Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 382–394

WEITZENBÖCK FORMULAS ON POISSON PROBABILITY SPACES

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Abstract. This paper surveys and compares some recent approaches to stochastic infinite-dimensional geometry on the space Γ of configurations (i. e. locally finite subsets) of a Riemannian manifold M under Poisson measures. In particular, different approaches to Bochner-Weitzenböck formulas are considered. A unitary transform is also introduced by mapping functions of n configuration points to their multiple stochastic integral.

1. Weitzenböck Formula under a Measure

Let M be a Riemannian manifold with volume measure $\mathrm{d}x$, covariant derivative ∇, and exterior derivative d. Let ∇_μ^* and d_μ^* denote the adjoints of ∇ and d under a measure μ on M of the form $\mu(\mathrm{d}x) = \mathrm{e}^{\phi(x)}\,\mathrm{d}x$. The classical Weitzenböck formula under the measure μ states that

$$\mathrm{d}_\mu^*\,\mathrm{d} + \mathrm{d}\,\mathrm{d}_\mu^* = \nabla_\mu^*\nabla + R - \mathrm{Hess}\,\phi\,,$$

where R denotes the Ricci tensor on M. In terms of the de Rham Laplacian $H_R = \mathrm{d}_\mu^*\,\mathrm{d} + \mathrm{d}\,\mathrm{d}_\mu^*$ and of the Bochner Laplacian $H_B = \nabla_\mu^*\nabla$ we have

$$H_R = H_B + R - \mathrm{Hess}\,\phi\,.$$

In particular the term $\mathrm{Hess}\,\phi$ plays the role of a curvature under the measure μ.

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2. Probability: Poisson Space

In this section we recall some facts on random functionals on Poisson space. The Poisson probability measure on $\mathbb{N}$ can be introduced by considering N independent $\{0,1\}$-valued Bernoulli random variables $X_1,\dots,X_N$, with parameter λ/N, $\lambda>0$. Then $X_1+\cdots+X_N$ has a binomial law, and

$$P(X_1+\cdots+X_N=k)=\binom{N}{k}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k}$$

converges to $\frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}$ as N goes to infinity. This defines a probability measure π_λ on $\mathbb{N}$ as

$$\pi_\lambda(\{k\})=\frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}\,.$$

Let X be a metric space with a σ-finite Borel measure σ. The measure π_λ has the convolution property $\pi_\lambda\star\pi_\mu=\pi_{\lambda+\mu}$, which allows to construct the Poisson measure π_σ with intensity σ on

$$\Gamma=\left\{\gamma=\sum_{k=1}^{k=n}\delta_{x_k};\ x_1,\dots,x_n\in X,\ n\in\mathbb{N}\cup\{\infty\}\right\}$$

by letting

$$\begin{aligned}&\pi_\sigma(\{\gamma\in\Gamma;\ \gamma(A_1)=k_1,\dots,\gamma(A_n)=k_n\})\\&\qquad=\frac{\sigma(A_1)^{k_1}}{k_1!}\,\mathrm{e}^{-\sigma(A_1)}\cdots\frac{\sigma(A_n)^{k_n}}{k_n!}\,\mathrm{e}^{-\sigma(A_n)}\,,\end{aligned}$$

where $A_1,\dots,A_n$ are disjoint compact subsets of X. This measure is characterized by its Fourier transform

$$\int_\Gamma \mathrm{e}^{\mathrm{i}\int_X f(x)\,\mathrm{d}\gamma(x)}\,\mathrm{d}\pi(\gamma)=\exp\left(\int_X(\mathrm{e}^{\mathrm{i}f(x)}-1)\,\mathrm{d}\sigma(x)\right).$$

If $\gamma\in\Gamma$ is finite with cardinal $|\gamma|=n$ we write

$$\gamma=\sum_{i=1}^{i=n}\delta_{x_i}\,.$$

For a given compact subset Λ we consider $F\colon\Gamma\to\mathbb{R}$ such that $F(\gamma)=F(\gamma\cap\Lambda)$, and written as

$$F(\gamma)=F(\gamma\cap\Lambda)=\mathrm{e}^{\sigma(\Lambda)/2}\sum_{n=0}^{\infty}1_{\{|\gamma\cap\Lambda|=n\}}n!f_n(x_1,\dots,x_n)=\sum_{n=0}^{\infty}J_n(f_n)$$

where f_n is a symmetric function with support in Λ^n, with

$$J_n(f_n)(\gamma)=J_n(f_n)(\gamma\cap\Lambda)=n!1_{\{|\gamma\cap\Lambda|=n\}}\,\mathrm{e}^{\sigma(\Lambda)/2}f_n(x_1,\dots,x_n)\,,\quad n\geq1\,.$$

