Notre Dame Journal of Formal Logic
Volume 56, Number 3, 2015

Pseudofinite and Pseudocompact Metric Structures

Isaac Goldbring and Vinicius Cifú Lopes

Abstract The definition of a pseudofinite structure can be translated verbatim into continuous logic, but it also gives rise to a stronger notion and to two parallel concepts of pseudocompactness. Our purpose is to investigate the relationship between these four concepts and establish or refute each of them for several basic theories in continuous logic. Pseudofiniteness and pseudocompactness turn out to be equivalent for relational languages with constant symbols, and the four notions coincide with the standard pseudofiniteness in the case of classical structures, but the details appear to be slightly more important here than in the usual translation of definitions from classical logic. We also prove that injective "formula-definable" endofunctions are surjective, and conversely, in strongly pseudofinite omega-saturated structures.

1 Introduction

Pseudofiniteness is a well-known, interesting, and useful notion in classical logic (see, e.g., Ax [1], Felgner [12], Hrushovski [14], Väänänen [17]). Our goal is to introduce a concept in continuous logic in the setting of Ben Yaacov, Berenstein, Henson, and Usvyatsov in [3] and [5] which corresponds as much as possible to the classical one. This should also provide conceptual framework for Farah, Hart, and Sherman [11, Section 5] (note that their *pseudofiniteness* is our *pseudocompactness*) and Moore [15].

However, in view of the lack of actual negations in the formal language, two different notions arise in our study. We choose to define them after the names of *pseudofiniteness* and *strong pseudofiniteness*; each resembles a different aspect of the classical notion.

We also introduce the related concept of *pseudocompactness* (and a corresponding stronger version), because compact structures in many cases appear to be the right counterpart in continuous logic of finite structures in classical logic: they are saturated, have no proper ultrapowers, and indeed are totally categorical. (We dedicate

Received September 5, 2012; accepted April 10, 2013
2010 Mathematics Subject Classification: Primary 03C52; Secondary 03C20, 03C13
Keywords: metric structure, pseudofinite, injective-surjective
 10.1215/00294527-3132833

an appendix to this fact.) Although we show that pseudocompactness is equivalent to pseudofiniteness in many cases, the stronger versions are distinct in an essential way, and thus we end up considering three different concepts.

Section 2 defines those four properties and proves some elementary results about them. We present detailed proofs for statements which correspond to trivial or well-known properties of classical pseudofiniteness, in order to highlight the nuances of continuous logic.

In Section 3, we show that, in the case of classical languages and structures, the four notions coincide with the original one; this seems to require unusual attention.

Section 4 introduces some examples and questions based on the fundamental theories of continuous logic.

Section 5 proves the equivalence of pseudofiniteness and pseudocompactness for relational languages and argues considerably in favor of a conjecture of general equivalence. Currently, we need to recourse to *almost structures*, which satisfy a weaker clause of modulus of continuity for each nonlogical symbol.

In Section 6, we discuss the injectivity-surjectivity of definable endofunctions, that is, whether injective definable functions of the form $X \to X$ are surjective, and conversely. In classical logic, that property is a straightforward consequence of (and one main source of interest in) pseudofiniteness, although it holds independently of the latter as well. In continuous logic, work is more complex and requires strong pseudofiniteness and a strong assumption on the definable function; it also helps to distinguish between pseudofiniteness and strong pseudofiniteness.

The basics of (bounded) continuous logic are established rigorously in [5] and the comprehensive monograph [3]. In contrast to classical first-order logic, continuous logic substitutes a closed bounded interval (assumed to be $[0, 1]$ in this discussion for sake of simplicity) for $\{\top, \bot\}$ as the set of truth values. Each sort of a structure is a bounded metric space (of diameter at most 1 in this discussion), which can assumed to be complete without changing the intended semantics. Since $x = y \Leftrightarrow d(x, y) = 0$, 0 is the truth value that mimics “true” while positive values represent approximations.

Predicate symbols are interpreted as uniformly continuous functions into $[0, 1]$, and function symbols are interpreted as uniformly continuous functions of the appropriate arity. One adopts all continuous functions $[0, 1]^n \to [0, 1]$ as connectives, and supremum and infimum play the role of quantifiers in the following way: if φ is a formula with free variables $x, \vec{y}$, say, then $\sup_x \varphi$ and $\inf_x \varphi$ are formulas with free variables $\vec{y}$. Now given a structure M, any formula φ induces a function φ^M into $[0, 1]$ which is defined on the appropriate product or power of sorts of M.

Moreover, every metric signature must specify a *modulus of uniform continuity* for each of its function and predicate symbols: given $\Delta\colon (0, 1] \to (0, 1]$, metric spaces M, N, and $f\colon M \to N$, we say that f has Δ as a modulus of uniform continuity if for all $\epsilon \in (0, 1]$ and $x, y \in M$,

$$d_M(x, y) < \Delta(\epsilon) \Rightarrow d_N\big(f(x), f(y)\big) \le \epsilon.$$

Doing so ensures that every formula induces a uniformly continuous function in a uniform way across structures; for example, this is needed to ensure that an ultraproduct of structures is again a structure.

Finally, since every continuous $f\colon [0, 1] \to [0, 1]$ which is 0 on $(0, 1]$ also satisfies $f(0) = 0$, we realize that there is no proper negation connective in continuous logic, whence one cannot *a priori* express implications. However, we will repeatedly use the following well-known fact to circumvent this issue.

Fact 1.1 ([3, Proposition 7.14]) Suppose that L is a 1-bounded continuous signature, M is an ω-saturated L-structure, and $\varphi(x)$ and $\psi(x)$ are two L-formulas, where x is an n-tuple of variables. Then the following are equivalent.

1. For all $a \in M^n$, if $\varphi^M(a) = 0$, then $\psi^M(a) = 0$.
2. There is an increasing, continuous function $\alpha\colon [0,1] \to [0,1]$ satisfying $\alpha(0) = 0$ so that, for all $a \in M^n$, we have $\psi^M(a) \leq \alpha(\varphi^M(a))$.

The import of this fact is that the second condition is indeed expressible by the condition $\sup_x(\psi(x) \dot{-} \alpha(\varphi(x))) = 0$.

2 Definitions and Basic Properties

Until further notice, L is a 1-bounded metric signature (assumed to be one-sorted for simplicity) and M is an L-structure.

