Chapter 5

Stochastic differential equations in Hilbert space

Throughout this chapter, H will be a separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $\|\cdot\|$. $L(H,H)$ will denote the class of all continuous linear operators on H and $L_2(H,H)$ the class of all Hilbert-Schmidt operators. For an operator $A \in L_2(H,H)$, the Hilbert-Schmidt norm will be denoted by $\|\cdot\|_2$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a given filtration $(\mathcal{F}_t)$ assumed to satisfy the usual conditions. Let (W_t) be an $(\mathcal{F}_t)$-cylindrical Brownian motion (c.B.m) on H and let (B_t) be an $(\mathcal{F}_t)$-adapted H-valued Brownian with covariance Σ (cf. Section 3.2 for definition).

5.1 Diffusion equations in Hilbert spaces

Suppose that $A : H \to H$ and $G : H \to L(H,H)$ are two continuous mappings. We consider the following SDE on H:

$$X_t = X_0 + \int_0^t A(X_s)ds + \int_0^t G(X_s)dB_s. \tag{5.1.1}$$

It is possible to establish a unique solution for (5.1.1) by making use of weak convergence techniques and by following the method which will be developed in Chapter 6, i.e., first we obtain a solution for the corresponding martingale problem by approximation and then get a weak solution by the representation theorems given in Chapter 3; finally we establish a unique strong solution by the Yamada-Watanabe argument. However, in this section, we shall adopt the approach given by Leha and Ritter [37] to establish a unique strong directly.

Definition 5.1.1 *$\{X_t\}$ is called a* **strong solution** *of (5.1.1) with explosion time τ if*
(i)

$$\limsup_{t\to\tau} \|X_t\| = \infty \qquad \text{on the set } \{\tau < \infty\}.$$

(ii) There exists a sequence $\{\tau_n\}$ of stopping times, increasing to τ, such that
(a)

$$E\int_0^{t\wedge\tau_n} \|A(X_s)\|ds < \infty.$$

(b)

$$E\int_0^{t\wedge\tau_n} \|G(X_s)\|^2_{L(H,H)}ds < \infty.$$

(c)

$$X_{t\wedge\tau_n} = X_0 + \int_0^{t\wedge\tau_n} A(X_s)ds + \int_0^{t\wedge\tau_n} G(X_s)dB_s$$

where the stochastic integral is defined as $I_t(f)$ (cf. Section 3.3) with

$$f(s,\omega) = G(X_s)\Sigma^{\frac{1}{2}}1_{s\le\tau_n} \in L_{(2)}(H,H).$$

As

$$E\int_0^t \|f(s,\omega)\|^2_{(2)}ds \le \|\Sigma\|_{(1)}E\int_0^{t\wedge\tau_n} \|G(X_s)\|^2_{L(H,H)}ds < \infty,$$

$I_t(f)$ is well-defined, where $\|\cdot\|_{(1)}$ denotes the nuclear norm of nuclear operators on H.

Theorem 5.1.1 (Leha-Ritter) *Suppose that X_0 has a finite second moment and A, G satisfy Lipschitz conditions on bounded sets, i.e., $\forall n$, $\exists L_n$ such that $\forall x, y \in H$, $\|x\| \le n$, $\|y\| \le n$, we have*

$$\|A(x) - A(y)\| + \|G(x) - G(y)\|_{L(H,H)} \le L_n\|x - y\|.$$

Then there is a unique strong solution to the SDE (5.1.1).

Proof: First assume the global Lipschitz conditions for A and G, i.e. $L_n = L$. We construct a Picard sequence as follows:

$$\begin{aligned} X_t^0 &\equiv X_0, \\ X_t^{m+1} &\equiv X_0 + \int_0^t A(X_s^m)ds + \int_0^t G(X_s^m)dB_s. \end{aligned} \tag{5.1.2}$$

By induction, it is easy to show that $\forall t \ge 0$

$$E\int_0^t \|X_s^m\|^2 ds < \infty \quad \forall m \ge 0,$$

and consequently, $\{X_t^m\}$ is well-defined by (5.1.2).

Note that, by Theorem 3.3.2,

$$\begin{aligned}
& E \sup_{0\le t\le r} \left\| \int_0^t \left(G(X_s^m) - G(X_s^{m-1}) \right) dB_s \right\|^2 \\
\le\ & 4E \int_0^r \left\| \left(G(X_s^m) - G(X_s^{m-1}) \right) \Sigma^{\frac{1}{2}} \right\|_2^2 ds \\
\le\ & 4\|\Sigma\|_{(1)} L^2 \int_0^r \|X_s^m - X_s^{m-1}\|^2 ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
D_r^m &\equiv E \sup_{0\le t\le r} \|X_s^m - X_s^{m-1}\|^2 \\
&\le 2E \sup_{0\le t\le r} \left\| \int_0^t \left(G(X_s^m) - G(X_s^{m-1}) \right) dB_s \right\|^2 \\
&\quad +2E \sup_{0\le t\le r} \left\| \int_0^t \left(A(X_s^m) - A(X_s^{m-1}) \right) ds \right\|^2 \\
&\le 2L^2(4\|\Sigma\|_{(1)} + r) \int_0^r D_s^{m-1} ds. \qquad (5.1.3)
\end{aligned}$$

Let $K(r) = 2L^2(4\|\Sigma\|_{(1)} + r)$. Then

$$D_r^m \le D_r^0 \frac{(K(r)r)^m}{m!} \qquad (5.1.4)$$

where

$$D_r^0 \le 2E \sup_{0\le t\le r} \left\| \int_0^t G(X_0) dB_s \right\|^2 + 2E \sup_{0\le t\le r} \left\| \int_0^t A(X_0) ds \right\|^2 < \infty.$$

$\forall r > 0$, let

$$\Omega_r = \left\{ \omega : \sum_{m=1}^\infty \sup_{0\le t\le r} \|X_t^m - X_t^{m-1}\| < \infty \right\}.$$

As

$$\begin{aligned}
E \sum_{m=1}^\infty \sup_{0\le t\le r} \|X_t^m - X_t^{m-1}\| &\le \sum_{m=1}^\infty \sqrt{E \sup_{0\le t\le r} \|X_t^m - X_t^{m-1}\|^2} \\
&\le \sum_{m=1}^\infty \sqrt{D_r^0 \frac{(K(r)r)^m}{m!}} < \infty,
\end{aligned}$$

$P(\Omega_r) = 1$ and hence, $P(\Omega') = 1$ where $\Omega' = \cup_{r=1}^\infty \Omega_r$. It is clear that $\forall \omega \in \Omega'$, $\exists X(\omega) \in C([0,\infty), H)$ s.t. $\forall T > 0$

$$\sup_{0\le t\le T} \|X_t^m(\omega) - X_t(\omega)\| \to 0. \qquad (5.1.5)$$

By (5.1.4), it is easy to show that, $\forall r > 0$

$$K_1(r) \equiv \sup_{m\geq 0} E \sup_{0\leq t\leq r} \|X_t^m\|^2 < \infty \tag{5.1.6}$$

and

$$\lim_{m\to\infty} E \sup_{0\leq t\leq r} \|X_t^m - X_t\|^2 \to 0. \tag{5.1.7}$$

By Fatou's lemma, we then have

$$E \sup_{0\leq t\leq r} \|X_t\|^2 \leq K_1(r). \tag{5.1.8}$$

Now we show that $\{X_t\}$ satisfies the conditions of Definition 5.1.1 with $\tau \equiv \infty$. (i) is trivially true. For (ii), (a) and (b) follows from (5.1.8) and the global Lipschitz conditions on A and G. (c) follows from (5.1.7) and (5.1.2). Hence X is a strong solution of (5.1.1).

