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A MEASURE OF THE CONFORMITY OF A PARAMETER SET TO A TREND: THE PARTIALLY ORDERED CASE[1]

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Inferences concerning order restrictions on a collection of parameters, $\theta_1, \theta_2, \ldots, \theta_k$, are considered with the order restrinctions of the form, $\theta_i \leq \theta_j$ for $i \preceq j$ where $\preceq$ is a partial order on $1, 2, \ldots, k$. Clearly, some parameter sets conform more closely to these order restrinctions than others. We are interested in measures of the degree of conformity. Some of the measures available in the literature for the totally ordered case are generalized to the partially ordered case and the theory developed is applied in several tests of order restricted hypotheses.

1. Introduction. In various situations, one is interested in a collection of parameters $\theta_1, \theta_2, \ldots, \theta_k$ which are believed to satisfy certain known order restrictions and inference procedures which make use of this ordering information are preferred. We consider order restrictions that are induced by partial orders on $\Omega = \{1, 2, \ldots, k\}$. That is, suppose that $\preceq$ is a partial order on Ω and that the order restrictions are $\theta_i \leq \theta_j$ when $i \preceq j$. Such a vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ is said to be isotone (with respect to $\preceq$). In studying such inference procedures it is helpful to have a measure of the degree of conformity to the order restrictions. For instance, a test of H_0: θ is constant versus H_1: θ is isotone, but not constant should have power that increases with the degree of conformity. For a non-simple null hypothesis such a concept could be useful in identifying a least favorable configuration. In a Bayesian approach, priors which assign larger probabilities to parameters conforming more closely to the order restrictions would be sought.

Barlow, Bartholomew, Bremner and Brunk (1972) contains a thorough discussion of order restricted inference. Robertson and Wright (1982) develop several measures of conformity for the totally ordered case, ie. $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k (1 \preceq 2 \preceq \ldots \preceq k)$. In considering unimodal structures, partial orders of the type $1 \preceq 2 \preceq \ldots \preceq r \succeq r+1 \succeq \ldots \succeq k$ arise and when making one-sided comparisons of several treatments with a common control, the partial order $1 \preceq i$ for $i = 2, 3, \ldots, k$ occurs. (See Bartholomew (1959) and Robertson and Wright (1981).) Suppose that a dependent variable has mean $\theta(i,j)$ when the first independent variable is fixed at level i, $1 \leq i \leq r$, and the second independent variable is fixed at level j, $1 \leq j \leq c$. If the levels are increasing and if $\theta(\cdot,\cdot)$ increases with each independent variable as the other is held fixed, then the order restrictions are $\theta(i,j) \leq \theta(s,t)$ for $i \leq s$ and $j \leq t$. This is another example of a partial order that is not total. We extend the measures of conformity in Robertson and Wright (1982) to the partially ordered case.

A set $L \subset \Omega$ is a lower layer provided $i \in L$ whenever $i \preceq j$ and $j \in L$. We denote the collection of lower layers by $\mathcal{L}$. To allow for different weights on the parameters, let $\mathbf{w}$ be a positive weight function defined on Ω, ie. $\mathbf{w} = (w_1, w_2, \ldots, w_k)$. For situations in which the degree of conformity should be translation invariant, we consider the relationship

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$>>$, defined on Euclidean space $\mathcal{R}^k$, by $\mathbf{x} = (x_1, x_2, \ldots, x_k) >> \mathbf{y} = (y_1, y_2, \ldots, y_k)$ if and only if

$$\Sigma_{i \in L} w_i(x_i - m(\mathbf{x})) \leq \Sigma_{i \in L} w_i(y_i - m(\mathbf{y})) \quad \text{for each } L \in \mathcal{B},$$

with $m(\mathbf{x}) = \Sigma_{i=1}^k w_i x_i / \Sigma_{i=1}^k w_i$. Robertson and Wright (1982) argue that $>>$ is appropriate for normal means, but for Poisson means a more appropriate measure is the following: $\mathbf{x} >> {}^*\mathbf{y}$ if and only if

$$\Sigma_{i \in L} w_i x_i \leq \Sigma_{i \in L} w_i y_i \text{ for each } L \in \mathcal{B} \text{ and } \Sigma_{i=1}^k w_i x_i = \Sigma_{i=1}^k w_i y_i.$$