Definition 2.1 We say that M is *pseudofinite* (resp., *pseudocompact*) if $\sigma^M = 0$ for any L-sentence σ such that $\sigma^A = 0$ for all finite (resp., compact) L-structures A.

Remark 2.2 The term *pseudocompact* is already in use in the topology literature: a space X is said to be pseudocompact if any continuous function $X \to \mathbb{R}$ is bounded. There is no relationship between our notion and the previous notion, and thus this should not be a source of confusion.

Remark 2.3 If L is a many-sorted language, then an L-structure is said to be *finite* (resp., *compact*) if the underlying universe of each sort is finite (resp., compact), regardless of the number of sorts. Then one defines pseudofinite and pseudocompact L-structures exactly as in the above definition.

Clearly pseudofinite structures are pseudocompact.

Lemma 2.4 *The following are equivalent:*

1. *M is pseudofinite (resp., pseudocompact);*
2. *for any L-sentence σ, if $\sigma^M = 0$, then for any $\epsilon > 0$, there is a finite (resp., compact) L-structure A such that $\sigma^A \leq \epsilon$;*
3. *there is a set $\{A_i : i \in I\}$ of finite (resp., compact) L-structures and an ultrafilter $\mathcal{U}$ on I such that $M \equiv \prod_{\mathcal{U}} A_i$.*

As usual, (3) together with the Keisler–Shelah theorem (see Henson and Iovino [13, Chapter 10] for a proof in the context of the approximate semantics predating continuous logic), provides an algebraic characterization of pseudofinite (resp., pseudocompact) structures, namely, as those which have some ultrapower isomorphic to some ultraproduct of finite (resp., compact) structures.

Proof (1) $\Rightarrow$ (2): Suppose that (1) holds but (2) fails. Then there is an L-sentence σ and $\epsilon > 0$ such that $\sigma^M = 0$ but $\sigma^A \geq \epsilon$ for all finite (resp., compact) L-structures A. But then $(\epsilon \dot{-} \sigma)^A = 0$ for all such A, whence $\sigma^M \geq \epsilon$ by (1), which is a contradiction. Note for future reference that the converse is similar. Suppose that (2) holds and that $\sigma^M =: r > 0$. Consider $\epsilon \in (0, r)$; then $|\sigma^A - r| \leq \epsilon$ for some such A, whence $\sigma^A > 0$.

(2) $\Rightarrow$ (3): Assume that (2) holds. Let $T = \mathrm{Th}(M)$, and let J be the collection of finite subsets of T. For each $\Delta \in J$ and $k \in \mathbb{N}^{>0}$, let $A_{\Delta,k}$ be a finite (resp., compact) L-structure such that $A_{\Delta,k} \models \max \Delta \leq \frac{1}{k}$, and let

$$X_{\Delta,k} = \left\{(\Gamma, l) \in J \times \mathbb{N}^{>0} \colon A_{\Gamma,l} \models \max \Delta \leq \frac{1}{k}\right\}.$$

Note that $\{X_{\Delta,k} \colon (\Delta, k) \in J \times \mathbb{N}^{>0}\}$ has the finite intersection property: given $\Delta_1, \ldots, \Delta_m \in J$ and $k_1, \ldots, k_m \in \mathbb{N}^{>0}$, we have that

$$\big(\Delta_1 \cup \cdots \cup \Delta_m, \max(k_1, \ldots, k_m)\big) \in X_{\Delta_1,k_1} \cap \cdots \cap X_{\Delta_m,k_m}.$$

Let $\mathcal{U}$ be an ultrafilter on $J \times \mathbb{N}^{>0}$ extending $\{X_{\Delta,k} \colon (\Delta, k) \in J \times \mathbb{N}^{>0}\}$, and set $N := \prod_{\mathcal{U}} A_{\Delta,k}$. To show that $M \equiv N$, assume that $\sigma^M = 0$, and take any $k > 0$. Then $X_{\{\sigma\},k} \in \mathcal{U}$, so that $\sigma^N \leq \frac{1}{k}$. Since $k > 0$ is arbitrary, we have that $\sigma^N = 0$.

That (3) $\Rightarrow$ (1) is clear. □

Lemma 2.5 *Any ultraproduct of pseudofinite L-structures is pseudofinite. If M is pseudofinite, so is any L-structure elementarily equivalent to M, any reduct of M to a sublanguage of L, and any expansion of M by constants. The analogous statements for pseudocompactness also hold.*

Proof The only statement whose proof is not identical to that in classical logic is the one about expansion by constants, yet the procedure is similar. Given a sentence σ in the expanded language, replace the new constant symbols by fresh variables x, thus obtaining an L-formula $\varphi(x)$. Assume that $\sigma^B = 0$ for every finite structure B in the expanded language, that is, $\varphi^A(a) = 0$ for every L-structure A and every adequate sequence of parameters a from A. Note then $[\sup_x \varphi(x)]^A = 0$ for any such A, hence $[\sup_x \varphi(x)]^M = 0$, and so $\sigma^{M'} = 0$ for any expansion M' of M with interpretations for the new constant symbols. □

In the classical analogue of Lemma 2.4, item (2) is replaced by the following statement: Whenever $M \models \sigma$, then $A \models \sigma$ for some finite structure A. (Equivalence holds due to the use of the negation connective.) This motivates the following definition.

Definition 2.6 We say that M is *strongly pseudofinite* (resp., *strongly pseudocompact*) if for any L-sentence σ such that $\sigma^M = 0$, there is a finite (resp., compact) L-structure A such that $\sigma^A = 0$.

Strongly pseudofinite structures are strongly pseudocompact, and Lemma 2.4 yields that each strong concept implies its corresponding plain version (see Examples 4.2 and 6.5 for proved distinctions).

Observe also that a finitely axiomatizable theory with no finite models cannot be strongly pseudofinite.

The following preservation lemma is almost as bold as the previous one: our examples (especially Example 6.5) will show that strong pseudofiniteness is not preserved under ultraproducts.

