IMS Collections
Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz
Vol. 4 (2008) 99–106

DOI: 10.1214/074921708000000327

Stochastic Equations Driven by a Cauchy Process

Vladimir P. Kurenok[1]

University of Wisconsin–Green Bay

Abstract: Using the method of Krylov's estimates, we prove the existence of (weak) solutions of the one-dimensional stochastic equation $dX_t = b(X_{t-})dZ_t + a(X_t)dt$ with arbitrary initial value $x_0 \in \mathbb{R}$ and the driven symmetric Cauchy process Z. The bounded coefficient b is assumed to be of non-degenerate form and the drift a to satisfy the condition $|a(x)| \leq (1/2)|b(x)|$ for all $x \in \mathbb{R}$.

1. Introduction

The goal of this note is to prove the existence of (weak) solutions for stochastic differential equations (SDE's) of the form

$$dX_t = b(X_{t-})dZ_t + a(X_t)dt, \quad X_0 = x_0 \in \mathbb{R}, \tag{1.1}$$

where $a, b : \mathbb{R} \to \mathbb{R}$ are measurable functions and Z is a one-dimensional symmetric Cauchy process.

Since a symmetric Cauchy process is a semimartingale, the equation (1.1) is a particular case of SDE's driven by a semimartingale for which the general existence results are known (cf. [2], Theorem 6.2.3). Those results are subject to some condition of Lipshitz continuity of the coefficients and the condition of their linear growth. The natural question arises whether one can improve those general results in the case of equation (1.1)?

The equation (1.1) without drift ($a = 0$) but with the time-dependent coefficient b was considered in [5] where one proved the existence of solutions under some conditions of the local integrability of b. A slightly different sufficient condition in the case of time-independent coefficient b when $a = 0$ was provided in [10]. The approaches in [5] and [10] were similar and essentially based on using a time-change method. Another particular case of the equation (1.1), when $b = 1$, was considered in [9] where one constructed a solution under the assumption that $\sup_x |a(x)| < 1/2$. The method used there was a purely analytical one relying on Markov properties of the solution X satisfying (1.1). To our knowledge, for the equation (1.1) in its general form, there are no known weaker existence conditions than conditions of Lipshitz continuity of the coefficients mentioned above.

We shall prove here the existence of a solution of the equation (1.1) for bounded b satisfying a non-degenerate condition and the drift a such that $|a(x)| \leq (1/2)|b(x)|$ for all $x \in \mathbb{R}$. In contrast to [9], we shall use a probabilistic technique based on some variants of Krylov's estimates for solutions of SDE's driven by a symmetric Cauchy process. It is similar to the approach exploited in [8] where the equation

[1]Department of Natural and Applied Sciences, University of Wisconsin–Green Bay, 2420 Nicolet Drive, Green Bay, WI 54311-7001, USA, e-mail: kurenokv@uwgb.edu

AMS 2000 subject classifications: Primary 60H10, 60J60, 60J65

Keywords and phrases: Cauchy process, stochastic differential equations, Krylov's estimates, weak convergence

of the form (1.1) driven by a symmetric stable process Z of index $1 < \alpha < 2$ was considered.

2. Some integral estimates

Let $\mathbb{D}_{[0,\infty)}(\mathbb{R})$ be the Skorokhod space, i.e. the set of all real-valued functions $x(\cdot) : [0,\infty) \to \mathbb{R}$ with right-continuous trajectories and with finite left limits. For simplicity, we shall write $\mathbb{D}$ instead of $\mathbb{D}_{[0,\infty)}(\mathbb{R})$. We will equip $\mathbb{D}$ with the σ-algebra $\mathcal{D}$ generated by the Skorokhod topology. Under $\mathbb{D}^n, n \geq 1$, we will understand the n-dimensional Skorokhod space defined as $\mathbb{D}^n = \mathbb{D} \times \ldots \times \mathbb{D}$ with the corresponding σ-algebra $\mathcal{D}^n$ being the direct product of n one-dimensional σ-algebras $\mathcal{D}$.

