Multivariate Analysis and Its Applications
IMS Lecture Notes – Monograph Series (1994) Volume 24

UNLINKING THEOREM FOR SYMMETRIC CONVEX FUNCTIONS†

BY S. K. BHANDARI AND S. DASGUPTA

Indian Statistical Institute, Calcutta, India

In this paper the authors have proved the following result: Suppose U and V are two centrally symmetric convex functions of X, when X is an $n \times 1$ random vector distributed as $N(0, I_n)$ such that $\mathrm{Cov}(U(X), V(X)) = 0$. Then, under certain conditions, there exists an orthogonal transformation $Y = LX$ such that U and V can be expressed as functions of two different sets of components of Y. This provides a partial answer to Linnik's question on unlinking two given functions of X.

1. Introduction.

Kagan et al. [1] have considered the following problem. let X be an $n \times 1$ random vector distributed as $N(0, I_n)$. Suppose $P(X)$ and $Q(X)$ are two independently distributed polynomial functions. Is it possible to find an orthogonal transformation $Y = LX$ such that P and Q could be expressed as functions of different sets of components of Y? If the answer to this question is in the affirmative, then the functions P and Q are said to be unlinked. Partial answers to this question are given in Chapter II of [1].

We have shown in this paper that two statistics $U(X)$ and $V(X)$ could be unlinked when both U and V are centrally symmetric convex functions and $\mathrm{Cov}(U(X), V(X)) = 0$ under certain conditions on U and V. Our result depends on the validity of a probability inequality given in lemma 3.

2. Preliminary Results.

LEMMA 1. *Let g be a convex function on $\mathbb{R}$ to $\mathbb{R}$. Suppose there exists λ_1, λ_2 in $\mathbb{R}$ such that $g(\lambda_1) \neq g(\lambda_2)$. Then at least one of the following holds.*

(a) *There exists λ_0 such that $g(u) < g(v)$ for $\lambda_0 \leq u < v$ and $g(\lambda) \to \infty$ as $\lambda \to \infty$.*

(b) *There exists λ_0 such that $g(u) < g(v)$ for $v < u \leq \lambda_0$ and $g(\lambda) \to +\infty$ as $\lambda \to -\infty$.*

---

† Research supported by Australian Research Council Grant A6 8931380

AMS 1980 Subject Classifications: Primary 62H10, Secondary 52A20.

Key words and phrases: Convex symmetric functions and Linnik's unlinking proposition.

PROOF. Suppose $\lambda_1 < \lambda_2$ and $g(\lambda_1) < g(\lambda_2)$. Define $h(\lambda) = g(\lambda) - g(\lambda_1)$. Then for $\lambda_2 < \lambda$

$$\begin{aligned} h(\lambda_2) \leq & \frac{\lambda - \lambda_2}{\lambda - \lambda_1} h(\lambda_1) + \frac{\lambda_2 - \lambda_1}{\lambda - \lambda_1} h(\lambda) \\ = & \frac{\lambda_2 - \lambda_1}{\lambda - \lambda_1} h(\lambda). \end{aligned}$$

Thus

$$h(\lambda) \geq \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} h(\lambda_2) > h(\lambda_2).$$

The above shows that $h(\lambda)$, as well as $g(\lambda)$, strictly increases on (λ_2, ∞) and tends to ∞ as $\lambda \to \infty$. Now take $\lambda_0 > \lambda_2$ to satisfy (a).

If $g(\lambda_1) > g(\lambda_2)$, then the above method of proof yields (b).

COROLLARY 1.1. *Let g be a convex function on $\mathbb{R}$ to $\mathbb{R}$. If g is bounded above, then g must be a constant function.*

COROLLARY 1.2. *Let U be a convex function on $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that for some fixed vector $\alpha \in \mathbb{R}^n, U(\lambda \cdot \alpha)$ is a constant function of λ. Then for any fixed vector $b \in \mathbb{R}^n, U(b + \lambda\alpha)$ is a constant function of λ.*

PROOF. Note that

$$U(b + \lambda\alpha) \leq \frac{1}{2} U(2b) + \frac{1}{2} U(2\lambda\alpha).$$

Thus $g(\lambda) \equiv U(b + \lambda\alpha)$ is bounded above. Now Corollary 1.1 yields the result.