Suppose that $\tilde{X}$ is another solution and let

$$\tilde{D}_r = E \sup_{0\leq t\leq r} \|X_s - \tilde{X}_s\|^2.$$

As in (5.1.3) we have

$$\tilde{D}_r \leq 2L^2(4\|\Sigma\|_{(1)} + r) \int_0^r \tilde{D}_s ds$$

and hence $\tilde{D} \equiv 0$. This proves the uniqueness of the solution.

Finally, we return to the general case. Define

$$G_n(x) = \begin{cases} G(x) & \text{if } \|x\| \leq n \\ G\left(\frac{nx}{\|x\|}\right) & \text{otherwise.} \end{cases}$$

A_n can be defined similarly. Then A_n, G_n satisfy the global Lipschitz conditions and hence by the first part of the proof there is a unique strong solution ξ^n for (5.1.1) with A, G replaced by A_n, G_n respectively. Let τ_n be the first exit time of ξ^n from $\{x \in H : \|x\| \leq n\}$. Then $\{\tau_n\}$ is a non-decreasing sequence of stopping times and

$$\xi_t^{n+1} = \xi_t^n \quad \forall t \leq \tau_n.$$

Let $\tau = \sup_n \tau_n$ and

$$X_t = \xi_t^n \quad \forall t \leq \tau_n.$$

Then

$$\limsup_{t\to\tau} \|X_t\| \geq \lim_{n\to\infty} \|X_{\tau_n}\| = \infty.$$

This proves (i) of Definition 5.1.1. The condition (ii) follows directly from the construction of $\{X_t\}$. Hence X is a solution of (5.1.1) upto time τ. ∎

It follows from the proof of the above theorem that $\tau = \infty$ a.s. if A and G satisfy the global Lipschitz conditions. The following theorem gives the same result under weaker conditions.

Theorem 5.1.2 (Leha-Ritter) *If, in addition to the conditions of Theorem 5.1.1, A and G satisfy the following: There exists a positive constant K such that for any $x \in H$,*

$$< x, A(x) > \leq K(1 + \|x\|^2)$$

and

$$\|G(x)\|^2_{L(H,H)} \leq K(1 + \|x\|^2),$$

then $\{X_t\}$ has infinite explosion time, i.e., $\tau = \infty$ a.s.

Proof: We use the same notation as in the proof of Theorem 5.1.1. It follows from Itô's formula that

$$\begin{aligned} \|\xi^n_{t\wedge\tau_n}\|^2 &= \|X_0\|^2 + 2\int_0^{t\wedge\tau_n} < G(\xi^n_s)^*\xi^n_s, dB_s > \\ &\quad + 2\int_0^{t\wedge\tau_n} < \xi^n_s, A(\xi^n_s) > ds + \int_0^{t\wedge\tau_n} \left\|G(\xi^n_s)\Sigma^{\frac{1}{2}}\right\|_2^2 ds \end{aligned}$$

where $G(\xi^n_s)^*$ denotes the adjoint operator of $G(\xi^n_s)$. Therefore

$$\begin{aligned} E\|\xi^n_{t\wedge\tau_n}\|^2 &\leq E\|X_0\|^2 + 2K\int_0^t (1 + E\|\xi^n_{s\wedge\tau_n}\|^2)ds \\ &\quad + \|\Sigma\|_{(1)}K\int_0^t (1 + E\|\xi^n_{s\wedge\tau_n}\|^2)ds. \end{aligned}$$

By Gronwall's inequality, we have

$$E\|\xi^n_{t\wedge\tau_n}\|^2 \leq (1 + E\|X_0\|^2)e^{\|\Sigma\|_{(1)}Kt} \equiv g(t) < \infty, \quad \forall t > 0.$$

Hence

$$P(\tau \leq t) \leq P(\tau_n \leq t) \leq P(\|\xi^n_{t\wedge\tau_n}\| \geq n) \leq n^{-2}g(t),$$

i.e. $P(\tau \leq t) = 0$ $\forall t > 0$ and hence, $\tau = \infty$ a.s. ∎

5.2 Stochastic evolution equations in Hilbert space

We are going to consider the following SDE

$$dX_t = -LX_t dt + G(t, X_t)dW_t + A(t, X_t)dt \tag{5.2.1}$$

where X_0 is independent of (W_t). Here the operator L is assumed to satisfy the following conditions:

$$T_t \equiv e^{-tL} \quad \text{is a contraction semigroup on H,} \tag{5.2.2}$$

$$L^{-1} \quad \text{is a bounded self-adjoint operator with discrete spectrum.} \tag{5.2.3}$$

Let $\{\phi_k\}$ be the eigenfunctions of L, which constitutes a CONS in H and let $\{\lambda_k\}$ be the corresponding eigenvalues. We assume also that $A : [0,T] \times H \to H$ and $G : [0,T] \times H \to L(H, H)$ are continuous functions satisfying

$$|\langle A(t,h), \phi_k\rangle| \leq a_k(1 + \|h\|^2)^{\frac{1}{2}} \tag{5.2.4}$$

$$\|G^*(t,h)\phi_k\| \leq b_k(1 + \|h\|^2)^{\frac{1}{2}} \tag{5.2.5}$$

$$|\langle A(t,h_1) - A(t,h_2), \phi_k\rangle| \leq a_k\|h_1 - h_2\| \tag{5.2.6}$$

$$\|(G^*(t,h_1) - G^*(t,h_2))\phi_k\| \leq b_k\|h_1 - h_2\| \tag{5.2.7}$$

for all $k \geq 1, t \in [0,T], h, h_1, h_2 \in H$, where G^* is the adjoint of the operator G and $\{a_k\}$, $\{b_k\}$ satisfy

$$\sum_{k=1}^{\infty} a_k^2 \lambda_k^{-1} \equiv C_{2,1} < \infty \tag{5.2.8}$$

$$\sum_{k=1}^{\infty} b_k^2 \lambda_k^{-1} \equiv C_{2,2} < \infty. \tag{5.2.9}$$

Under these conditions the stochastic integral $\int_0^t G(s, X_s)dW_s$ may not be defined. However, for any predictable process (X_t),

$$\begin{aligned}\int_0^t \|T_{t-s}G(s,X_s)\|_2^2 ds &\leq \int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k} b_k^2(1 + \|X_s\|^2)ds \\ &= \int_0^t f_G(t-s)(1 + \|X_s\|^2)ds \end{aligned} \tag{5.2.10}$$

where

$$f_G(t) = \sum_{k=1}^{\infty} e^{-2\lambda_k t} b_k^2. \tag{5.2.11}$$

Since $\int_0^t f_G(u)du \leq \sum_{k=1}^{\infty} b_k^2 \lambda_k^{-1} = C_{2,2}$ it follows that the stochastic integral referred to above exists if

$$\int_0^T \|X_s\|^2 ds < \infty \quad a.s. \tag{5.2.12}$$

Similarly

$$\begin{aligned}\left[\int_0^t \|T_{t-s}A(s,X_s)\|ds\right]^2 &\leq T\int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k} a_k^2 (1+\|X_s\|^2)ds \\ &= \int_0^t f_A(t-s)(1+\|X_s\|^2)ds \end{aligned}\tag{5.2.13}$$

where

$$f_A(t) = T\sum_{k=1}^{\infty} e^{-2\lambda_k t} a_k^2 \tag{5.2.14}$$

and again we have that $\int_0^t f_A(u)du \leq TC_{2,1}$. Thus for every ω such that (5.2.12) holds, we also have that the integral

$$\int_0^t T_{t-s}A(s,X_s)ds \tag{5.2.15}$$

is well defined.