The multiple Poisson stochastic integral of f_n is defined as

$$I_n(f_n) = \int\limits_{\substack{(x_1,\ldots,x_n)\in X^n \\ x_i\neq x_j,\ i\neq j}} f_n(x_1,\ldots,x_n)(\gamma-\sigma)(\,\mathrm{d}x_1)\cdots(\gamma-\sigma)(\,\mathrm{d}x_n)\,,$$

and extends to $f_n \in L^2_\sigma(X)^{\circ n}$ via the well-known isometry

$$\int\limits_\Gamma I_n(f_n)I_m(g_m)d\pi = n!1_{\{n=m\}}\langle f_n, g_m\rangle_{L^2_\sigma(X)^{\circ n}}\,,$$

$$f_n \in L^2_\sigma(X)^{\circ n}\,,\quad g_m \in L^2_\sigma(X)^{\circ m}\,.$$

We introduce a combinatorial transform $\tilde{K}$ which has some similarities with the K-transform, cf. [6] and references therein. The transform $\tilde{K}$ identifies the functional $J_n(f_n)$, which makes sense only in finite volume, to $I_n(f_n)$ which is defined for all square-integrable f_n.

Proposition 2.1. *The operator $\tilde{K}$ defined by*

$$\tilde{K}J_n(f_n) = I_n(f_n)\,,\quad f_n \ \textit{symmetric in}\ \mathcal{C}_c(\Lambda^n)\,,\, n\in\mathbb{N}\,,$$

is unitary on $L^2_{\pi_\sigma}(\Gamma)$. Moreover, $\tilde{K}$ satisfies

$$\tilde{K}F(\gamma) = \sum_{\eta\subset\gamma}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\int\limits_{X^k} F(\eta\cup\{y_1,\ldots,y_k\})\sigma(\,\mathrm{d}y_1)\cdots\sigma(\,\mathrm{d}y_k).$$

Proof: We have

$$\begin{aligned}
&\int\limits_\Gamma J_n(f_n)J_m(g_m)\,\mathrm{d}\pi \\
&\quad = n!^2 1_{\{n=m\}}\,\mathrm{e}^{\sigma(\Lambda)}\int\limits_\Gamma 1_{\{|\gamma\cap\Lambda|=n\}}f_n(x_1,\ldots,x_n)g_n(x_1,\ldots,x_n)\,\mathrm{d}\pi(\gamma) \\
&\quad = \frac{n!^2}{\sigma(\Lambda)^n}\,\mathrm{e}^{\sigma(\Lambda)}\pi_\sigma(|\gamma\cap\Lambda|=n)\langle f_n,g_n\rangle_{L^2_\sigma(X)^{\otimes n}} \\
&\quad = n!1_{\{n=m\}}\langle f_n,g_n\rangle_{L^2_\sigma(X)^{\otimes n}}\,,
\end{aligned}$$

which shows the first statement. On the other hand we have

$$\begin{aligned}
&\tilde{K}J_n(f_n)(\gamma) \\
&\quad = \sum_{\eta\subset\gamma\cap\Lambda}\sum_{k=0}^\infty(-1)^k\frac{n!}{k!}\int\limits_X 1_{\{|\eta\cup\{y_1,\ldots,y_k\}|=n\}}f_n(\eta\cup\{y_1,\ldots,y_k\}) \\
&\qquad\qquad \times\sigma(\,\mathrm{d}y_1)\cdots\sigma(\,\mathrm{d}y_k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{k=n} (-1)^k \frac{n!}{k!} \sum_{\substack{\eta \subset \gamma \cap \Lambda \\ |\eta| = n-k}} \int_X f_n(\eta \cup \{y_1, \ldots, y_k\}) \sigma(\mathrm{d}y_1) \cdots \sigma(\mathrm{d}y_k) \\
&= \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \sum_{\substack{x_1, \ldots, x_{n-k} \in \gamma \cap \Lambda \\ x_i \neq x_j,\ i \neq j}} \int_X f_n(\{x_1, \ldots, x_{n-k}, y_1, \ldots, y_k\}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \sigma(\mathrm{d}y_1) \cdots \sigma(\mathrm{d}y_k) \\
&= I_n(f_n)(\gamma),
\end{aligned}
$$

the last relation follows e. g. from Prop. 4.1 of [9]. □

If Λ is compact and $F(\gamma) = F(\gamma \cap \Lambda)$ we have

$$
\int_\Gamma F(\gamma)\, \mathrm{d}\pi(\gamma) = \mathrm{e}^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_X \cdots \int_X f_n(x_1, \ldots, x_n) \sigma(\mathrm{d}x_1) \cdots \sigma(\mathrm{d}x_n).
$$

