$$\mathfrak{m}\mathfrak{n} \leqq \mathfrak{m} + \mathfrak{n}.$$

However we have proved earlier that if $\mathfrak{m}$ and $\mathfrak{n}$ are $\geqq 2$, then $\mathfrak{m} + \mathfrak{n} \leqq \mathfrak{m} \cdot \mathfrak{n}$. Thus we obtain $\mathfrak{m}\mathfrak{n} = \mathfrak{m} + \mathfrak{n}$.

6. Some remarks on functions of ordinal numbers

A function f(x) is called monotonic, if $(x < y) \to (f(x) \leqq f(y))$. It is called strictly increasing, if

$$(x < y) \to (f(x) < f(y)).$$

The function is called seminormal, if it is monotonic and continuous, that is if $f(\lim \alpha_\lambda) = \lim f(\alpha_\lambda)$, λ here indicating a sequence with ordinal number of the second kind, i.e., without immediate predecessor, while $(\lambda_1 < \lambda_2) \to (\alpha_{\lambda_1} < \alpha_{\lambda_2})$.

The function is called normal, if it is strictly increasing and continuous; ξ is called a critical number for f, if $f(\xi) = \xi$.

Theorem 17. *Every normal function possesses critical numbers and indeed such numbers $>$ any α.*

Proof: Let α be chosen arbitrarily and let us consider the sequence α, $f(\alpha)$, $f^2(\alpha)$,.... Then if $\alpha_\omega = \lim_{n<\omega} f^n(\alpha)$, we have $f(\alpha_\omega) = f\,(\lim\,(f^n(\alpha)) = \lim f^{n+1}(\alpha) = \alpha_\omega$, that is, α_ω is a critical number for f.

Examples.

1) The function $1 + x$ is normal. Critical numbers are all $x = \omega + \alpha$, α arbitrary.

2) The function $2x$ is normal. Critical numbers are all of the form $\omega\alpha$, α arbitrary.

3) The function ω^x is normal. Critical numbers of this function are called ε-numbers. The least of them is the limit of the sequence $\omega,\ \omega^\omega,\ \omega^{(\omega^\omega)}, \ldots..$

I will mention the quite trivial fact that every increasing function f is such that $f(x) \geqq x$ for every x.

Theorem 18. *Let $g(x) \geqq x$ for all x and α be an arbitrary ordinal; then there is a unique semi-normal function f such that*

$$f(0) = \alpha,\ f(x+1) = g(f(x)).$$

Proof clear by transfinite induction.

Theorem 19. *If f is a semi-normal function and β is an ordinal which is not a value of f, while f possesses values $<\beta$ and values $>\beta$, then there is among the x such that $f(x) < \beta$ a maximal one x_0 such that $f(x_0) < \beta < f(x_0 + 1)$.*

Proof trivial, because if $f(x_\lambda) < \beta$ for all λ in a sequence without last element, then

$$f(\lim x_\lambda) = \lim f(x_\lambda) \leqq \beta,$$

but the equality sign is excluded.

Let A be a set of ordinal numbers without maximal element. A subset B is said to be closed in A, if every limit of a sequence in B is ϵ B, if it is ϵ A. If B is closed in A and cofinal with A it is called a band of A.

Remark. Every band consists of the values of a normal function, and the inverse is true, if the set of the arguments is cofinal with A.

Theorem 20. *If M and N are bands of A, so is M $\cup$ N.*

Proof. Of course M $\cup$ N is cofinal with A. An arbitrary sequence S in M $\cup$ N without last element is either such that from a certain point on all elements belong to M say, then the limit is in M; or there are always greater elements both in M and in N, and then there is a common limit in M and N.

Theorem 21. *If M and N are bands of A and A is as already indicated without last element, but not cofinal with ω, then M $\cap$ N is a band of A.*

Proof. We assume that after a certain $\alpha_0 \epsilon$ M there are no common elements in M and N. Then we have an increasing sequence thus:

α_{2n+1} is the first element of N which is $> \alpha_{2n}$

α_{2n+2} . . . . M which is $> \alpha_{2n+1}$.

