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MULTIVARIATE MAJORIZATION BY POSITIVE COMBINATIONS[1]

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Multivariate majorization orderings are used to compare matrices according to their dispersiveness. When applied to matrices whose rows represent distributions of different resources, the ordering that appears to be most useful is called majorization by positive linear combinations (PC-majorization). Two matrices are PC-majorized if all positive linear combinations of the rows are ordered by ordinary vector majorization. Properties of PC-majorization are derived; an algorithm is given to determine whether or not one matrix is PC-majorized by another; and elementary operations that reduce a matrix in the PC-ordering are explained.

1. Introduction

The key idea of majorization is to pre-order vectors according to a universal standard of dispersiveness. That is, any reasonable measure of dispersiveness of the components of a vector should imply an ordering that is consistent with the pre-ordering of majorization. The universality of the majorization ordering is well illustrated by the hundreds of applications mentioned in Marshall and Olkin (1979), and many other sources.

Several attempts have been made to extend majorization to a pre-ordering of matrices. However, there appears to be no 'universal' extension, but rather several different extensions that are useful for different purposes. For example, Joe (1985) uses a 'vectorized' generalization to describe association in contingency tables, and Tong (1989) uses uniform majorization (described below) to obtain probability inequalities for rectangles. Several other multivariate majorization orderings may be found in the books by Marshall and Olkin (1979) and Arnold (1987).

In this paper, we study multivariate majorization orderings that can be interpreted as orderings of distribution of wealth of several resources, with lower in the ordering meaning closer to equal division of the resources. Our

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