Remark 1.1. The relationship $>>$ *and* $>>^*$ *are transitive and symmetric,* $>>^*$ *is reflexive, and* $\mathbf{x} << \mathbf{y}$ *and* $\mathbf{x} >> \mathbf{y}$ *imply that* $\mathbf{x}-\mathbf{y}$ *is a constant vector.*

Proof. The first conclusion is obvious and because $\mathbf{x} >> \mathbf{y}$ is equivalent to $\mathbf{x}-m(\mathbf{x}) >>^* \mathbf{y}-m(\mathbf{y})$, it suffices to show that $>>^*$ is reflexive.

Suppose $\mathbf{x} << {}^*\mathbf{y}$ and $\mathbf{x} >> {}^*\mathbf{y}$. Let $L_0 = \phi$ and inductively define L_α to consist of those $j \in \Omega$ for which $i \preceq j$ and $i \neq j$ *imply that* $i \in L_{\alpha-1}$. Observe that $L_{\alpha-1} \subset L_\alpha$, $L_\alpha - L_{\alpha-1} \neq \phi$, and because Ω is finite, there is an integer h for which $\phi = L_0 \subset L_1 \subset \ldots \subset L_h = \Omega$. For each $j \in L_1$, $\{j\} \in \mathcal{B}$ and so $x_j = y_j$. Next, for $j \in L_2$, $L(j) = \{i \in \Omega : i \preceq j\} \in \mathcal{B}$, $L(j) - L_1 = \{j\}$ and so $x_j = y_j$. Continuing we see that $\mathbf{x}=\mathbf{y}$ and the proof is completed. □

If one identifies vectors $\mathbf{x}$ and $\mathbf{y}$ which differ by a constant vector, then $>>$ induces a partial order on the equivalence classes which is essentially $>>^*$.

Let $C = \{\mathbf{x} \in \mathcal{R}^k : \mathbf{x} \text{ is isotone with respect to } \preceq\}$ and note that the apriori belief concerning θ is that $\theta \in C$. Typically, estimates of θ are obtained by projecting initial estimates onto C, and test statistics are related to the distance from the initial estimates to the projections. The above measures of conformity can be characterized in terms of the Fenchel dual of C, which is defined by

$$C^{*\mathbf{w}} = \{\mathbf{y} \in \mathcal{R}^k : \Sigma_{i=1}^k w_i x_i y_i \leq 0 \text{ for all } \mathbf{x} \in C\}.$$

(If $\mathbf{w}$ is constant we denote the dual cone by C^*.) Barlow and Brunk (1972) and Dykstra (1981) discuss some of the implications of duality theory in order restricted inference. The following result is proved in the former reference (cf. Section 4).

Remark 1.2. With $\mathbf{x},\mathbf{y} \in \mathcal{R}^k$, *the following are equivalent:*

(A) $\mathbf{x} >> \mathbf{y}\ (\mathbf{x} >>^* \mathbf{y})$;

(B) $\mathbf{y}-m(\mathbf{y})-\mathbf{x}+m(\mathbf{x}) \in C^{*\mathbf{w}} (\mathbf{y}-\mathbf{x} \in C^{*\mathbf{w}})$; *and*

(C) $\Sigma_{i=1}^k w_i(y_i - m(\mathbf{y}) - x_i + m(\mathbf{x}))z_i \leq 0 (\Sigma_{i=1}^k w_i(y_i - x_i)z_i \leq 0)$
for each $\mathbf{z} \in C$.