LEMMA 2. *Let U be a convex function on $\mathbb{R}^n \to \mathbb{R}$ with $U(0) = 0$. Let*

$$S_U = \{\alpha : U(\lambda\alpha) = 0 \text{ for all } \lambda \in \mathbb{R}\}.$$

Then S_U is a vector subspace of $\mathbb{R}^n$.

PROOF. Suppose α_1, α_2 are in S_U. For c_1, c_2 in $\mathbb{R}$,

$$U(c_1\alpha_1 + c_2\alpha_2) = U(c_1\alpha_1) = 0$$

by Corollary 1.2. Thus $c_1\alpha_1 + c_2\alpha_2$ is in S_U.

LEMMA 3. Let X be an $r \times 1$ random vector distributed as $N(0, I_r)$. Let A and B be centrally symmetric (i.e., $A = -A, B = -B$) convex sets in $\mathbb{R}^r$. Then, for $r \leq 2$,

$$P[X \in A \cap B] \geq P[X \in A], P[X \in B].$$

The above lemma is trivially true for $r = 1$. Pitt has proved it when $r = 2$ (Theorem 2 in [3]).

3. The Main Result. Suppose U and V are two convex functions on $\mathbb{R}^n \to \mathbb{R}$ such that $U(0) = V(0) = 0$. Define S_U and S_V as in Lemma 2.

DEFINITION. *U and V are said to be concordant of order r, if*

$$\dim(S_U^{\perp}) - \dim(S_U^{\perp} \cap S_V) = r.$$

Note that this definition is symmetric in U and V, since

$$\begin{aligned} r =& \dim(S_U^{\perp}) - \dim(S_U^{\perp} \cap S_V) \\ =& n - \dim(S_U) - n + \dim(S_U + S_V^{\perp}) \\ =& \dim(S_U + S_V^{\perp}) - \dim(S_U) \\ =& \dim(S_V^{\perp}) - \dim(S_U \cap S_V^{\perp}). \end{aligned}$$

We now state the main result.

THEOREM. *Let X be an $n \times 1$ random vector distributed as $N(0, I_n)$. Let U and V be two centrally symmetric (i.e., $U(X) = U(-X), V(X) = V(-X)$) convex functions of X such that $\mathrm{Cov}(U(X), V(X)) = 0$. Furthermore, assume that $U(0) = 0 = V(0)$, and U and V are concordant of order $r \leq 2$. Then there exists an orthogonal transformation $Y = LX$ such that U and V can be expressed as functions of two different sets of components of Y.*

PROOF. Let $\{\alpha_1, \cdots, \alpha_{r+t}\}, \{\alpha_{r+1}, \cdots, \alpha_{r+t}\}, \{\alpha_1, \cdots, \alpha_{r+t+m}\}$, and $\{\alpha_1, \cdots, \alpha_n\}$ be orthonormal bases of $S_U^{\perp}, S_U^{\perp} \cap S_V, S_U^{\perp} + S_V^{\perp}$, and $\mathbb{R}^n$, respectively. We shall show that $\mathrm{Cov}(U(X), V(X)) > 0$ if $r \neq 0$; otherwise $U(X)$ and $V(X)$ could be unlinked. Note that Y_i's defined by $X = \sum_1^n Y_i \alpha_i$ are i.i.d. as $N(0,1)$. By Corollary 1.2,

$$U(X) = U\left(\sum_1^n Y_i \alpha_i\right) = U\left(\sum_1^r Y_i \alpha_i + \sum_{r+1}^{r+t} Y_i \alpha_i\right), \tag{3.1}$$

$$V(X) = V\left(\sum_1^n Y_i \alpha_i\right) = V\left(\sum_1^r Y_i \alpha_i + \sum_{r+t+1}^{r+t+m} Y_i \alpha_i\right). \tag{3.2}$$

If $r = 0$, we are done. In the following we shall assume $r > 0$. Let $y^* = (y_1, \cdots, y_r)'$. Define

$$U^*(y^*) = E\left(U\left(\sum_1^r y_i \alpha_i + \sum_{r+1}^{r+t} Y_i \alpha_i\right)\right), \tag{3.3}$$

$$V^*(y^*) = E\left(V\left(\sum_1^r y_i \alpha_i + \sum_{r+t+1}^{r+t+m} Y_i \alpha_i\right)\right), \tag{3.4}$$

Note that both U^* and V^* are centrally symmetric convex functions of y^*.