Lemma 2.7 *If M is strongly pseudofinite, so is any L-structure elementarily equivalent to M, any reduct of M to a sublanguage of L, and any expansion of M by constants. The analogous statements for pseudocompactness also hold.*

Proof Again, we only deal with the expansion by constants. Given any sentence in the expanded language, replace the new constant symbols by fresh variables, and quantify over each of them using the inf quantifier, then recall that inf quantifiers are actually realized in finite or compact structures. $\square$

In the next results, we follow the approach to imaginaries and the notation from [3, Section 5], which we briefly recall here. Given a family $(\varphi_n(x, y_n) : n \in \mathbb{N})$ of L-formulas, we consider the definable predicate $\psi(x, Y) := \mathcal{F}\lim \varphi_n$, which is the forced limit of these formulas. For each such $\psi(x, Y)$, we add an imaginary sort (S_ψ, d_ψ) for canonical parameters of instances of ψ; given a tuple b of sort Y in some L-structure M, we let $[b]_\psi$ denote its image in M^{eq} of sort S_ψ. We also add a predicate symbol $P_\psi(x, z)$ (where z is of sort S_ψ), whose interpretation will satisfy $P_\psi(x, [b]_\psi) = \psi(x, b)$, and predicate symbols $\gamma_{\psi,n}$ which are approximations to the graph of the quotient map between tuples and their equivalence classes in S_ψ. We let L^{eq} denote the resulting language and let T_0^{eq} denote the L^{eq}-theory axiomatizing the properties of the new symbols (so the models of T_0^{eq} are precisely the eq-expansions of L-structures). *We emphasize that this approach to imaginaries is independent of any ambient L-structure.*

Given a finite or countable tuple b and $m > 0$, we set $b|_m$ to be the truncation of b to the first m elements; we also refer to $b|_m$ simply as a *truncation* of b.

Lemma 2.8 *Let $\varphi(u, v_1, \ldots, v_n)$ be an L^{eq}-formula, where u is a tuple of variables from L and $v_1, \ldots, v_n$ are variables from imaginary sorts. Given $\epsilon > 0$, there is an L-formula $\varphi'(u, v^1, \ldots, v^n)$ such that for all L-structures M and $[b_i] \in M^{\mathrm{eq}}$ of the same sort as v_i, there are truncations b'_i such that*

$$M^{\mathrm{eq}} \models \sup_u \bigl|\varphi\bigl(u, [b_1], \ldots, [b_n]\bigr) - \varphi'(u, b'_1, \ldots, b'_n)\bigr| \le \epsilon.$$

In particular, for any L^{eq}-sentence σ and any $\epsilon > 0$, there is an L-sentence σ' such that $T_0^{\mathrm{eq}} \models |\sigma - \sigma'| \le \epsilon$.

Proof Proceed by induction on the complexity of φ, which we assume is not an L-formula. First suppose that φ is atomic. Thus, there is a definable predicate $\psi(x, Y) = \mathcal{F}\lim(\varphi_n(x, y_n))$ such that φ has one of the forms

$$P_\psi(x, z), \qquad d_\psi(z, z^*), \qquad \text{or} \qquad \gamma_{\varphi_n,\psi}(y_n, z).$$

In the first case, choose N such that $2^{-N} \le \epsilon$. Then

$$M^{\mathrm{eq}} \models \sup_x \bigl| P_\psi\bigl(x, [b]\bigr) - \varphi_N(x, b|_N)\bigr| \le \epsilon.$$

In the second case, choose N so that $2^{-N+1} \le \epsilon$. Then

$$M^{\mathrm{eq}} \models \Bigl| d\bigl([b], [b^*]\bigr) - \sup_x \bigl|\varphi_N(x, b|_N) - \varphi_N(x, b^*|_N)\bigr|\Bigr| \le \epsilon.$$

Finally, for the third case, choose N so that $2^{-N} \le \epsilon$. Then

$$M^{\mathrm{eq}} \models \sup_{y_n} \Bigl|\gamma_{\varphi_n}\bigl(y_n, [b]\bigr) - \sup_x \bigl|\varphi_n(x, y_n) - \varphi_N(x, b|_N)\bigr|\Bigr| \le \epsilon.$$

The connective step of the proof follows immediately from uniform continuity. There are two quantifier cases to consider. First, suppose that $\varphi(u, v_1, \ldots, v_n) = \inf_w \chi(u,$

$w, v_1, \dots, v_n)$, where w is a variable of L. Let $\chi'(u, w, v^1, \dots, v^n)$ be as in the conclusion of the lemma for χ and ϵ. Then set

$$\varphi'(u, v^1, \dots, v^n) := \inf_w \chi'(u, w, v^1, \dots, v^n).$$

Now assume that $\varphi(u, v_1, \dots, v_n) = \inf_w \chi(u, v_1, \dots, v_n, w)$, where w is an imaginary variable. Let $\chi'(u, v^1, \dots, v^n, w')$ be an L-formula satisfying the conclusion of the lemma for χ and ϵ. Then set $\varphi'(u, v^1, \dots, v^n) := \inf_{w'} \chi'(u, v^1, \dots, v^n, w')$. The supremum is dealt with similarly. □

Now we modify the eq-construction for finite structures slightly. Suppose that A is a finite L-structure and that $\psi(x, Y)$ is a definable predicate, where Y is a countable tuple of parameter variables. Observe that then there is a *finitary* definable predicate $\widetilde{\psi}(x, y)$ (i.e., y is a finite tuple) such that ψ and $\widetilde{\psi}$ are logically equivalent. Consequently, there is no need to add the sort S_ψ if one adds the sort $S_{\widetilde{\psi}}$. Thus, we insist that the eq-construction for finite structures only add sorts for canonical parameters of finitary imaginaries. In this way, A^{eq} is once again a finite structure. Observe that if A is a compact structure, then A^{eq} is also a compact structure.

Proposition 2.9 *If M is pseudofinite (resp., pseudocompact), then so is M^{eq}.*

Proof Suppose that M is pseudofinite and that $M^{\mathrm{eq}} \models \sigma = 0$. Fix $\epsilon > 0$, and let σ' be as in Lemma 2.8, so $M \models \sigma' \leq \epsilon$. Then there is a finite (resp., compact) L-structure A such that $A \models \sigma' \leq 2\epsilon$, whence $A^{\mathrm{eq}} \models \sigma \leq 3\epsilon$. □

Question 2.10 Are the notions of strong pseudofiniteness and strong pseudocompactness preserved when adding imaginaries? More generally, how natural is strong pseudofiniteness in continuous logic, which relies heavily on approximations?

