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Diffusion Processes on Manifolds

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Abstract: This is an informal introduction to stochastic analysis on both Riemannian and Lorentzian manifolds. We review the basics underlying the construction of diffusions on manifolds, highlighting the important differences between the Riemannian and Lorentzian cases. We also discuss a few recent applications which range from biophysics to cosmology.

1. Introduction

The aim of the present contribution is to offer an informal and self-contained introduction to stochastic analysis on manifolds and to highlight some of its most recent applications to physics. The interest in diffusions on manifolds dates back at least to Itô [29, 30], who first extended stochastic calculus to deal with processes defined on Riemannian manifolds. Section 2 is therefore devoted to diffusions process on Riemannian manifolds. Section 2.1 reviews the basic construction of these processes, highlighting the fact that these diffusions are most simply defined through their generators. Section 2.2 offers an extension which deals with diffusions defined on a fixed base manifold equipped with a statistical ensemble of Riemannian metrics. This extension is used in Section 2.3 to model realistically the influence of metric fluctuations on the so-called lateral diffusions occuring on 2-D biophysical interfaces. Section 3 is devoted to processes defined on Lorentzian manifolds and, more particularly, with diffusions in 4-D relativistic space-times. Section 3.1 deals with diffusions in flat Minkowski space-time, insisting on the important conceptual and technical differences displayed by the constructions of stochastic processes on Riemannian and Lorentzian manifolds. Diffusions on curved Lorentzian manifolds are presented in Section 3.2, along with an H-theorem obeyed by at least the simplest of these processes and an application to diffusions in an expanding universe. A final section reviews some processes which have been recently considered in the mathematical and physical literature and which our text had hitherto not explicitly mentioned. We then suggest a few lines along which we believe research on stochastic analysis on manifolds could fruitfully develop.

2. Diffusions on Riemannian Manifolds

2.1. Basics

A Riemannian manifold is defined [18] as a real base manifold $\mathcal{B}$ equipped with a Riemannian metric g. This metric defines a canonical volume measure on $\mathcal{B}$, which is usually denoted by dVol_g. Diffusions on Riemannian manifolds are best defined

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through their generators [22, 27, 33, 35]. Let L be a second order elliptic operator which acts on functions defined on $I \times \mathcal{B}$, where I is some interval of the positive real axis. Consider the equation

$$\partial_t n = Ln, \tag{2.1}$$

and suppose that this conserves the integral of n with respect to dVol_g. The idea underlying the construction of stochastic processes on $\mathcal{B}$ is to consider equation (2.1) as a forward Kolmogorov equation. More precisely, given a local chart (x) covering $U \subset \mathcal{B}$, the adjoint $L^\dagger$ of L can be locally decomposed on the basis $(\partial_{x^i}, \partial_{x^i}\partial_{x^j})$, $i, j = 1, \cdots, d$, where d is the dimension of $\mathcal{B}$; let $L^\dagger = a^i(x)\partial_{x^i} + b^{ij}(x)\partial_{x^i}\partial_{x^j}$. The operator $L^\dagger$ is the generator of the stochastic process $dx_t^i = a^i(x)dt + \sigma^{ij}(x)d(B_j)_t$ with $(b^{ij})(x) = \frac{1}{2}\left(\sigma(x)\sigma^T(x)\right)^{ij}$, $i, j = 1, \cdots, d$. This x-process defines a stochastic process on $U \subset \mathcal{B}$ at all times t inferior to the exit time from U. It can be shown that this local definition can be extended into a global and intrinsic (*i.e.* coordinate-free) definition [33].

In particular, a Brownian motion B_t^g on $\mathcal{B}$ is defined by choosing $L = \Delta_g$, the Laplace-Beltrami operator [18] associated with the metric g. In coordinates,

$$\Delta_g n = \frac{1}{\sqrt{\det g_{ij}}}\, \partial_i \left(\sqrt{\det g_{ij}}\, g^{ij} \partial_j n\right), \tag{2.2}$$

where g_{ij} are the components of the metric g in the chart x and g^{ij} are the components of the matrix inverse to g_{ij}. Equation (2.1) then conserves the normalization of n with respect to dVol_g because $\mathrm{dVol}_g = \sqrt{\det g_{ij}}\, d^n x$.

The above construction can be extended to include cases in which the metric g explicitly depends on time t [6, 10]. The Brownian motion $B_t^{g(t)}$ is then defined through the forward Kolmogorov equation

$$\frac{1}{\mu_{g(t)/h}} \partial_t \left(\mu_{g(t)/h}\; n\right) = \Delta_{g(t)} n, \tag{2.3}$$

where $\mu_{g(t)/h}$ is the density of $\mathrm{dVol}_{g(t)}$ with respect to dVol_h, h being an arbitrary time-independent metric on the manifold. Note that this definition of $B_t^{g(t)}$ does not depend on the choice of h; it is also the minimal extension of the standard definition which ensures that $\int n\, \mathrm{dVol}_{g(t)}$ is conserved in time, even for time-dependent metrics. Let us also mention that conditional entropy currents for these Brownian motions have been constructed in [10].

