Chapter 4

Algebraic K3 surfaces case

From now, our focus moves on to the collapsing of Ricci-flat *K3 surfaces*. We begin by reviewing and setting up the basic background.

4.1 Moduli of polarized K3 surfaces

Let us first review the well-known construction of moduli of polarized K3 surfaces, and set our notation (cf. e.g. [Huy16]). This description uses the Torelli theorem of algebraic K3 surfaces ([PiShaSha71]) and surjectivity of the period maps ([Kul77], [PP81]).[1] Let $U = \mathbb{Z}e_0 \oplus \mathbb{Z}f_0$ be a lattice of rank two with a symmetric bilinear form given by $(e_0, e_0) = (f_0, f_0) = 0$ and $(e_0, f_0) = 1$. Define $\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8^{\oplus 2}$ the even unimodular lattice of rank 22. Here, E_8 denotes the negative definite E_8-lattice so Λ_{K3} has signature $(3, 19)$. Fix a positive integer d. A primitive element $\lambda \in \Lambda_{\mathrm{K3}}$ with $(\lambda, \lambda) = 2d$ is unique up to automorphisms of Λ_{K3}. We fix $\lambda = de_0 + f_0$ contained in one of U and then put $\Lambda_{2d} := \lambda^{\perp} \subset \Lambda_{\mathrm{K3}}$. The lattice Λ_{2d} has signature $(2, 19)$ and $\Lambda_{2d} \simeq \mathbb{Z}(de_0 - f_0) \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$.

Let $\mathcal{F}_{2d}$ be the moduli space of K3 surfaces allowing possible ADE singularities, with primitive polarizations of degree $2d$. Recall that a polarization (ample line bundle) L is called primitive if it can not be written as $M^{\otimes m}$ with $m > 1$. Its structure is well-known as follows. Let us set

$$\Omega(\Lambda_{2d}) := \{[w] \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) \mid (w, w) = 0,\ (w, \bar{w}) > 0\},$$

which has two connected components. Note there is a natural involution $\iota\colon [w] \mapsto [\bar{w}]$, which interchanges the two components. We choose one of its connected components and denote by $\mathcal{D}_{\Lambda_{2d}}$. Also, $\Omega(\Lambda_{2d})$ can be identified with the set of positive definite oriented two-dimensional planes in $\Lambda_{2d} \otimes \mathbb{R}$ by assigning $[w]$ to $\mathbb{R}(\operatorname{Re} w) \oplus \mathbb{R}(\operatorname{Im} w)$. The choice of one connected component $\mathcal{D}_{\Lambda_{2d}}$ corresponds to giving orientations on all the positive definite planes in $\Lambda_{2d} \otimes \mathbb{R}$.

Let $O(\Lambda_{\mathrm{K3}})$ be the automorphism group of the lattice Λ preserving the bilinear form. Let $\tilde{O}(\Lambda_{2d}) = \{g|_{\Lambda_{2d}} : g \in O(\Lambda_{\mathrm{K3}}),\, g(\lambda) = \lambda\}$ (cf. [Nik79, 1.5.2, 1.6.1]). The group $\tilde{O}(\Lambda_{2d})$ naturally acts on $\Omega(\Lambda_{2d})$. Let $\tilde{O}^{+}(\Lambda_{2d}) \subset \tilde{O}(\Lambda_{2d})$ be the index two subgroup consisting of the elements which preserve each connected component of $\Omega(\Lambda_{2d})$. The following is well-known:

Fact 4.1. *We have an isomorphism*

$$\mathcal{F}_{2d} \simeq \tilde{O}(\Lambda_{2d})\backslash\Omega(\Lambda_{2d}) (\simeq \tilde{O}^{+}(\Lambda_{2d})\backslash\mathcal{D}_{\Lambda_{2d}}).$$

[1] Their Kähler versions are due to Burns-Rapoport [BR75] (Torelli theorem) and Todorov [Tod79], [Tod80] (surjectivity) respectively.