We will prove the existence and uniqueness of the above equation (5.2.1). The details are taken from Bhatt, Kallianpur, Karandikar and Xiong [1].

Definition 5.2.1 *A predictable process (X_t) is said to be a* **mild solution** *or* **evolution solution** *to (5.2.1) if (5.2.12) holds and for every t*

$$X_t = T_t X_0 + \int_0^t T_{t-s}G(s,X_s)dW_s + \int_0^t T_{t-s}A(s,X_s)ds \;\; a.s. \tag{5.2.16}$$

Note that the predictability of (X_t) implies that X_0 is independent of (W_t). It is easy to see that if (X_t) is a solution and $(X_t^{'})$ is a predictable modification of (X_t), i.e. $P(X_t = X_t^{'}) = 1$ for all t, then $(X_t^{'})$ is also a solution to (5.2.1).

It is convenient to define a new probability measure $\tilde{P}$ on $\mathcal{F}$,

$$\tilde{P}(C) = \int_C \exp\{-\|X_0\|\}dP \Big/ \int \exp\{-\|X_0\|\}dP. \tag{5.2.17}$$

Clearly, $\tilde{P}$ and P are mutually absolutely continuous and the Radon-Nikodym derivative $\frac{d\tilde{P}}{dP}$ is $\mathcal{F}_0$ measurable. Hence (W_t) is again a c.B.m on $(\Omega, \mathcal{F}, \tilde{P})$. If $M_t = \int_0^t F_s dW_s$ on $(\Omega, \mathcal{F}, P)$ and $\tilde{M} = \int_0^t F_s dW_s$ on $(\Omega, \mathcal{F}, \tilde{P})$ where $\int_0^T \|F_s\|_2^2 ds < \infty$ a.s. (P or $\tilde{P}$), then

$$P(M_t = \tilde{M} \text{ for all t}) = \tilde{P}(M_t = \tilde{M} \text{ for all t}) = 1.$$

Thus (X_t) is a solution to (5.2.1) on $(\Omega, \mathcal{F}, P)$ if and only if (X_t) is a solution to (5.2.1) on $(\Omega, \mathcal{F}, \tilde{P})$. Further, we have for all $p < \infty$,

$$E^{\tilde{P}}\|X_0\|^p < \infty.$$

Here is a version of Gronwall's lemma which will be used in proving existence and uniqueness results for the solution.

Lemma 5.2.1 *i) Let f, g and δ be nonnegative functions on $[0,T]$. Let $\alpha \in [0,\infty)$ such that $\int_0^T e^{-\alpha t} f(t)dt \leq \frac{1}{2}$. Suppose that either g is bounded or g is integrable and δ is bounded. If for all $t \leq T$,*

$$g(t) \leq c + \int_0^t f(s)\{g(t-s) + \delta(t-s)\}ds, \tag{5.2.18}$$

then there exists a nonnegative Borel measure μ on $[0,T]$ such that $\mu[0,t] \leq e^{\alpha t}$ and

$$g(t) \leq c(1+e^{\alpha t}) + \int_0^t \delta(t-s)\mu(ds). \tag{5.2.19}$$

ii) Let f, g be positive functions on $\{0,1,\cdots,n\}$. Let $\alpha \in [0,\infty)$ such that $\sum_{i=1}^n e^{-\alpha i} f(i) \leq \frac{1}{2}$. If for all $0 \leq i \leq n$

$$g(i) \leq c + \sum_{j=1}^{i} f(j) g(i-j), \tag{5.2.20}$$

then

$$g(i) \leq c(1+e^{\alpha i}). \tag{5.2.21}$$

Proof: Iterating the inequality (5.2.18) we get

$$\begin{aligned}
g(t) &\leq c + \int_0^t f(s_1)\delta(t-s_1)ds_1 + \qquad (5.2.22)\\
&\quad \int_0^t f(s_1)\left[c + \int_0^{t-s_1} f(s_2)\{g(t-s_1-s_2)+\delta(t-s_1-s_2)\}ds_2\right]ds_1\\
&= c + \int_0^t \{c+\delta(t-s_1)\}f(s_1)ds_1\\
&\quad + \int_0^t\int_0^t \{g(t-s_1-s_2)+\delta(t-s_1-s_2)\}f(s_1)f(s_2)1_{s_1+s_2\leq t}ds_1ds_2\\
&\leq \cdots\cdots\\
&\leq c + \sum_{j=1}^{k}\int_0^t\{c+\delta(t-s)\}\mu_j(ds) - c\mu_k([0,t]) + \int_0^t g(t-s)\mu_k(ds)
\end{aligned}$$

where

$$\mu_j([0,t]) = \int_0^t \cdots \int_0^t f(s_1)\cdots f(s_j)1_{s_1+\cdots+s_j\leq t}ds_1\cdots ds_j.$$

As

$$\begin{aligned}\mu_j([0,t]) &\leq e^{\alpha t}\int_0^t \cdots \int_0^t e^{-\alpha(s_1+\cdots+s_j)} f(s_1)\cdots f(s_j)ds_1\cdots ds_j \\ &\leq e^{\alpha t}\left(\frac{1}{2}\right)^j\end{aligned}$$

$\mu(C) \equiv \sum_{j=1}^{\infty}\mu_j(C)$, $C \in \mathcal{B}([0,T])$ is a well-defined nonnegative Borel measure on [0,T] such that $\mu[0,t] \leq e^{\alpha t}$. Letting $k \to \infty$ on the right hand side of (5.2.22), we have

$$g(t) \leq c(1+e^{\alpha t}) + \int_0^t \delta(t-s)\mu(ds) + \liminf_{k\to\infty}\int_0^t g(t-s)\mu_k(ds). \quad (5.2.23)$$

If g is bounded, then $\liminf_{k\to\infty}\int_0^t g(t-s)\mu_k(ds) = 0$ and hence (5.2.19) holds. If g is integrable and δ is bounded, then

$$\begin{aligned}\int_0^T \liminf_{k\to\infty}\int_0^t g(t-s)\mu_k(ds)dt &\leq \liminf_{k\to\infty}\int_0^T\int_0^t g(t-s)\mu_k(ds)dt \\ &\leq \int_0^T g(t)dt \liminf_{k\to\infty}\mu_k([0,T]) = 0,\end{aligned}$$

i.e. $\liminf_{k\to\infty}\int_0^t g(t-s)\mu_k(ds) = 0$ for a.e. $t \in [0,T]$ and hence, for a.e. $t \in [0,T]$

$$g(t) \leq c(1+e^{\alpha t}) + \|\delta\|_\infty e^{\alpha t}.$$

By (5.2.18), $\forall t \in [0,T]$

$$g(t) \leq c + \int_0^T f(s)ds\,(c+\|\delta\|_\infty)\left(1+e^{\alpha T}\right)$$

i.e. g is bounded and hence (5.2.19) holds. (5.2.21) can be proved similarly. ∎

We will now obtain an estimate on the second moment of a solution.

Theorem 5.2.1 *If (X_t) is a solution to (5.2.1) satisfying $E\|X_0\|^2 < \infty$, then*

$$\sup_{t\leq T} E\|X_t\|^2 \leq C_{2,3}[1+E\|X_0\|^2] \quad (5.2.24)$$

where $C_{2,3}$ is a constant depending only on the constants $C_{2,1}$, $C_{2,2}$.