3 The Case of a Classical Structure

In this section, we let L denote a signature for classical logic. In order to treat L as a signature for continuous logic, we interpret each predicate symbol as a function into $\{0, 1\}$ (where 0, 1 replace $\top$, $\perp$, resp.), and we specify $\Delta(\epsilon) := \epsilon$ as a modulus of uniform continuity for each function and predicate symbol. Now any classical L-structure, when equipped with the discrete metric, is a metric L-structure.

For the rest of this section, let M denote a classical L-structure. If M is pseudofinite in the classical sense, that is, M satisfies the first-order theory of finite structures, then we will say that M is *classically pseudofinite*. Thus, when we say that M is pseudofinite, we are considering M as a metric L-structure. It is not immediately clear that those two notions agree, as there are metric L-structures that are not the result of viewing classical L-structures as metric structures, plus the two syntaxes are different. Nonetheless, in this section, we will prove the following.

Theorem 3.1 *Given a classical L-structure, the five notions* classically pseudofinite*,* pseudofinite*,* strongly pseudofinite*,* pseudocompact*, and* strongly pseudocompact *coincide.*

Toward proving this result, note that, since all five notions are invariant under elementary equivalence, we may replace M with an elementary extension, thus reducing to the case that M is ω-saturated (as a metric structure).

One proves the next lemma by induction on the complexity of formulas (see also [3, Remark 9.21]), which is uniform over all classical structures.

Lemma 3.2 *For any continuous L-formula $\varphi(x)$, there is finite $R_\varphi \subseteq [0,1]$ such that, for any* classical *L-structure A and any adequate tuple a from A, one has $\varphi^A(a) \in R_\varphi$. Moreover, for each $r \in R_\varphi$ there is a classical L-formula $\varphi_r(x)$ such that, for all those pairs (A, a), we have $\varphi^A(a) = r \Leftrightarrow A \models \varphi_r(a)$.*

Corollary 3.3 *If M is classically pseudofinite, then M is strongly pseudofinite.*

Proof Assume that M is classically pseudofinite, let σ be a continuous L-sentence such that $\sigma^M = 0$, and take σ_0 given by Lemma 3.2. Since $M \models \sigma_0$ there is a finite classical L-structure A such that $A \models \sigma_0$, whence $\sigma^A = 0$. $\square$

Remark 3.4 Lemma 3.2 also provides a way to see that if $A \preccurlyeq B$ (resp., $A \equiv B$) as classical structures, then $A \preccurlyeq B$ (resp., $A \equiv B$) as metric structures.

Now note that, for all $a, b \in M$, we have $d(a,b) \leq \frac{1}{2} \Rightarrow d(a,b) = 0$. Thus, since M is ω-saturated, there is an increasing, continuous $\alpha\colon [0,1] \to [0,1]$ such that $\alpha(0) = 0$ and $d(a,b) \leq \alpha(d(a,b) \dot{-} \frac{1}{2})$ for all $a, b \in M$.

Lemma 3.5 *If M is strongly pseudocompact, then M is strongly pseudofinite.*

Proof Suppose that M is strongly pseudocompact and that $\sigma^M = 0$. Then

$$M \models \max\Big(\sigma, \sup_{x,y}\Big(d(x,y) \dot{-} \alpha\Big(d(x,y) \dot{-} \frac{1}{2}\Big)\Big)\Big) = 0.$$

Thus the displayed sentence has value 0 in some compact L-structure A. However, the second "conjunct" forces A to be discrete, whence finite. $\square$

It remains to prove that if M is pseudocompact, then it is classically pseudofinite. Towards this end, given a classical L-formula $\varphi(x)$, we define its *continuous transform* $\widetilde{\varphi}(x)$ by recursion as follows:

- if $\varphi(x)$ is $t_1(x) = t_2(x)$, then $\widetilde{\varphi}(x)$ is $d(t_1(x), t_2(x))$;
- if $\varphi(x)$ is $P(t_1(x), \ldots, t_n(x))$, then $\widetilde{\varphi}(x)$ is $\varphi(x)$ (which will correspond to a function into $\{0,1\}$);
- $[\neg\varphi]^\sim$ is $1 \dot{-} \widetilde{\varphi}$;
- $[\varphi \wedge \psi]^\sim$ is $\max(\widetilde{\varphi}, \widetilde{\psi})$;
- $[\exists y\, \psi(x,y)]^\sim$ is $\inf_y \widetilde{\psi}(x,y)$.

Now for any *classical* L-structure B, any classical L-formula $\varphi(x)$, and any b from B, we have $\widetilde{\varphi}^B(b) \in \{0,1\}$; moreover, $B \models \varphi(b) \Leftrightarrow \widetilde{\varphi}^B(b) = 0$.

Now suppose that M is pseudocompact and that σ is a classical L-sentence such that $M \models \sigma$. We will find a finite classical L-structure B such that $B \models \sigma$.

For every predicate symbol P in σ and every suitable tuple a from M, we have $P^M(a) \leq \frac{1}{2} \Rightarrow P^M(a) = 0$. Thus, there is an increasing, continuous $\alpha_P\colon [0,1] \to [0,1]$ with $\alpha_P(0) = 0$ such that $\sigma_P^M = 0$, where

$$\sigma_P := \sup_x\Big(P(x) \dot{-} \alpha_P\Big(P(x) \dot{-} \frac{1}{2}\Big)\Big).$$

Similarly, letting $\tau := \sup_{x,y}(d(x,y) \dot{-} \alpha(d(x,y) \dot{-} \frac{1}{2}))$ with α as above, we also have that $\tau^M = 0$. Let $\sigma' := \max(\widetilde{\sigma}, (\sigma_P)_P, \tau)$, where P ranges over the predicate symbols appearing in σ.

