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Club-Isomorphisms of Aronszajn Trees in the Extension with a Suslin Tree

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Abstract We show that, under PFA(S), a coherent Suslin tree forces that every two Aronszajn trees are club-isomorphic.

1 Introduction

S. Todorčević [27] introduced the proper forcing axiom PFA(S). This axiom asserts that there exists a coherent Suslin tree S such that the forcing axiom holds for every proper forcing notion $\mathbb{P}$ which preserves that S is Suslin, that is, for any set $\{D_\alpha; \alpha \in \omega_1\}$ of $\aleph_1$-many dense open subsets of $\mathbb{P}$, there exists a filter of $\mathbb{P}$ which intersects all of the D_α's. We note that, under the existence of a supercompact cardinal, PFA(S) can be forced by the use of the theorem due to Miyamoto [16, Theorem 1.3] (see Theorem 2.1 below).

Larson and Todorčević [11] introduced the forcing axiom $\mathsf{MA}_{\aleph_1}(S)$, which is analogous to PFA(S) replacing "proper" with "countable chain condition" (ccc), to give the consistency of the affirmative answer to Katětov's problem. In particular, they introduced the axiom $\mathcal{K}_2(\mathrm{rec})$, which is a fragment of $\mathsf{MA}_{\aleph_1}$, and proved that $\mathcal{K}_2(\mathrm{rec})$ holds in the extension with a coherent Suslin tree S (which witnesses the axiom $\mathsf{MA}_{\aleph_1}(S)$). Later, Todorčević [27] proved that, under PFA(S), a coherent Suslin tree S (which witnesses PFA(S)) forces that every compact hereditarily normal space satisfying the countable chain condition is hereditarily separable and hereditarily Lindelöf. A. Fischer, P. B. Larson, C. Martinez-Ranero, F. D. Tall, and S. Todorčević developed the investigation of PFA(S) for set-theoretic topology in Fischer, Tall, and Todorčević [5], [11], Larson and Tall [12], Martinez-Ranero [15], and Tall [20]–[22].

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Todorčević said that, under $\mathsf{PFA}(S)$, the forcing extension of a coherent Suslin tree is a model which mixes a part of the forcing axiom with a part of $\diamondsuit$. A basic, but remarkable, fact is that a Suslin tree forces the tower number to be $\aleph_1$ (see Farah [4]). This is one of the consequences of $\diamondsuit$. Larson and Todorčević [13], [14] demonstrated some consequences of $\diamondsuit$ which hold in the extension with a Suslin tree. These consequences are implied by $\diamondsuit(\mathbb{R}, \neq)$, which is also forced with a Suslin tree. (This is a result due to J. T. Moore, M. Hrušák, and M. Džamonja [17, Theorems 6.15, 6.16].)

A proof of I. Farah [4] showed that, under $\mathsf{PFA}(S)$, a coherent Suslin tree S forces that the open coloring axiom holds. This is one of the consequences of the proper forcing axiom PFA. For other examples, under $\mathsf{PFA}(S)$, a coherent Suslin tree S forces that the continuum is $\aleph_2$ (in [11]), and the P-ideal dichotomy holds (in [27]). The author [30] gave a fragment of $\mathsf{MA}_{\aleph_1}$, which is a stronger axiom than $\mathcal{K}_2(\mathrm{rec})$ and holds in the extension with a coherent Suslin tree S over a ground model which satisfies $\mathsf{MA}_{\aleph_1}(S)$. So, for example, every Aronszajn tree is special in its extension (see also Abraham and Todorčević [2], Todorčević [25]). Raghavan and Yorioka [19] proved that, under $\mathsf{PFA}(S)$, a coherent Suslin tree S forces that there are no ω_2-Aronszajn trees. In this paper, we demonstrate a set-theoretic statement which is one of the consequences of PFA and is forced with a coherent Suslin tree.

Abraham and Shelah [1] investigated the isomorphism types of Aronszajn trees. Aronszajn trees T and U are called *club-isomorphic* if there exists a club C on ω_1 such that the set $T\restriction C$, which is the set of nodes of T of heights in C, is order-isomorphic to the set $U\restriction C$. Abraham and Shelah proved that the weak diamond ($2^{\aleph_0} < 2^{\aleph_1}$), which is weaker than $\diamondsuit$, implies that there are $2^{\aleph_1}$ pairwise non-club-isomorphic Aronszajn trees, and PFA implies that every two Aronszajn trees are club-isomorphic (see [1, Section 5]; see also Todorčević [23, Section 5]). In this paper, the following theorem is proved.

Theorem Under $\mathsf{PFA}(S)$, a coherent Suslin tree S forces that every two Aronszajn trees are club-isomorphic.

2 Preliminaries

In this section, we present all prerequisites. Notation and terminology are standard as in Jech [6] and Kunen [8].

An *Aronszajn tree* is a tree of height ω_1 such that, for every $\alpha \in \omega_1$, the set of nodes of height α is countable, and there are no uncountable chains. In this paper, we focus on Aronszajn trees such that every node splits into infinitely many successors, every node extends to any higher height, and different nodes have different sets of predecessors. A *Suslin tree* is an Aronszajn tree which has no uncountable antichains, and $\omega^{<\omega_1}$ is the set of all functions from some countable ordinal into ω (or all sequences in ω of countable length). We note that every Aronszajn tree is embedded into the order structure $\langle \omega^{<\omega_1}, \subseteq\rangle$, which can be considered as a tree. So in this paper, to simplify notation, we may assume that every Aronszajn tree $\langle T, \leq_T\rangle$ is a suborder of the structure $\langle \omega^{<\omega_1}, \subseteq\rangle$, that is, $T \subseteq \omega^{<\omega_1}$ and, for every s and t in T, $s \leq_S t$ if and only if $s \subseteq t$. Moreover, we may assume that an Aronszajn tree T is closed under taking initial segments, that is, for every $t \in T$, $\mathrm{ht}(t)$, which is the height of the node t, is equal to the length of t. For an Aronszajn tree T and a subset I of ω_1, $T\restriction I$ is the set of all $t \in T$ such that $\mathrm{ht}(t) \in I$.

A *coherent* Suslin tree is a Suslin tree S as a subtree of the tree $\omega^{<\omega_1}$ such that

- S is closed under taking initial segments,
- for any s and t in S, the set
$$\{\alpha \in \min\{\mathsf{ht}(s), \mathsf{ht}(t)\}; s(\alpha) \neq t(\alpha)\}$$
is finite.

Moreover, we add the following property to a coherent Suslin tree in this paper:

- for any $s \in S$ and $t \in \omega^{\mathsf{ht}(s)}$, if the set
$$\{\alpha \in \mathsf{ht}(s); s(\alpha) \neq t(\alpha)\}$$
is finite, then $t \in S$.