In the particular case $X = \mathbb{R}_+$ with σ the Lebesgue measure, the standard Poisson process is defined as

$$
N_t(\gamma) = \gamma([0, t]) = \sum_{k=1}^{\infty} 1_{[T_k, \infty]}(t), \qquad t > 0,
$$

i. e. every configuration $\gamma \in \Gamma$ can be viewed as the ordered sequence $\gamma = (T_k)_{k \geq 1}$ of jump times of $(N_t)_{t \in \mathbb{R}_+}$ on $\mathbb{R}_+$. Let $f_n \in \mathcal{C}_c([0, \lambda]^n)$ be symmetric. Then

$$
\begin{aligned}
\int_\Gamma f_n(T_1, \ldots, T_n) d\pi(\gamma) &= \mathrm{e}^{-\lambda} \sum_{k=n}^{\infty} \frac{1}{k!} \int_0^\lambda \cdots \int_0^\lambda f_n(t_1, \ldots, t_n) \sigma(\mathrm{d}t_1) \cdots \sigma(\mathrm{d}t_k) \\
&= \sum_{k=n}^{\infty} \mathrm{e}^{-\lambda} \int_0^\lambda \int_0^{t_k} \cdots \int_0^{t_1} f_n(t_1, \ldots, t_n) \sigma(\mathrm{d}t_1) \cdots \sigma(\mathrm{d}t_k) \\
&= \sum_{k=n}^{\infty} \mathrm{e}^{-\lambda} \int_0^\lambda \frac{(\lambda - t_n)^{k-n}}{(k-n)!} \int_0^{t_n} \cdots \int_0^{t_1} f_n(t_1, \ldots, t_n) \sigma(\mathrm{d}t_1) \cdots \sigma(\mathrm{d}t_n) \\
&= \int_0^\infty \mathrm{e}^{-t_n} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n .
\end{aligned}
$$

This formula extends to f bounded and measurable.

3. Geometry

We recall the construction of [1,2] in the case of 1-forms, see also [3] for the case of n-forms. We assume that X is a Riemannian manifold. The tangent space at $\gamma \in \Gamma$ is taken to be

$$L^2(X;TX,\gamma) = \oplus_{x\in\gamma} T_x X \,.$$

A differential form of order n maps $\gamma \in \Gamma$ into the antisymmetric tensor product

$$\wedge^n(T_\gamma \Gamma) = \wedge^n(\oplus_{x\in\gamma} T_x X) \,.$$

Bochner and de Rham Laplacians on differential forms over configuration spaces are then constructed from their counterparts at the level of the manifold X. Let d_x^X be the exterior differential on X, let ∇_x^X, Δ_x^X be the natural covariant derivative and Bochner Laplacian on the bundle $T_{\gamma\setminus\{x\}\cup\{y\}}\Gamma \to y \in \mathcal{O}_{\gamma,x}$, where $\mathcal{O}_{\gamma,x}$ is an open set in X such that $\bar{\mathcal{O}}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \emptyset$. The covariant derivative of the smooth differential 1-form W is defined as

$$(\nabla_x W_x(\gamma,x))_{x\in\gamma} \in T_\gamma\Gamma \otimes T_\gamma\Gamma \,,$$

where $W_x(\gamma,y) = W((\gamma\setminus\{x\}) \cup \{y\})$, $x,y \in X$. The Bochner Laplacian H^B on Γ is defined as

$$H^B W(\gamma) = -\sum_{x\in\gamma} \Delta_x^X W_x(\gamma,x) \,.$$

The exterior derivative d^Γ is defined as

$$\mathrm{d}^\Gamma W = \sum_{x\in\gamma}\sum_{y\in\gamma} \mathrm{d}_x^X W_x(\gamma,x)_y \,,$$

where $W_x(\gamma,x)_y$ is the component of $W_x(\gamma,x)$ of index $y \in \gamma$, with adjoint