Since M is pseudocompact, there is a compact structure A such that $(\sigma')^A \leq \frac{1}{4}$. Define a binary relation E on A by $E(x,y) :\Leftrightarrow d(x,y) \leq \frac{1}{4}$. Clearly E is reflexive and symmetric. However, E is also transitive: indeed, if $d(x,y), d(y,z) \leq \frac{1}{4}$, then $d(x,z) \leq \frac{1}{2}$; since $\tau^A \leq \frac{1}{4}$, we get $d(x,z) \leq \frac{1}{4}$. We now define B as follows: Its underlying set is the set of E-equivalence classes of A. Since A has a finite $\frac{1}{4}$-net, B is finite. We only define the symbols appearing in σ. Given a predicate symbol P, first we check that if $d(a_i, a_i') \leq \frac{1}{4}$ for each i, then $P^A(a) \leq \frac{1}{2} \Leftrightarrow P^A(a') \leq \frac{1}{2}$. Indeed, $(\sigma_P)^A \leq \frac{1}{4}$, so $P^A(a) \leq \frac{1}{2}$ implies $P^A(a) \leq \frac{1}{4}$; since $\Delta_P(\epsilon) = \epsilon$, we obtain $|P^A(a) - P^A(a')| \leq \frac{1}{4}$. Therefore, we declare $[a] \in P^B$ if and only if $P^A(a) \leq \frac{1}{2}$. Similarly, for a function symbol f, we can define $f^B([a]) := [f^A(a)]$ because $[a] = [a']$ implies $|f^A(a) - f^A(a')| \leq \frac{1}{4}$.

Let L' be the reduct of L that only contains the symbols of σ. One proves by induction on the complexity of formulas, and using that inf quantifiers are realized in compact structures, the following.

Lemma 3.6 *Suppose that $\varphi(x)$ is a classical L'-formula. Then for any suitable tuple a in A, we have the following.*

1. *If $\widetilde{\varphi}^A(a) \leq \frac{1}{4}$, then $B \models \varphi([a])$.*
2. *If $\widetilde{\varphi}^A(a) \geq \frac{3}{4}$, then $B \models \neg\varphi([a])$.*

Therefore, since $\widetilde{\sigma}^A \leq \frac{1}{4}$, we have $B \models \sigma$.

This finishes the proof of Theorem 3.1. We should remark that the above discussion is unusual in the sense that when one generalizes a notion to continuous logic, it is often immediate that it agrees with the classical notion on classical structures. For example, if T is a classical theory, then it is immediate to see that T is stable as a classical theory if and only if T is stable as a continuous theory. Pseudofiniteness appears to be the first notion where some work is required to show that the notions agree on classical structures.

4 Examples and Questions

Example 4.1 Let L be the signature naturally used for closed unit balls of inner product spaces (see [3, Example 2.1]). For $n \geq 1$, let B_n denote the closed unit ball of $\mathbb{R}^n$, let $\mathcal{U}$ be any nonprincipal ultrafilter on $\mathbb{N}$, and let $H = \prod_{\mathcal{U}} B_n$. Clearly H is the closed unit ball of an infinite-dimensional Hilbert space. By completeness, *any closed unit ball of a Hilbert space is pseudocompact.* We will be able to show that H is not strongly pseudofinite: the function $H \to H$, $x \mapsto \frac{1}{2}x$, is injective but not surjective.

There are two relevant pseudocompact expansions of H. First, consider $(H, P) = \prod_{\mathcal{U}}(B_n, P_n)$, where $P_n\colon \mathbb{R}^n \to \mathbb{R}^n$ is a projection operator onto an $\lfloor n/2 \rfloor$-dimensional subspace of $\mathbb{R}^n$. Then (H, P) is a model of the theory of *beautiful pairs* of Hilbert spaces, namely, an infinite-dimensional Hilbert space equipped with a projection with infinite-dimensional image and infinite-dimensional orthogonal complement. (See Berenstein and Villaveces [7].)

Next, let $\{z_i : i < \omega\}$ be a countable dense subset of the circle $\{z \in \mathbb{C} : |z| = 1\}$. Let $(H, U) = \prod_{\mathcal{U}}(B_n, U_n)$, where $U_n\colon \mathbb{C}^n \to \mathbb{C}^n$ is a unitary operator with eigenvalues $\{z_0, \ldots, z_n\}$. Then for any $m \geq n$, we have

$$(B_m, U_m) \models \inf_x \max\left(\left|\langle x, x\rangle - 1\right|, \left\|U(x) - z_n x\right\|\right) = 0.$$

It follows from the work in Ben Yaacov, Usvyatsov, and Zadka [6] that (H, U) is a model of the theory of Hilbert spaces equipped with a generic automorphism.

Example 4.2 Let L be the signature for probability structures (see [3, Section 16]). Let $\mathcal{B}_n$ be the probability structure with event algebra 2^n and counting measure μ_n. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $\mathcal{B} = \prod_{\mathcal{U}} \mathcal{B}_n$. We claim that $\mathcal{B}$ is an atomless probability structure; by completeness, it will follow that *any atomless probability structure is pseudofinite*, but not strongly pseudofinite since its theory is finitely axiomatizable. Towards this end, suppose that $x = [(x_n)] \in \mathcal{B}$ is such that $\mu(x) > 0$. Set $m_n := |x_n|$; then $m_n > 1$ for almost all n, because otherwise $\mu_n(x_n) \leq \frac{1}{n}$ and hence $\mu(x) = \lim_{\mathcal{U}} \mu_n(x_n) = 0$. For such n, let $y_n \subseteq x_n$ be such that $|y_n| = \frac{1}{2}m_n$ if m_n is even or $|y_n| = \frac{1}{2}(m_n - 1)$ if m_n is odd. Then $\mathcal{B}_n \models \inf_y |\mu(x_n \cap y) - \mu(x_n \cap y^c)| \leq \frac{1}{n}$ for almost all n, whence $\mathcal{B} \models \inf_y |\mu(x \cap y) - \mu(x \cap y^c)| = 0$. It follows that $\mathcal{B}$ is atomless.