Then $\lim\limits_{n<\omega} \alpha_n$ is ϵ A and therefore ϵ M and ϵ N which is contrary to the assumption.

Theorem 22. *Let $f(\alpha,\beta)$ be normal with respect to β. Then it is not an always increasing function with respect to α.*

Proof. If $\alpha_1 < \alpha_2$, then the normal functions $f(\alpha_1,\beta)$ and $f(\alpha_2,\beta)$ of β have a common critical value ξ according to the last theorem so that $f(\alpha_1,\xi) = f(\alpha_2,\xi) = \xi$.

Let us however, following E. Jacobsthal, consider the functions having the following two properties:

1) $f(\alpha,\beta)$ is for constant α a normal function of β

2) $f(\alpha,\beta)$ is for constant β a monotonic function of α with $f(\alpha,\beta) > \alpha$.

Further let us call f_1 a generating function for f when

$$f(\alpha,\beta+1) = f_1(f(\alpha,\beta), \alpha).$$

This equation together with $f(\alpha,0)$ defines f when f is continuous.

Theorem 23. *If f_1 has for $\alpha > 1$, $\beta > 1$ the property 2) and is monotonic in β, while f is continuous and $f(\alpha,1)$ increasing in α, then f satisfies 1) and 2).*

Proof. When $\alpha > 1$, one has $f(\alpha,1) > 1$, namely $f(\alpha,1) \geqq \alpha > 1$. If, for $\alpha > 1$ and $\beta \geqq 1$, $f(\alpha,\beta)$ is monotonic in α and $f(\alpha,\beta) > 1$, then because of the definition of f above $f(\alpha, \beta + 1)$ is monotonic in α and $f(\alpha, \beta + 1) = f_1(f(\alpha,\beta), \alpha) > f(\alpha,\beta)$ (see 2)). If λ is a limit number, and if, for $\alpha > 1$ and $1 < \beta < \lambda$, $f(\alpha,\beta)$ monotonic in α, then $f(\alpha,\lambda)$ is monotonic in α. Thus for $\alpha > 1$ and $\beta > 1$ we have that $f(\alpha,\beta)$ is monotonic in α and a normal function in β. Further, for $\alpha > 1$ we have, because of $f(\alpha,1) > \alpha$, also $f(\alpha,\beta) > \alpha$ for $\beta > 1$.

Now, if one starts with $\phi_0(\alpha,\beta) = \alpha + 1$ and defines $\phi_{r+1}(\alpha,\beta)$ by using ϕ_r as generating function for $r = 0,1,2$ putting $\phi_1(\alpha, 0) = \alpha$, $\phi_2(\alpha, 0) = 0$, $\phi_3(\alpha, 0) = 1$, then we obtain

$$\phi_1(\alpha,\beta) = \alpha + \beta, \quad \phi_2(\alpha,\beta) = \alpha \cdot \beta, \quad \phi_3(\alpha,\beta) = \alpha^\beta.$$

An immediate result is that these functions have the properties 1) and 2).

Definitions: 1) Let us say that f with generating function f_1 satisfies a generalized distributive law when a function f_2 exists such that

$$(1) \quad f_1(f(\alpha,\beta), f(\alpha,\gamma)) = f(\alpha, f_2(\beta,\gamma)).$$

If $f_2 = f_1$, we say that f satisfies the special distributive law.

2) We may say that f fulfills a generalized associative law, if a function f_3 exists such that

$$(2) \quad f(f(\alpha, \beta), \gamma) = f(\alpha, f_3(\beta, \gamma)).$$

If $f_3 = f$, f satisfies the special associative law.

Theorem 24. *If f satisfies the general associative law, then f_3 satisfies the special associative law.*

Proof. If in the formula (2) we put $\alpha = f(\xi, \alpha')$, $\beta = \beta'$, $\gamma = \gamma'$, the formula (2) yields

$$f(f(f(\xi, \alpha'), \beta'), \gamma') = f(f(\xi, \alpha'), f_3(\beta', \gamma'))$$

and by application of (2) twice on the left and once on the right side we get

$$f(f(\xi, f_3(\alpha',\beta')), \gamma') = f(\xi, f_3(f_3(\alpha',\beta'), \gamma')) = f(\xi, f_3(\alpha', f_3(\alpha', \gamma'))).$$

whence because $f(\xi, \beta)$ is increasing in β.

$$f_3(f_3(\alpha',\beta'), \gamma') = f_3(\alpha', f_3(\beta', \gamma')).$$

and that is the special associative law for f_3.