Let Z be a process with $Z_0 = 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\mathbb{F} = (\mathcal{F}_t)$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. We use the notation $(Z, \mathbb{F})$ to express that Z is adapted to the filtration $\mathbb{F}$. A process $(Z, \mathbb{F})$ is said to be a *symmetric Cauchy process* if trajectories of Z belong to $\mathbb{D}$ and

$$\mathbf{E}\left(e^{i\xi(Z_t - Z_s)} | \mathcal{F}_s\right) = e^{-(t-s)|\xi|}$$

for all $t > s \geq 0, \xi \in \mathbb{R}$.

We are dealing here with the existence of solutions of equation (1.1) in the weak sense (cf. [6], chapter 4).

It is well-known that a symmetric Cauchy process Z is a Markov process; hence it can be characterized in terms of its infinitesimal generator $\mathcal{L}$ defined as

$$(\mathcal{L}g)(x) = \int\limits_{\mathbb{R}\setminus\{0\}} \left(g(x+z) - g(x) - \mathbf{1}_{\{|z|<1\}} g_x(x) z\right) \frac{c_1}{|z|^2} dz$$

with c_1 being a constant depending on Z only and $g \in C^2$, where C^2 is the set of all bounded and twice continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$. Here g_x denotes the derivative of g.

We also recall some known facts about the Fourier transforms of functions $\mathcal{L}g$ and g_x. Let $g \in L_1(\mathbb{R})$ and

$$Fg(x) := \int\limits_{\mathbb{R}} e^{izx} g(z) dz$$

be the Fourier transform of g. Then the following facts are true (cf. Proposition 9, page 24 in [3]):

(i) Assume that $g \in C^2$ and $\mathcal{L}g \in L_1$. Then, it holds that

$$F(\mathcal{L}g)(x) = -|x| Fg(x).$$

ii) Assume that g is absolutely continuous on every compact subset of $\mathbb{R}$ and $g_x \in L_1(\mathbb{R})$. Then

$$Fg_x(x) = -ixFg(x).$$

Let f be a nonnegative, measurable function such that $f \in C_0^\infty(\mathbb{R})$ where $C_0^\infty(\mathbb{R})$ denotes the class of all infinitely many times differentiable real-valued functions with compact support defined on $\mathbb{R}$. Assume further that $\mathcal{T}$ is the class of all $\mathbb{F}$-predictable one-dimensional processes (a_t) such that $|a_t| \leq 1/2$.

Consider the controlled processes X^a of the form

$$dX_t^a = dZ_t + a_t dt$$

and, for any $\lambda > 0$, define the corresponding value function $v(x), x \in \mathbb{R}$, by

$$v(x) = \sup_{a \in \mathcal{T}} \mathbf{E} \int_0^\infty e^{-\lambda s} f(x + X_s^a) ds.$$

Lemma 2.1 *It holds that*

$$\sup_x v(x) \le N \|f\|_2, \tag{2.1}$$

where $\|f\|_2 := \left(\int_{\mathbb{R}} f^2(y) dy \right)^{1/2}$.

Proof. We provide only a sketch of the proof because it follows similar steps as the proof of Lemma 3.1 in [8].

First, using the Bellman's principle of optimality (cf. [7], chapter 1), it is proven that

$$\mathcal{L}v - \lambda v + (1/2)|v_x| + f = 0 \tag{2.2}$$

a.e. in $\mathbb{R}$.

Now, let $q(x)$ be a nonnegative function such that $q \in C_0^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} q(x) dx = 1$. For any measurable function $h : \mathbb{R} \to [0, \infty)$ and any $\varepsilon > 0$ we define

$$h^{(\varepsilon)}(x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} q \left(\frac{x - y}{\varepsilon} \right) h(y) dy$$

to be the ε-convolution of h with q. Set

$$f_{(\varepsilon)} := \lambda v^{(\varepsilon)} - \mathcal{L}v^{(\varepsilon)} - (1/2)|v_x^{(\varepsilon)}|.$$