$$\mathrm{d}^{\Gamma *} W = \sum_{x\in\gamma}\sum_{y\in\gamma} \mathrm{d}_x^{X*} W_x(\gamma,x)_{x,y} \,,$$

where $W_x(\gamma,x)_{x,y}$ is the component of $W_x(\gamma,x)$ of index (x,y) and d_x^{X*} is the adjoint of d_x^X under the volume element σ on X. A Weitzenböck formula is stated in [1,3] as

$$H^R = H^B + R, \tag{3.1}$$

where H^R is the de Rham Laplacian $H^R = \mathrm{d}^\Gamma\, \mathrm{d}^{\Gamma *} + \mathrm{d}^{\Gamma *}\, \mathrm{d}^\Gamma$ and the curvature term

$$R(\gamma) = \sum_{x\in\gamma} R(\gamma,x)$$

has the explicit expression

$$R(\gamma, x)(V(\gamma)_y) = 1_{\{x=y\}} \sum_{i,j=1}^{d} \mathrm{Ric}_{ij}(x) e_i \langle V(\gamma)_x, e_j \rangle_x \,,$$

where $(e_j)_{j=1}^{j=d}$ is an orthonormal basis of $T_x X$. Formula (3.1) can be viewed as the lifting to Γ of the Weitzenböck formula on X.

Note that in the above construction the curvature term in (3.1) is essentially due to the curvature of X, in particular it vanishes if $X = \mathbb{R}^d$ and no curvature term is induced from the Poisson measure itself.

In this paper we present a different geometry on the infinite-dimensional space Γ, in which the Ricci curvature tensor under the Poisson measure appears to be the identity operator when $X = \mathbb{R}_+$, see [8] when X is a more general Riemannian manifold.

Lifting of Differential Structure

Let $\mathcal{S}$ denote the space of cylindrical functionals of the form

$$F(\gamma) = f(T_1, \dots, T_n)\,, \qquad f \in \mathcal{C}_b^\infty(\mathbb{R}^n)\,. \tag{3.2}$$

Let $\mathcal{U}$ denote the space of smooth processes of the form

$$\begin{aligned} u(\gamma, x) = \sum_{i=1}^{i=n} F_i(\gamma) h_i(x)\,, \qquad (\gamma, x) \in \Gamma \times \mathbb{R}_+\,, \\ h_i \in \mathcal{C}_c^\infty(\mathbb{R}_+)\,, \qquad F_i \in \mathcal{S}\,, \qquad i = 1, \dots, n\,. \end{aligned} \tag{3.3}$$

The differential geometric objects to be introduced below have finite dimensional counterparts, and each of them has a stochastic interpretation. The following table describes the correspondence between geometry and probability.

	Geometry	Probability
γ	element of Γ	point measure on $\mathbb{R}_+$
$\mathcal{C}_c^\infty(\mathbb{R}_+)$	tangent vectors to Γ	test functions on $\mathbb{R}_+$
σ	Riemannian metric on Γ	Lebesgue measure
d	gradient on Γ	stochastic gradient
$\mathcal{U}$	vector fields on Γ	stochastic processes
$\mathrm{d}u$	exterior derivative of $u \in \mathcal{U}$	two-parameter process
$\{\cdot,\cdot\}$	bracket of vector fields on Γ	bracket on $\mathcal{U} \times \mathcal{U}$
Ω	curvature tensor on Γ	trilinear mapping on $\mathcal{U}$
d^*	divergence on Γ	stochastic integral operator

Divergence Operator

The definition of the following gradient operator goes back to [4].

Definition 3.1. *Given* $F \in \mathcal{S}$, $F = f(T_1, \ldots, T_d)$, *let*

$$\mathrm{d}_t F(\gamma) = -\sum_{k=1}^{k=d} 1_{[0,T_k]}(t)\partial_k f(T_1, \ldots, T_d), \qquad t \geq 0.$$

The following is a finite-dimensional integration by parts formula for d.