Real valued functions which are nondecreasing with respect to these orderings are of interest. If $f: \mathcal{R}^k \to \mathcal{R}$ and $f(\mathbf{x}) \geq f(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathcal{R}^k$ with $\mathbf{x} >> \mathbf{y}$ ($\mathbf{x} >>^* \mathbf{y}$), then f is said to be ISO(ISO*). The next result is immediate.

Remark 1.3. A function $f: \mathcal{R}^k \to R$ *is ISO if and only if it is ISO* and* $f(\mathbf{x} + c\mathbf{e}_k) = f(\mathbf{x})$ *for all* $\mathbf{x} \in \mathcal{R}^k$ *and* $c \in R$, *where* $\mathbf{e}_k$ *is a k-dimensional vector of ones.*

Remark 1.4. Let $\mathbf{x},\mathbf{y} \in \mathcal{R}^k$. $\mathbf{x} >> \mathbf{y}$ ($\mathbf{x} >>^* \mathbf{y}$) *if and only if* $f(\mathbf{x}) \geq f(\mathbf{y})$ *for all f which are ISO(ISO*).*

Proof. The result is an easy consequence of the definitions of ISO and ISO* and the following facts: $f_L(\mathbf{x}) = -\Sigma_{i \in L} w_i(x_i - m(\mathbf{x}))$ is ISO for each $L \in \mathcal{B}$, $g_L(\mathbf{x}) = -\Sigma_{i \in L} w_i x_i$ is ISO* for each $L \in \mathcal{B}$ and $\Sigma_{i=1}^k w_i x_i$ is ISO*. □

The partial ordering $>>^*$ is a cone ordering as discussed in Marshall, Walkup and Wets

(1967) and the following result is contained in their work. However, its proof is so simple it is included here.

THEOREM 1.5. *Let $f\colon \mathcal{R}^k \to R$ be differentiable and let $f_i(\mathbf{x}) = (\partial/\partial x_i)f(\mathbf{x})$ for $i = 1,2, \ldots, k$. If $f_i(\mathbf{x})/w_i \leq f_j(\mathbf{x})/w_j$ for all i and j with $i \preceq j$ and all $\mathbf{x} \in \mathcal{R}^k$, then f is ISO*.*

Proof. Suppose $\mathbf{x} >>^* \mathbf{y}$. Using the mean value theorem there is a point $\mathbf{z}$ on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ for which

$$f(\mathbf{y}) - f(\mathbf{x}) = \Sigma_{i=1}^k (y_i - x_i) f_i(\mathbf{z}) = \Sigma_{i=1}^k w_i (y_i - x_i)(f_{ij}(\mathbf{z})/w_i)$$

and the latter sum is non-positive since $\mathbf{y} - \mathbf{x} \in C^{*\mathbf{w}}$ and $(f_1(\mathbf{z})/w_1, \ldots, f_k(\mathbf{z})/w_k) \in C$ by hypothesis. □

2. Preservation Theorems. In this section, we establish results which say that if $\mathbf{X}$ is a set of observations, $f(\mathbf{X})$ is a statistic with f ISO(ISO*) and $h(\theta) = E_\theta f(\mathbf{X})$, then h is ISO(ISO*). The first result deals with a multinomial setting. Let $\mathbf{w} = \mathbf{e}_k$, let $A_n = \{\mathbf{x} \in \mathcal{R}^k$: each x_i is a nonnegative integer and $\Sigma_{i=1}^k x_i = n\}$, let $B = \{\mathbf{p} \in \mathcal{R}^k$: each $p_i \geq 0$ and $\Sigma_{i=1}^k p_i = 1\}$ and let $\mathbf{X} = (X_1, X_2, \ldots, X_k)$ be a multinomial vector with parameters n and $\mathbf{p} = (p_1, p_2, \ldots, p_k)$.