This property is called *homogeneity*. In most cases, the property is equipped with the coherent Suslin tree when we apply $\mathsf{PFA}(S)$. For $s \in S$, we let
$$\mathrm{Cone}(S, s) := \{u \in S; s \leq_S u\}.$$

We note that, using $\diamondsuit$ or adding a Cohen real, we can construct a coherent Suslin tree (see, e.g., Devlin and Johnsbråten [3], Larson [9, Lemma 1.2], Todorčević [24, (6.9)], Todorčević and Farah [28, Section 3], Todorčević [26, Section 3.2]). A coherent Suslin tree has canonical commutative isomorphisms. Let s and t be nodes in S with the same height. Then we define a function $\psi_{s,t}$ from $\mathrm{Cone}(S, s)$ into $\mathrm{Cone}(S, t)$ such that, for each $v \in \mathrm{Cone}(S, s)$,
$$\psi_{s,t}(v) := t \cup \left(v \restriction \left[\mathsf{ht}(s), \mathsf{ht}(v)\right)\right).$$

(Here, $v \restriction [\mathsf{ht}(s), \mathsf{ht}(v))$ is the function v restricted to the domain $[\mathsf{ht}(s), \mathsf{ht}(v))$.) We note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S with the same height, then $\psi_{s,t}$, $\psi_{t,u}$, and $\psi_{s,u}$ commute (for a coherent Suslin tree, see, e.g., König [7], [13]). For s and t in S, when s and t are incomparable in S, let
$$\Delta(s, t) := \min\{\xi \in \min\{\mathsf{ht}(s), \mathsf{ht}(t)\}; s(\xi) \neq t(\xi)\},$$
and define
$$s \wedge t := s \restriction \Delta(s, t) = t \restriction \Delta(s, t).$$

In this paper, for a forcing notion $\mathbb{P}$ and conditions p and q in $\mathbb{P}$, $p \leq_{\mathbb{P}} q$ means that p is an extension of q in $\mathbb{P}$. For each s and t in the Suslin tree S, $s \leq_S t$ means that t is an extension of s in S as a forcing notion. So the order of the product forcing of $\mathbb{P}$ and S is defined as follows: for each $\langle p, s\rangle$ and $\langle q, t\rangle$ in $\mathbb{P} \times S$,
$$\langle p, s\rangle \leq_{\mathbb{P}\times S} \langle q, t\rangle \iff p \leq_{\mathbb{P}} q \ \&\ s \geq_S t.$$

Let S be a coherent Suslin tree, and let $\dot{T}$ and $\dot{U}$ be S-names for Aronszajn trees. To show the Theorem in Section 1, it suffices to find a proper forcing $\mathbb{P}$ such that $\mathbb{P}$ preserves that S is Suslin, and $\mathbb{P}$ adds an S-name $\dot{I}$ for an uncountable subset of ω_1 and an S-name $\dot{f}$ for a function such that
$$\Vdash_S \text{“}\dot{f} \text{ is an order-isomorphism from } \dot{T} \restriction \dot{I} \text{ onto } \dot{U} \restriction \dot{I}.\text{”}$$

Then in the extension by S, $\dot{f}$ can be extended uniquely to the closure of $\dot{I}$ in ω_1. This is the same scenario as the one in [1, Section 5] and [23, Section 5]. To demonstrate that a proper forcing preserves a Suslin tree, we use the following condition from Miyamoto.

Theorem 2.1 (Miyamoto [16, Proposition 1.1]) *For a Suslin tree S and a proper forcing $\mathbb{P}$, $\mathbb{P}$ preserves that S is Suslin if and only if, for any sufficiently large regular cardinal θ, any countable elementary substructure N of $H(\theta)$ which contains $\mathbb{P}$ and S as members, any $(\mathbb{P}, N)$-generic p, and any $t \in S$ of level $\omega_1 \cap N$, the pair $\langle p, t\rangle$ is $(\mathbb{P} \times S, N)$-generic.*

Proof We show here only the if case, which is the necessary implication in this note. Suppose that $p \in \mathbb{P}$, and let $\dot{A}$ be an S-name for a maximal antichain in S.

We take a sufficiently large regular cardinal θ and a countable elementary substructure N of $H(\theta)$ which contains $\mathbb{P}$, S, p, and $\dot{A}$ as members. Then the set

$$D := \left\{\langle q, s\rangle \in \mathbb{P} \times S ; q \Vdash_{\mathbb{P}} \text{“}s \in \dot{A}\text{”}\right\}$$

is predense in $\mathbb{P} \times S$ and is a member of the model N. Let $q \in \mathbb{P}$ be $(\mathbb{P}, N)$-generic.

Then by our assumption, for every $t \in S_{\omega_1 \cap N}$, the pair $\langle q, t\rangle$ is $(\mathbb{P}\times S, N)$-generic. Therefore, for every $t \in S_{\omega_1\cap N}$, $D \cap N$ is predense below $\langle q, t\rangle$ in $\mathbb{P} \times S$. Thus, for every $t \in S_{\omega_1\cap N}$, there exists $s \in S$ such that $s <_S t$ and

$$q \Vdash_{\mathbb{P}} \text{“}s \in \dot{A}.\text{”}$$

This says that

$$q \Vdash_{\mathbb{P}} \text{“}\dot{A} \subseteq S \cap \omega^{<\omega_1\cap N}\text{”};$$

hence, q forces that $\dot{A}$ is countable. This finishes the proof. □

Moreover, to show that our forcing notion is proper and preserves a Suslin tree, we use the following property of an Aronszajn tree. This property is defined in Yorioka [29, Definition 2.6].

Proposition 2.2 ([29, Propositions 2.5, 2.7]) *For an Aronszajn tree T, a countable elementary submodel N of $H(\theta)$ which contains T as a member (where θ is a large enough regular cardinal), an uncountable set I of finite antichains in T such that $I \in N$, and a finite antichain σ in T, if I forms a Δ-system with root $\sigma \cap N$, then there exists $I' \in [I]^{\aleph_1} \cap N$ such that, for any $\tau \in I'$, $\tau \cup \sigma$ is still an antichain in T.*[1]

Proof We note that T is an Aronszajn tree if and only if, for every uncountable subset J of T, there are two nodes s_0 and s_1 such that s_0 and s_1 are incomparable in T and the sets $\{u \in J; s_0 <_T u\}$ and $\{u \in J; s_1 <_T u\}$ are both uncountable. So for any Aronszajn tree T, any countable elementary submodel N of $H(\theta)$ which contains T as a member, any uncountable subset J of T with $J \in N$, and $t \in T \setminus N$, there exists $s \in T \cap N$ such that the set $\{u \in J; s <_T u\}$ is uncountable and s is incomparable with t in T. Then every element of the set $\{u \in J; s <_T u\}$ is also incomparable with t in T.

Suppose that T is an Aronszajn tree, N is a countable elementary submodel of $H(\theta)$ which contains T as a member, I is an uncountable subset of finite antichains in T such that $I \in N$, and σ is a finite antichain in T such that I forms a Δ-system with root $\sigma \cap N$. By shrinking I in the model N if necessary, we may assume that the size of elements of I is constant, say, n. Then since T is Aronszajn, by applying the above observation finitely many times, there exists an uncountable subset I' of I with $I' \in N$ such that, for each $i < n - |\sigma \cap N|$, $j < |\sigma \setminus N|$, and $\tau \in I'$, the ith element of $\tau \setminus (\sigma \cap N)$ is incompatible with the jth element of $\sigma \setminus N$. Then for every $\tau \in I'$, $\tau \cup \sigma$ is still an antichain in T. □

At last in this section, we demonstrate that a Suslin tree (which is not necessarily coherent) forces some consequence of $\diamondsuit$. Larson and Todorčević [13, Section 6] proved that a Suslin tree forces that every ladder system has a coloring which cannot be uniformized and there are no Q-sequences, and so on. Moore, Hrušák, and Džamonja [17, Theorem 6.15] proved that a Suslin tree forces $\diamondsuit(\mathbb{R}, \mathbb{R}, \neq)$. These proofs are essentially the same. The proof below summarizes them.