Lemma 3.1. *We have for* $F = f(T_1, \ldots, T_d)$ *and* $h \in \mathcal{C}_c(\mathbb{R}_+)$*:*

$$\int_\Gamma \langle \mathrm{d}F, h\rangle_{L^2(\mathbb{R}_+)} \,\mathrm{d}\pi(\gamma) = \int_\Gamma F(\gamma)\left(\sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t)\,\mathrm{d}t\right)\mathrm{d}\pi(\gamma).$$

Proof: All $\mathcal{C}^\infty$ functions on $\Delta_d = \{(t_1, \ldots, t_d);\quad 0 \leq t_1 < \cdots < t_d\}$ are extended by continuity to the closure of Δ_d. We have

$$\begin{aligned}
&\int_\Gamma \langle \mathrm{d}F(\gamma), h\rangle_{L^2(\mathbb{R}_+)} \,\mathrm{d}\pi(\gamma)\\
&= -\sum_{k=1}^{k=d}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_2}\int_0^{t_k} h(s)\,\mathrm{d}s\partial_k f(t_1, \ldots, t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d\\
&= \int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_2} h(t_1) f(t_1, \ldots, t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d\\
&\quad - \int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_3}\int_0^{t_2} h(s)\,\mathrm{d}s f(t_2, t_2, \ldots, t_d)\,\mathrm{d}t_2\ldots\mathrm{d}t_d\\
&\quad + \sum_{k=2}^{k=d}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_2} h(t_k) f(t_1, \ldots, t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d\\
&\quad - \int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d} h(s)\,\mathrm{d}s\int_0^{t_d}\cdots\int_0^{t_2} f(t_1, \ldots, t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d\\
&\quad - \sum_{k=2}^{k=d-1}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_{k+1}}\int_0^{t_{k-1}}\cdots\int_0^{t_2}\\
&\qquad \int_0^{t_{k+1}} h(s)\,\mathrm{d}s f(t_1, ., t_{k-1}, t_{k+1}, t_{k+1}, ., t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_{k+1}\,\mathrm{d}t_{k-1}\ldots\mathrm{d}t_d
\end{aligned}$$

$$+\sum_{k=2}^{k=d}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_k}\int_0^{t_{k-2}}\cdots\int_0^{t_2}\int_0^{t_k} h(s)\,\mathrm{d}s f(t_1,.,t_{k-2},t_k,t_k,.,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d$$

$$=\sum_{k=1}^{k=d}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_2} h(t_k)f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d$$

$$-\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d} h(s)\,\mathrm{d}s\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_d$$

$$=\int_\Gamma F(\gamma)\left(\sum_{k=1}^{k=d}h(T_k)-\int_0^{T_d}h(t)\,\mathrm{d}t\right)\mathrm{d}\pi(\gamma)\,.$$

□

The following definition of the divergence coincides with the compensated Poisson stochastic integral with respect to $(N_t - t)_{t\in\mathbb{R}_+}$ on the adapted square-integrable processes.

Definition 3.2. *We define* d^*_π *on* $\mathcal{U}$ *by*

$$\mathrm{d}^*_\pi(hG)=\int_0^\infty h(t)(\gamma(\,\mathrm{d}t)-\,\mathrm{d}t)-\langle h,\,\mathrm{d}G\rangle_{L^2(\mathbb{R}_+)}\,,\qquad G\in\mathcal{S},\ h\in L^2(\mathbb{R}_+)\,.$$

Using this definition, an integration by parts formula can be obtained independently of the dimension.

Proposition 3.1. *The divergence operator* $\mathrm{d}^*_\pi\colon L^2(\Gamma\times\mathbb{R}_+)\to L^2(\Gamma)$ *is the closable adjoint of* d, *i. e.*

$$\int_\Gamma F\,\mathrm{d}^*_\pi u\,\mathrm{d}\pi(\gamma)=\int_\Gamma\langle\,\mathrm{d}F,u\rangle_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma)\,,\qquad F\in\mathcal{S}\,,\ u\in\mathcal{U}\,.\quad(3.4)$$

Proof: Given Lemma 3.1 it suffices to notice that if $k>d$,

$$\int_\Gamma F(\gamma)h(T_k)d\pi(\gamma)=\int_0^\infty \mathrm{e}^{-t_k}h(t_k)\int_0^{t_k}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_k$$

$$=\int_0^\infty \mathrm{e}^{-t_k}\int_0^{t_k}h(s)\,\mathrm{d}s\int_0^{t_k}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_k$$

$$-\int_0^\infty \mathrm{e}^{-t_{k-1}}\int_0^{t_{k-1}}h(s)\,\mathrm{d}s\int_0^{t_{k-1}}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)\,\mathrm{d}t_1\ldots\mathrm{d}t_{k-1}$$

