# OPTIMUM MULTIVARIATE DESIGNS

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## 1. Introduction

1.1. *Notation and preliminaries.* This paper is concerned with the computation of optimum designs in certain multivariate polynomial regression settings.

Let $f = (f_1, \cdots, f_k)$ be a vector of $k$ real-valued continuous linearly independent functions on a compact set $X$. We shall work in the realm of the approximate theory discussed in many of the references, wherein a design is a probability measure $\xi$ (which can be taken to be discrete) on $X$. The information matrix $M(\xi)$ of the design $\xi$ for problems where the regression function is $\sum_1^k \theta_i f_i(x)$ (with $\theta = (\theta_1, \cdots, \theta_k)$ unknown and with uncorrelated homoscedastic observations and quadratic loss considerations of best linear unbiased estimators) has elements $m_{ij}(\xi) = \int f_i f_j \, d\xi$. Thus, det $M^{-1}(\xi)$ is proportional to the generalized variance of the best linear estimators of all $\theta_i$. We denote by $\Gamma$ the space of all $M(\xi)$. We shall have occasion to consider the set of all *distinct* functions of the form $f_i f_j$, $i \geq j$, and shall write them as $\{\phi_t, 1 \leq t \leq p\}$. We then write $\mu_t(\xi) = \int \phi_t \, d\xi$. Whether or not some $\phi_t$ is a nonzero constant (as it is in our polynomial examples), we define $\phi_0(x) \equiv 1$ and $\mu_0 = 1$.

The main results of this paper characterize, for certain $X$ and $f$, some designs $\xi^*$ which are $D$-optimum; that is, for which

$$\det M(\xi^*) = \max_{\xi} \det M(\xi). \tag{1.1}$$

Define, for $M(\xi)$ nonsingular,

$$\begin{aligned} d(x, \xi) &= f(x) M^{-1}(\xi) f(x)', \\ \bar{d}(\xi) &= \max_{x \in X} d(x, \xi). \end{aligned} \tag{1.2}$$

The quantity $d(x, \xi)$ is proportional to the variance of the best linear estimator of the regression $f(x)\theta'$ at $x$. A result of Kiefer and Wolfowitz [8] is that $\xi^*$ satisfies (1.1) if and only if it satisfies the $G$-(global-) optimality criterion

$$\bar{d}(\xi^*) = \min_{\xi} \bar{d}(\xi), \tag{1.3}$$

and that (1.1) and (1.3) are satisfied if and only if

$$\bar{d}(\xi^*) = k. \tag{1.4}$$

If the support of an optimum design is exactly $k$ points, then $\xi$ is uniform on those points. Our main way of finding $D$- and $G$-optimum (hereafter simply

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