THEOREM 2.1. *If $f\colon A_n \to R$ is ISO, then $h(\mathbf{p}) = Ef(\mathbf{X})$ is ISO on B.*

Proof. As in Robertson and Wright (1982), $h_i(\mathbf{p}) - h_j(\mathbf{p}) =$

$$\Sigma_{\mathbf{y} \in A_{n-1}} (f(\mathbf{y}+\delta_i) - f(\mathbf{y}+\delta_j)) n! \Pi_{i=1}^k (p_i^{y_i}/y_i!),$$

where δ_r is a k-dimensional vector with sth coordinate zero unless $s = r$ and the rth coordinate is one. Suppose $i \preceq j$ and let $L \in \mathcal{L}$. If $i \notin L$ then $\Sigma_{r \in L}(\mathbf{y}+\delta_i)_r = \Sigma_{r \in L}(\mathbf{y}+\delta_j)_r$; if $i \in L$ and $j \notin L$, *then* $\Sigma_{r \in L}(\mathbf{y}+\delta_i)_r \geq \Sigma_{r \in L}(\mathbf{y}+\delta_j)_r$; and if $i,j \in L$, *then* $\Sigma_{r \in L}(\mathbf{y}+\delta_i)_r = \Sigma_{r \in L}(\mathbf{y}+\delta_j)_r$. The proof is completed by applying Theorem 1.5. □

Chacko (1966) and Robertson (1978) considered testing H_0: $\mathbf{p} = k^{-1}\mathbf{e}_k$ with the alternative restricted by the trend, H_1: $\mathbf{p}$ is isotone with respect to $\preceq$. Chacko considered the totally ordered case and Robertson the partially ordered case. The likelihood ratio test statistic is $T_{01} = -2 \ln \lambda = 2\Sigma_{i=1}^k X_i \ln (P(\mathbf{X}|C)_i) - 2n \ln n + 2n \ln k$ where $P(\mathbf{X}|C)$ is the projection of $\mathbf{X}$ onto C, which is characterized by

$$\Sigma_{i=1}^k (X_i - P(\mathbf{X}|C)_i) P(\mathbf{X}|C)_i = 0 \text{ and } \Sigma_{i=1}^k (X_i - P(\mathbf{X}|C)_i) z_i \leq 0$$

for all $\mathbf{z} \in C$. (See Barlow, Bartholomew, Bremner, and Brunk (1972, p. 28). Computation algorithms for $P(\mathbf{X}|C)$ are also discussed in their Chapter 2.) We first show that $f(\mathbf{x}) = \Sigma_{i=1}^k x_i \ln(P(\mathbf{x}|C)_i)$ is ISO on A_n, then note that this implies that $I_{[T_{01} \geq t]}$ is, for fixed t, ISO on A_n and applying Theorem 2.1, we see that the power function of T_{01}, $EI_{[T_{01} \geq t]}$, is ISO on B.

Suppose $\mathbf{x} >> \mathbf{y}$ with $x,y \in A_n$, then $\mathbf{y} - \mathbf{x} \in C^*$ (we omit the superscript w since it is constant) and so

$$\Sigma_{i=1}^k y_i \ln(P(\mathbf{y}|C)_i) = \Sigma_{i=1}^k x_i \ln(P(\mathbf{y}|C)_i) + \Sigma_{i=1}^k (y_i - x_i) \ln(P(\mathbf{y}|C)_i).$$

The second term on the r.h.s. is nonpositive since $y - x \in C^*$ and $P(\mathbf{y}|C) \in C$. Furthermore, $P(\mathbf{x}|C)/n$ maximizes $\Sigma_{i=1}^k x_i \ln p_i$ with $\mathbf{p} \in C$ and so $\Sigma_{i=1}^k x_i \ln(P(\mathbf{y}|C)_i) \leq \Sigma_{i=1}^k x_i \ln(P(\mathbf{x}|C)_i)$. Hence, $\Sigma_{i=1}^k y_i \ln(P(\mathbf{y}|C)_i) \leq \Sigma_{i=1}^k x_i \ln(P(\mathbf{x}|C)_i)$, or f is ISO on A_n.