2.2. Comparing diffusions in different metrics

Many situations of physical or biophysical interest involve diffusions on interfaces [4, 11, 15, 34, 38] and it is convenient, for most purposes, to model these interfaces as 2-D Riemannian manifolds. In such a model, the base manifold $\mathcal{B}$ codes for the topology of the interface and the metric $g(t)$ defined on $\mathcal{B}$ codes for the possible time-dependent local geometry of the interface. In practice, the topology of a 'real' physical interface is often known with great precision, but its local geometry is not, our knowledge being for example limited by the finite temporal and spatial resolutions of the experiments one wants to model. The natural way to take this into account is to change the mathematical model of the interface [6, 10].

First, one still encodes the topology of the interface in the choice of a certain base manifold $\mathcal{B}$, but one introduces a new structure Σ on $\mathcal{B}$, which will code for the finite spatial resolution of a given observational set-up; the simplest possible choice for Σ is an atlas of $\mathcal{B}$, with typical chart $(x) = (x^1, x^2)$, where the finite spatial resolution of the observational set-up is represented by a certain typical spatial cutoff scale x^*. By convention all scales much larger than the cutoff scale will be called large scales and scales smaller than the cutoff scale will be called small scales. One similarly introduces a temporal cutoff t^* on the real axis and defines large and small temporal scales accordingly.

Second, one introduces on $\mathcal{B}$, not a single metric $g(t)$, but a collection of metrics $g(t, \omega)$, $\omega \in \Omega$, where Ω is an at this stage arbitrary probability space. The observational constraint is that these metrics all coincide with a certain effective metric $\bar{g}(t)$ on large spatial and temporal scales, but differ from $\bar{g}(t)$ on small scales. We make two further hypotheses. The first one, which is not restrictive, is that $\bar{g}(t)$ varies on large temporal and spatial scales only; the second one is restrictive, but computationaly convenient, and states that $< g(t, \omega) >= \bar{g}(t)$, where the angular brackets denote the averaging over ω.

Consider now a 'real' diffusion on the physical interface. This 'real' diffusion is best modeled by a collection of diffusions $M_t(\omega)$, $M_t(\omega) \in \mathcal{B}$ for all values of $(t, \omega) \in \mathbb{R}_+ \times \Omega$. It is natural to wonder if there is a sense in which this collection of diffusions averages into a diffusion $\bar{M}_t$ defined on $\mathcal{B}$ endowed with the effective metric $\bar{g}$. We will now present a preliminary result which strongly suggests this is not the case. The conjecture is thus that small scale variations of the local geometry cannot generically be neglected and that diffusions in g do not average into diffusions in $\bar{g}$.

Let us focus on the simplest diffusions, *i.e.* Brownian motions. Let h be an arbitrary time-independent Riemannian metric on $\mathcal{B}$ and let $G(t)$ be an arbitrary ω-independent Riemannian metric on $\mathcal{B}$. Let also $N_i \mathrm{dVol}_{G(t=0)}$ be an arbitrary probability measure on $\mathcal{B}$. This probability measure can be considered as an initial condition and we will call $N(t, \omega)\mathrm{dVol}_{G(t)}$ the probability measure generated from $N_i \mathrm{dVol}_{G(t=0)}$ by the Brownian motion $B_t^{g(t,\omega)}$. The density $N(t, \omega)$ satisfies the following forward Kolmogorov equation:

$$\frac{1}{\mu_{g/h}} \partial_t \left(\mu_{g/h} \frac{N(t, \omega)}{\mu_{g/G}} \right) = \Delta_{g(t,\omega)} \left(\frac{N(t, \omega)}{\mu_{g/G}} \right), \tag{2.4}$$

which ensures that $\int_{\mathcal{B}} N(t, \omega)\mathrm{dVol}_{G(t)} = 1$ at all times. There is a canonical choice for $G(t)$, namely $G(t) = \bar{g}(t)$, and it will be retained for the remainder of this presentation.

We will say that the collection of diffusions $M_t(\omega)$ averages into a diffusion in $\bar{g}$ if there exists a diffusion $\bar{M}_t$ in $\bar{g}(t)$, with initial density N_i with respect to $\mathrm{dVol}_{\bar{g}(t=0)}$, whose density $\bar{N}(t)$ at time t with respect to $\mathrm{dVol}_{\bar{g}(t)}$ coincides with $< N(t, \omega) >$, where angular brackets denote averages over ω. We will now present a perturbative treatment of diffusions in nearly flat metrics which strongly suggests that collections of diffusions generally do not average into Itô processes.