Similarly, any atomless probability structure equipped with a *generic (or aperiodic) automorphism* is also pseudofinite, but not strongly pseudofinite. Indeed, the theory of atomless probability structures equipped with a generic automorphism (denoted APAA in [3, Section 18]) is axiomatized (in the language of probability structures expanded by a unary function symbol τ) by the axioms of atomless probability structures equipped with an automorphism together with, for each $n \geq 1$, the axiom

$$\inf_e \max\Big(\Big|\frac{1}{n} - \mu(e)\Big|, \mu\big(e \cap \tau(e)\big), \dots, \mu\big(e \cap \tau^{n-1}(e)\big)\Big) = 0.$$

We equip the above probability structures $\mathcal{B}_m$ with the automorphisms τ_m induced by the point map $x \mapsto x + 1 \mod m$. Fix $n \geq 1$, and suppose that $m > n$. Choose $k \in \{1, \dots, m\}$ maximal with respect to $(k-1)/(m-1) \leq \frac{1}{n}$. Let $e = \{1, 1+n, \dots, 1+(k-1)n\} \in \mathcal{B}_m$. Observe that $|e| = k$, so

$$\Big|\mu(e) - \frac{1}{n}\Big| \leq \Big|\frac{k}{m} - \frac{k}{m-1}\Big| + \Big|\frac{k}{m-1} - \frac{k-1}{m-1}\Big| + \Big|\frac{k-1}{m-1} - \frac{1}{n}\Big| \leq \frac{3}{m-1}.$$

Furthermore, observe that for $i \in \{0, 1, \dots, n-1\}$ we have $\tau_m^i(e) \cap e \subseteq \{1\}$, so $\mu(\tau_m^i(e) \cap e) \leq \frac{1}{m}$ for each such i. Consequently,

$$(\mathcal{B}_m, \tau_m) \models \inf_e \max\Big(\Big|\frac{1}{n} - \mu(e)\Big|, \mu\big(e \cap \tau(e)\big), \dots, \mu\big(e \cap \tau^{n-1}(e)\big)\Big) \leq \frac{3}{m-1}$$

for each $m \geq n$. Let $\tau_\infty = \lim_{\mathcal{U}} \tau_m$. It follows that

$$(\mathcal{B}, \tau_\infty) \models \inf_e \max\Big(\Big|\frac{1}{n} - \mu(e)\Big|, \mu\big(e \cap \tau(e)\big), \dots, \mu\big(e \cap \tau^{n-1}(e)\big)\Big) = 0.$$

Since $n \geq 1$ was arbitrary, we have that $(\mathcal{B}, \tau_\infty) \models$ APAA.

Example 4.3 More generally, let L be a countable classical signature: we claim that if M is a pseudofinite L-structure, then any of its *Keisler randomizations* are pseudofinite metric structures. (See Ben Yaacov and Keisler [4] for information on this notion; the theory of atomless probability algebras is the theory of the Keisler randomization of a two-element set.) To see this, suppose that $M \equiv \prod_{\mathcal{U}} M_n$, where $(M_n \colon n \in \mathbb{N})$ is a family of finite L-structures and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$; observe that since L is countable, we may always find such a countable family of finite structures. We consider the probability structures $\mathcal{B}_n$ from the previous example, and we let $\mathcal{K}_n := M_n^{\{1,\dots,n\}}$. We claim that $(\mathcal{K}, \mathcal{B}) := \prod_{\mathcal{U}}(\mathcal{K}_n, \mathcal{B}_n) \models T^R$, where $T := \mathrm{Th}(M)$; since T^R is complete, it suffices to prove this claim. The validity, Boolean, distance, fullness, event, and measure axioms are clearly true in each of the factor structures, and hence true in the ultraproduct. As above, the ultraproduct satisfies the atomless axiom. It remains to verify the transfer axiom; namely, for every sentence $\sigma \in T$, we need $(\mathcal{K}, \mathcal{B}) \models d([\![\sigma]\!], \top) = 0$. Note that $d^{\mathcal{B}_n}([\![\sigma]\!], \top) = 0$ if and only if $M_n \models \sigma$; since $\sigma \in T$, then $M_n \models \sigma$ for almost all n, whence $d^{\mathcal{B}}([\![\sigma]\!], \top) = \lim_{\mathcal{U}} d^{\mathcal{B}_n}([\![\sigma]\!], \top) = 0$. Note that this example provides us with our first examples of unstable pseudofinite theories (other than the classical ones). Indeed, if T is not stable, then T^R is unstable (see [4]). Also, by a result of Ben Yaacov (see [2, Corollary 4.15]), if T is simple unstable, then T^R is *not* simple. Thus, if T is the theory of the random graph or the theory of pseudofinite fields, then T^R is pseudofinite but not simple.

Example 4.4 Let (X, d) be a *proper* metric space (i.e., its closed balls are compact). Fix a base point $p \in X$, and consider (X, d, p) as a many-sorted structure in the natural way. (See Carlisle [8] for all the concepts.) Then the asymptotic cone of (X, d, p) with respect to any nonprincipal ultrafilter on $\mathbb{N}$ will be pseudocompact, and hence pseudofinite by Theorem 5.1 below. Of particular interest is the case when (X, d) is the Cayley graph of a finitely generated group G. In the case when G is hyperbolic, this asymptotic cone will be an $\mathbb{R}$-tree. If G is hyperbolic and nonelementary (i.e., it does not contain an infinite cyclic group of finite index), for example $G = \mathbb{F}_2$ the free group on two generators, then the $\mathbb{R}$-tree is *richly branching*. Again by completeness, every richly branching $\mathbb{R}$-tree is pseudofinite.

Question 4.5 Let $\mathfrak{U}$ denote the *bounded Urysohn space*, that is, the unique Polish metric space of diameter 1 which is ultrahomogeneous and contains an isometric copy of every Polish metric space of diameter bounded by 1. Let $T_{\mathfrak{U}}$ denote the theory of $\mathfrak{U}$ in the metric signature containing only the distance symbol. Is $\mathfrak{U}$ pseudofinite?

The Urysohn space is the continuous analogue of the random graph, and so one might expect the answer to be positive. The model theory of $\mathfrak{U}$ is quite well understood; $T_{\mathfrak{U}}$ is complete, admits quantifier elimination, is $\aleph_0$-categorical, and is rosy with respect to finitary imaginaries, but it is not simple (see Usvyatsov [16], Ealy and Goldbring [10]). In [16], an axiomatization for $T_{\mathfrak{U}}$ is given by writing down conditions in continuous logic describing a collection of "extension axioms." Thus, the following lemma (and Theorem 5.1) might prove useful to decide the question.