The next result is an adaptation of Theorem 1.1 of Proschan and Sethuraman (1977). Let $\phi(\theta,x)$ be a nonnegative function defined on $(0,\infty) \times [0,\infty)$ satisfying the semigroup property,

$$\phi(\theta_1+\theta_2,x) = \int_0^\infty \phi(\theta_1,x-y)\phi(\theta_2,y)d\mu(y),$$

with μ either Lebesgue measure on $[0,\infty)$ or counting measure on the nonnegative integers.

THEOREM 2.2. *Let ϕ be as above, let f: $\mathcal{R}^k \to R$ be ISO* and let h be defined on $(0,\infty)^k$ by*

$$h(\theta) = \int_{[0,\infty)}\int_{[0,\infty)} \cdots \int_{[0,\infty)} f(\mathbf{x})\Pi_{i=1}^{k}\phi(\theta_i,x_i)d\mu(x_1) \ \dots \ d\mu(x_k),$$

where the integral is assumed finite. Then h is ISO.*

LEMMA. *For $i,j \in \Omega$, set $\delta_{ij} = \delta_i/w_i - \delta_j/w_j$. $C^{*\mathbf{w}}$, the dual of the cone of isotone vectors, and K, the collection of vectors $\mathbf{x} = \Sigma_{\{(i,j)\in\Omega^2:\ i\neq j\ i\lesssim j\}}c_{ij}\delta_{ij}$ with the $c_{ij} \geq 0$ are equal.*

Proof. A proof similar to that given for the Remark on p. 49 of Barlow, Bartholomew, Bremner and Brunk (1972) shows that

$$C^{*\mathbf{w}} = \{\mathbf{y}: \Sigma_{i\in L}w_iy_i \geq 0 \text{ for every } L\in\mathcal{L} \text{ and } \Sigma_{i=1}^{k}w_iy_i = 0\}.$$

For $L\in\mathcal{L}$, $\alpha,\beta \in \Omega$ with $\alpha\lesssim b$ and $\alpha \neq \beta$,

$$\Sigma_{i\in L}(\delta_{\alpha,\beta})_iw_i = \begin{cases} 0 & \text{if } \alpha\notin L \\ 1 & \text{if } \alpha\in L \text{ but } \beta\notin L \\ 0 & \text{if } \alpha,\beta \in L. \end{cases}$$

So $K \subset C^{*\mathbf{w}}$ and hence $K^{*\mathbf{w}} \supset (C^{*\mathbf{w}})^{*\mathbf{w}}$. As Dykstra (1981) observed, $(C^{*\mathbf{w}})^{*\mathbf{w}} = C$ if C is a closed convex cone. This can also be shown using the following: the result holds when $\mathbf{w} = \mathbf{e}_k$, ie. $(C^*)^* = C$ for C closed, (cf. Rockafeller (1970, p. 121)) and $C^{*\mathbf{w}} = \{(y_1/w_1, \ \dots \ , y_k/w_k): \mathbf{y} \in C^*\}$ (cf. Barlow and Brunk (1972)). Suppose that $\mathbf{z} \in K^{*\mathbf{w}} - C$, that is $\mathbf{z}$ is not isotone and $\Sigma_{i=1}^{k}w_iz_ix_i \leq 0$ for each $\mathbf{x} \in K$. Now if $\mathbf{z}$ is not isotone there exist $\alpha,b \in \Omega$ with $\alpha\lesssim\beta$, $\alpha \neq \beta$ and $z_\alpha > z_\beta$ and so $\Sigma_{i=1}^{k}w_iz_i(\delta_{\alpha,\beta})_i = z_\alpha - z_\beta > 0$. This contradiction implies that $K^{*\mathbf{w}} = C$ or $C^{*\mathbf{w}} = (K^{*\mathbf{w}})^{*\mathbf{w}} = K$. □