2.3. Application

We choose $\mathcal{B} = \mathbb{R}^2$ and introduce a global chart $x = (x^1, x^2)$ which plays the role of the extra structure Σ introduced in Section 2.2. The spatial and temporal cutoffs are still denoted by x^* and t^* respectively. We then consider the collection of metrics $g(t, \omega)$ defined by

$$g(t, x, \omega) = \eta + \varepsilon \gamma(t, x, \omega), \tag{2.5}$$

where η is the flat metric, with components $\eta^{ij} = \text{diag}(1,1)$ in this chart, and $\varepsilon\gamma$, $\varepsilon \ll 1$, is a perturbation around η, with components:

$$\gamma^{ij}(t,x,\omega) = \sum_{nn'} A^{ij}_{nn'} \cos\left(K_n.x + \Omega_{n'}t + \phi_{nn'} + \omega\right). \tag{2.6}$$

We choose $\Omega = [0, 2\pi[$, equipped with the uniform probability measure $dp_\omega = \frac{1}{2\pi}d\omega$. Averaging over ω thus yields: $\langle\gamma(t,x,\omega)\rangle = 0$, which implies that $\bar{g} = \eta$. In accordance with the discussion in Section 2.2, we finally impose that all wave numbers K_n (*resp.* all pulsations $\Omega_{n'}$) are superior to $K_* = 2\pi/x^*$ (*resp.* $\Omega_* = 2\pi/t^*$).

Equation (2.4) transcribes into

$$\partial_t\left(N(t,x,\omega)\right) = \partial_i\left(\sqrt{\text{det}g_{ij}(t,x,\omega)}\,g^{ij}(t,x,\omega)\partial_j\left(\frac{N(t,x,\omega)}{\sqrt{\text{det}g_{ij}(t,x,\omega)}}\right)\right), \tag{2.7}$$

which conserves the normalization of $N(t,\omega)$ with respect to $\text{dVol}_\eta = d^2x$ (*i.e.* $\int_{\mathbb{R}^2} N(t,x,\omega)d^2x = 1$ at all times).

Formally expand (2.7) in powers of ε and average the obtained expansion over ω. The result takes the form $\partial_t\bar{N} = L\bar{N}$, where the operator L is a formal series in ε; this last equation is a formal transport equation for the average density $\bar{N}$. Let $\mathcal{F}(\bar{N})$ be the spatial Fourier transport of $\bar{N}$; the transport equation can be transcribed [6, 10] into an equation of the form:

$$\partial_t\mathcal{F}(\bar{N}) = -k^2\left(1 + \varepsilon^2 F(t,k)\right)\mathcal{F}(\bar{N}), \tag{2.8}$$

where $F(t,k)$ depends exponentially on k. Thus, if $\bar{N}$ is analytic in ε, it cannot be the density of an Itô process.

The exact expression for F is reproduced in [6]. Inspecting this expression reveals that, at fixed k, F grows exponentially in time if there exist wave numbers K_n and K_p of the perturbation for which either $(K_n - K_p)^2 + 2k.(K_n - K_p)$ or $(K_n - K_p)^2 - 2k.(K_n - K_p)$ is positive. This is for example the case if one can find an (n,p) such that $k = K_n - K_p$. The term $\varepsilon^2 F(t,k)$ then becomes of order unity after a typical time $\tau(\varepsilon,k)$ which scales as $|\ln\varepsilon|/k^2$ [6]. This result suggests that the small scale variations of the metric have a cumulative effect on the large scale aspects of Brownian motions and that Brownian motions on a nearly flat surface generally do not even remain 'close' to Brownian motions on a plane.

3. Relativistic diffusions as examples of diffusions on Lorentzian manifolds

3.1. *Diffusions on flat Lorentzian manifolds*

3.1.1. Special Relativistic space-time

Relativistic space-time is a real 4-D Lorentzian manifold [39]. In the absence of gravitational field, this manifold is isomorphic to $\mathbb{R}^4$, can thus be covered by single chart atlases, and is equipped with the flat Minkowski metric η. It is possible to choose charts[1] $x^\mu = (ct, \mathbf{r})$ in which the metric components read: $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$; these charts are called Lorentzian (or inertial) charts. It is also customary to call

[1] c stands for the velocity of light in vacuo

Lorentzian or inertial frame a set of inertial charts which only differ by isometries in $\mathbf{r}$-space. As done by most authors, we will hereafter only use global inertial charts to map the flat 4-D Minkowski space-time.

Motions of point masses on the space-time manifold are represented by parametrized curves. It is customary, and often convenient, to choose as parameter the so-called proper-time σ along the motion. In a given chart, a motion is thus represented by a set of four functions $x^\mu(\sigma)$, $\mu = 0, 1, 2, 3$, σ itself being defined by $d\sigma^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$. This parameter is real because 3-D velocities of point masses are bounded by c. The 4-velocity u of a point mass m is then given by $u^\mu = dx^\mu/d\sigma$ and its 4-momentum p by $p_\mu = mc\eta_{\mu\nu}u^\nu$.