Proof: Let (X_t) be a solution to (5.2.1) satisfying (5.2.12). Then it follows that

$$\begin{aligned}\langle X_t,\phi_k\rangle &= e^{-\lambda_k t}\langle X_0,\phi_k\rangle + \int_0^t \left\langle e^{-\lambda_k(t-s)}G^*(s,X_s)\phi_k, dW_s\right\rangle \\ &\quad + \int_0^t e^{-\lambda_k(t-s)}\langle A(s,X_s),\phi_k\rangle ds \quad (5.2.25)\end{aligned}$$

and hence that

$$d\langle X_t, \phi_k\rangle = \langle G^*(t, X_t)\phi_k, dW_t\rangle + \langle A(t, X_t) - \lambda_k X_t, \phi_k\rangle\, dt. \tag{5.2.26}$$

Fix n and define a stopping time τ_n by

$$\tau_n = \inf\left\{t \geq 0 : \int_0^t \|X_s\|^2 ds \geq n\right\} \wedge T \tag{5.2.27}$$

and let

$$\xi_t^k \equiv e^{\lambda_k(t\wedge\tau_n)} \langle X_{t\wedge\tau_n}, \phi_k\rangle .$$

Note that $\tau_n \to T$ since (X_t) is assumed to satisfy (5.2.12). It is easy to see that

$$\begin{aligned} \xi_t^k &= \xi_0^k + \int_0^{t\wedge\tau_n} e^{\lambda_k s} \langle G^*(s, X_s)\phi_k, dW_s\rangle \\ &\quad + \int_0^{t\wedge\tau_n} e^{\lambda_k s} \langle A(s, X_s), \phi_k\rangle\, ds \end{aligned}$$

and hence from (5.2.4) and (5.2.5) we have

$$\begin{aligned} E|\xi_t^k|^2 &\leq 3E\left[|\xi_0^k|^2 + \int_0^{t\wedge\tau_n} e^{2\lambda_k s}\|G^*(s, X_s)\phi_k\|^2 ds \right.\\ &\quad \left. + t\int_0^{t\wedge\tau_n} e^{2\lambda_k s} \langle A(s, X_s), \phi_k\rangle^2 ds\right] \\ &\leq 3\left[E|\xi_0^k|^2 + \int_0^t e^{2\lambda_k s}(b_k^2 + Ta_k^2)E\left\{(1 + \|X_s\|^2)1_{s<\tau_n}\right\} ds\right]. \end{aligned}$$

From the inequality $E[\|X_t\|^2 1_{t<\tau_n}] \leq \sum_k e^{-2\lambda_k t} E|\xi_t^k|^2$ we get

$$\begin{aligned} & E[\|X_t\|^2 1_{t<\tau_n}] \\ \leq\ & 3\left[E\|X_0\|^2 + \int_0^t \sum_k e^{-2\lambda_k(t-s)}(b_k^2 + Ta_k^2)E\left\{(1 + \|X_s\|^2)1_{s<\tau_n}\right\} ds\right] \\ \leq\ & 3\left[E\|X_0\|^2 + TC_{2,1} + C_{2,2} + \int_0^t f_0(t-s)E\{\|X_s\|^2 1_{s<\tau_n}\}ds\right] \end{aligned}$$

where $f_0(u) = f_G(u) + f_A(u)$ is an integrable function (see (5.2.11), (5.2.14)). Since $\int_0^T E[\|X_s\|^2 1_{s<\tau_n}]ds \leq n$ by the choice of τ_n, $\delta = 0$ and there exists α such that

$$3\sum_{k=1}^{\infty} \frac{Ta_k^2 + b_k^2}{\alpha + 2\lambda_k} \leq \frac{1}{2}, \tag{5.2.28}$$

we can use Lemma 5.2.1i) to conclude that

$$E[\|X_t\|^2 1_{t<\tau_n}] \leq C[1 + E\|X_0\|^2]$$

where the constant C does not depend on n. Now the result follows from Fatou's lemma by letting $n \to \infty$. ∎

The next result proves the existence and uniqueness of the solution to (5.2.1).

Theorem 5.2.2 *Suppose that L, A, G satisfy (5.2.2)-(5.2.9). Let X_0 be an $\mathcal{F}_0$-measurable H-valued random variable and let (W_t) be an $(\mathcal{F}_t)$-cylindrical Brownian motion. Then*
(i) There exists a solution $(\hat{X}_t)$ of (5.2.1) satisfying (5.2.12) with $\hat{X}_0 = X_0$.
(ii) If $\{X_t\}$ and $\{U_t\}$ are solutions to (5.2.1) satisfying (5.2.12) such that $X_0 = U_0$, then

$$P(X_t = U_t) = 1 \quad \textit{for all } t. \tag{5.2.29}$$

Proof: (i) Let $\tilde{P}$ be defined by (5.2.17). It suffices to construct a solution on $(\Omega, \mathcal{F}, \tilde{P})$. For $n \geq 1$, let $t_i^n = \frac{i}{n}T$, $0 \leq i \leq n$. Let $X_0^n = X_0$ and define $\{X_t^n, t_i^n < t \leq t_{i+1}^n\}$ $i \geq 0$ inductively as follows. For $t_i^n < t \leq t_{i+1}^n$, let

$$\begin{aligned} X_t^n &= T_{t-t_i^n} X_{t_i^n}^n + \int_{t_i^n}^t T_{t-u} G(u, X_{t_i^n}^n) dW_u \\ &\quad + \int_{t_i^n}^t T_{t-u} A(u, X_{t_i^n}^n) du. \end{aligned} \tag{5.2.30}$$

As in (5.2.10), (5.2.13), $\forall t_i^n < t \leq t_{i+1}^n$, we have

$$\begin{aligned} \tilde{E}\|X_t^n\|^2 &\leq 3\left[\tilde{E}\|X_{t_i^n}^n\|^2 + \int_{t_i^n}^t \sum_k e^{-2\lambda_k(t-s)}(b_k^2 + Ta_k^2)\tilde{E}(1 + \|X_{t_i^n}^n\|^2)ds\right] \\ &\leq 3(1 + TC_{2,1} + C_{2,2})(1 + \tilde{E}\|X_{t_i^n}^n\|^2). \end{aligned} \tag{5.2.31}$$

Let $Y_t^n = X_{t_i^n}^n$ for $t_i^n < t \leq t_{i+1}^n$. Then

$$X_t^n = T_t X_0 + \int_0^t T_{t-u} G(u, Y_u^n) dW_u + \int_0^t T_{t-u} A(u, Y_u^n) du. \tag{5.2.32}$$

Proceeding as in (5.2.10), (5.2.13), it follows that

$$\tilde{E}\|X_t^n\|^2 \leq 3\left[\tilde{E}\|X_0\|^2 + TC_{2,1} + C_{2,2} + \int_0^t f_0(t-s)\tilde{E}\|Y_s^n\|^2 ds\right] \tag{5.2.33}$$

where $f_0 = f_A + f_G$. Let $g_n(i) = \tilde{E}\|X_{t_i^n}^n\|^2$, $0 \leq i \leq n$. By (5.2.31) and induction in i, it is easy to show that $g_n(i)$ is a finite valued function on $i \in \{0, 1, \cdots, n\}$. It follows from (5.2.33) that

$$g_n(i) \leq 3\Big[\tilde{E}\|X_0\|^2 + TC_{2,1} + C_{2,2}$$

$$
\begin{aligned}
&+\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n}\sum_k (b_k^2+Ta_k^2)e^{-2\lambda_k(t_i^n-s)}ds g_n(j)\Big] \\
\le\ & 3(\tilde{E}\|X_0\|^2+TC_{2,1}+C_{2,2})+\sum_{j=1}^{i} f_n(j)g_n(i-j) \quad (5.2.34)
\end{aligned}
$$

where

$$f_n(i)=3\sum_k \frac{b_k^2+Ta_k^2}{2\lambda_k}e^{-2\lambda_k t_i^n}\left(e^{\frac{2\lambda_k}{n}T}-1\right).$$