Proof. (Theorem 2.2) Let $\mathbf{w} = \mathbf{e}_k$ and consider $\theta'' >>^* \theta'$, then $\theta' - \theta'' \in C^*$. Hence, $\theta' = \theta'' + \Sigma_{\{i\leq j, i\neq j\}}c_{ij}\delta_{ij}$ with $c_{ij} \geq 0$. So it suffices to show that for arbitrary θ, $h(\theta + c_{ij}\delta_{ij}) \leq h(\theta)$, but this can be shown using the proof of Theorem 3.3 of Robertson and Wright (1982). □

Suppose that k independent Poisson processes are each observed for T units of time and that the intensity of the ith process is θ_i. The likelihood ratio test of $\theta_1 = \theta_2 = \ \dots \ = \theta_k$ when the alternative is restricted by the trend, θ is isotone, rejects for large values of

$$T_{01} = -2\ln\lambda = 2\{\Sigma_{i=1}^{k}X_i\ln(P(\mathbf{X}|C)_i) - (\Sigma_{i=1}^{k}X_i)\ln(\Sigma_{i=1}^{k}X_i/k)\}$$

where λ is the likelihood ratio and $\mathbf{X} = (X_1,X_2, \ \dots \ , X_k)$ with the X_i independent Poisson variables and $E(X_i) = \theta_iT$. The family of Poisson densities satisfies the semigroup property with μ counting measure on $\{0, 1, \dots\}$, $-(\Sigma_{i=1}^{k}X_i)\ln(\Sigma_{i=1}^{k}x_i/k)$ is ISO* and we have seen earlier that $\Sigma_{i=1}^{k}x_i\ln(P(\mathbf{x}|C)_i)$ is ISO*. Hence, Theorem 2.2 shows that this test has power function that is ISO*. This result could also have been obtained from Theorem 2.1 since conditioning on the total number of occurrences, $\Sigma_{i=1}^{k}X_i$, leads to a multinomial testing situation. However, this approach is more direct.

THEOREM 2.3. *Suppose $\{P_\theta: \theta \in \Theta\}$ is a family of probability measures on the Borel subsets of $\mathcal{R}^k$ with $\Theta \subset \mathcal{R}^k$ and suppose that if $\mathbf{X}$ has distribution P_θ then $\mathbf{X}-\theta$ has the distribution Q which is independent of θ. If f: $R^k \to R$ is ISO and h: $\Theta \to R$ is defined by $h(\theta) = \int f(\mathbf{x})dP_\theta(\mathbf{x})$ (which is assumed finite for each $\theta \in \Theta$), then h is ISO on Θ.*

The proof of Theorem 2.3 is just like that given for the totally ordered case (cf. Robertson and Wright (1982)) and in fact, the result holds for any cone ordering (cf. Marshall, Walkup and Wets (1967)).

Suppose X_{ij}, $j = 1,2, \ldots, n$ and $i = 1,2, \ldots, k$, are independent normal variables with mean θ_i and common variance σ^2. The estimator $\hat{\sigma}^2 = \Sigma_{i=1}^{k}\Sigma_{j=1}^{n}(X_{ij}-\bar{X}_i)^2/(k(n-1))$ for σ^2 is independent of $\hat{\theta}_i = \bar{X}_i = \Sigma_{j=1}^{n}X_{ij}/n$. To test $\theta_1 = \theta_2 = \ldots = \theta_k$ with the alternative restricted by, θ is isotone, one could use $T = \Sigma_{\{i \preceq j, i \neq j\}}(\hat{\theta}_j - \hat{\theta}_i)/\hat{\sigma}$, or more generally

$$T_c = \sqrt{n}\ \Sigma_{i=1}^{k}c_i\hat{\theta}_i/[(\Sigma_{i=1}^{k}c_i^2)^{1/2}\hat{\sigma}] \text{ with } \Sigma_{i=1}^{k}c_i = 0.$$