$$= \int_\Gamma F(\gamma) \int_{x_{k-1}}^{T_k} h(t)\,\mathrm{d}t\,\mathrm{d}\pi(\gamma),$$

in other terms the discrete-time process

$$\left(\sum_{k=1}^{k=n} h(T_k) - \int_0^{T_k} h(t)\,\mathrm{d}t \right)_{k\geq 1} = \left(\int_0^{T_k} h(t)\,\mathrm{d}(N_t - t) \right)_{k\geq 1}$$

is a martingale. Hence relation (3.4) also implies that for $F, G \in \mathcal{S}$,

$$\begin{aligned} \int_\Gamma \langle \mathrm{d}F, hG\rangle_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) &= \int_\Gamma \langle \mathrm{d}(FG), h\rangle_{L^2(\mathbb{R}_+)} - F\langle \mathrm{d}G, h\rangle_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) \\ &= \int_\Gamma F\left(G\int_0^\infty h(t)(\gamma(\mathrm{d}t) - \mathrm{d}t) - \langle h, \mathrm{d}G\rangle_{L^2(\mathbb{R}_+)} \right)\mathrm{d}\pi(\gamma) \\ &= \int_\Gamma F\,\mathrm{d}_\pi^*(hG)\,\mathrm{d}\pi(\gamma)\,. \end{aligned} \tag{3.5}$$

□

Covariant Derivative

Given $u \in \mathcal{U}$ we define the covariant derivative $\nabla_u v$ in the direction $u \in L^2(\mathbb{R}_+)$ of the vector field $v = \sum_{i=1}^{i=n} F_i h_i \in \mathcal{U}$ as

$$\nabla_u v(t) = \sum_{i=1}^{i=n} h_i(t)\,\mathrm{d}_u F_i - F_i \dot{h}_i(t) \int_0^t u(s)\,\mathrm{d}s\,, \qquad t \in \mathbb{R}_+\,, \tag{3.6}$$

where

$$\mathrm{d}_u F = \langle \mathrm{d}F, u\rangle_{L^2(\mathbb{R}_+)}\,, \qquad F \in \mathcal{S}.$$

We have

$$\nabla_{uF}(vG) = Fv\,\mathrm{d}_u G + FG\nabla_u v\,, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)\,, \quad F, G \in \mathcal{S}\,. \tag{3.7}$$

We also let

$$\nabla_s v(t) = \sum_{i=1}^{i=n} h_i(t)\,\mathrm{d}_s F_i - F_i \dot{h}_i(t) 1_{[0,t]}(s)\,, \qquad s, t \in \mathbb{R}_+\,,$$

in order to write

$$\nabla_u v(t) = \int_0^\infty u(s)\nabla_s v(t)\,\mathrm{d}s\,, \qquad t \in \mathbb{R}_+\,, \qquad u, v \in \mathcal{U}\,.$$

Lie-Poisson Bracket

Definition 3.3. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, *is defined to be the unique element of* $\mathcal{C}_c^\infty(\mathbb{R}_+)$ *satisfying* $(\mathrm{d}_u \mathrm{d}_v - \mathrm{d}_v \mathrm{d}_u)F = \mathrm{d}_w F$, $F \in \mathcal{S}$.

The bracket $\{u, v\}$ is defined for $u, v \in \mathcal{U}$ with

$$\{Fu, Gv\}(x) = FG\{u, v\}(x) + v(x)F \,\mathrm{d}_u G - u(x)G \,\mathrm{d}_v F\,, \qquad x \in \mathbb{R}_+\,,$$

$u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F, G \in \mathcal{S}$.

Vanishing of Torsion

Proposition 3.2. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{U}$ *satisfies*

$$\{u, v\} = \nabla_u v - \nabla_v u\,,$$

i. e. the connection defined by ∇ *has a vanishing torsion.*

Proof: We have $F(\gamma) = T_n$. If $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ we have

$$\begin{aligned}
(\mathrm{d}_u \mathrm{d}_v - \mathrm{d}_v \mathrm{d}_u)T_n &= -\mathrm{d}_u \int_0^{T_n} v(s)\,\mathrm{d}s + \mathrm{d}_v \int_0^{T_n} u(s)\,\mathrm{d}s \\
&= v(T_n) \int_0^{T_n} u(s)\,\mathrm{d}s - u(T_n) \int_0^{T_n} v(s)\,\mathrm{d}s \\
&= \int_0^{T_n} \left(\dot{v}(t) \int_0^t u(s)\,\mathrm{d}s - \dot{u}(t) \int_0^t v(s)\,\mathrm{d}s \right) \mathrm{d}t \\
&= \mathrm{d}_{\nabla_u v - \nabla_v u} T_n\,.
\end{aligned}$$