Let α be given by (5.2.28). Then

$$
\begin{aligned}
\sum_{i=1}^{n} f_n(i)e^{-\frac{\alpha T}{n}i} &= \sum_{i=1}^{n}3\sum_k \frac{b_k^2+Ta_k^2}{2\lambda_k}e^{-\frac{2\lambda_k T+\alpha T}{n}i}\left(e^{\frac{2\lambda_k}{n}T}-1\right) \\
&\le 3\sum_{k=1}^{\infty}\frac{Ta_k^2+b_k^2}{2\lambda_k}\frac{e^{\frac{2\lambda_k T}{n}}-1}{e^{\frac{2\lambda_k+\alpha}{n}T}-1} \\
&\le 3\sum_{k=1}^{\infty}\frac{Ta_k^2+b_k^2}{\alpha+2\lambda_k}\le\frac{1}{2}
\end{aligned}
$$

and hence, by (5.2.34) and Lemma 5.2.1(ii)

$$\tilde{E}\|X_{t_i^n}^n\|^2=g_n(i)\le 3(\tilde{E}\|X_0\|^2+TC_{2,1}+C_{2,2})(1+e^{\frac{\alpha T}{n}i}).$$

It then follows from (5.2.31) again that

$$\sup_{n\ge 1}\sup_{0\le t\le T}\tilde{E}\|X_t^n\|^2\le C'[1+\tilde{E}\|X_0\|^2]=C''. \quad (5.2.35)$$

Using (5.2.32) for n, m and using the Lipschitz conditions on A, G we get (the calculations are similar to those in (5.2.10), (5.2.13))

$$
\begin{aligned}
\tilde{E}\|X_t^n-X_t^m\|^2 &\le 2\tilde{E}\left\{\int_0^t\|T_{t-u}(G(u,Y_u^n)-G(u,Y_u^m))\|_2^2du\right. \\
&\quad\left.+T\int_0^t\|T_{t-u}(A(u,Y_u^n)-A(u,Y_u^m))\|^2du\right\} \\
&\le 2\int_0^t f_0(t-u)\tilde{E}\|Y_u^n-Y_u^m\|^2du.
\end{aligned}
$$

Let $g_{n,m}(t)=\tilde{E}\|X_t^n-X_t^m\|^2$ and $\delta_{n,m}(t)=\tilde{E}\|X_t^n-Y_t^n\|^2+\tilde{E}\|X_t^m-Y_t^m\|^2$. Then $g_{n,m}$, $\delta_{n,m}$ are uniformly bounded (by (5.2.35)) and

$$
\begin{aligned}
&g_{n,m}(t) \\
\le\ & 2\int_0^t f_0(t-u)3(\tilde{E}\|Y_u^n-X_u^n\|^2+\|X_u^n-X_u^m\|^2+\|X_u^m-Y_u^m\|^2)du \\
\le\ & \int_0^t 6f_0(t-u)\{g_{n,m}(u)+\delta_{n,m}(u)\}du.
\end{aligned}
$$

Similar to (5.2.31), it follows from (5.2.30) and (5.2.35) that, for $t_i^n < t \leq t_{i+1}^n$

$$\begin{aligned} \tilde{E}\|X_t^n - Y_t^n\|^2 &= \tilde{E}\|X_t^n - X_{t_i^n}^n\|^2 \\ &\leq 3\sum_{k=1}^{\infty}\left(e^{-\frac{\lambda_k T}{n}} - 1\right)^2 \tilde{E} < X_{t_i^n}^n, \phi_k >^2 \\ &\quad +3(1+C'')\sum_{k=1}^{\infty}\frac{Ta_k^2 + b_k^2}{2\lambda_k}\left(1 - e^{-2\frac{\lambda_k T}{n}}\right). \end{aligned} \tag{5.2.36}$$

It follows from (5.2.32) that

$$\begin{aligned} \langle X_t^n, \phi_k\rangle &= e^{-\lambda_k t}\langle X_0, \phi_k\rangle + \int_0^t e^{-\lambda_k(t-u)}\langle G(u, Y_u^n)^*\phi_k, dW_u\rangle \\ &\quad + \int_0^t e^{-\lambda_k(t-u)}\langle A(u, Y_u^n), \phi_k\rangle\, du \end{aligned}$$

and then

$$\begin{aligned} \tilde{E}\langle X_t^n, \phi_k\rangle^2 &\leq 3\tilde{E}\langle X_0, \phi_k\rangle^2 \\ &\quad +3\int_0^t (b_k^2 + Ta_k^2)e^{-2\lambda_k(t-u)}(1 + \tilde{E}\|Y_u^n\|^2)du \\ &\leq 3\tilde{E}\langle X_0, \phi_k\rangle^2 + 3(1+C'')\frac{b_k^2 + Ta_k^2}{2\lambda_k}. \end{aligned}$$

Hence, by the dominated convergence theorem, it follows from (5.2.36) that $\delta_{n,m}(t) \to 0$. By Lemma 5.2.1(i) and the dominated convergence theorem again,

$$g_{n,m}(t) \leq \int_0^T \delta_{n,m}(t)\mu(dt) \to 0.$$

Therefore

$$\sup_{t\leq T}\tilde{E}\|X_t^n - X_t^m\|^2 \to 0, \quad \sup_{t\leq T}\tilde{E}\|Y_t^n - Y_t^m\|^2 \to 0. \tag{5.2.37}$$

Note that since Y^n is a piecewise constant, left-continuous, adapted process it is predictable. In view of (5.2.37) we can choose a subsequence $\{n_k\}$ such that $Z_s^k \equiv Y_s^{n_k}$ satisfies

$$\sup_{s\leq T}\tilde{E}\|Z_s^k - Z_s^{k+1}\|^2 \leq 2^{-k}.$$

Then it follows that $\sum_k \|Z_s^k - Z_s^{k+1}\| < \infty$ a.s. for all s. Thus Z_s^k converges a.s. for each s. Define

$$\hat{X}_s(\omega) = \begin{cases} \lim_{k\to\infty} Z_s^k(\omega) & \text{if it exists in H} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{X}_s$ is a predictable process. Further, it follows from (5.2.37) that

$$\sup_{s\leq T}\tilde{E}\|Y_s^n - \hat{X}_s\|^2 \to 0, \quad \sup_{s\leq T}\tilde{E}\|X_s^n - \hat{X}_s\|^2 \to 0.$$

From this, it can be verified that $\hat{X}$ is a solution to (5.2.1) (on $(\Omega, \mathcal{F}, \tilde{P})$) with $\hat{X}_0 = X_0$ and that (5.2.12) holds. This completes the proof of (i).