A given observer will however wish to represent motions in the usual intuitive way, *i.e.* as 3-D functions $\mathbf{r}(t)$ of his/her own time t. Let $\mathbf{v}(t) = d\mathbf{r}/dt$ be the usual 3-D velocity and $\gamma(\mathbf{v}) = \sqrt{1 - \mathbf{v} \cdot \mathbf{v}/c^2}$ be the so-called Lorentz factor (here, $\cdot$ stands for the usual 3-D euclidean scalar product). A direct calculation shows that $u^0 = \gamma(\mathbf{v})$ and $\mathbf{u} = \gamma(\mathbf{v})\mathbf{v}/c$. The energy E and the 3-D momentum $\mathbf{p}$ of the point mass are defined by $p_\mu = (-E/c, \mathbf{p})$. One thus has $E = \gamma mc^2$ and $\mathbf{p} = m\gamma(\mathbf{v})\mathbf{v}$. Note that the square root in the definition of γ traces the fact that the Euclidean norm of the 3-D velocity $\mathbf{v}$ of any point mass is bounded by c. At any point of space-time, the four components p_μ are not independent, but rather satisfy $\mathbf{p}^2 - p_0^2/c^2 = -m^2c^2$. Thus, at any point of space-time, momentum space is actually not 4 dimensional, but is rather a 3-D hyperboloid naturally embedded in 4-D real euclidean space. The phase space $\mathcal{P}$ of a relativistic point mass is thus a 7-D submanifold of the 8-D bundle cotangent to space-time. Any chart $(x^\mu) = (ct, \mathbf{r})$ of the 4-D space-time manifold induces a chart of the 8-D bundle cotangent to space-time, with coordinates $(x^\mu, p_\nu) = (ct, \mathbf{r}, p_0, \mathbf{p})$. This chart in turn induces a chart $(ct, \mathbf{r}, \mathbf{p})$ of the phase-space $\mathcal{P}$ of a special relativistic point mass.

3.1.2. Why a naive construction of stochastic processes fails on Lorentzian manifolds

Suppose one is interested in modeling stochastic motions of relativistic point masses in flat 4-D space-time. It seems quite natural to start by trying to construct a relativistic equivalent of the usual 3-D Brownian motion $\mathbf{B}_t$. Typically, one might wish to construct a 4-D Brownian motion B_σ, using the proper-time σ along the motion as parameter. Such an object, however, cannot exist on a Lorentzian manifold.

Indeed, the jump probabilities of such a B_σ would have to be proportional to $\exp\left(-\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu/\Delta\sigma\right)$, where Δx represents the jump in space-time made by the point mass during the proper time interval $\Delta\sigma$. But, since the manifold is Lorentzian, $\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$ is not positive definite. Indeed, in the flat case discussed here, and in inertial coordinates, $\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = (\Delta\mathbf{r})^2 - c^2(\Delta t)^2$, and the jump probability could not be normalized in the time jump Δt.

The same problem can be looked at from the point of view of partial differential equations, by considering what the generator of B_σ would be. The Kolmogorov equation associated to B_σ would read $\partial_s n = \Delta_\eta n$; in inertial coordinates, this gives $\partial_s n = \left(\partial^2/\partial\mathbf{r}^2 - \partial^2/\partial(ct)^2\right) n = \Box n$, where $\Box$ is the standard D'Alembert operator, which is obviously hyperbolic, and not elliptic. Finally, the density n associated to B_σ would be a σ-dependent function defined on the space-time manifold, and, at any proper time σ, this function could be integrated on 4-D domains of space-time against the natural 4-D volume measure dVol_η. In an inertial chart, n would thus appear as a function of the five variables, $(\sigma, ct, \mathbf{r})$ and, for all values of σ, could be integrated against $d(ct) \wedge d^3r$ in 4-D domains of space-time. Thus, n would not be a density in the usual sense of the word, since standard densities are represented by functions of the four real variables $(ct, \mathbf{r})$ only, and can only be integrated at fixed time t in 3-D domains of the space-time.

Thus, the standard Riemannian construction of stochastic processes cannot be extended in a naive way to the Lorentzian case.

3.1.3. The Relativistic Ornstein–Uhlenbeck Process (ROUP)

The way out of these difficulties is to construct a diffusion in momentum space and to use this diffusion to generate a diffusion in physical space-time [19]. In other words, one constructs a diffusion in the whole phase-space $\mathcal{P}$ of a relativistic point mass, and not on the space-time manifold only. This idea underlies the construction of the ROUP [14], which models how a special relativistic point mass diffuses through its interaction with a fluid in thermodynamical equilibrium. The state of this fluid is characterized by its temperature T and its 4-velocity U, which are both uniform and constant. One can, without any loss of generality, decide to work in global inertial charts of space-time which belong to the rest frame of the fluid and, in these charts, $U = (1, 0)$.

Given any of these charts $(ct, \mathbf{r})$, consider the following set of stochastic differential equations:

$$\begin{aligned} d\mathbf{r}_t &= \frac{1}{m}\,\frac{\mathbf{p}_t}{\gamma(\mathbf{p}_t)}dt \\ d\mathbf{p}_t &= -\alpha\,\frac{\mathbf{p}_t}{\gamma(\mathbf{p}_t)}dt + \sqrt{2D}d\mathbf{B}_t, \end{aligned} \tag{3.1}$$

where $\gamma(\mathbf{p}) = \sqrt{1 + \mathbf{p}^2/m^2c^2}$ is the Lorentz factor, conceived as a function of $\mathbf{p}$, and α and D are two strictly positive real constants. The first equation in (3.1) is simply the definition of the usual 3-D velocity $\mathbf{v}$ in terms of $\mathbf{p}$. The second equation states that the force experienced by the diffusing particle is made up of two contributions. The first one is a deterministic frictional force and the second one is what physicists call a Gaussian white noise.