It follows then that

$$(\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)})^2 = ((1/2)|v_x^{(\varepsilon)}| + f_{(\varepsilon)})^2$$

and

$$\begin{aligned} \int_{\mathbb{R}} \left(\mathcal{L}v^{(\varepsilon)}(x) - \lambda v^{(\varepsilon)}(x) \right)^2 dx &= \int_{\mathbb{R}} \left((1/2)|v_x^{(\varepsilon)}|(x) + f_{(\varepsilon)}(x) \right)^2 dx \\ &\le \frac{1}{2} \int_{\mathbb{R}} \left(v_x^{(\varepsilon)}(x) \right)^2 dx + 2 \int_{\mathbb{R}} \left(f_{(\varepsilon)}(x) \right)^2 dx. \end{aligned} \tag{2.3}$$

Using the Parseval-Plancherel equality

$$\int_{\mathbb{R}} (v^{(\varepsilon)}(x))^2 dx = \frac{1}{2\pi} \int_{\mathbb{R}} |Fv^{(\varepsilon)}(z)|^2 dz,$$

and integration by parts, the relation (2.3) can be rewritten in terms of Fourier transforms as

$$\int_{\mathbb{R}} |Fv^{(\varepsilon)}(x)|^2 (|x| + \lambda)^2 dx \le \frac{1}{2} \int_{\mathbb{R}} |Fv^{(\varepsilon)}(x)|^2 |x|^2 dx + 2 \int_{\mathbb{R}} \left(Ff_{(\varepsilon)}(x) \right)^2 dx. \tag{2.4}$$

Since $(|x|+\lambda) > |x|$ for all $x \in \mathbb{R}$, one has

$$\frac{1}{2}\int\limits_{\mathbb{R}} |Fv^{(\varepsilon)}(x)|^2(|x|+\lambda)^2 dx \leq 2\int\limits_{\mathbb{R}} \Big(Ff_{(\varepsilon)}(x)\Big)^2 dx.$$

Applying the inverse Fourier transform and the Cauchy-Schwarz inequality, we finally obtain from (2.4) for all $y \in \mathbb{R}$ and $\lambda > 0$

$$\begin{aligned}\Big(v^{(\varepsilon)}(y)\Big)^2 &\leq \frac{1}{4\pi^2}\int\limits_{\mathbb{R}} |Fv^{(\varepsilon)}(x)|^2\Big(|x|+\lambda\Big)^2 dx \int\limits_{\mathbb{R}} \Big(|x|+\lambda\Big)^{-2} dx \\ &\leq N\int\limits_{\mathbb{R}} \Big(f_{(\varepsilon)}(x)\Big)^2 dx,\end{aligned}$$

where

$$N = \frac{1}{\pi^2}\int\limits_{\mathbb{R}} (|x|+\lambda)^{-2} dx < \infty.$$

The desired estimate follows then by taking the limit $\varepsilon \to 0$ in the above inequality and using the Lebesgue dominated convergence theorem.

Now, let X be a solution of the equation (1.1) and

$$\tau_m(X) = \inf\{t \geq 0 : |X_t| > m\}.$$

For $m \in \mathbb{N}$, define $\|f\|_{2,m} := (\int_{[-m,m]} |f(x)|^2 dx)^{\frac{1}{2}}$ as the L_2-norm of f on $[-m,m]$. We are interested in L_2 - estimates of the form

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} e^{-\lambda\psi_u}\varphi_u f(x+X_u)du \leq N\|f\|_{2,m},$$

where ψ and φ are some nonnegative, predictable processes. We shall assume that

$$|a(x)| \leq (1/2)|b(x)| \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.5}$$

Theorem 2.2 *Suppose X is a solution of the equation (1.1) driven by a symmetric Cauchy process and the condition (2.5) is satisfied. Then, for any $x \in \mathbb{R}, \lambda > 0, t \geq 0$, and any measurable function $f : \mathbb{R} \to [0,\infty)$, it holds that*

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} e^{-\lambda\psi_u}|b(X_u)|f(x+X_u)du \leq N\|f\|_{2,m}, \tag{2.6}$$

where $\psi_t = \int_0^t |b(X_s)|ds$ and the constant N depends on m and t only.