For (ii), again, let $\tilde{P}$ be given by (5.2.17). Then $\{X_t\}$ and $\{U_t\}$ are solutions to (5.2.1) on $(\Omega, \mathcal{F}, \tilde{P})$ and in view of Theorem 5.2.1, $\int_0^T \tilde{E}\|X_s - U_s\|^2 ds < \infty$. Using the Lipschitz conditions on A, G, we deduce that

$$\tilde{E}\|X_t - U_t\|^2 \leq 2\left[\int_0^t f_0(t-s)\tilde{E}\|X_s - U_s\|^2 ds\right].$$

An application of Lemma 5.2.1, with $c = 0$ and $\delta = 0$, yields

$$\tilde{E}\|X_t - U_t\|^2 = 0$$

for all t. Thus $\tilde{P}(X_t = U_t) = 1$ and hence (5.2.29) follows. ∎

We are now in a position to obtain an estimate on the growth of the p^{th} moment of the solution.

Theorem 5.2.3 *Let $\{X_t\}$ be a solution to (5.2.1) satisfying (5.2.12). Then for $p \geq 2$, there exists a constant C_p' depending only on the constant C_p in Theorem 3.3.2 and on $C_{2,1}$, $C_{2,2}$ such that if $E\|X_0\|^p < \infty$, then*

$$\sup_{s\leq T} E\|X_s\|^p \leq C_p'[1 + E\|X_0\|^p]. \tag{5.2.38}$$

Proof: Let X_t^n be the approximation constructed in the proof of the previous theorem. Using Theorem 3.3.2, it follows from (5.2.30) that for $t_i^n < t \leq t_{i+1}^n$,

$$\begin{aligned} E\|X_t^n\|^p &\leq 3^{p-1}\left[E\|X_{t_i^n}^n\|^p + C_p E\left(\int_{t_i^n}^t f_G(t-s)ds(1+\|X_{t_i^n}^n\|^2)\right)^{\frac{p}{2}}\right. \\ &\quad \left. + E\left(\int_{t_i^n}^t f_A(t-s)ds(1+\|X_{t_i^n}^n\|^2)\right)^{\frac{p}{2}}\right] \qquad (5.2.39) \\ &\leq 3^{p-1}\left[E\|X_{t_i^n}^n\|^p + (C_p C_{2,2}^{\frac{p}{2}} + (TC_{2,1})^{\frac{p}{2}})E(1+\|X_{t_i^n}^n\|^2)^{\frac{p}{2}}\right]. \end{aligned}$$

Let $h_n(i) = \tilde{E}\|X_{t_i^n}^n\|^p$, $0 \leq i \leq n$. By (5.2.39) and by induction in i, we see that $h_n(\cdot)$ is a finite valued function. By (5.2.32), proceeding as in (5.2.39), we have

$$\begin{aligned} E\|X_t^n\|^p &\leq 3^{p-1}\left[E\|X_0\|^p + C_p E\left(\int_0^t f_G(t-s)(1+\|Y_s^n\|^2)ds\right)^{\frac{p}{2}}\right. \\ &\quad \left. + E\left(\int_0^t f_A(t-s)(1+\|Y_s^n\|^2)ds\right)^{\frac{p}{2}}\right]. \qquad (5.2.40) \end{aligned}$$

Using Hölder's inequality for the ds integrals, we get

$$\begin{aligned}E\|X_t^n\|^p &\le 3^{p-1}\Big[E\|X_0\|^p \\ &+C_p\left(\int_0^t f_G(t-s)ds\right)^{\frac{p}{2}-1} E\left(\int_0^t f_G(t-s)(1+\|Y_s^n\|^2)^{\frac{p}{2}}ds\right) \\ &+\left(\int_0^t f_A(t-s)ds\right)^{\frac{p}{2}-1} E\left(\int_0^t f_A(t-s)(1+\|Y_s^n\|^2)^{\frac{p}{2}}ds\right)\Big].\end{aligned} \tag{5.2.41}$$

It then follows from similar arguments as in (5.2.33)-(5.2.35) that there exists a constant C_p' depending only on p and on $C_{2,1}$, $C_{2,2}$ such that

$$\sup_{n\ge 1}\sup_{0\le t\le T} E\|X_t^n\|^p \le C_p'[1+E\|X_0\|^p]. \tag{5.2.42}$$

As noted in the previous result, a subsequence of X_t^n converges to $\hat{X}_s$, where $\hat{X}$ is a solution to (5.2.1). Hence, using Fatou's lemma, it follows that the required moment estimate holds for $\hat{X}$. The result follows from this as $\hat{X}$, X have the same finite dimensional distributions by the uniqueness part of the previous theorem. ∎

We now look at regularity of paths of the solution to (5.2.1).

In order to prove sample continuity of the solution, we impose a stronger condition than (5.2.9):

$$\sum_{k=1}^{\infty} b_k^2\lambda_k^{-\theta} \equiv C_{2,4} < \infty \tag{5.2.43}$$

for some θ, $0<\theta<1$.

Theorem 5.2.4 *Let (X_t) be a solution to (5.2.1). Then (X_t) admits a continuous modification, which is of course, a solution to (5.2.1).*

Proof: Let $\tilde{P}$ be defined by (5.2.17). It suffices to prove that X has a continuous modification on $(\Omega,\mathcal{F},\tilde{P})$. Let us write

$$X_t = T_tX_0 + Y_t + Z_t$$

where $Y_t = \int_0^t T_{t-u}G(u,X_u)dW_u$ and $Z_t = \int_0^t T_{t-u}A(u,X_u)du$. Clearly, $T_tX_0(\omega)$ is continuous for all ω. For $0\le s\le t\le T$,

$$\begin{aligned}&\|Z_t - Z_s\|^2 \\ = &\left\|\int_0^s (T_{t-u}-T_{s-u})A(u,X_u)du + \int_s^t T_{t-u}A(u,X_u)du\right\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left[\int_0^s \|(T_{t-u}-T_{s-u})A(u,X_u)\|du\right]^2 + 2\left[\int_s^t \|T_{t-u}A(u,X_u)\|du\right]^2 \\
&\leq 2\left[\int_0^s \left\{\sum_k \left(e^{-\lambda_k(t-u)} - e^{-\lambda_k(s-u)}\right)^2 a_k^2(1+\|X_u\|^2)\right\}^{\frac{1}{2}} du\right]^2 \\
&\quad +2\left[\int_s^t \left\{\sum_k e^{-2\lambda_k(t-u)} a_k^2(1+\|X_u\|^2)\right\}^{\frac{1}{2}} du\right]^2 \\
&\leq 2\left[\int_0^T (1+\|X_u\|^2)du\right]\alpha(s,t) \qquad (5.2.44)
\end{aligned}
$$

by Hölder's inequality where

$$
\begin{aligned}
\alpha(s,t) &= \int_0^s \sum_k \left(e^{-\lambda_k(t-u)} - e^{-\lambda_k(s-u)}\right)^2 a_k^2 du \\
&\quad + \int_s^t \sum_k e^{-2\lambda_k(t-u)} a_k^2 du.
\end{aligned}
$$

It is easy to verify that $\alpha(s,t) \leq \beta(t-s)$ where

$$
\beta(\delta) \equiv \sum_{k=1}^{\infty} \frac{a_k^2}{2\lambda_k}\left[\left(1-e^{-\delta\lambda_k}\right)^2 + \left(1-e^{-2\delta\lambda_k}\right)\right]. \qquad (5.2.45)
$$

Clearly (5.2.8) implies $\beta(\delta) \to 0$ as $\delta \to 0$. Using (5.2.12), it follows that

$$
\lim_{\delta\to 0} \sup_{0\leq t-s\leq\delta} \|Z_t - Z_s\|^2 = 0 \quad a.s.
$$

Thus $\{Z_t\}$ is continuous a.s.