The forward Kolmogorov equation associated with (3.1) reads:

$$\partial_t \Pi + \partial_{\mathbf{r}} \,.\, \left(\frac{\mathbf{p}}{m\gamma}\,\Pi\right) + \partial_{\mathbf{p}} \,.\, \left(-\alpha\frac{\mathbf{p}}{\gamma}\,\Pi\right) = D\ \Delta_{\mathbf{p}}\Pi, \tag{3.2}$$

where $\Pi(t, \mathbf{r}, \mathbf{p})$ represents the density of the process (at given initial condition) with respect to the Lebesgue measure d^3rd^3p. The $\mathbf{p}$-process admits an invariant measure, whose density Π_{inv} with respect to d^3p is given by:

$$\Pi_{inv}(\mathbf{p}) = \frac{1}{4\pi(mc)^3}\frac{k_B/(mc^2\beta)}{K_2(mc^2\beta/k_B)}\exp\left(-\frac{mc^2\beta}{k_B}\gamma(\mathbf{p})\right), \tag{3.3}$$

with $\beta\, mc^2/k_B = D/\alpha$. This invariant measure is called the Jüttner distribution and is associated to a thermal equilibrium at temperature $1/\beta$. Since the diffusion is induced by the interaction of the particle with a fluid at temperature T, one gets the following special relativistic fluctuation-dissipation theorem:

$$k_BT = mc^2\frac{D}{\alpha}. \tag{3.4}$$

The so-called Galilean limit is obtained by formally letting the velocity of light c tends to infinity. The $\mathbf{p}$-process (3.1) then degenerates into the usual Ornstein–Uhlenbeck process and the Jüttner distribution becomes a Maxwellian at temperature $1/\beta$. The fluctuation-dissipation theorem conserves its form (3.4).

The stochastic equations (3.1) define a process on $\mathbb{R}^6$. We will now construct a stochastic process on the phase space $\mathcal{P}$ of a point mass. Let us again pick an arbitrary global chart $(ct, \mathbf{r})$ belonging to the rest frame of the fluid surrounding the diffusing particle. As already mentioned, this chart induces a global chart $(ct, \mathbf{r}, \mathbf{p})$ on the one particle phase space $\mathcal{P}$. The system (3.1) implies the following set of SDE:

$$\begin{aligned} d(ct_\sigma) &= \gamma(\mathbf{p}_\sigma) d\sigma \\ d\mathbf{r}_\sigma &= \frac{\mathbf{p}_\sigma}{mc} d\sigma \\ d\mathbf{p}_\sigma &= -\alpha\, \frac{\mathbf{p}_\sigma}{c} d\sigma + \sqrt{\frac{\gamma(\mathbf{p}_\sigma)}{c}} \sqrt{2D} d\mathbf{B}_\sigma . \end{aligned} \tag{3.5}$$

Note that the proper-time σ is now used as parameter in (3.5). The new system (3.5) fixes the stochastic proper-time evolution of the $(ct, \mathbf{r}, \mathbf{p})$-coordinates of a point of $\mathcal{P}$. Since the chart $(ct, \mathbf{r}, \mathbf{p})$ is global, (3.5) also fixes the proper-time evolution of the coordinates of the same point in any (not necessarily global nor inertial) other chart of $\mathcal{P}$. The SDE (3.5) thus defines a stochastic process on $\mathcal{P}$. This process is the ROUP.

3.1.4. Manifestly covariant Kolmogorov equation for the ROUP

Equation (3.2) is the forward Kolomogorov equation associated to the ROUP in any chart of $\mathcal{P}$ associated to a space-time chart belonging to the proper frame of the fluid surrounding the diffusing particle. It is not clear, however, how the transport equation reads in other charts. One might think that the above system (3.5) of SDEs furnishes a straightforward answer, but this is not so. The problem lies in the very definition of the densities associated to (3.2). The definition of the proper-time σ (*i.e.* $d\sigma^2 = c^2 dt^2 - \mathbf{dr}^2$) entails that the increments of t and $\mathbf{r}$ are not independent from the increments of the parameter σ. Any density would thus have to be a function of only $4 + 3 = 7$ of the 8 variables $(\sigma, ct, \mathbf{r}, \mathbf{p})$. It may at first glance seem natural to retain $\sigma, \mathbf{r}$ and $\mathbf{p}$, but, even if were mathematically interesting, the density of relativistic diffusion at fixed value of the proper time σ would not be a physically measurable object anyway, since only densities at fixed values of a time-coordinate on the manifold are accessible to observations.

