3 The First Variation

Let $X : M \hookrightarrow \mathbf{R}^3$ be a regular surface and $(U, (x, y))$ be a coordinate neighbourhood. Let $X_1 = X_x$, $X_2 = X_y$, $g_{ij} = X_i \bullet X_j$, and $g = \det(g_{ij})$. Then

$$dA := \sqrt{g}\, dx \wedge dy$$

is a well defined two form on M and $dA \neq 0$ everywhere.

Let $f : M \to \mathbf{R}$ be a continuous function of compact support, or suppose f does not change sign on M, then the integral of f on M is defined by

$$\int_M f := \int_M f\, dA.$$

When M is precompact and $f \equiv 1$, $\int_M dA$ is the area of the surface $X : M \hookrightarrow \mathbf{R}^3$.

The adjective "minimal" of minimal surfaces comes from the fact that at any point of the surface there exists a neighbourhood such that the surface in that neighbourhood has the least area among all surfaces with the same boundary.

To be precise, let $\Omega \subset M$ be a precompact domain and $X : \Omega \to \mathbf{R}^3$ be a surface. Let $X(t) : \Omega \to \mathbf{R}^3$, $-1 < t < 1$ and $X(0) = X$, such that $X(t)|_{\partial\Omega} = X|_{\partial\Omega}$, and $X(t, p) = X(t)(p)$ is C^2 on $\Omega \times (-1, 1)$. Such a family of surfaces is called a *variation of* X.

Consider the *area functional*

$$A(t) = \int_\Omega dA_t,$$

where dA_t is the area form induced by $X(t)$. The definition of minimal surface from the point view of the calculus of variations is that for any variation family $X(t)$,

$$\frac{dA(t)}{dt}\Big|_{t=0} = 0. \tag{3.2}$$

We will prove that this is another equivalent definition of minimal surface.

Without loss of generality, we may assume that X is conformal. Let $p \in \Omega$ and $p \in U \subset \Omega$ be an isothermal coordinate neighbourhood of p for X. On U, dA_t is expressed as

$$dA_t = \sqrt{\det[g_{ij}(t)]}\, dx \wedge dy,$$

where $z = x + iy$ is the isothermal coordinate and $g_{ij}(t) = X_i(t) \bullet X_j(t)$ (note that z may not be an isothermal coordinate for $X(t)$). Hence

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \int_U dA_t &= \int_U \frac{d}{dt}\Big|_{t=0} dA_t = \int_U \frac{d\sqrt{\det[g_{ij}(t)]}}{dt}\Big|_{t=0} dx \wedge dy \\ &= \frac{1}{2} \int_U \frac{d\det[g_{ij}(t)]}{dt}\Big|_{t=0} \{\det[g_{ij}(0)]\}^{-1/2} dx \wedge dy.\end{aligned}$$