] V. Markovic, Biholomorphic maps between Teichmüller spaces, *Duke Math. J.* **120** (2003), 405–431.

[29] V. Markovic, Quasisymmetric groups, *J. Amer. Math. Soc.* **19** (2006), 673–715.

[30] K. Matsuzaki, Symmetric groups that are not the symmetric conjugates of Fuchsian groups, in *In the tradition of Ahlfors-Bers, V*, Contemp. Math. **510**, Amer. Math. Soc., Providence, RI, 2010, 239–247.

[31] K. Matsuzaki, The universal Teichmüller space and diffeomorphisms of the circle with Hölder continuous derivatives, in *Handbook of Group Actions*, Advanced Lectures in Mathematics **31**, Higher Education Press and International Press, Somerville, 2015, 333–372.

[32] K. Matsuzaki, Uniform convexity, normal structure and the fixed point property of metric spaces, *Topology Appl.* **196** (2015), part B, 684–695.

[33] K. Matsuzaki, *Teichmüller spaces of circle diffeomorphisms with Hölder continuous derivatives*, arXiv:1607.06300.

[34] K. Matsuzaki, *Rigidity of groups of circle diffeomorphisms and Teichmüller spaces*, arXiv:1607.06316.

[35] S. Nag, *The complex analytic theory of Teichmüller spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, New York, 1988.

[36] S. Nag and A. Verjovsky, Diff(S^1) and the Teichmüller spaces, *Comm. Math. Phys.* **130** (1990), 123–138.

[37] A. Naor and L. Silberman, Poincaré inequalities, embeddings, and wild groups, *Compos. Math.* **147** (2011), 1546–1572.

[38] A. Navas, On uniformly quasisymmetric groups of circle diffeomorphisms, *Ann. Acad. Sci. Fenn. Math.* **31** (2006), 437–462.

[39] Y. Shen, *Weil–Petersson Teichmüller Space*, arXiv:1304.3197.

[40] D. Sullivan, On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud. **97**, Princeton Univ. Press, Princeton, NJ, 1981, 465–496.

[41] L. A. Takhtajan and L.-P. Teo, Weil–Petersson metric on the universal Teichmüller space, *Mem. Amer. Math. Soc.* **183** (2006).

[42] L.-F. Tam and T. Y. H. Wan, Quasi-conformal harmonic diffeomorphism and the universal Teichmüller space, *J. Differential Geom.* **42** (1995), 368–410.

[43] S. Tang, Some characterizations of the integrable Teichmüller space, *Sci. China Math.* **56** (2013), 541–551.
[44] P. Tukia, On two-dimensional quasiconformal groups, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **5** (1980), 73–78.
[45] P. Tukia, Homeomorphic conjugates of Fuchsian groups, *J. Reine Angew. Math.* **391** (1988), 1–54.
[46] S. Wolpert, Noncompleteness of the Weil–Petersson metric for Teichmüller space, *Pacific J. Math.* **61** (1975), 573–577.
[47] M. Yanagishita, Introduction of a complex structure on the p-integrable Teichmüller space, *Ann. Acad. Sci. Fenn. Math.* **39** (2014), 947–971.
[48] J. M. Anderson, A. Cantón and J. L. Fernández, On smoothness of symmetric mappings, *Complex Var. Theory Appl.* **37** (1998), 161–169.
[49] E. Dyn'kin, Estimates for asymptotically conformal mappings, *Ann. Acad. Sci. Fenn. Math.* **22** (1997), 275–304.
[50] M. A. Khamsi, On metric spaces with uniform normal structure, *Proc. Amer. Math. Soc.* **106** (1989), 723–726.
[51] K. Matsuzaki, The Teichmüller space of group invariant symmetric structures on the circle, *Ann. Acad. Sci. Fenn. Math.*, to appear.
[52] K. Matsuzaki, *Continuity of the barycentric extension of circle diffeomorphisms of Hölder continuous derivatives*, arXiv:1607.06310.

Katsuhiko Matsuzaki
Department of Mathematics, School of Education, Waseda University
Shinjuku, Tokyo 169-8050, Japan
E-mail address: matsuzak@waseda.jp