Since $U\left(\lambda \sum_1^r y_i\alpha_i\right)$ is not identically 0 as a function of $\lambda, U\left(\lambda \sum_1^r y_i\alpha_i + \sum_{r+1}^{r+t} y_i\alpha_i\right)$ is a non-constant function of λ (use Corollary 1.2). By Lemma 1

$$U\left(\lambda \sum_1^r y_i\alpha_i + \sum_{r+1}^{r+t} y_i\alpha_i\right) + U\left(-\lambda \sum_1^r y_i\alpha_i + \sum_{r+1}^{r+t} y_i\alpha_i\right) \tag{3.5}$$

tends to ∞ as $\lambda \to \infty$. Taking the expectation of (3.5) with respect to $Y_{r+1}, \cdots, Y_{r+t}$ and using Egoroff's theorem [2], we get

$$U^*(\lambda y^*) \to \infty \text{ as } \lambda \to \infty. \tag{3.6}$$

Similarly

$$V^*(\lambda y^*) \to \infty \text{ as } \lambda \to \infty. \tag{3.7}$$

Note that

$$\begin{aligned}
\mathrm{Cov}(U(X), V(X)) =& EU(X)V(X) - E[U(X)]E[V(X)] \\
=& EU^*(Y^*)V^*(Y^*) - E[U^*(Y^*)]E[V^*(Y^*)] \\
=& \int_0^\infty \int_0^\infty [P(A_{k_1}^c \cap B_{k_2}^c) - P(A_{k_1}^c)P(B_{k_2}^c)]dk_1 dk_2 \\
=& \int_0^\infty \int_0^\infty [P(A_{k_1} \cap B_{k_2}) - P(A_{k_1})P(B_{k_2})]dk_1 dk_2
\end{aligned} \tag{3.8}$$

where

$$A_{k_1} = \{y^* : U^*(y^*) \le k_1\}, \tag{3.9}$$

$$B_{k_2} = \{y^* : V^*(y^*) \le k_2\}. \tag{3.10}$$

From (3.6), (3.7) and Lemma 1 we can assert that there exist k_1, k_2 sufficiently large, such that

$$A_{k_1} \subset B_{k_2}, P(B_{k_2}^c) > 0, P(A_{k_1}) > 0. \tag{3.11}$$

Now Lemma 3 and (3.11) yield

$$\mathrm{Cov}(U(X), V(X)) > 0, \tag{3.12}$$

since there would exist a set of values of k_1, k_2 with positive Lebesgue measure for which the integrand in (3.8) is strictly positive. Since (3.12) contradicts the assumption we must have $r = 0$.

NOTE 1. In the above theorem we have assumed that $U(0) = V(0) = 0$. In general, $U(X) \ge U(0)$ and $V(X) \ge V(0)$. So the above assumption can be made without any loss of generality.

NOTE 2. The above theorem also holds for any $r > 2$, provided Lemma 3 holds for that r. There will be no change in the proof for general r. However, it is not known whether Lemma 3 holds for $r > 2$. It may be noted that the theorem is always true when $n = 2$.

Acknowledgement. The authors are grateful to Dr. Aloke Goswami for some helpful discussions.

REFERENCES

KAGAN, A. M., LINNIK, YU. V., and RAO, C. R. (1973). *Characterization Problems in Mathematical Statistics.* Wiley, New York.

MUNROE, M. E. (1952). *Introduction to Measure and Integration*, Addison-Wesley, Cambridge, Mass.

PITT, L. D. (1977). A Gaussian inequality for symmetric convex sets. *Ann. Probability* **5**, 470–474.

INDIAN STATISTICAL INSTITUTE
203 B.T. BD
CALCUTTA 700035
INDIA