Definition 2.3 ([17]) (1) An invariant is a triple (A, B, E) such that
- both A and B have cardinality at most $\mathfrak{c}$,
- $E \subseteq A \times B$ is a relation,
- $\forall a \in A\ \exists b \in B\ (\langle a, b\rangle \in E)$, and
- $\forall b \in B\ \exists a \in A\ (\langle a, b\rangle \notin E)$.

(2) For an invariant (A, B, E), define its evaluation $\langle A, B, E\rangle$ by

$$\langle A, B, E\rangle := \min\big\{|X|; X \subseteq B\ \&\ \forall a \in A\ \exists b \in X\ \big(\langle a, b\rangle \in E\big)\big\}.$$

(3) For an invariant (A, B, E), define the diamond principle for (A, B, E) as follows:

$$\diamondsuit(A, B, E) \equiv \forall \text{ Borel } F : 2^{<\omega_1} \to A\ \exists g : \omega_1 \to B\ \forall f : \omega_1 \to 2,$$
$$\big\{\alpha < \omega_1; \big\langle F(f \restriction \alpha), g(\alpha)\big\rangle \in E\big\} \text{ is stationary},$$

where we call a function F Borel if $F \restriction 2^{\alpha}$ is a Borel function for every countable ordinal α.

Proposition 2.4 *If $\langle A, B, E\rangle = \aleph_0$, then a Suslin tree forces $\diamondsuit(A, B, E)$.*

There are some applications of invariants of the countable evaluations, for example, in [17] and Morgan and da Silva [18].

Proof Let S be a Suslin tree, let $\{b_k; k \in \omega\}$ be a witness that the evaluation of the invariant (A, B, E) is equal to $\aleph_0$, and let $\dot{F}$ be an S-name for a Borel function from $2^{<\omega_1}$ to A. We note that a Suslin tree is ccc and does not add new reals, and a Borel function on 2^{α}, for some $\alpha \in \omega_1$, is coded by a real.

So for each $\alpha \in \omega_1$, there exists $\delta_\alpha \in \omega_1$ such that $\delta_\alpha \geq \alpha$ and every node in S of height δ_α decides the S-name $\dot{F} \restriction 2^{\alpha}$ as a Borel function coded in the ground model. We define an S-name $\dot{g}$ for a function from ω_1 into ω such that, for each $\alpha \in \omega_1$,
- every node in S of height at most δ_α does not decide the value $\dot{g}(\alpha)$, and
- for every node t in S of height δ_α and $k \in \omega$,

$$t^\frown\langle k\rangle \Vdash_S \text{“}\dot{g}(\alpha) = b_k.\text{”}$$

We show that

$$\Vdash_S \text{“for all } \dot{f} : \omega_1 \to 2,\ \text{the set } \big\{\alpha \in \omega_1; \big\langle \dot{F}(\dot{f} \restriction \alpha), \dot{g}(\alpha)\big\rangle \in E\big\} \text{ is stationary.”}$$

Let $t \in S$, let $\dot{f}$ be an S-name for a function from ω_1 into 2, and let C be a club subset of ω_1. It suffices to show that there are $t_1 \in S$ and $\alpha \in C$ such that $t \leq_S t_1$ and

$$t_1 \Vdash_S \text{“}\big\langle \dot{F}(\dot{f} \restriction \alpha), \dot{g}(\alpha)\big\rangle \in E,\text{”}$$

because for any S-name for a club set, there exists a club set in the ground model which is forced (by the weakest condition in S) to be a subset of the given S-name. Since S is ccc, there exists $\alpha \in C$ such that every node in S of height α decides the value of $\dot{f}(\gamma)$ for all $\gamma < \alpha$.

We take a node t_0 in S of height δ_α such that $t \leq_S t_0$. Then by the property of δ_α, t_0 decides the value of the S-name $\dot{F}(\dot{f}\restriction\alpha)$, say, x. So there exists $k \in \omega$ such that $\langle x, b_k\rangle \in E$. Therefore, it follows that

$$t_0 \Vdash_S \text{“}\big\langle \dot{F}(\dot{f}\restriction\alpha), b_k\big\rangle \in E.\text{”}$$

We let

$$t_1 := t_0{}^\frown\langle k\rangle,$$

which is also a node in S. Then

$$t_1 \Vdash_S \text{“}\big\langle \dot{F}(\dot{f}\restriction\alpha), \dot{g}(\alpha)\big\rangle \in E,\text{”}$$

which finishes the proof. □

3 Proof of the Theorem

This section is devoted to the proof of the Theorem which is given in Section 1.

Let S be a coherent Suslin tree, and let $\dot{T}$ and $\dot{U}$ be S-names for Aronszajn subtrees of $\omega^{<\omega_1}$ which are closed under initial segments. Then $\mathbb{P}$ consists of the functions p such that

- $\mathrm{dom}(p)$ is a finite $\in$-chain of countable elementary submodels of the set $H((2^{\aleph_1})^+)$ which contain the sets S, $\dot{T}$, and $\dot{U}$ as members;
- for each $M \in \mathrm{dom}(p)$, $p(M) = \langle t_M, f_M\rangle$, where $t_M \in S$ and f_M is a nonempty finite partial injection from ω^{α_M} into ω^{α_M} for some $\alpha_M < \mathsf{ht}(t_M)$;
- the sets $\{t_M; M \in \mathrm{dom}(p)\}$ and $\{\alpha_M; M \in \mathrm{dom}(p)\}$ are separated by $\mathrm{dom}(p)$, that is, for each $M \in \mathrm{dom}(p)$ and $M' \in \mathrm{dom}(p) \cap M$,
$$t_M \notin M, \qquad t_{M'} \in M, \qquad \alpha_M \notin M, \qquad \text{and} \qquad \alpha_{M'} \in M;$$
- for each $M \in \mathrm{dom}(p)$,
  - every node in S of height $\mathsf{ht}(t_M)$ decides the S-names $\dot{T} \cap \omega^{\leq\alpha_M}$ and $\dot{U} \cap \omega^{\leq\alpha_M}$,
  - $t_M \Vdash_S$ “$\mathrm{dom}(f_M) \subseteq \dot{T}$ & $\mathrm{ran}(f_M) \subseteq \dot{U}$,” and
  - for any $M' \in \mathrm{dom}(p) \cap M$, if $t_{M'} <_S t_M$, then the statement
  $$f_{M'} \subseteq \big\{\langle x\restriction\alpha_{M'}, y\restriction\alpha_{M'}\rangle; \langle x, y\rangle \in f_M\big\}$$
  holds; and

  for each $p = \langle\langle t^p_M, f^p_M\rangle; M \in \mathrm{dom}(p)\rangle$ and $q = \langle\langle t^q_M, f^q_M\rangle; M \in \mathrm{dom}(q)\rangle$ in $\mathbb{P}$,

$$p \leq_{\mathbb{P}} q :\iff \mathrm{dom}(p) \supseteq \mathrm{dom}(q) \;\&\; \forall M \in \mathrm{dom}(q)\; (t^p_M = t^q_M \;\&\; f^p_M \supseteq f^q_M).$$