It remains to show that $\{Y_t\}$ admits a continuous modification. We shall achieve this via the Kolmogorov criterion. Choose p such that $(1-\theta)p > 2$, where θ is as in (5.2.43). Recall that by the choice of $\tilde{P}$, $\tilde{E}\|X_0\|^p < \infty$ and hence by Theorem 5.2.3, $\sup_{s\leq T}\tilde{E}\|X_s\|^p < \infty$. As before, $\tilde{E}$ stands for the integral with respect to $\tilde{P}$. For $s \leq t \leq T$, writing

$$
Y_t - Y_s = \int_0^s (T_{t-u}-T_{s-u})G(u,X_u)dW_u + \int_s^t T_{t-u}G(u,X_u)dW_u
$$

and using Theorem 3.3.2, we get

$$
\begin{aligned}
&\tilde{E}\|Y_t - Y_s\|^p \\
&= 2^{p-1}C_p\tilde{E}\left[\left\{\int_0^s \|(T_{t-u}-T_{s-u})G^*(u,X_u)\|_2^2 du\right\}^{\frac{p}{2}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\int_s^t \|T_{t-u}G^*(u,X_u)\|_2^2du\right\}^{\frac{p}{2}}\Bigg] \\
= &\ 2^{p-1}C_p\tilde{E}\left[\left\{\int_0^s \sum_k \left(e^{-\lambda_k(t-u)}-e^{-\lambda_k(s-u)}\right)^2 b_k^2(1+\|X_u\|^2)du\right\}^{\frac{p}{2}}\right. \\
&\left.+\left\{\int_s^t \sum_k e^{-2\lambda_k(t-u)}b_k^2(1+\|X_u\|^2)du\right\}^{\frac{p}{2}}\right]. \qquad (5.2.46)
\end{aligned}
$$

Let us write

$$\psi_1(u)=\sum_k \left(e^{-\lambda_k(t-u)}-e^{-\lambda_k(s-u)}\right)^2 b_k^2$$

and

$$\psi_2(u)=\sum_k e^{-2\lambda_k(t-u)}b_k^2.$$

Now

$$
\begin{aligned}
&\tilde{E}\left[\int_0^s \psi_1(u)(1+\|X_u\|^2)du\right]^{\frac{p}{2}} \\
\leq &\ \tilde{E}\left[\left(\int_0^s \psi_1(u)du\right)^{\frac{p}{2}-1}\int_0^s \psi_1(u)(1+\|X_u\|^2)^p du\right] \\
\leq &\ C_p'(1+\tilde{E}\|X_0\|^p)\left(\int_0^s \psi_1(u)du\right)^{\frac{p}{2}}
\end{aligned}
$$

by Hölder's inequality and (5.2.38). Similarly, estimating the second term in (5.2.46), we get

$$\tilde{E}\|Y_t-Y_s\|^p \leq C_p'(1+\tilde{E}\|X_0\|^p)\left[\left(\int_0^s \psi_1(u)du\right)^{\frac{p}{2}}+\left(\int_0^s \psi_2(u)du\right)^{\frac{p}{2}}\right]. \qquad (5.2.47)$$

Evaluating the integrals, one obtains

$$
\begin{aligned}
\tilde{E}\|Y_t-Y_s\|^p \leq &\ C_p'\tilde{E}(1+\|X_0\|)^p\left[\left(\sum_k \frac{b_k^2}{2\lambda_k}\left(1-e^{-\lambda_k(t-s)}\right)^2\right)^{\frac{p}{2}}\right. \\
&\left.+\left(\sum_k \frac{b_k^2}{2\lambda_k}\left(1-e^{-2\lambda_k(t-s)}\right)\right)^{\frac{p}{2}}\right].
\end{aligned}
$$

Now using the obvious inequality $1-e^x \leq x\wedge 1 \leq x^\delta$ for $x>0$, $0<\delta\leq 1$, for $\delta=\frac{1-\theta}{2}$ and $\delta=1-\theta$ respectively, we get

$$\tilde{E}\|Y_t-Y_s\|^p$$

$$\begin{aligned}
&\le C_p'\tilde{E}(1+\|X_0\|)^p\left[\left(\sum_k \frac{b_k^2}{2\lambda_k}(\lambda_k(t-s))^{1-\theta}\right)^{\frac{p}{2}}\right.\\
&\quad\left.+\left(\sum_k \frac{b_k^2}{2\lambda_k}(2\lambda_k(t-s))^{1-\theta}\right)^{\frac{p}{2}}\right]\\
&\le C_p'\tilde{E}(1+\|X_0\|)^p\left(\frac{1}{2^{p/2}}+\frac{1}{2^{p\theta/2}}\right)\left(\sum_k \frac{b_k^2}{\lambda_k^\theta}\right)^{\frac{p}{2}}(t-s)^{(1-\theta)p/2}.
\end{aligned}$$

Recalling the assumption (5.2.43) and noting that by our choice of p, $\frac{p}{2}(1-\theta)>1$, we conclude that

$$\tilde{E}\|Y_t-Y_s\|^p \le C_{2,5}|t-s|^{1+\delta} \tag{5.2.48}$$

with $\delta=\frac{p}{2}(1-\theta)-1$, where $C_{2,5}$ depends only on p, $C_{2,4}$. Thus $\{Y_t\}$ has a continuous modification. ∎

Now the existence and uniqueness result, Theorem 5.2.2, can be recast as follows.

Theorem 5.2.5 *There exists a continuous solution X to the SDE (5.2.1). Further, if X' is any other solution to (5.2.1) with continuous paths, then*

$$P(X_t=X_t' \text{ for all } t,\ 0\le t\le T)=1.$$

Our next step is to prove uniqueness in law of solutions to (5.2.1).

Theorem 5.2.6 *Let $\{X_t\}$ be a solution to (5.2.1) [on $(\Omega,\mathcal{F},P)$] and let $\{X_t'\}$ be a solution to (5.2.1) on $(\Omega',\mathcal{F}',P')$ with respect to some P'-c.B.m. on H. Suppose that X, X' have continuous paths and suppose $P\circ X_0^{-1}=P'\circ X_0^{-1}$. Then*

$$P\circ X^{-1}=P'\circ X^{-1}. \tag{5.2.49}$$

Proof: Let $\{X_t^n\}$ be the approximation constructed in the previous theorem and let $\{V_t^n\}$ be the approximation defined analogously on $(\Omega',\mathcal{F}',P')$ (with X_0' in place of X_0 and $\{W_t'\}$ in place of $\{W_t\}$ in (5.2.30)). It is easy to see that the finite dimensional distributions of $\{X_t^n\}$ and $\{V_t^n\}$ are the same. Now $\tilde{E}\|X_t^n-X_t\|^2\to 0$ implies that $P(\|X_t^n-X_t\|>\delta)\to 0$ for all $\delta>0$. Similarly, $P'(\|V_t^n-X_t'\|>\delta)\to 0$. Thus the finite dimensional distributions of $\{X_t\}$ and $\{X_t'\}$ are the same. Since X, X' have continuous paths, this yields (5.2.49). ∎

We will now consider the martingale problem corresponding to (5.2.1). For $f\in C_0^2(\mathbf{R}^n)$, $n\ge 1$, let $U_nf: H\to\mathbf{R}$ be defined by

$$(U_nf)(h)=f(\langle h,\phi_1\rangle,\cdots,\langle h,\phi_n\rangle). \tag{5.2.50}$$