It turns out that the easiest way to solve the problem is to consider how a change of charts acts on the original SDEs. Let $(ct, \mathbf{r}, \mathbf{p})$ be the chart of $\mathcal{P}$ in which (3.1) applies, and let $(ct', \mathbf{r}', \mathbf{p}')$ be another chart of $\mathcal{P}$ corresponding to an inertial chart $(ct', \mathbf{r}')$ of the space-time. It is possible to obtain from (3.1) the stochastic equations governing the evolution of $(\mathbf{r}', \mathbf{p}')$ with the time coordinate t'. It is then possible to derive directly from these new stochastic equations the Kolmogorov equation which describes the transport in the chart $(ct', \mathbf{r}', \mathbf{p}')$. The main conclusion of these computations [3] is that the density Π of the ROUP is actually a function defined on the phase-space $\mathcal{P}$. Given any inertial chart $(ct, \mathbf{r}, \mathbf{p})$, this function transcribes into a function $\Pi(t, \mathbf{x}, \mathbf{p})$, which represents the density of the process with respect to the volume measure $d^3x d^3p$ associated to the chart under consideration. The

Kolmogorov equation verified by $\Pi(t, \mathbf{r}, \mathbf{p})$ is however rather complicated, unless of course $(ct, \mathbf{r})$ belongs to the inertial frame of the fluid (in which case the Kolmogorov equation simply is (3.2)).

There is nevertheless a technique which helps simplify computations, when these are carried out in charts which do not belong to the proper frame of the fluid surrounding the particle [28]. The idea is to extend the function Π into a function, say f, defined on the whole 8-D cotangent bundle and to also extend the original Kolmogorov equation for Π into a new, computationaly simpler transport equation for f. Here are the results one obtains. Let (x^μ) be a global inertial chart of the space-time and (x^μ, p_ν) its associated chart of the cotangent bundle. In this chart, the function f appears as a function $f(x, p)$ and the simplest transport equation which extends to the full 8-D cotangent bundle the original 7-D dynamics of the ROUP reads [2]:

$$\partial_{x^\mu}(p^\mu f) + \partial_{p^\mu}(mc\ F_d^\mu f) + DK^{\mu\rho\beta\nu}\partial_{p^\rho}\left(\frac{p_\mu p_\beta}{p.U}\ \partial_{p^\nu} f\right) = 0, \tag{3.6}$$

with

$$F_d^\mu = -\lambda^\mu_\nu p^\nu\ \frac{p^2}{m^2c^2} + \lambda^\alpha_\beta\ \frac{p_\alpha p^\beta}{m^2c^2}\ p^\mu, \tag{3.7}$$

$$\lambda^\mu_\nu = \frac{\alpha(mc)^2}{(p.U)^2}\ \Delta^\mu_\nu, \tag{3.8}$$

$$\Delta_{\mu\nu} = \eta_{\mu\nu} - U_\mu U_\nu, \tag{3.9}$$

and

$$K^{\mu\rho\beta\nu} = U^\mu U^\beta \Delta^{\rho\nu} - U^\mu U^\nu \Delta^{\rho\beta} + U^\rho U^\nu \Delta^{\mu\beta} - U^\rho U^\beta \Delta^{\mu\nu}. \tag{3.10}$$

3.2. Diffusions on curved Lorentzian manifolds

3.2.1. General construction

Relativistic gravitation is mathematically described by endowing the 4-D space-time manifold with an arbitrary, possibly curved Lorentzian metric g [18, 39]. The derivative operator ∇ used on the space-time is built from the Levi-Civita connection of g; this connection is commonly represented in a given chart by the so-called Christoffel symbols $\Gamma^\mu_{\alpha\beta}$. The phase-space $\mathcal{P}$ of a relativistic point-mass is still a 7-D submanifold of the cotangent bundle, and is defined by the so-called mass-shell relations $p.p = -m^2c^2$, $p_0 < 0$ where $\cdot$ denotes the scalar product defined by g.

Let α and D be two smooth, positive definite functions on $\mathcal{P}$ and U a vector-field on space-time verifying $U.U = -1$. One can define [12] a stochastic process on $\mathcal{P}$ through the following transport equation:

$$\mathcal{D}_\mu(g^{\mu\nu}(x)p_\nu f) + \frac{\partial}{\partial p_\mu}(mc\ F_{d\mu} f) + K^\mu{}_\rho{}^\beta{}_\nu\ \partial_{p_\rho}\left(D\ \frac{p_\mu p_\beta}{p.U}\ \partial_{p_\nu} f\right) = 0. \tag{3.11}$$

Here, f is a function defined on the cotangent bundle and, given any chart $(ct, \mathbf{r}, p_0, \mathbf{p})$ of this bundle, the restriction Π of f to the mass-shell, defined by

$$\Pi(t, \mathbf{r}, \mathbf{p}) = \int 2p^0 f(t, \mathbf{r}, p_0, \mathbf{p})\delta(p.p + m^2c^2)\theta(-p_0)dp_0 \tag{3.12}$$

represents the density of the stochastic process with respect to the volume measure d^3rd^3p associated with the chart (the symbol θ in (3.12) denotes the Heaviside step function). The definitions of the deterministic 4-force F_d and of the tensors λ and K are formally identical to the flat space-time definitions (3.7), (3.8) and (3.10). The curved space-time definition of the projector Δ reads:

$$\Delta_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu \tag{3.13}$$

and the operator $\mathcal{D}_\mu$ is defined by:

$$\mathcal{D}_\mu = \nabla_\mu + \Gamma^\alpha_{\mu\nu} p_\alpha \frac{\partial}{\partial p_\nu}, \tag{3.14}$$

where ∇_μ stands for the usual covariant derivative operator with respect to space-time degrees of freedom. The operator $\mathcal{D}$ is usually called the horizontal derivative at momentum p covariantly constant. The so-called vertical derivative is simply $\partial/\partial p$.