For a condition $p \in \mathbb{P}$ and $M \in \mathrm{dom}(p)$, we let α^p_M be such that f^p_M is a nonempty finite partial injection from $\omega^{\alpha^p_M}$ into $\omega^{\alpha^p_M}$. We note that, for each $p \in \mathbb{P}$ and $M \in \mathrm{dom}(p)$,

$$\begin{aligned} t_M \Vdash_S \text{“}&\big\{\langle x\restriction\alpha^p_{M'}, y\restriction\alpha^p_{M'}\rangle; M' \in \big(\mathrm{dom}(p) \cap M\big) \cup \{M\}, t^p_{M'} \leq_S t^p_M,\\ &\langle x, y\rangle \in f^p_{M'}\big\}\\ &\text{is a finite partial height-preserving isomorphism from } \dot{T} \text{ into } \dot{U}.\text{”}\end{aligned}$$

This definition is an S-name version of one in [1, Definition 5.2]. See also [23, Theorem 5.10].

By the definition of $\mathbb{P}$, we notice that the following sets are dense in $\mathbb{P}$:

- for every $s \in S$ and $\alpha \in \omega_1$, the set
$$\{p \in \mathbb{P};\ \exists M \in \mathrm{dom}(p) \text{ such that } s \le_S t^p_M \ \&\ \alpha \le \alpha^p_M\};$$
- for every $s \in S$ and $x \in \omega^{<\omega_1}$ such that $\mathsf{ht}(x) < \mathsf{ht}(s)$ and
$$s \Vdash_S \text{“}x \in \dot{T},\text{”}$$
the set
$$\Big\{p \in \mathbb{P};\ \text{either } p \Vdash_{\mathbb{P}} \text{“}\mathsf{ht}(x) \notin \bigcup_{p\in\dot{G}} \{\alpha^p_N; N \in \mathrm{dom}(p)\}\text{”}$$
$$\text{or there are } M \in \mathrm{dom}(p),\ M' \in \big(\mathrm{dom}(p) \cap M\big) \cup \{M\}, \text{ and } \langle z, w\rangle \in f^p_M$$
$$\text{such that } s \le_S t^p_M,\ t^p_{M'} \le_S t^p_M, \text{ and } x = z \restriction \alpha^p_{M'}\Big\};$$
- for every $s \in S$ and $y \in \omega^{<\omega_1}$ such that $\mathsf{ht}(y) < \mathsf{ht}(s)$ and
$$s \Vdash_S \text{“}y \in \dot{U},\text{”}$$
the set
$$\Big\{p \in \mathbb{P};\ \text{either } p \Vdash_{\mathbb{P}} \text{“}\mathsf{ht}(y) \notin \bigcup_{p\in\dot{G}} \{\alpha^p_N; N \in \mathrm{dom}(p)\}\text{”}$$
$$\text{or there are } M \in \mathrm{dom}(p),\ M' \in \big(\mathrm{dom}(p) \cap M\big) \cup \{M\}, \text{ and } \langle z, w\rangle \in f^p_M$$
$$\text{such that } s \le_S t^p_M,\ t^p_{M'} \le_S t^p_M, \text{ and } y = w \restriction \alpha^p_{M'}\Big\}.$$

For a $\mathbb{P}$-generic $G_{\mathbb{P}}$, we define S-names $\dot{I}_{G_{\mathbb{P}}}$ and $\dot{f}_{G_{\mathbb{P}}}$ such that, by letting $\dot{G}_S$ be a canonical S-generic name over the extension by $G_{\mathbb{P}}$,
$$\Vdash_S \text{“}\dot{I}_{G_{\mathbb{P}}} := \{\alpha^p_M; p \in G_{\mathbb{P}} \ \&\ M \in \mathrm{dom}(p) \ \&\ t^p_M \in \dot{G}_S\}\text{”}$$
and
$$\begin{aligned}\Vdash_S \text{“}\dot{f}_{G_{\mathbb{P}}} := \big\{\langle x \restriction \alpha^p_{M'}, y \restriction \alpha^p_{M'}\rangle;\ & \text{there exists } M \in \mathrm{dom}(p) \text{ such that } p \in G_{\mathbb{P}},\\ & t^p_M \in \dot{G}_S, \langle x, y\rangle \in f^p_M, M' \in \big(\mathrm{dom}(p) \cap M\big) \cup \{M\},\\ & \text{and } t^p_{M'} \le_S t^p_M\big\}.\text{”}\end{aligned}$$
By the genericity of $G_{\mathbb{P}}$, $\dot{I}_{G_{\mathbb{P}}}$ is an S-name for an uncountable subset of ω_1 and $\dot{f}_{G_{\mathbb{P}}}$ is an S-name for an isomorphism from the set $\{x \in \dot{T}; \mathsf{ht}(x) \in \dot{I}_{G_{\mathbb{P}}}\}$ onto the set $\{y \in \dot{U}; \mathsf{ht}(y) \in \dot{I}_{G_{\mathbb{P}}}\}$.

We show that $\mathbb{P}$ is proper and preserves that S is Suslin. Let θ be a large enough regular cardinal, let N be a countable elementary submodel of the set $H(\theta)$ which contains the sets S, $\dot{T}$, $\dot{U}$, and $H((2^{\aleph_1})^+)$ as members, and let $p_0 \in \mathbb{P} \cap N$. We let

- $N' := N \cap H((2^{\aleph_1})^+)$, which is an elementary submodel of $H((2^{\aleph_1})^+)$,
- $\alpha^{p_1}_{N'} := \omega_1 \cap N$,
- $t^{p_1}_{N'} \in S \setminus N'$ such that
  - $\mathsf{ht}(t^{p_1}_{N'}) > \omega_1 \cap N$,
  - every node in S of height $\mathsf{ht}(t^{p_1}_{N'})$ decides the S-names $\dot{T} \cap \omega^{\le\alpha^{p_1}_{N'}}$ and $\dot{U} \cap \omega^{\le\alpha^{p_1}_{N'}}$,
  - for every $M \in \mathrm{dom}(p_0)$, $t^{p_0}_M$ and $t^{p_1}_{N'}$ are incomparable in S, and

- $f^{p_1}_{N'}$ be an arbitrary nonempty finite partial injection from $\omega^{\alpha^{p_1}_{N'}}$ into $\omega^{\alpha^{p_1}_{N'}}$ such that

$$t^{p_1}_{N'} \Vdash_S \text{“}\mathrm{dom}(f^{p_1}_{N'}) \subseteq \dot{T} \text{ and } \mathrm{ran}(f^{p_1}_{N'}) \subseteq \dot{U}.\text{”}$$

Moreover, we let

$$p_1 := p_0 \cup \bigl\{\bigl\langle N', \langle t^{p_1}_{N'}, f^{p_1}_{N'}\rangle\bigr\rangle\bigr\}.$$

We note that p_1 is a condition of $\mathbb{P}$ and an extension of p_0.

