Representations of Derivatives of Functions in Sobolev Spaces in Terms of Finite Differences

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1. Introduction

In numerical analysis, one can approximate $f'(x)$, the derivative of a given function f at x, by a single difference $\frac{f(x)-f(x-h)}{h}$. However, such an approximation is not generally identical to $f'(x)$ if f is not a linear function. On the other hand, if we approximate $f'(x)$ by a sum of the two finite differences $[\frac{f(x+h)-f(x)}{2h}+\frac{f(x)-f(x-h)}{2h}]$, this approximation is identical to $f'(x)$ for each quadratic function f. This raises the questions: under what conditions can the derivative of a given function be expressed as a sum of finite differences and how many terms in such a sum of finite differences is sufficient?

In [2] p.187, it was shown that if f is a twice continuously differentiable function on R, there are constants a_1, a_2 and continuous functions f_1, f_2 such that

$$f'(x)=\sum_{j=1}^{2}[f_j(x)-f_j(x-a_j)], \text{ for all } x \text{ in R.}$$

In this paper, we apply some results of difference subspaces of $L^2(R)$ in [4] to show that for each positive integer m, if f is a function in the Sobolev space $H^{m+2}(R)$, the subspace of functions in $L^2(R)$ whose distributional derivatives up to order m are also in $L^2(R)$, and if f' is the distributional derivative of f, then there are constants a_1, a_2 and functions f_1, f_2 in $H^m(R)$ such that

$$f'=\sum_{j=1}^{2}[f_j-\delta_{a_j}*f_j] \text{ a.e.} \qquad (1)$$