The processes defined by (3.11) describe the diffusion of a relativistic point mass immersed in a fluid characterized by a 4-velocity field U, a friction coefficient α and a diffusion coefficient D in momentum space; these processes also take into account through the metric g the action of a possible gravitational field on the point mass. Given a base manifold, these processes thus depend on α, D, U and g; note that the choices for U and g are not completely independent since the condition $U.U = -1$ must be satisfied.

3.2.2. An H-theorem on curved Lorentzian manifolds

Keep U and g arbitrary but impose that α and D are both constant. The corresponding processes satisfy a very simple H-theorem. Let f and h be two solutions of the extended transport equation (3.11). Let (x, p) be a chart of the extended 8-D phase space. The current $S_{f/h}$ of conditional entropy of f with respect to h is defined by:

$$S_{f/h}(x) = -2 \int_X pf \ln\left(\frac{f}{h}\right) D^4p, \tag{3.15}$$

where D^4p is defined by:

$$D^4p = \frac{1}{\sqrt{\det g_{\mu\nu}}} \delta(p.p + m^2c^2)\theta(-p_0)dp_0d^3p, \tag{3.16}$$

and X is the region of the 4-D space-time accessible to the particle [12, 37]. Note that the measure D^4p is invariant under a change of chart, so that $S_{f/g}$ is a vector field cotangent to the space-time manifold. It has been proven [37] that $\nabla \cdot S_{f/h} \geq 0$ for all f's and h's and for all possible choices of U and g. This constitutes an H-theorem for the processes under consideration. It seems quite remarkable that this theorem is true, even if the metric g allows for the existence of closed time-like curves [26] on the space-time manifold, and even if the velocity field U of the fluid surrounding the diffusing particle is tangent to one of those curves [2]. In other words, the irreversibility of the considered stochastic processes is always stronger than any possible acausal behaviour generated by the geometry of the space-time itself.

[2] Think for example of a particle diffusing in the Gödel universe [25]

3.2.3. Fluctuation-Dissipation relations in an expanding universe

Keep now α and D arbitrary, but choose for g the standard Friedman-Robertson-Walker (FRW) metric of a spatially flat, homogeneous isotropic cosmological model [36, 39] and take as U the large-scale 4-velocity field of the matter in this universe. One can then find a chart $(ct, \mathbf{r})$ of the space-time where $ds^2 = -c^2dt^2 + a^2(t)d\mathbf{r}^2$ and $(U^\mu) = (1, 0)$; the function $a(t)$, which is called the expansion factor, increases with t if the universe is expanding. The range of the time coordinate t is $\mathbb{R}_+$ and the range of $\mathbf{r}$ is $\mathbb{R}^3$. The large-scale flow of matter is further characterized by a time-dependent thermal equilibrium at temperature $T(t)$ [31]. In accordance with the homogeneity and isotropy of the cosmological model, we require that α and D do not depend on the coordinates $\mathbf{r}$ and depend on $\mathbf{p}$ only through $\mid \mathbf{p} \mid$. The most practical choice which implements these requirements is to consider both α and D as functions of t and $\gamma(t, \mathbf{p}) = \sqrt{1 + \frac{\mathbf{p}^2}{a^2(t)m^2c^2}}$.

The forward Kolmogorov equation for the density $\Pi(t, \mathbf{r}, \mathbf{p})$ then reads:

$$\partial_t \Pi = \frac{1}{a^2(t)} \partial_{\mathbf{r}} \cdot \left(\frac{\mathbf{p}}{m\,\gamma(t, \mathbf{p})} \Pi \right) + \partial_{\mathbf{p}} \cdot \left(\alpha(t, \gamma) \frac{\mathbf{p}}{\gamma} \Pi + D(t, \gamma)\, a^2(t) \partial_{\mathbf{p}} \Pi \right), \tag{3.17}$$

where $\cdot$ denotes the standard Euclidean scalar product in (p_1, p_2, p_3)-space. The stochastic differential equations associated to equation (3.17) are:

$$d\mathbf{r} = \frac{1}{a^2(t)} \frac{\mathbf{p}}{m\,\gamma(t, \mathbf{p})} dt \tag{3.18}$$

$$d\mathbf{p} = -\alpha_I(t, \gamma)\, \frac{\mathbf{p}}{\gamma(t, \mathbf{p})} dt + \sqrt{2\, D(t, \gamma)}\, a(t) d\mathbf{B}_t, \tag{3.19}$$

where $\alpha_I(t, \gamma) = \alpha(t, \gamma) - \frac{\partial_\gamma D}{m^2c^2}$.