Let $s_1 \in S$ be of height $\omega_1 \cap N$. We show that the pair $\langle p_1, s_1\rangle$ is $(\mathbb{P} \times S, N)$-generic. By ignoring the second coordinate in the product $\mathbb{P} \times S$, the following argument shows that $\mathbb{P}$ is proper.[2] Therefore, by Theorem 2.1, $\mathbb{P}$ preserves that S is Suslin, which finishes the proof.

Let $\mathcal{D} \in N$ be a dense open subset of $\mathbb{P} \times S$, and let $r \leq_{\mathbb{P}} p_1$ and $u \geq_S s_1$ be such that $\langle r, u\rangle \in \mathcal{D}$ (which means that $\langle r, u\rangle$ is an extension of $\langle p_1, s_1\rangle$ in $\mathbb{P} \times S$ and $\langle r, u\rangle \in \mathcal{D}$). By the definition of $\mathbb{P}$, $r \restriction (\mathrm{dom}(r) \cap N) = r \cap N$, $r \cap N \in \mathbb{P}$, and $r \leq_{\mathbb{P}} r \cap N$. By extending u if necessary, we may assume that $\mathsf{ht}(u) \geq \mathsf{ht}(t^r_M)$ holds for every $M \in \mathrm{dom}(r)$. By the coherency of the tree S, we can take $\gamma_0 \in \omega_1 \cap N$ such that, for every $M \in \mathrm{dom}(r)$,

$$\bigl\{\xi \in \mathsf{ht}(t^r_M) \cap N; t^r_M(\xi) \neq u(\xi)\bigr\} = \bigl\{\xi \in \mathsf{ht}(t^r_M) \cap N; t^r_M(\xi) \neq s_1(\xi)\bigr\} \subseteq \gamma_0.$$

Let $\{M^r_i; i \in n\}$ be the $\in$-increasing enumeration of the set $\mathrm{dom}(r) \setminus N$, and let M^r_{-1} be the maximal model in $\mathrm{dom}(r) \cap N$. We note that $\mathrm{dom}(r) \setminus N = \mathrm{dom}(r) \setminus \mathrm{dom}(r \cap N)$. For each $v \in S$, we define the set H^n_v which consists of the sequences

$$\bigl\langle f^q_M; M \in \mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N)\bigr\rangle$$

such that

- $q \in \mathbb{P}$,
- $q \leq_{\mathbb{P}} r \cap N$ and q end-extends $r \cap N$, that is, $\mathrm{dom}(q)$ end-extends $\mathrm{dom}(r \cap N)$ and, for each $M \in \mathrm{dom}(r \cap N)$, $r(M) = q(M)$,
- $\langle q, v\rangle \in \mathcal{D}$,
- $|q| = |r|$ and, say, $\mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N) := \{M^q_i; i \in n\}$ according to its $\in$-increasing enumeration,
- for every $M \in \mathrm{dom}(q)$, $\mathsf{ht}(t^q_M) \leq \mathsf{ht}(v)$, and
- for every $i \in n$,

$$\begin{gathered} t^q_{M^q_i} \restriction \gamma_0 = t^r_{M^r_i} \restriction \gamma_0, \\ t^q_{M^q_i} \restriction \bigl[\gamma_0, \mathsf{ht}(t^q_{M^q_i})\bigr) = v \restriction \bigl[\gamma_0, \mathsf{ht}(t^q_{M^q_i})\bigr) \iff \\ t^r_{M^r_i} \restriction \bigl[\gamma_0, \mathsf{ht}(t^r_{M^r_i})\bigr) = u \restriction \bigl[\gamma_0, \mathsf{ht}(t^r_{M^r_i})\bigr), \end{gathered}$$

  and

$$\bigl\{\langle x \restriction \alpha^r_{M^r_{-1}}, y \restriction \alpha^r_{M^r_{-1}}\rangle; \langle x, y\rangle \in f^q_{M^q_i}\bigr\} = \bigl\{\langle x \restriction \alpha^r_{M^r_{-1}}, y \restriction \alpha^r_{M^r_{-1}}\rangle; \langle x, y\rangle \in f^r_{M^r_i}\bigr\}.$$

We note that the sequence $\langle H^n_v; v \in S\rangle$ belongs to the model N, and for every $v, v' \in S$, if $v \leq_S v'$, then $H^n_v \subseteq H^n_{v'}$. We consider each H^n_v as a tree which consists of all initial segments of its members. For simplicity of notation, for each $i < n$, let

$$\psi_i := \psi_{v \restriction \gamma_0, t^r_{M^r_i} \restriction \gamma_0},$$

and let
$$\mathrm{a} := \{i < n; t^r_{M^r_i} \restriction [\gamma_0, \mathrm{ht}(t^r_{M^r_i})) = u \restriction [\gamma_0, \mathrm{ht}(t^r_{M^r_i}))\}.$$
We note that both the sequence $\langle \psi_i ; i < n \rangle$ and the set a are in N. By induction on the reverse order on $i < n$, we define the sequence $\langle A^i_v, H^i_v ; v \in S \rangle$ as follows. We start from the sequence $\langle H^n_v ; v \in S \rangle$, which has already been defined, and we do not define the A^n_v's. Let $i < n$, and suppose that $\langle H^{i+1}_v ; v \in S \rangle$ has already been built as above. If $i \notin \mathrm{a}$, then for each $v \in S$, we define
$$A^i_v := \emptyset \qquad \text{and} \qquad H^i_v := H^{i+1}_v.$$
If $i \in \mathrm{a}$, then for each $v \in S$, we define
$$\begin{aligned} A^i_v := \{\sigma \in H^{i+1}_v ; |\sigma| = i \ \& \\ &\psi_i(v) \nVdash_S \text{“}\{\beta \in \omega_1;\ \exists t \in \dot{G}_S\ \exists f : \dot{T} \cap \omega^\beta \to \dot{U} \cap \omega^\beta \text{ finite partial} \\ &\text{such that } \sigma^\frown \langle f \rangle \in H^{i+1}_t\} \text{ is uncountable”}\} \end{aligned}$$
(here, $\dot{G}_S$ is the canonical S-generic name) and
$$H^i_v := H^{i+1}_v \setminus \{\tau \in H^{i+1}_v ;\ \exists \sigma \in A^i_v \text{ such that } \sigma \subseteq \tau\}.$$
Since the sequence of length 0 means the empty sequence, the set A^0_v is either the empty set or the set $\{\emptyset\}$. So if A^0_v satisfies the former case, then $H^0_v = H^1_v$ holds, and if A^0_v is the latter case, then the set H^0_v is the empty set. By the definition, we note that, for each $v \in S$ and $\sigma \in H^0_v$ which is not a terminal such that $|\sigma| \in \mathrm{a}$,
$$\begin{aligned} \psi_{|\sigma|}(v) \Vdash_S \text{“}\{\beta \in \omega_1;\ \exists t \in \dot{G}_S\ \exists f : \dot{T} \cap \omega^\beta \to \dot{U} \cap \omega^\beta \text{ finite partial} \\ \text{such that } \sigma^\frown \langle f \rangle \in H^0_t\} \text{ is uncountable.”} \end{aligned}$$
We note that, for every $i < n$, the sequence $\langle A^i_v, H^i_v ; v \in S \rangle$ also belongs to the model N. By the definition, if $i < n$, $v, v' \in S$, and $v \leq_S v'$, then $A^i_v \supseteq A^i_{v'}$ holds; hence, $H^i_v \subseteq H^i_{v'}$ holds.