Proof. Assume first that $f \in C_0^\infty(\mathbb{R})$ so that there is a solution v of equation (2.2) satisfying the inequality (2.1). By taking the ε-convolution on both sides of (2.2), we obtain

$$\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + (1/2)|v_x^{(\varepsilon)}| + f^{(\varepsilon)} \leq 0.$$

Then, for all $s \in [0, \tau_m(X))$, applying the Itô's formula to the expression

$$v^{(\varepsilon)}(x+X_s)e^{-\lambda\psi_s},$$

yields

$$\begin{aligned}
&\mathbf{E}v^{(\varepsilon)}(x+X_s)e^{-\lambda\psi_s} - v^{(\varepsilon)}(x)\\
&\quad = \mathbf{E}\int_0^s e^{-\lambda\psi_u}\Big[|b(X_u)|\mathcal{L}v^{(\varepsilon)} - |b(X_u)|\lambda v^{(\varepsilon)} + a(X_u)v_x^{(\varepsilon)}\Big](x+X_u)du\\
&\quad \le \mathbf{E}\int_0^s e^{-\lambda\psi_u}|b(X_u)|\Big[\mathcal{L}v^{(\varepsilon)} - \lambda v^{(\varepsilon)} + (1/2)|v_x^{(\varepsilon)}|\Big](x+X_u)du\\
&\quad \le -\mathbf{E}\int_0^s e^{-\lambda\psi_u}|b(X_u)|f^{(\varepsilon)}(x+X_u)du. \qquad (2.7)
\end{aligned}$$

Hence using Lemma 2.1 we obtain

$$\mathbf{E}\int_0^s e^{-\lambda\psi_u}|b(X_u)|f^{(\varepsilon)}(x+X_u)du \le \sup_{x\in[-m,m]} v^{(\varepsilon)}(x) \le N\|f^{(\varepsilon)}\|_{2,m}.$$

Letting $\varepsilon \to 0$ and $s \to t$ and using the Fatou's lemma, we arrive at

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} e^{-\lambda\psi_u}|b(X_u)|f(x+X_u)du \le N\|f\|_{2,m}.$$

The latter inequality can be extended in a standard way first to any function $f \in L_2(\mathbb{R})$ and then to any nonnegative, measurable function using the monotone class theorem arguments (see, for example, [4], Theorem 20).

Corollary 2.3 *Let X be a solution of the equation (1.1) with $b = 1$ and $|a(x)| \le 1/2$. Then, for any $t \ge 0, m \in \mathbb{N}$, and any nonnegative, measurable function f, it holds that*

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} f(X_u)du \le N\|f\|_{2,m},$$

where N is a constant depending on m and t only.

3. Existence of solutions

Here we apply the L_2-estimates obtained in the previous section to construct a solution of SDE's driven by a symmetric Cauchy process. We start first with the equation

$$dX_t = dZ_t + a(X_t)dt, \quad X_0 = x_0 \in \mathbb{R}, \quad t \ge 0. \qquad (3.1)$$

Theorem 3.1 *Assume that $|a(x)| \le 1/2$ for all $x \in \mathbb{R}$. Then, for any $x_0 \in \mathbb{R}$, there exists a solution of the equation (3.1).*

Proof. The proof is similar to the proof of Theorem 4.1 in [8] so that we only provide the outline of it.

By standard arguments, there exists a sequence of uniformly bounded (by the constant 1/2) and Lipshitz continuous functions $a_n, n = 1, 2, \ldots$ such that $a_n \to a$ as $n \to \infty$ pointwise. By Theorem 6.2.3 in [2], for any fixed n and a given symmetric Cauchy process Z defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, there is a unique solution X^n of the equation

$$X_t^n = x_0 + Z_t + \int_0^t a_n(X_s^n)ds.$$

Set $Y_t^n := \int_0^t a_n(X_s^n)ds$ and consider the sequence of 3-dimensional processes $(X^n, Z, Y^n), n \in \mathbb{N}$. It is a simple consequence of the uniform boundness of the functions a_n that the sequence satisfies the assumptions of Aldous's criterion of weak convergence ([1]). Furtheremore, using the famous embedding principle of Skorokhod (cf. Theorem 2.7 in [6]), we conclude that there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and the processes $(\tilde{X}^n, \tilde{Z}, \tilde{Y}^n)$, $(\tilde{X}, \tilde{Z}, \tilde{Y})$ on it such that the distributions of the processes (X^n, Z, Y^n) and $(\tilde{X}^n, \tilde{Z}, \tilde{Y}^n)$ coincide for any $n \in \mathbb{N}$ and

$$(\tilde{X}^n, \tilde{Z}, \tilde{Y}^n) \to (\tilde{X}, \tilde{Z}, \tilde{Y}) \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Similarly as in [7] (chapter 2), one shows that, for all $n \in \mathbb{N}$,

$$\tilde{X}_t^n = x_0 + \tilde{Z}_t + \int_0^t a_n(\tilde{X}_s^n)ds, \quad t \geq 0$$

with probability one.