Lemma 4.6 *Suppose that* $\{\gamma = 0 \colon \gamma \in \Gamma\} \models \mathrm{Th}(M)$ *for some collection* Γ *of* L*-sentences, and suppose that, for every* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *and every* $\epsilon > 0$*, there is a finite (resp., compact)* L*-structure* A *such that* $A \models \max(\gamma_1, \ldots, \gamma_n) \leq \epsilon$*. Then* M *is pseudofinite (resp., pseudocompact).*

Proof Let σ be an L-sentence such that $\sigma^M = 0$, and set $\delta > 0$. Then by compactness, there is $\epsilon > 0$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\{\max(\gamma_1, \ldots, \gamma_n) \leq \epsilon, \sigma \geq \delta\}$ is unsatisfiable, so, for A as in the statement, we obtain $\sigma^A < \delta$. □

We should remark that the Urysohn space is not the underlying metric space of any Keisler randomization, which eliminates a strategy for proving its pseudofiniteness. Indeed, the metric space consisting of the vertices of an equilateral triangle together with the centroid cannot be realized in a randomization. (We thank Julien Melleray and Todor Tsankov for pointing this out to us.)

Question 4.7 Is there an example of an *essentially continuous* strongly pseudofinite (resp., strongly pseudocompact) structure that is not finite (resp., compact)?

While the term *essentially continuous* is admittedly vague (although there have been attempts by others to make this notion precise), we use it to preclude discrete/classical examples. In fact, since continuous logic is an approximate logic, we conjecture that the answer to the above question might be negative. It would even be interesting to settle this question under some extra set-theoretic hypotheses.

We should remark that an exercise from classical logic also works in continuous logic, and thus it fails to construct a nontrivial strongly pseudofinite structure. Suppose that $(A_i : i \in I)$ is a family of finite (resp., compact) L-structures, and that $\mathcal{U}$ is an ω_1-complete ultrafilter on I (see Chang and Keisler [9, Section 4.2]), and let $M := \prod_{\mathcal{U}} A_i$. Such an M is strongly pseudofinite (resp., strongly pseudocompact). Indeed, suppose that σ is a sentence such that $\sigma^M = 0$. Let $I_n := \{i \in I : A_i \models \sigma^{A_i} \leq \frac{1}{n}\} \in \mathcal{U}$. Then by assumption, there is $i \in \bigcap_{n \geq 1} I_n$, and $\sigma^{A_i} = 0$ for this i. However, M is actually finite (resp., compact). In fact, let

$$D_n := \{i \in I : |A_i| \geq n\}.$$

If each $D_n \in \mathcal{U}$, then their intersection is also in $\mathcal{U}$, contradicting that each A_i is finite. Consequently, $I \setminus D_n \in \mathcal{U}$ for some n, whence $|M| < n$. (For compactness, given $\epsilon > 0$, take

$$D_n := \{i \in I : A_i \text{ does not have an } \epsilon\text{-net of size } \leq n\}.)$$

5 Relationship Between Pseudofiniteness and Pseudocompactness

In this section, L denotes a 1-bounded, one-sorted metric signature. We will prove the following.

Theorem 5.1 *Suppose that L contains only predicate and constant symbols. Then the two notions* pseudofinite *and* pseudocompact *coincide.*

We will also obtain a similar result for languages with function symbols.

In this section, M denotes a compact L-structure. For each $m \geq 1$, let $X_m \subseteq M$ be a finite $\frac{1}{m}$-net for M.

Suppose first that L is relational. View each X_m as a substructure of M, and let $N := \prod_{\mathcal{U}} X_m$, where $\mathcal{U}$ is some nonprincipal ultrafilter on $\mathbb{N}$. We denote sequences from $\prod X_m$ as (a^m), and we write $[a^m]$ for the corresponding equivalence class in N.

Suppose next that c is a constant symbol in L; choose $c^{X_m} \in X_m$ so that $d(c^M, c^{X_m}) < \frac{1}{m}$. Now X_m continues to be an L-structure, though it need not be a substructure of M.

Finally, if f is a function symbol in L and a is a suitable tuple in X_m, define $f^{X_m}(a) \in X_m$ so that $d(f^{X_m}(a), f^M(a)) < \frac{1}{m}$. Observe that X_m may not be an L-structure for the singular reason that it may not respect the modulus of uniform continuity for f specified by L. However, X_m is an *almost L-structure* in the following sense.

Definition 5.2 An *almost L-structure* X is defined as an L-structure, except that the clause of modulus of continuity for each function symbol f is weakened thus: for sufficiently small ϵ and every a, b from X,

$$d(a,b) < \Delta_f(\epsilon) \Rightarrow d\big(f^X(a), f^X(b)\big) \leq \epsilon,$$

and the clause of modulus of continuity for each predicate symbol is weakened analogously. (Every L-structure is an almost L-structure.)

Indeed, since X_m is finite, there is $r_m > 0$ such that $d(a,b) \geq r_m$ for each pair of distinct $a,b \in X_m$, in which case, when $\Delta_f(\epsilon) < r_m$, we have that $d(a,b) < \Delta_f(\epsilon) \Rightarrow a = b$. (We assume here that $\lim_{\epsilon\to 0^+} \Delta_f(\epsilon) = 0$, which is usually the case.)

Once again, let $N := \prod_{\mathcal{U}} X_m$. Observe that N is an actual L-structure. Indeed, fix $\epsilon > 0$, and suppose that $[a^m], [b^m]$ are tuples in N such that $d([a^m],[b^m]) < \Delta_f(\epsilon)$. Then for almost all m, we have $d(a^m, b^m) < \Delta_f(\epsilon)$, whence $d(f^{X_m}(a^m), f^{X_m}(b^m)) \leq \epsilon + \frac{2}{m}$. Consequently, $d(f^N([a^m]), f^N([b^m])) \leq \epsilon$.

Lemma 5.3 *For any $\epsilon > 0$ and L-term $t(x)$, there is $K \in \mathbb{N}$ so that for all $m \geq K$ and all suitable a from X_m, we have $d(t^M(a), t^{X_m}(a)) < \epsilon$.*

Proof We prove by induction on the complexity of t. The basic cases follow from the definitions of interpretation in X_m. Now suppose that $t(x) = f(t_1(x), \ldots, t_k(x))$. Choose K' so that $\frac{1}{K'} < \Delta_f(\frac{\epsilon}{2})$, and then choose $K > \frac{2}{\epsilon}$ so that our claim holds for $t_1, \ldots, t_k$ with $\frac{1}{K'}$ in place of ϵ. Suppose that $m \geq K$ and that a lies in X_m. Then $d(t_i^M(a), t_i^{X_m}(a)) < \frac{1}{K'}$ for each i, whence

$$d\big(f^M\big(t_1^M(a), \ldots, t_k^M(a)\big), f^M\big(t_1^{X_m}(a), \ldots, t_k^{X_m}(a)\big)\big) \leq \frac{\epsilon}{2},$$

whence

$$d\big(f^M\big(t_1^M(a), \ldots, t_k^M(a)\big), f^{X_m}\big(t_1^{X_m}(a), \ldots, t_k^{X_m}(a)\big)\big) \leq \frac{\epsilon}{2} + \frac{1}{m} < \epsilon. \qquad \square$$