Claim 3.1 *The sequence $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle$ is a cofinal chain through H^0_u.*

Proof At first, we notice that $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle \in H^n_u$ holds. We show, by induction on the reverse order on $i < n$, that the initial segment of the sequence $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle$ of length i does not belong to the set A^i_u.

Suppose to the contrary that $i < n$ satisfies that the initial segment of the sequence $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle$ of length $i+1$ does not belong to the set A^{i+1}_u, but the initial segment of the sequence $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle$ of length i belongs to the set A^i_u. We denote the initial segment of the sequence $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle$ of length i by σ and then denote the next coordinate of σ in the sequence $\langle f^r_M ; M \in \mathrm{dom}(r \setminus N) \rangle$ by $f^r_{M^r_i}$. Then we note that

- $i \in \mathrm{a}$, that is, $t^r_{M^r_i} \restriction [\gamma_0, \mathrm{ht}(t^r_{M^r_i})) = u \restriction [\gamma_0, \mathrm{ht}(t^r_{M^r_i}))$ (because then $A^i_u \neq \emptyset$),
- $\sigma \in M^r_i$,
- $f^r_{M^r_i} \notin M^r_i$,
- $\sigma^\frown \langle f^r_{M^r_i} \rangle \in H^{i+1}_u$, and
- $t^r_{M^r_i} \Vdash_S$ “$f^r_{M^r_i}$ is a finite partial function from $\dot{T} \cap \omega^{\alpha^r_{M^r_i}}$ into $\dot{U} \cap \omega^{\alpha^r_{M^r_i}}$.”

Then by the definition of $\mathbb{P}$, $\alpha^r_{M^r_i} \notin M^r_i$. Since $\mathrm{ht}(u) \geq \mathrm{ht}(t^r_{M^r_i}) \geq \omega_1 \cap M^r_i$, u is (S, M^r_i)-generic. (Remember here that S is a Suslin tree.) Our assumption says that

$$\psi_i(u) \not\Vdash_S \text{“}\big\{\alpha \in \omega_1;\ \exists t \in \dot{G}_S\ \exists g : \dot{T} \cap \omega^\alpha \to \dot{U} \cap \omega^\alpha \text{ finite partial such that } \sigma^\frown\langle g\rangle \in H^{i+1}_t\big\} \text{ is uncountable.”}$$

Then some extension of $\psi_i(u)$ in S forces that

$$\text{“}\big\{\alpha \in \omega_1;\ \exists t \in \dot{G}_S\ \exists g : \dot{T} \cap \omega^\alpha \to \dot{U} \cap \omega^\alpha \text{ finite partial such that } \sigma^\frown\langle g\rangle \in H^{i+1}_t\big\} \text{ is countable.”}$$

This statement can be expressed in the model M^r_i, because S, $\dot{T}$, $\dot{U}$, and $\langle H^i_v; v \in S\rangle$ are in the model $N' = N \cap H((2^{\aleph_1})^+)$ and $N' \subseteq M^r_i$. So since such an extension is also (S, M^r_i)-generic, there exists $w \in S \cap M^r_i$ such that $w <_S u$ and

$$\psi_i(w) \Vdash_S \text{“}\big\{\alpha \in \omega_1;\ \exists t \in \dot{G}_S\ \exists g : \dot{T} \cap \omega^\alpha \to \dot{U} \cap \omega^\alpha \text{ finite partial such that } \sigma^\frown\langle g\rangle \in H^{i+1}_t\big\} \text{ is countable.”}$$

In particular, since $w \in M^r_i$, it follows that

$$\psi_i(w) \Vdash_S \text{“}\big\{\alpha \in \omega_1;\ \exists t \in \dot{G}_S\ \exists g : \dot{T} \cap \omega^\alpha \to \dot{U} \cap \omega^\alpha \text{ finite partial such that } \sigma^\frown\langle g\rangle \in H^{i+1}_t\big\} \subseteq M^r_i.\text{”}$$

This is a contradiction. Because now it holds that

$$t^r_{M^r_i} \restriction (\omega_1 \cap N) <_S \psi_i(u) \restriction (\omega_1 \cap N),$$

$w <_S u$, and $\mathrm{ht}(w) < \omega_1 \cap N$, and so $\psi_i(w) \leq_S t^r_{M^r_i}$ holds. Since $f^r_{M^r_i} \notin M^r_i$, it follows that

$$t^r_{M^r_i} \Vdash_S \text{“}\alpha^r_{M^r_i} \in \big\{\alpha \in \omega_1;\ \exists t \in \dot{G}_S\ \exists g : \dot{T} \cap \omega^\alpha \to \dot{U} \cap \omega^\alpha \text{ finite partial such that } \sigma^\frown\langle g\rangle \in H^{i+1}_t\big\} \setminus M^r_i.\text{”} \qquad \square$$

Therefore, the set

$$\big\{v \in S; u \restriction \gamma_0 \leq_S v \;\&\; H^0_v \text{ is of height } |r \setminus N|\big\}$$

is not empty and, in particular, contains u as a member. We note that this set belongs to the model N. Thus, since u is (S, N)-generic, there exists $v_{-1} \in S \cap N$ such that $u \restriction \gamma_0 \leq_S v_{-1} \leq_S u$ (then $v_{-1} \leq_S s_1$) and $H^0_{v_{-1}}$ has a cofinal branch of length $|r \setminus N|$.[3]

Claim 3.2 *There exists the sequence $\langle f_j, v_j; j < n\rangle$ in N such that*

- *for each $j < n-1$, $v_j \leq_S v_{j+1} \leq_S s_1$,*
- *for each $j < n$, the sequence $\langle f_k; k \leq j\rangle$ is a chain through $H^0_{v_j}$, and*
- *for each $j \in \mathrm{a}$ and $k < n$ with $t^r_{M^r_k} \restriction \gamma_0 = t^r_{M^r_j} \restriction \gamma_0$, $\mathrm{dom}(f_j) \cup \mathrm{dom}(f^r_{M^r_k})$ is an antichain in $\omega^{<\omega_1}$ and $\mathrm{ran}(f_j) \cup \mathrm{ran}(f^r_{M^r_k})$ is an antichain in $\omega^{<\omega_1}$.*

Proof We build the sequence by induction on $j < n$. Suppose that we have built $\langle v_l, f_l; l < j\rangle$ as above. If $j \notin \mathrm{a}$, then let $v_j := v_{j-1}$ and pick $f_j \in N$ such that $\langle f_k; k \leq j\rangle \in H^0_{v_j}$. This can be done because, in this case, $A^j_{v_j} = A^j_{v_{j-1}} = \emptyset$ and so, since $\langle f_k; k < j\rangle \in H^0_{v_j}$, the set of the successors of $\langle f_k; k < j\rangle$ in $H^0_{v_{j-1}}$ is equal to the set of the successors of $\langle f_k; k < j\rangle$ in $H^j_{v_j} = H^j_{v_{j-1}}$.