Since the convergence in probability implies the existence of a subsequence for which the convergence almost surely follows, we observe that in order to finish the proof it is enough to verify that, for all $t \geq 0$,

$$\int_0^t a_n(\tilde{X}_s^n)ds \to \int_0^t a(\tilde{X}_s)ds \quad \text{as} \quad n \to \infty$$

in probability.

For any $t \geq 0, m, k \in \mathbb{N}$, and $\varepsilon > 0$, one has

$$\begin{aligned} &\tilde{\mathbf{P}}\Big(|\int_0^t a_n(\tilde{X}_s^n)ds - \int_0^t a(\tilde{X}_s)ds| > \varepsilon\Big) \\ &\quad \leq \tilde{\mathbf{P}}\left(\left|\int_0^t a_k(\tilde{X}_s^n)ds - \int_0^t a_k(\tilde{X}_s)ds\right| > \varepsilon/3\right) \\ &\qquad + \tilde{\mathbf{P}}\left(\int_0^{t\wedge\tau_m(\tilde{X}^n)} |a_k - a_n|(\tilde{X}_s^n)ds > \varepsilon/3\right) \\ &\qquad + \tilde{\mathbf{P}}\left(\int_0^{t\wedge\tau_m(\tilde{X})} |a_k - a|(\tilde{X}_s)ds > \varepsilon/3\right) \\ &\qquad + \tilde{\mathbf{P}}\big(\tau_m(\tilde{X}^n) < t\big) + \tilde{\mathbf{P}}\big(\tau_m(\tilde{X}) < t\big). \end{aligned}$$

The convergence of the first term on the right-hand side of the last inequality to zero as $n \to \infty$ is trivial because a_k is a smooth, bounded function. Secondly, the last two terms can be made arbitrary small uniformly for all n by choosing large enough m. To verify the convergence to zero of the remaining two terms, we apply the Chebyshev's inequality and Krylov's estimates to obtain

$$\tilde{\mathbf{P}}\left(\int_0^{t\wedge\tau_m(\tilde{X}^n)} |a_k - a_n|(\tilde{X}_s^n)ds > \varepsilon/3\right) \leq \frac{3N}{\varepsilon}\|a_k - a_n\|_{2,m} \tag{3.2}$$

and

$$\tilde{\mathbf{P}}\left(\int_0^{t\wedge\tau_m(\tilde{X})} |a_k - a|(\tilde{X}_s)ds > \varepsilon/3\right) \leq \frac{3N}{\varepsilon}\|a_k - a\|_{2,m}, \tag{3.3}$$

where in the inequality (3.3) we used the Krylov's estimate for the limit process $\tilde{X}$ that can be obtained from Krylov's estimates for the processes $\tilde{X}^n$ (cf. Lemma 4.2 in [8]).

By letting $n, k \to \infty$ we obtain that the right-hand sides of (3.2) and (3.3) converge to zero.

Theorem 3.2 *Suppose that*

i) there exists $\delta_1 > 0$ *and* $\delta_2 > 0$ *such that* $\delta_1 \le |b(x)| \le \delta_2$ *for all* $x \in \mathbb{R}$*;*
ii) $|a(x)| \le (1/2)|b(x)|$ *for all* $x \in \mathbb{R}$.

Then, for any $x_0 \in \mathbb{R}$*, there exists a solution of the equation (1.1).*

Proof. Consider the stochastic equation

$$Y_t = x_0 + \bar{Z}_t + \int_0^t A(Y_s)ds, \tag{3.4}$$

where $A = |b|^{-1}a$ and $\bar{Z}$ is a symmetric Cauchy process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It follows from the conditions $i)$ and $ii)$ that $|A(x)| \le 1/2$ for all $x \in \mathbb{R}$. Therefore, by Theorem 3.1, the equation (3.4) has a solution for any initial value $x_0 \in \mathbb{R}$.