The Jüttner distribution at the time-dependent temperature $T(t)$ is imposed as a solution to (3.17). This leads to the following differential equation [5, 7]:

$$\partial_\gamma \Delta \,+\, f(t, \gamma)\, \Delta \,=\, g(t, \gamma); \tag{3.20}$$

where Δ is defined by: $\Delta(t, \gamma) \,=\, \alpha(t, \gamma) - \frac{\beta(t) D(t, \gamma)}{m^2c^2}$, and the functions f and g read: $f(t, \gamma) \,=\, \frac{3\,\gamma}{\gamma^2 - 1} - \beta(t) - \frac{1}{\gamma}$, $g(t, \gamma) \,=\, \frac{\gamma^2}{\gamma^2 - 1} \left(-3\frac{\dot{a}}{a} + \frac{\dot{\beta}}{\beta} - \dot{\beta} \frac{K_2'(\beta)}{K_2(\beta)} - \dot{\beta}\gamma + \beta \frac{\dot{a}}{a} \frac{\gamma^2 - 1}{\gamma} \right)$, with $\beta(t) \,=\, mc^2/(k_B T(t))$. Equation (3.20) constitutes a fluctuation-dissipation theorem in differential form, valid for diffusions in an homogeneous and isotropic expanding universe. For vanishing expansion factor a and constant temperature T, one recovers the special relativistic fluctuation-dissipation theorem given by (3.4). It is actually possible to solve (3.20) exactly in $\Delta(t, \gamma)$ for arbitrary $a(t)$ and $\beta(t)$ and, thus, to obtain an integral form of the fluctuation-dissipation theorem. The obtained expression for Δ is the sum of two contributions [5]; the first one is a time-dependent entire series in $1/\gamma$ and the second one involves the exponential integral function Ei taken at point $mc^2/k_B T(t)$.

Standard cosmological models suppose that $a(t)$ is proportional to $\beta(t)$ [31]; we have solved equation (3.20) numerically in this case (see Figure (1)).

One can see that $\Delta_* = H^{-1}\Delta$, where $H^{-1} = a/\dot{a}$, is always negative. This can be interpreted as follows: in an expanding universe ($H^{-1} < 0$), the amplitude of the noise $D(t)$ which, when associated to a certain friction coefficient $\alpha(t)$, ensures that the time-dependent thermal equilibrium at temperature $T(t)$ is a possible measure of the process is, at any time t_0, superior to the amplitude of the noise which would have to be associated to $\alpha(t_0)$ in flat space-time to ensure that the time-independent thermal equilibrium at temperature $T(t_0)$ is an invariant measure of the ROUP.

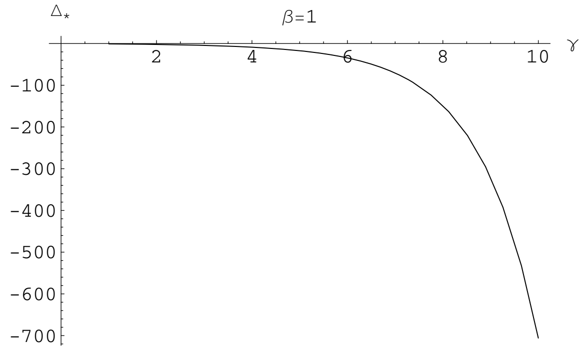

FIGURE 1. *Evolution of* $\Delta_* = H^{-1}\Delta$ *with* γ*, for* $\beta = 1$.

3.2.4. Franchi and LeJan's relativistic Brownian motion

J. Franchi and Y. LeJan have constructed a stochastic process on 4-D general relativistic space-times and studied this new process on various manifolds, including a Schwarzschild black hole [1, 13, 23, 24]. This process is actually a diffusion in the one particle phase-space $\mathcal{P}$, as are all processes presented in Section 3.1 of this contribution. But the only structure which enters the definition of this process is the Lorentzian space-time metric; in particular, no velocity field U is used to define the diffusion. It follows that the process constructed by Franchi and LeJan *cannot* be considered as modeling the motion of a relativistic particle diffusing through its interaction with a surrounding fluid. It has been suggested that the Franchi-LeJan process models the stochastic motion of a relativistic point mass diffusing through its interaction with the quantum degrees of freedom of the gravitational field [17].

J. Dunkel and P. Hänggi [13, 20, 21] have used Franchi and LeJan's process to construct another, physically realistic process modeling the relativistic diffusion of a special relativistic particle immersed in a fluid in a state of thermal equilibrium. It has been shown recently [8, 9] that the ROUP, its general relativistic extensions introduced in Section 3.2.1, the Franchi-LeJan and the Dunkel-Hänggi processes are particular members of a very general class characterized by a simple property of the noise used in stochastic equations of motion. All processes in this class satisfy an H-theorem [9] which extends the one described in Section 3.2.2.

4. Conclusion

Let us now conclude by mentioning a few possible extensions of the work presented in this contribution. We think a proper study of the effects that metric fluctuations have on both classical and quantum diffusions is now long overdue. This problem is rich in new, non trivial mathematical issues, and offers an incredibly large field of applications, which range from biophysics to cosmology. In a different direction, the notion of hydrodynamical limit is still poorly understood mathematically, and a physical apprehension of its applications on Lorentzian manifolds is certainly partial at best [16, 28]. Of particular interest are also stochastic processes on general relativistic space-times with horizons. Finally, one should try and construct stochastic geometries *i.e.* stochastic metrics defined on fixed base manifolds and, ultimately, stochastic manifolds.

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