We consider the case in which $j \in \mathtt{a}$. Then we note that $\psi_j(v_{j-1}) \leq_S t^r_{M^r_j}$, and

$$\psi_j(v_{j-1}) \Vdash_S \text{“}\big\{\beta \in \omega_1;\ \exists t \in \dot{G}_S\ \exists f : \dot{T} \cap \omega^\beta \to \dot{U} \cap \omega^\beta \text{ finite partial such that } \langle f_k; k < j\rangle^\frown\langle f\rangle \in H^0_t\big\} \text{ is uncountable.”}$$

Since the S-name

$$\big\{\beta \in \omega_1;\ \exists t \in \dot{G}_S\ \exists f : \dot{T} \cap \omega^\beta \to \dot{U} \cap \omega^\beta \text{ finite partial such that } \langle f_k; k < j\rangle^\frown\langle f\rangle \in H^0_t\big\}$$

belongs to the model N, the uncountable set

$$I_j := \big\{\beta \in \omega_1;\ \exists w \geq_S \psi_j(v_{j-1})\ \exists g : \omega^\beta \to \omega^\beta \text{ finite partial such that } \langle f_k; k < j\rangle^\frown\langle g\rangle \in H^0_w\big\}$$

also belongs to the model N. So, we can take a sequence $\langle B^j_\beta, g^j_\beta; \beta \in I_j\rangle$ in N such that, for each $\beta \in I_j$,

- B^j_β forms a maximal antichain in S above $\psi_j(v_{j-1})$, and
- g^j_β is a function with $\mathrm{dom}(g^j_\beta) \subseteq B^j_\beta$ such that, for each $w \in B^j_\beta$, either there are no finite partial functions $d : \omega^\beta \to \omega^\beta$ such that $\langle f_k; k < j\rangle^\frown\langle d\rangle \in H^0_w$ (and in this case $w \notin \mathrm{dom}(g^j_\beta)$), or
$$\langle f_k; k < j\rangle^\frown\big\langle g^j_\beta(w)\big\rangle \in H^0_w.$$

Then we note that

$$\psi_j(v_{j-1}) \Vdash_S \text{“}\dot{I}^j_{-1} := \big\{\beta \in I_j; \mathrm{dom}(g^j_\beta) \cap \dot{G}_S \neq \emptyset\big\} \text{ is uncountable.”}$$

Let $\mathtt{b}_j := \{k^j_h; h < l_j\}$ be an enumeration of the index set

$$\{k < n; t^r_{M^r_k} \restriction \gamma_0 = t^r_{M^r_j} \restriction \gamma_0\}.$$

By induction on $h < l_j$, we will build an S-name $\dot{I}^j_h$ for an uncountable subset of ω_1 such that, for each $h < l$, $\dot{I}^j_h \in N$,

$$\psi_j(v_{j-1}) \Vdash_S \text{“}\dot{I}^j_h \subseteq \dot{I}^j_{h-1} \text{ and, for every } \beta \in \dot{I}^j_h,\ \mathrm{dom}(g^j_\beta) \cap \dot{G}_S \neq \emptyset,\text{”}$$

and

$$\begin{aligned} t^r_{M^r_{k^j_h}} \Vdash_S \text{“}&\text{for every } \beta \in \dot{I}^j_h \text{ and } w \in \mathrm{dom}(g^j_\beta) \cap \dot{G}_S, \\ &\mathrm{dom}\big(g^j_\beta(w)\big) \cup \mathrm{dom}(f^r_{M^r_{k^j_h}}) \text{ is an antichain in } \dot{T} \\ &\text{and } \mathrm{ran}\big(g^j_\beta(w)\big) \cup \mathrm{ran}(f^r_{M^r_{k^j_h}}) \text{ is an antichain in } \dot{U}.\text{”} \end{aligned}$$

Suppose that $h < l_j$, and suppose that we have built $\dot{I}^j_{h-1}$. Since $t^r_{M^r_{k^j_h}} \restriction \gamma_0 = t^r_{M^r_j} \restriction \gamma_0$, $v_{j-1} \leq_S s_1 \leq_s u$, and $\mathsf{ht}(v_{j-1}) < \omega_1 \cap N$, by the property of γ_0, it follows that $\psi_j(v_{j-1}) \leq_S t^r_{M^r_{k^j_h}}$. Thus,

$$t^r_{M^r_{k^j_h}} \Vdash_S \text{“}\big\{\mathrm{dom}\big(g^j_\beta(w)\big); \beta \in \dot{I}^j_{h-1} \text{ and } w \in \mathrm{dom}(g^j_\beta) \cap \dot{G}_S\big\} \in N'[\dot{G}_s]$$

is an uncountable set of finite antichains in $\dot{T}$, all of whose members are pairwise disjoint,
and $\{\operatorname{ran}(g^j_\beta(w)); \beta \in \dot{I}^j_{h-1}$ and $w \in \operatorname{dom}(g^j_\beta) \cap \dot{G}_S\} \in N'[\dot{G}_s]$ is an uncountable set of finite antichains in $\dot{U}$, all of whose members are pairwise disjoint."

We note that

$$\operatorname{dom}(f^r_{M^r_{k^j_h}}) \cap N' = \operatorname{ran}(f^r_{M^r_{k^j_h}}) \cap N' = \emptyset,$$

because $N' \in M^r_j$, $f^r_{M^r_j} \subseteq 2^{\alpha^r_{M^r_j}}$, and $\alpha^r_{M^r_j} \notin M^r_j$. So by applying Proposition 2.2 twice, since S is a proper forcing, we can take an S-name $\dot{I}^j_h \in N'$ for an uncountable subset of $\dot{I}^j_{h-1}$ such that

$$\begin{aligned} t^r_{M^r_{k^j_h}} \Vdash_S \text{"for every } \beta \in \dot{I}^j_h \text{ and } w \in \operatorname{dom}(g^j_\beta) \cap \dot{G}_S,\\ \operatorname{dom}\big(g^j_\beta(w)\big) \cup \operatorname{dom}(f^r_{M^r_{k^j_h}}) \text{ is an antichain in } \dot{T}\\ \text{and } \operatorname{ran}\big(g^j_\beta(w)\big) \cup \operatorname{ran}(f^r_{M^r_{k^j_h}}) \text{ is an antichain in } \dot{U}\text{,"} \end{aligned}$$

which finishes the construction of $\dot{I}^j_h$.