Since d is a derivation, this shows that

$$\mathrm{d}_u \mathrm{d}_v - \mathrm{d}_v \mathrm{d}_u = \mathrm{d}_{\nabla_u v - \nabla_v u}\,, \qquad u, v \in \mathcal{U}\,.$$

The extension to $u, v \in \mathcal{U}$ follows from (3.7). □

Vanishing of Curvature

Proposition 3.3. *The curvature tensor* Ω *of* ∇ *vanishes on* $\mathcal{U}$, *i. e.*

$$\Omega(u, v)h := [\nabla_u, \nabla_v]h - \nabla_{\{u,v\}}h = 0\,, \qquad u, v, h \in \mathcal{U}\,,$$

and $\mathcal{U}$ *is a Lie algebra under the bracket* $\{\cdot\,,\cdot\}$.

Proof: We have, letting $\tilde{u}(t) = -\int_0^t u(s)\,\mathrm{d}s$:

$$[\nabla_u, \nabla_v]h = \tilde{u}\dot{\overline{\nabla_v h}} - \tilde{v}\dot{\overline{\nabla_u h}} = \tilde{u}\,\dot{\overline{\tilde{v}\dot{h}}} - \tilde{v}\,\dot{\overline{\tilde{u}\dot{h}}} = -\tilde{u}v\dot{h} + \tilde{v}u\dot{h}\,,$$

and

$$\nabla_{\{u,v\}}h = \nabla_{\tilde{u}\dot{v}-\tilde{v}\dot{u}}h = (\widetilde{\tilde{u}\dot{v} - \tilde{v}\dot{u}})\dot{h} = (u\tilde{v} - v\tilde{u})\dot{h}\,,$$

hence $\Omega(u,v)h = 0$, $h, u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension of the result to $\mathcal{U}$ follows again from (3.7). The Lie algebra property follows from the vanishing of Ω. □

Exterior Derivative

The exterior derivative $\mathrm{d}u$ of a smooth vector field $u \in \mathcal{U}$ is defined from

$$\langle\, \mathrm{d}u, h_1 \wedge h_2 \rangle_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} = \langle \nabla_{h_1} u, h_2 \rangle_{L^2(\mathbb{R}_+)} - \langle \nabla_{h_2} u, h_1 \rangle_{L^2(\mathbb{R}_+)}\,,$$

$h_1, h_2 \in \mathcal{U}$. We have

$$\|\mathrm{d}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} = 2\int_0^\infty\int_0^\infty (\,\mathrm{d}u(s,t))^2\,\mathrm{d}s\,\mathrm{d}t\,, \tag{3.8}$$

where

$$\mathrm{d}u(s,t) = \frac{1}{2}(\nabla_s u(t) - \nabla_t u(s))\,, \qquad s,t \in \mathbb{R}_+\,,\ u \in \mathcal{U}\,.$$

Isometry Formula

Lemma 3.2. *We have for $u \in \mathcal{U}$:*

$$\int_\Gamma (\,\mathrm{d}^*_\pi u)^2\,\mathrm{d}\pi(\gamma) = \int_\Gamma \|u\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) + \int_\Gamma\int_0^\infty\int_0^\infty \nabla_s u(t)\nabla_t u(s)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\,. \tag{3.9}$$

Proof: (cf. [8, 7] and the proof of [5] for path spaces over Lie groups). Given $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ we have