For $f \in C_0^2(\mathbf{R}^n)$, we will write $f_i = (\partial/\partial x_i)f$ and $f_{ij} = (\partial/\partial x_j)f_i$. Let

$$\mathcal{D} = \{U_n f : f \in C_0^2(\mathbf{R}^n), n \geq 1\}. \tag{5.2.51}$$

Define $\mathbf{L}_t$ on $\mathcal{D}$ by

$$\begin{aligned} \mathbf{L}_t(U_n f)(h) &= \frac{1}{2}\sum_{i,j=1}^{n} \langle G^*(t,h)\phi_i, G^*(t,h)\phi_j\rangle\,(U_n f_{ij})(h) \\ &\quad + \sum_{i=1}^{n} \langle A(t,h) - \lambda_i h, \phi_i\rangle\,(U_n f_i)(h). \end{aligned} \tag{5.2.52}$$

If $\{X_t\}$ is a solution to (5.2.1), then we have seen that (5.2.26) holds and hence it follows that for all $g \in \mathcal{D}$,

$$g(X_t) - g(X_0) - \int_0^t (\mathbf{L}_s g)(X_s)ds \tag{5.2.53}$$

is also a martingale. In other words, if $\{X_t\}$ is a solution to (5.2.1) then $\{X_t\}$ is a solution to the $\{\mathbf{L}_t\}$-martingale problem. The converse is also true is proved next.

Theorem 5.2.7 *Let (X_t) be a predictable process satisfying (5.2.12) such that (5.2.52) is a martingale for all $g \in \mathcal{D}$. Then on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ of the stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ there exists a H-cylindrical Brownian motion (W_t) such that (a) (X_t) is $(\tilde{\mathcal{F}}_t)$-predictable and (b) (X_t) is a solution to (5.2.1).*

Proof: Using (5.2.52) for $g = U_n f$, $f \in C_0^2(\mathbf{R}^n)$, we can first conclude that $(< X_t, \phi_i >, 1 \leq i \leq n)$ has a r.c.l.l. modification and then further it has a continuous modification. (this follows using arguments in Theorem IV 3.6 in [9] and exercise 4.6.3 in [53].) Let us denote the continuous version of $< X_t, \phi_i >$ by Y^i. Then we also deduce that

$$M_t^i = Y_t^i - Y_0^i - \int_0^t \lambda_i Y_s^i ds - \int_0^t < A(s, X_s), \phi_i > ds$$

is a continuous local martingale and that

$$\left\langle M^i, M^j \right\rangle_t = \int_0^t \langle G^*(s,X_s)\phi_i, G^*(s,X_s)\phi_j\rangle\, ds.$$

As a consequence, recalling the definition (5.2.27) of τ_n, and using (5.2.9) we have

$$E \sup_{t \leq \tau_n} |M_t^k|^2 \leq 4E \left\langle M^k, M^k \right\rangle_{\tau_n} \leq b_k^2(1+n). \tag{5.2.54}$$

Let $N_t^k \equiv \lambda_k^{-1/2} M_t^k$. Then using (5.2.9) and (5.2.54) we get

$$E \sup_{t \le \tau_n} \left\| \sum_{k=m}^{r} N_t^k \phi_k \right\|^2 \to 0 \qquad m, r \to \infty.$$

Hence $N_t \equiv \sum_{k=1}^{\infty} N_t^k \phi_k$ is an H-valued continuous local martingale. Hence

$$\begin{aligned} \left\langle N^k, N^j \right\rangle_t &= \int_0^t \lambda_k^{-1/2} \lambda_j^{-1/2} \langle G^*(s, X_s)\phi_k, G^*(s, X_s)\phi_j \rangle \, ds \\ &= \int_0^t \langle f_s^* \phi_k, f_s^* \phi_j \rangle \, ds \end{aligned}$$

where $f_s(\omega) = L^{-1/2} G(s, X_s)$. Note that

$$\int_0^T \|f_s(\omega)\|_2^2 ds < \infty$$

in view of the assumption (5.2.9). It follows from Theorem 3.3.5 that on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t)$ of $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, there exists an H-c.B.m W_t such that

$$N_t = \int_0^t f_s dW_s.$$

Then $N_t^k = (N_t, \phi_k) = \int_0^t < \lambda_k^{-1/2} G^*(s, X_s)\phi_k, dW_s >$ and hence

$$M_t^k = \int_0^t \langle G^*(s, X_s)\phi_k, dW_s \rangle \, .$$

From here, it follows that $\{X_t\}$ satisfies (5.2.26) and hence $\{X_t\}$ is a solution to (5.2.1). ∎

In the light of Theorem 5.2.5, some of the results concerning the equation (5.2.1) proved earlier can be recast for the $\{\mathbf{L}_t\}$-martingale problem as follows.

Theorem 5.2.8 *(a) Let (X_t) be a predictable process satisfying (5.2.12) and suppose that (X_t) is a solution to the $\{\mathbf{L}_t\}$-martingale problem. Then (X_t) admits a continuous modification.*
(b) For all $\mu \in \mathcal{P}(H)$, there exists a continuous process (X_t) such that (5.2.53) is a martingale for every $g \in \mathcal{D}$ and such that the law of X_0 is μ. Further, the law of the process X is uniquely determined.
(c) For $0 \le s \le T$, $x \in H$, there is a unique measure $P_{s,x}$ on $C([0,T], H)$ such that (writing the co-ordinate process on $C([0,T], H)$ as η_t),
(i) $P_{s,x}(\eta(u) = x, 0 \le u \le s) = 1$.
(ii) $g(\eta_t) - \int_s^t (\mathbf{L}_u g)(\eta_u) du$, $t \ge s$ is a $P_{s,x}$-martingale.

(d) Further, (η_t) is a time inhomogeneous Markov process on the probability space $(\Omega', \mathcal{F}', P_{s,x})$ (where Ω' is $C([0,T], H)$ and $\mathcal{F}'$ is the Borel σ-field on Ω') for each $(s,x) \in [0,T] \times H$. The (common) transition probability function $P(r,y,t,C)$ is given by

$$P(r,y,t,C) = P_{r,y}(\eta_t \in C)$$

for $r \leq t \leq T$, $y \in H$, C Borel in H.

Proof: (a), (b) follow from Theorem 5.2.2, 5.2.4, 5.2.5 and 5.2.7. (c) is the same as (b)-with a change of origin from 0 to s in the time variable. For (d), let us note that if for each n, $\mathcal{C}_n$ is a countable dense subset of $C_0^2(\mathbf{R}^n)$ (in the norm, $\|f\|_0 = \|f\| + \sum_i \|f_i\| + \sum_{ij} \|f_{ij}\|$, $\|\cdot\|$, being sup norm) then

$$\mathcal{D}_0 = \{U_n f : f \in \mathcal{C}_n\}$$

is a countable set and for every $g = U_n f \in \mathcal{D}$ we can get $g_k \in \mathcal{D}_0$ such that $g_k \to g$ and $\mathbf{L}_t g_k \to \mathbf{L}_t g$. Just take $g_k = U_n f_k$ where $f_k \in \mathcal{C}_n$ approximate f in $\|\cdot\|_0$ norm. Hence the Markov property of (η_t) under $\{P_{s,x}\}$ and the expression for the transition function follow from the uniqueness of solution to the martingale problem. (See Theorem 6.2.2 in Stroock and Varadhan [53]) ∎