Let

$$\begin{aligned} C := \big\{t \in S;\ \exists f \text{ such that } \langle f_k; k < j\rangle^\frown\langle f\rangle \in H^0_t\\ \text{and } \psi_j(t) \Vdash_S \text{"} f \in \big\{g^j_\beta(w); \beta \in \dot{I}^j_{l_j-1} \text{ and } w \in \operatorname{dom}(g^j_\beta) \cap \dot{G}_S\big\}\text{"}\big\}. \end{aligned}$$

We note that C belongs to the model N and C is dense above $\psi_j(v_{j-1})$. So there exists $v_j \in S \cap N$ such that $v_j \in C$ and $v_{j-1} \leq_S v_j <_S s_1$. Let f_j be a witness that $v_j \in C$. Then by our choice, for each $h < l_j$,

$$\begin{aligned} t^r_{M^r_{k^j_h}} \Vdash_S \text{"}\operatorname{dom}(f_j) \cup \operatorname{dom}(f^r_{M^r_{k^j_h}}) \text{ is an antichain in } \dot{T}\\ \text{and } \operatorname{ran}(f_j) \cup \operatorname{ran}(f^r_{M^r_{k^j_h}}) \text{ is an antichain in } \dot{U}.\text{"} \end{aligned}$$

Since both $\dot{T}$ and $\dot{U}$ are S-names for subtrees of $\langle \omega^{<\omega_1}, \subseteq\rangle$, we note that the set $\operatorname{dom}(f_j) \cup \operatorname{dom}(f^r_{M^r_j})$ is an antichain in $\omega^{<\omega_1}$ and the set $\operatorname{ran}(f_j) \cup \operatorname{ran}(f^r_{M^r_j})$ is an antichain in $\omega^{<\omega_1}$ in the ground model (because these statements are absolute). This finishes the proof. □

Let $q \in \mathbb{P} \cap N$ be a witness that the sequence $\langle f_j; j < n\rangle$ belongs to the set $H^0_{v_{n-1}}$. So then $\langle q, v_{n-1}\rangle$ is in $\mathcal{D} \cap N$. We will show that $\langle q, v_{n-1}\rangle$ and $\langle r, u\rangle$ are compatible in $\mathbb{P} \times S$.

We note that, for any j and k in n, $\alpha^q_{M^q_j} < \omega_1 \cap N \leq \alpha^r_{M^r_k}$. We should remember that, for each j and k in n, if $t^r_{M^r_j} \leq_S t^r_{M^r_k}$, then

$$f^r_{M^r_j} \subseteq \big\{\langle x \restriction \alpha^r_{M^r_j}, y \restriction \alpha^r_{M^r_j}\rangle; \langle x, y\rangle \in f^r_{M^r_k}\big\},$$

so in particular,

$$\begin{aligned}&\big\{\big\langle x\restriction(\omega_1\cap N), y\restriction(\omega_1\cap N)\big\rangle; \langle x,y\rangle \in f^r_{M^r_j}\big\}\\ &\quad\subseteq \big\{\big\langle x\restriction(\omega_1\cap N), y\restriction(\omega_1\cap N)\big\rangle; \langle x,y\rangle \in f^r_{M^r_k}\big\}.\end{aligned}$$

By the definition of the H^n_v's, we note that, for each $j \in n$,

- if $j \notin a$, then $t^q_{M^q_j} \not\leq_S t^r_{M^r_k}$ for every $k < n$, and
- if $j \in a$, then for every $k < n$, since $v_{n-1} \leq_S s_1 \leq_S u$,

$$t^q_{M^q_j} \leq_S t^r_{M^r_k} \iff t^q_{M^q_j}\restriction\gamma_0 = t^r_{M^r_k}\restriction\gamma_0.$$

Moreover, for each $j,k \in n$, if $t^q_{M^q_j} <_S t^r_{M^r_k}$, then it happens that $j \in \mathtt{a}$ and $k \in \mathtt{b}_j$; hence, both $\mathrm{dom}(f^q_{M^q_j}) \cup \mathrm{dom}(f^r_{M^r_k})$ and $\mathrm{ran}(f^q_{M^q_j}) \cup \mathrm{ran}(f^r_{M^r_k})$ form antichains (see Claim 3.2). Therefore, by the definition of the H^n_v's and the choice of q, we can find a finite subset X_k of ${}^{\alpha^r_{M^r_k}}\omega \times {}^{\alpha^r_{M^r_k}}\omega$, for each $k \in n$, such that

- $t^r_{M^r_k} \Vdash_S$ "$X_k \subseteq \dot{T} \times \dot{U}$,"
- both $\mathrm{dom}(X_k) \cup \mathrm{dom}(f^r_{M^r_k})$ and $\mathrm{ran}(X_k) \cup \mathrm{ran}(f^r_{M^r_k})$ form antichains in the set ${}^{\alpha^t_{M^r_k}}\omega$,
- for each $j \in n$ with $t^q_{M^q_j} <_S t^r_{M^r_k}$,

$$f_j\ (= f^q_{M^q_j}) \subseteq \big\{\langle x\restriction\alpha^q_{M^q_j}, y\restriction\alpha^q_{M^q_j}\rangle; \langle x,y\rangle \in X_k\big\},$$

  and
- for each $k' \in n$, if $t^r_{M^r_{k'}} <_S t^r_{M^r_k}$, then

$$X_{k'} \subseteq \big\{\langle x\restriction\alpha^r_{M^r_{k'}}, y\restriction\alpha^r_{M^r_{k'}}\rangle; \langle x,y\rangle \in X_k\big\}.$$

We define a function q' such that

- $\mathrm{dom}(q') := \mathrm{dom}(q) \cup \mathrm{dom}(r)$

$$= \mathrm{dom}(r\cap N) \cup \{M^q_j; j \in n\} \cup \{M^r_k; k \in n\},$$

- $q'\restriction(\mathrm{dom}(r)\cap N) = r \cap N$,
- for each $j \in n$, $q'(M^q_j) := q(M^q_j)$, and
- for each $k \in n$, $q'(M^r_k) := \langle t^r_{M^r_k}, f^r_{M^r_k} \cup X_k\rangle$.

By the above observations, we note that q' is a condition in $\mathbb{P}$, and hence, $q' \leq_{\mathbb{P}} q$ and $q' \leq_{\mathbb{P}} r$. So $\langle q', u\rangle$ is a common extension of $\langle r, u\rangle$ and $\langle q, v_{n-1}\rangle$. This finishes the proof.

Notes

1. In fact, we will use this only in the case in which $\sigma \cap N = \emptyset$ in this paper.

2. The following proof will be the proof that p_1 is $(\mathbb{P}, N)$-generic by taking that $u = t^r_M$ for the maximal M in $\mathrm{dom}(r) \setminus N$.

3. In general, if $A \in N \cap \mathcal{P}(S)$ contains u as a member, then there exists $v \in A \cap N$ with $v \leq_S u$, because the set $\{t \in S; \mathrm{Cone}_S(t) \cap A = \emptyset \text{ or } t \in A\}$ is in N and is dense in S. So there exists $v <_S u$ which belongs to this set. (We should remember that the set $\{v \in S; v <_S u\}$ is an (S, N)-generic filter.) Since $u \in A$, it has to be true that $v \in A$.

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