Theorem 25. *If f, being generated by f_1, satisfies both laws (1) and (2), then f, is generating function of f_3 and f_3 satisfies the special distributive law.*

Proof. We have

$$f(f(\alpha,\beta), \gamma+1) = f_1(f(f(\alpha,\beta), \gamma), f(\alpha,\beta)) = f_1(f(\alpha, f_3(\beta,\gamma)), \text{if } (\alpha,\beta)) = f(\alpha, f_2(f_3(\beta,\gamma), \beta))$$

and

$$f(f(\alpha,\beta), \gamma + 1) = f(\alpha, f_3(\beta, \gamma + 1)),$$

whence

$$f_3(\beta, \gamma + 1) = f_2(f_3(\beta,\gamma), \beta),$$

that is f_2 is generating function for f_3. Further, by (1)

$$f(\xi, f_2(f_3(\alpha,\beta), \quad f_3(\alpha,\gamma))) = f_1(f(\xi,f_3(\alpha,\beta)), \quad f(\xi, f_3(\alpha,\gamma)))$$

which by (2),(1),(2) successively yields

$$f_1(f(f(\xi,\alpha),\beta), \quad f(f(\xi,\alpha),\gamma)), \quad f(f(\xi,\alpha),f_2(\beta,\gamma)), \quad f(\xi, f_3(\alpha,f_2(\beta,\gamma))).$$

By comparison of the first and last expressions containing ξ one obtains

$$f_2(f_3(\alpha,\beta), \quad f_3(\alpha,\gamma)) = f_3(\alpha,f_2(\beta,\gamma)),$$

that is, f_3 satisfies the special distributive law.

Theorem 26. *If f is defined by f_1, $f(\alpha,0) = 0$ or 1, f satisfying the generalized distributive law, and if f_3 is defined as a continuous function with f_2 as generating function, by*

$$f_3(\alpha,0) = 0$$

$$f_3(\alpha,\beta + 1) = f_2(f_3(\alpha,\beta), \alpha),$$

then f satisfies the associative law (2).

Proof. This law (2) is valid for $\gamma = 0$, because $f(f(\alpha,\beta),0) = 0$ or 1 and $f(\alpha,f_3(\beta,0)) = f(\alpha,0) = 0$ or 1. If the law is valid for γ, then it is valid for $\gamma + 1$, because

$$f(f(\alpha,\beta), \gamma+1) = f_1(f(f(\alpha,\beta),\gamma), f(\alpha,\beta))$$

because of the supposition of induction $= f_1(f(\alpha,f_3(\alpha,\gamma)), f(\alpha,\beta)) = f(\alpha,f_2(f_3(\beta,\gamma), \beta)) = f(\alpha,f_3(\beta,\gamma+1))$. If the law is valid for all $\gamma < \gamma_0$, γ_0 a limit number, then it is true for γ_0, because

$$f(f(\alpha,\beta),\gamma_0) = \lim_{\gamma<\gamma_0} f(f(\alpha,\beta),\gamma) = \lim_{\gamma<\gamma_0} f(\alpha,f_3(\beta,\gamma)) = f(\alpha,f_3(\beta,\gamma_0)).$$

Theorem 27. *Let f be defined by f_1, $f(\alpha,0) = 0$, $f_1(\alpha,0) = \alpha$ or $f(\alpha,0) = 1$, $f_1(\alpha,1) = \alpha$, while the special associative law is valid for f_1, and f_1 is continuous in β; then f satisfies the distributive law (1) with $f_2(\alpha,\beta) = \alpha + \beta$.*