Now, let

$$T_t = \int_0^t |b(Y_s)|^{-1}ds, \quad t \ge 0. \tag{3.5}$$

It is easy to see that the process T is well-defined and is adapted to the filtration generated by the process Y. Moreover, $T_\infty = \lim_{t\to\infty} T_t = \infty$. Define by A_t the right-inverse of T. Because T is strictly increasing, A is a continuous time change with respect to the mentioned filtration and $A_\infty = \infty$. It can be immediately verified from (3.5) that

$$A_t = \int_0^t |b(Y_{A_s})|ds.$$

Define $X_t := Y_{A_t}$. By making the time change in the equation (3.4), one obtains

$$X_t = x_0 + \bar{Z}_{A_t} + \int_0^t a(X_s)ds.$$

Using similar arguments as in [5], one concludes that there exists a symmetric Cauchy process Z such that $\bar{Z}_{A_t} = \int_0^t b(X_{s-})dZ_s$ so that X is a solution of equation (1.1) with $X_0 = x_0$.

Corollary 3.3 *Assume that*

i) there exists a constant $\delta_2 > 0$ *such that* $0 < |b(x)| \le \delta_2$ *for all* $x \in \mathbb{R}$ *and* $|b|^{-1} \in L_2(\mathbb{R})$*;*
ii) $|a(x)| \le (1/2)|b(x)|$ *for all* $x \in \mathbb{R}$.

Then, for any $x_0 \in \mathbb{R}$*, there exists a solution of equation (1.1).*

Proof. The proof is similar to that of Theorem 3.2. The only difference is that we need other arguments to verify the finiteness of the integral functional T defined in (3.5). Since $|A| \le 1/2$, the application of Corollary 2.3 yields

$$\mathbf{E}\int_0^{t\wedge\tau_m(Y)} |b(Y_s)|^{-1}ds \le N\|b^{-1}\|_{2,m}.$$

Since $\tau_m(Y) \to \infty$ as $m \to \infty$, we obtain by letting $m \to \infty$ in the last inequality

$$\mathbf{E}\int_0^t |b(Y_s)|^{-1}ds \le N\|b^{-1}\|_2$$

and, consequentely,

$$\mathbf{E}T_t < \infty$$

for all $t \geq 0$ due to the assumption $|b|^{-1} \in L_2(\mathbb{R})$. Therefore, $T_t < \infty$ a.s. for all $t \geq 0$.

Acknowledgements

The author thanks the anonymous referee for helpful comments which improved the style of the paper.

References

[1] Aldous, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340.
[2] Appelbaum, D. (2004). *Lévy processes and stochastic calculus.* Cambridge University Press.
[3] Bertoin, J. (1999). *Lévy processes.* Cambridge University Press.
[4] Dellacherie, C. and Meyer, P. A. (1980). *Probabilities et potentiels B.* Hermann, Paris.
[5] Engelbert, H. J. and Kurenok, V. P. (2002). On one-dimensional stochastic equations driven by symmetric stable processes. In *Stochastic Processes and Related Topics* (R. Buckdahn, H. J. Engelbert and M. Yor, eds.) 81–110. Taylor and Francis.
[6] Ikeda, N. and Watanabe, S. (1989). *Stochastic Differential Equations and Diffusion Processes.* North-Holland Publ., Tokyo.
[7] Krylov, N. V. (1980). *Controlled Diffusion Processes.* Springer, New York.
[8] Kurenok, V. P. (2008). A note on L_2-estimates for stable integrals with drift. *Transactions of AMS* **360** (2) 925–936.
[9] Tsuchiya, M. (1970). On a small drift of a Cauchy process. *Journal Math. Kyoto University* **10** (3) 475–492.
[10] Zanzotto, P. A. (2002). On stochastic differential equations driven by Cauchy process and the other stable Lévy motions. *Ann. Probab.* **30** (2) 802–825.