Of course, this test rejects for $T_c \geq t$ where t is the $100(1-\alpha)$ percentile of the T distribution with $k(n-1)$ degrees of freedom. The power function is translation invariant, ie. the power is the same at θ and $\theta + c\mathbf{e}_k$, and so it is ISO if it is ISO*. The distribution of $\hat{\sigma}$ is independent of θ and the power at θ is given by

$$E(P_\theta[\Sigma_{i=1}^{k}c_i\hat{\theta}_i \geq t(\Sigma_{i=1}^{k}c_i^2)^{1/2}\hat{\sigma}/\sqrt{n}\,|\hat{\sigma}]).$$

So it suffices to show that for each positive a, $P_\theta[\Sigma_{i=1}^{k}c_i\hat{\theta}_i \geq at]$ is ISO*, but $\hat{\theta}' >>^* \hat{\theta}$ implies that $\hat{\theta} - \hat{\theta}' \in C^*$ and so $\Sigma_{i=1}^{k}c_i(\hat{\theta}_i' - \hat{\theta}_i) \geq 0$ if $\mathbf{c} \in C$. Hence if the vector $\mathbf{c}$ is isotone with respect to $\preceq$, then the power function is ISO.

In the case of T, c_i equals card. $\{\ell : \ell \preceq i\}$-card. $\{\ell : \ell \succeq i\}$ which is easily seen to be isotone. For the simple tree ordering, $1 \preceq i$, $i = 2, \ldots, k$, this choice of $\mathbf{c}$ is $(-k+1,1,1, \ldots ,1)$ and for the loop ordering, ie. $1 \preceq i \preceq k$ for $i = 2, \ldots, k-1$, this choice of $\mathbf{c}$ is $(-k+1,0, \ldots, 0,k-1)$. The test for the simple tree case is discussed in Barlow, Bartholomew, Bremner and Brunk (1972, p. 188) and it is argued there that this choice of $\mathbf{c}$ provides the optimum set of scores.

Robertson and Wright (1982) consider the likelihood ratio test for this testing problem with a total order, unequal sample sizes and known variances which are not necessarily equal. The arguments given there also show that the likelihood ratio test in the partially ordered case has power that is ISO.

Robertson and Wegman (1978) developed the likelihood ratio test for H_1: θ is isotone with respect to $\preceq$ versus H_2: $\sim H_1$ for exponential families. In the normal means case with known variances and $w_i = n_i/\sigma_i^2$, the test statistic is $T_{12} = \Sigma_{i=1}^{k}w_i(\hat{\theta}_i - P_\mathbf{w}(\hat{\theta}|C)_i)^2$ where $P_\mathbf{w}(\cdot|C)$ denotes the projection with respect to the distance function $d^2(\mathbf{x},\mathbf{y}) = \Sigma_{i=1}^{k}w_i(x_i - y_i)^2$. It is easy to show that neither T_{12} nor its negative is ISO*. As in Robertson and Wright (1982), we define another measure of conformity $\mathbf{x} \gtrsim \mathbf{y}$ provided $\mathbf{x} - \mathbf{y} \in C$. In the totally ordered case, $\mathbf{x} \gtrsim \mathbf{y}$ implies $\mathbf{x} >> \mathbf{y}$, but the converse is not true. However, in the partially ordered case this implication is not valid in general (For an example, consider $k=3$, the only order restriction is $2 \succeq 1$, $\mathbf{x} = (0,0,0)$, $\mathbf{y} = (1,1,-2)$ and $L = \{3\}$.) A function f: $\mathcal{R}^k \to R$ is ISO** provided $f(\mathbf{x}) \geq f(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathcal{R}^k$ with $\mathbf{x} \gtrsim \mathbf{y}$. The analogue of Remark 1.4, $x \gtrsim y$ if and only if $f(\mathbf{x}) \geq f(\mathbf{y})$ for all f which are ISO**, is easy to establish. (Note that $f(\mathbf{x}) = x_j - x_i$ is ISO** if $i \preceq j$.) Furthermore, since $\gtrsim$ is a cone ordering, Theorem 2.3 remains valid if ISO is changed to ISO**. Theorem 2.1 of Robertson and Wegman (1978) shows that the negative of $t_{12}(\mathbf{x}) = \Sigma_{i=1}^{k}w_i(x_i - P_\mathbf{w}(\mathbf{x}|C)_i)^2$ is ISO**. So the modification of Theorem 2.3 which applies to $\gtrsim$ shows that if $\theta \gtrsim \theta'$, then the power of T_{12} at θ' is at least as large as at θ. Furthermore, $\theta \in C$ and θ' imply that $\theta - \theta \in C$ or $\theta \gtrsim \theta'$. Hence, H_0: θ is constant, is least favorable within H_1 and Robertson and Wegman (1978) have shown that under H_0, T_{12} has a chi-bar-squared distribution.