Proof. The formula (1) is valid for $\gamma = 0$, because $f_1(f(\alpha,\beta), f(\alpha,0)) = f(\alpha,\beta)$. Let us assume its truth for γ. Then we have

$$f_1(f(\alpha,\beta), \quad f(\alpha, \gamma + 1)) = f_1(f(\alpha,\beta), \quad f_1(f(\alpha,\gamma), \alpha)),$$

and since the special associative law is valid for f this becomes

$$f_1(f_1(f(\alpha,\beta), \quad f(\alpha,\gamma)), \alpha) = f_1(f(\alpha,\beta + \gamma, \alpha) = f(\alpha,\beta + \gamma + 1).$$

If formula (1) with $f_2(\alpha,\beta) = \alpha + \beta$ is valid for all $\gamma < \gamma_0$, γ_0 a limit number, then it is valid for γ_0, because

$$f_1(f(\alpha,\beta), f(\alpha,\gamma_0)) = \lim_{\gamma<\gamma_0} f_1(f(\alpha,\beta), f(\alpha,\gamma)) = \lim_{\gamma<\gamma_0} f(\alpha,\beta+\gamma) = f(\alpha,\beta+\gamma_0).$$

Applying the last two theorems to the three elementary arithmetical operations, $\phi_1(\alpha,\beta) = \alpha + \beta$, $\phi_2(\alpha,\beta) = \alpha\beta$, $\phi_3(\alpha,\beta) = \alpha^\beta$, it is seen that the associative and distributive laws of these are all derivable from the special associative law of addition

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

Indeed, if we put $f_1 = \phi_1$, $f = \phi_2$ in Theorem 27 we get

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma),$$

and putting $f_1 = \phi_1$, $f_2 = \phi_1$, $f = \phi_2$, $f_3 = \phi_2$, Theorem 26 yields

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

Further, if we put $f_1 = \phi_2$, $f = \phi_3$, Theorem 27 yields

$$\alpha^{\beta} \cdot \alpha^{\gamma} = \alpha^{\beta+\gamma},$$

while putting $f_1 = \phi_2$, $f_2 = \phi_1$, $f = \phi_3$, $f_3 = \phi_2$ one obtains, according to Theorem 26,

$$(\alpha^{\beta})^{\gamma} = \alpha^{\beta\gamma}.$$

7. On the exponentiation of alephs

We have seen that an aleph is unchanged by elevation to a power with finite exponent. I shall add some remarks concerning the case of a transfinite exponent.

Since $2^{\aleph_0} > \aleph_0$, we have $(2^{\aleph_0})^{\aleph_0} \geqq \aleph_0{}^{\aleph_0}$, but $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0}$. On the other hand $2^{\aleph_0} \leqq \aleph_0{}^{\aleph_0}$. Hence

$$2^{\aleph_0} = \aleph_0{}^{\aleph_0}.$$

Of course we then have for arbitrary finite n

$$2^{\aleph_0} = n^{\aleph_0} = \aleph_0{}^{\aleph_0},$$

and not only that. Let namely $\aleph_0 < \mathfrak{m} \leqq 2^{\aleph_0}$. Then

$$2^{\aleph_0} = \aleph_0{}^{\aleph_0} \leqq \mathfrak{m}^{\aleph_0} \leqq 2^{\aleph_0},$$

whence

$$\mathfrak{m}^{\aleph_0} = 2^{\aleph_0},$$

In a similar way we obtain for an arbitrary $\aleph_\alpha$

$$2^{\aleph_\alpha} = \mathfrak{m}^{\aleph_\alpha}$$

for all $\mathfrak{m} > 1$ and $\leqq 2^{\aleph_\alpha}$.

From our axioms, in particular the axiom of choice, we have derived that every cardinal is an aleph. Therefore $2^{\aleph_\alpha}$ is an aleph. We can also prove by the axiom of choice that $2^{\aleph_\alpha} > \aleph_{\alpha+1}$ or perhaps $= \aleph_{\alpha+1}$. One has never succeeded in proving one of these two alternatives and according to a result of Gödel such a decision is impossible. However, in many applications of set theory it has been convenient to introduce the so-called generalized continuum hypothesis or aleph hypothesis, namely