$$\begin{aligned}
\int_\Gamma \mathrm{d}^*_\pi(h_i F_i)\,\mathrm{d}^*_\pi(h_j F_j)\,\mathrm{d}\pi(\gamma) &= \int_\Gamma F_i\,\mathrm{d}_{h_i}\,\mathrm{d}^*_\pi(h_j F_j)\,\mathrm{d}\pi(\gamma) \\
&= \int_\Gamma F_i\,\mathrm{d}_{h_i}\Big(F_j\,\mathrm{d}^*_\pi(h_j) - \mathrm{d}_{h_j}F_j\Big)\,\mathrm{d}\pi(\gamma) \\
&= \int_\Gamma \Big(F_i F_j\,\mathrm{d}_{h_i}\,\mathrm{d}^*_\pi h_j + F_i\,\mathrm{d}^*_\pi(h_j)\,\mathrm{d}_{h_i}F_j - F_i\,\mathrm{d}_{h_i}\,\mathrm{d}_{h_j}F_j\Big)\,\mathrm{d}\pi(\gamma)
\end{aligned}$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j \, \mathrm{d}_\pi^*(\nabla_{h_i} h_j) + F_i \, \mathrm{d}_\pi^*(h_j) \, \mathrm{d}_{h_i} F_j - F_i \, \mathrm{d}_{h_i} \, \mathrm{d}_{h_j} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + \mathrm{d}_{\nabla_{h_i} h_j}(F_i F_j) + \mathrm{d}_{h_j}(F_i \, \mathrm{d}_{h_i} F_j) - F_i \, \mathrm{d}_{h_i} \, \mathrm{d}_{h_j} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + \mathrm{d}_{\nabla_{h_i} h_j}(F_i F_j) + \mathrm{d}_{h_j} F_i \, \mathrm{d}_{h_i} F_j + F_i (\mathrm{d}_{h_j} \, \mathrm{d}_{h_i} F_j - \mathrm{d}_{h_i} \, \mathrm{d}_{h_j} F_j) \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + \mathrm{d}_{\nabla_{h_i} h_j}(F_i F_j) + \mathrm{d}_{h_j} F_i \, \mathrm{d}_{h_i} F_j + F_i \, \mathrm{d}_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Big(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \, \mathrm{d}_{\nabla_{h_i} h_j} F_i + F_i \, \mathrm{d}_{\nabla_{h_j} h_i} F_j + \mathrm{d}_{h_j} F_i \, \mathrm{d}_{h_i} F_j \Big) \, \mathrm{d}\pi(\gamma)$$

$$= \int_\Gamma \Bigg(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^\infty \mathrm{d}_s F_i \int_0^\infty \nabla_t h_j(s) h_i(t) \, \mathrm{d}t \, \mathrm{d}s + F_i \int_0^\infty \mathrm{d}_t F_j \int_0^\infty \nabla_s h_i(t) h_j(s) \, \mathrm{d}s \, \mathrm{d}t + \int_0^\infty h_i(t) \, \mathrm{d}_t F_j \int_0^\infty h_j(s) \, \mathrm{d}_s F_i \, \mathrm{d}s \, \mathrm{d}t \Bigg) \, \mathrm{d}\pi(\gamma) ,$$

where we used the commutation relation satisfied by the gradient d:

$$\mathrm{d}_u \, \mathrm{d}_\pi^* v = \mathrm{d}_\pi^* \nabla_u v + \langle u, v \rangle_{L^2(\mathbb{R}_+)} , \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+), \tag{3.10}$$

which can be proved as follows:

$$\mathrm{d}_u \, \mathrm{d}_\pi^* v = - \sum_{k=1}^\infty \dot{v}(T_k) \int_0^{T_k} u(s) \, \mathrm{d}s = - \mathrm{d}_\pi^* \left(v(\cdot) \int_0^{\cdot} u(s) \, \mathrm{d}s \right) - \int_0^\infty \dot{v}(t) \int_0^t u(s) \, \mathrm{d}s \, \mathrm{d}t = \mathrm{d}_\pi^* (\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)} .$$

□

Finally we state a Weitzenböck type identity on configuration space, that can be read as

$$\mathrm{d}\,\mathrm{d}_\pi^* + \mathrm{d}^*\,\mathrm{d} = \nabla^*\nabla + \mathrm{Id}_{L^2(\mathbb{R}_+)},$$

i. e. the Ricci tensor under the Poisson measure is the identity $\mathrm{Id}_{L^2(\mathbb{R}_+)}$ on $L^2(\mathbb{R}_+)$.

Theorem 3.1. *We have for* $u \in \mathcal{U}$:

$$\begin{aligned}\int_\Gamma (\,\mathrm{d}_\pi^* u)^2\,\mathrm{d}\pi(\gamma) &+ \int_\Gamma \|\,\mathrm{d}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma)\\ &= \int_\Gamma \|u\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma) + \int_\Gamma \|\nabla u\|^2_{L^2(\mathbb{R}_+)\otimes L^2(\mathbb{R}_+)}\,\mathrm{d}\pi(\gamma)\,.\end{aligned} \tag{3.11}$$

Proof: Relation (3.11) for $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ follows from (3.8) and Lemma 3.2. □

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