Perspectives in Mathematical Logic

Jon Barwise

Admissible Sets and Structures

Springer-Verlag
Berlin Heidelberg New York

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Ω-Group:
R. O. Gandy H. Hermes A. Levy G. H. Müller
G. E. Sacks D. S. Scott

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An Approach to Definability Theory

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To my mother
and the memory of my father

Preface to the Series

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book in with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R. O. Gandy, A. Levy, G. H. Müller, G. Sacks, D. S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F. K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of* *the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over-all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.*

Acknowledgements. *The confidence and support of Professor Martin Barner of the Mathematisches Forschungsinstitut at Oberwolfach and of Dr. Klaus Peters of Springer-Verlag made possible the first meeting and the preparation of a provisional plan. Encouraged by the Deutsche Forschungsgemeinschaft and the Heidelberger Akademie der Wissenschaften we submitted this plan to the Stiftung Volkswagenwerk where Dipl. Ing. Penschuck vetted our proposal; after careful investigation he became our adviser and advocate. We thank the Stiftung Volkswagenwerk for a generous grant (1970–73) which made our existence and our meetings possible.*

Since 1974 the work of the group has been supported by funds from the Heidelberg Academy; this was made possible by a special grant from the Kultusministerium von Baden-Württemberg (where Regierungsdirektor R. Goll was our counsellor). The success of the negotiations for this was largely due to the enthusiastic support of the former President of the Academy, Professor Wilhelm Doerr. We thank all those concerned.

Finally we thank the Oberwolfach Institute, which provides just the right atmosphere for our meetings, Drs. Ulrich Felgner and Klaus Gloede for all their help, and our indefatigable secretary Elfriede Ihrig.

Oberwolfach
September 1975

R. O. Gandy *H. Hermes*
A. Levy *G. H. Müller*
G. Sacks *D. S. Scott*

Author's Preface

It is only before or after a book is written that it makes sense to talk about *the* reason for writing it. In between, reasons are as numerous as the days. Looking back, though, I can see some motives that remained more or less constant in the writing of this book and that may not be completely obvious.

I wanted to write a book that would fill what I see as an artificial gap between model theory and recursion theory.

I wanted to write a companion volume to books by two friends, H. J. Keisler's *Model Theory for Infinitary Logic* and Y.N. Moschovakis' *Elementary Induction on Abstract Structures*, without assuming material from either.

I wanted to set forth the basic facts about admissible sets and admissible ordinals in a way that would, at long last, make them available to the logic student and specialist alike. I am convinced that the tools provided by admissible sets have an important role to play in the future of mathematical logic in general and definability theory in particular. This book contains much of what I wish every logician knew about admissible sets. It also contains some material that every logician ought to know about admissible sets.

Several courses have grown out of my desire to write this book. I thank the students of these courses for their interest, suggestions and corrections. A rough first draft was written at Stanford during the unforgettable winter and spring of 1973. The book was completed at Heatherton, Freeland, Oxfordshire during the academic year 1973—74 while I held a research grant from the University of Wisconsin and an SRC Fellowship at Oxford. I wish to thank colleagues at these three institutions who helped to make it possible for me to write this book, particularly Professors Feferman, Gandy, Keisler and Scott. I also appreciate the continued interest expressed in these topics over the past years by Professor G. Kreisel, and the support of the Ω-Group during the preparation of this book. I would like to thank Martha Kirtley and Judy Brickner for typing and John Schlipf, Matt Kaufmann and Azriel Levy for valuable comments on an earlier version of the manuscript. I owe a lot to Dana Scott for hours spent helping prepare the final manuscript. I would also like to thank Mrs. Nora Day and the other residents of Freeland for making our visit in England such a pleasant one.

A final but large measure of thanks goes to my family: to Melanie for allowing me to use her room as a study during the coal strike; to Jon Russell for help with the corrections; but most of all to Mary Ellen for her encouragement and patience. To Mary Ellen, on this our eleventh anniversary, I promise to write at most one book every eleven years.

September 19, 1975 K.J.B.
Santa Monica

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Major Dependencies

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\begin{tikzcd}
 & \mathrm{I} \arrow[d] & & & & \\
 & \mathrm{II} \arrow[d] & & & & \\
 & \mathrm{III} \arrow[dl] \arrow[dr] & & & & \\
\mathrm{IV} \arrow[dr] & & \mathrm{Va} \arrow[r] \arrow[dl] \arrow[dr] \arrow[drr] \arrow[drrr] & \mathrm{Vb} & & \\
 & \mathrm{VI} & & \mathrm{VIIa} \arrow[d] & \mathrm{VIIb} & \text{Appendix} \\
 & & & \mathrm{VIII} & &
\end{tikzcd}
$$

(Va denotes the first three §§ of Chapter V, similarly for VIIa.)