3. Comments. The problem of measuring the degree of conformity to an arbitrary partial order is a very broad one and in particular situations better measures may exist. In fact, we have noticed that none of the measures studied here are applicable to all the situations considered. In studying location parameters which are not related to the scale parameters, as in the normal case, $>>$ is preferred, but for cases such as that of Poisson means, where the location and scale parameters are related, $>>^*$ is more appropriate. We also found that $\gtrsim$ was useful when the null hypothesis stipulates that a collection of normal means satisfies a trend.

Because of the breadth of the problem it should not be surprising that in some special cases one can find a pair of parameter sets for which one of the orderings doesn't agree with our intuition. However, the measures studied here do seem to be useful in a variety of testing situations.

There are a couple of basic results in the totally ordered case which relate projections and the measures of conformity that are not true in the partially ordered case. Theorem 2.2 of Robertson and Wright (1982) states that

$$P_{\mathbf{w}}(\mathbf{y}|C) = \inf\{\mathbf{z} \in C : \mathbf{z} >>^* \mathbf{y}\}$$

and as a corollary $\mathbf{x} >>^* \mathbf{y}$ implies $P_{\mathbf{w}}(\mathbf{x}|C) >>^* P_{\mathbf{w}}(\mathbf{y}|C)$ and $\mathbf{x} >> \mathbf{y}$ implies that $P_{\mathbf{w}}(\mathbf{x}|C) >> P_{\mathbf{w}}(\mathbf{y}|C)$. The same example serves to show that these results are not valid in the general partially ordered case.

Example. *Suppose that* $k = 3$, $1 \preceq 2 \succeq 3$, $\mathbf{w} = \mathbf{e}_3$, $\mathbf{x} = (0,4.5,4.5)$ *and* $y = (1,3,5)$. *Observe that* $\mathbf{x} >>^* \mathbf{y}$ *(and of course,* $\mathbf{x} >> \mathbf{y}$*),* $P_{\mathbf{w}}(\mathbf{x}|C) = \mathbf{x}$, $P_{\mathbf{w}}(\mathbf{y}|C) = (1,4,4)$ *(one could use the lower sets algorithm discussed in Barlow, Bartholomew, Bremner and Brunk (1972)), but* $P_{\mathbf{w}}(\mathbf{x}|C) >> P_{\mathbf{w}}(\mathbf{y}|C)$ *is not true.*

The Remark on p. 1236 of that paper is also not valid for arbitrary partially ordered situations. It states that if $\phi \neq A \subset \mathcal{R}^k$ and A has a lower bound with respect to $>>(>>^*)$ then A has a greatest lower bound with respect to $>>(>>^*)$ and in the case of $>>^*$ the greatest lower bound is unique. It is not difficult to construct examples with A a set with two elements which has a lower bound with respect to $>>^*$ (and of course then with respect to $>>$) but not a greatest lower bound.

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