Lemma 5.4 *For any $\epsilon > 0$ and L-formula $\varphi(x)$, there is $K \in \mathbb{N}$ so that for all $m \geq K$ and all suitable a from X_m, we have $|\varphi^M(a) - \varphi^{X_m}(a)| < \epsilon$.*

Proof We induct on the complexity of φ. First, suppose that $\varphi(x)$ is the atomic formula $d(t_1(x), t_2(x))$. Note that

$$\begin{aligned}&\big|d\big(t_1^M(x), t_2^M(x)\big) - d\big(t_1^{X_m}(x), t_2^{X_m}(x)\big)\big| \\ &\quad \leq d\big(t_1^M(x), t_1^{X_m}(x)\big) + d\big(t_2^M(x), t_2^{X_m}(x)\big),\end{aligned}$$

and apply Lemma 5.3. Now suppose that $\varphi(x)$ is the atomic formula $P(t_1(x), \ldots, t_k(x))$. Then

$$\begin{aligned}&\big|P^M\big(t_1^M(x), \ldots, t_k^M(x)\big) - P^{X_m}\big(t_1^{X_m}(x), \ldots, t_k^{X_m}(x)\big)\big| \\ &\quad = \big|P^M\big(t_1^M(x), \ldots, t_k^M(x)\big) - P^M\big(t_1^{X_m}(x), \ldots, t_k^{X_m}(x)\big)\big| < \epsilon\end{aligned}$$

if m is sufficiently large so that $d(t_i^M(x), t_i^{X_m}(x)) < \Delta_P(\epsilon)$ for each i. Uniform continuity of connectives takes care of the connective case.

Finally, suppose that $\varphi(x) = \inf_y \psi(x,y)$. (The supremum case is analogous.) Let K' be as in the conclusion of the lemma for $\psi(x,y)$ and $\frac{\epsilon}{3}$. Choose $K \geq K'$ so that $\frac{1}{K} \leq \Delta_\psi(\frac{\epsilon}{3})$. Suppose that $m \geq K$ and that a lies in X_m. Let $r = \varphi^M(a)$, and let $s = \varphi^{X_m}(a)$. Let $b \in M$ be such that $\psi^M(a,b) < r + \frac{\epsilon}{3}$. Let $c \in X_m$ be such that $d(b,c) \leq \frac{1}{m}$. Then $\psi^M(a,c) \leq r + \frac{2\epsilon}{3}$, and so $\psi^{X_m}(a,c) < r + \epsilon$, hence $s < r + \epsilon$. On the other hand, let $e \in X_m$ be such that $\psi^{X_m}(a,e) < s + \frac{\epsilon}{3}$. Then $\psi^M(a,e) < s + \frac{2\epsilon}{3}$, so $r < s + \epsilon$. $\square$

In particular, for any sentence σ, we have $\sigma^M = \lim_{\mathcal{U}} \sigma^{X_m}$, and thus the following.

Corollary 5.5 *We have $M \equiv N$.*

This proves Theorem 5.1 because then every X_m will be a finite L-structure.

If we now assume that the language has function symbols and that M as above is locally finite, then we can arrange each X_m as above to be a finite substructure of M, proving that M is once again pseudofinite. If now A is a *uniformly* locally finite pseudocompact structure in a language with only finitely many function symbols, then it follows that A is elementarily equivalent to an ultraproduct of locally finite compact structures, proving the following.

Corollary 5.6 *For uniformly locally finite structures in languages with only finitely many function symbols, the two notions* pseudofinite *and* pseudocompact *coincide.*

To state the general result, we need the following concepts, which will be used also in the next section.

Definition 5.7 An L-structure Z is *almost pseudofinite* (resp., *almost pseudo-compact*) if whenever $\sigma^A = 0$ for an L-sentence σ and all finite (resp., compact) almost L-structures A, then $\sigma^Z = 0$.

Similarly, Z is *almost strongly pseudofinite* (resp., *almost strongly pseudocompact*) if whenever $\sigma^Z = 0$ for an L-sentence σ, then there is a finite (resp., compact) almost L-structure A such that $\sigma^A = 0$.

(In all four cases, the *almost* version of a property is a consequence of the property itself.)

Lemma 5.8 *Z is almost pseudofinite if and only if, whenever $\epsilon > 0$ and σ is an L-sentence such that $\sigma^Z = 0$, then there is a finite almost L-structure A such that $\sigma^A \le \epsilon$.*

Proof This is analogous to the equivalence of (1) and (2) in Lemma 2.4. □

Thus, since $M \equiv N$ and N is an ultraproduct of finite almost L-structures, we obtain the following.

Proposition 5.9 *Every compact L-structure is* almost *pseudofinite, and thus*

pseudocompactness $\Rightarrow$ almost pseudofiniteness $\Rightarrow$ almost pseudocompactness.

We conjecture that the notions of pseudofiniteness and pseudocompactness always agree. We will see in the next section (e.g., Example 6.5), however, that the strong versions are not equivalent.

6 Injectivity-Surjectivity of Endofunctions

In classical logic, if M is pseudofinite and $f : M \to M$ is definable, then f is injective if and only if it is surjective. In the continuous setting, already in the simplest case this seems to require a few more assumptions. Suppose that L is a 1-bounded, one-sorted metric signature, and suppose that M is a metric L-structure.

Definition 6.1 Say that $f : M \to M$ is *formula-definable* if there is an L-formula $\varphi(x, y, z)$, where z is a tuple of variables, and there is a tuple a from M such that $d(f(x), y) = \varphi(x, y, a)$ for all $x, y \in M$.

Theorem 6.2 *Suppose that M is ω-saturated and strongly pseudofinite. Suppose that $f : M \to M$ is a formula-definable function. Then f is injective if and only if it is surjective.*

Proof Suppose that f is injective but not surjective. By ω-saturation, there is $y \in M$ and an $\epsilon > 0$ such that $d(f(x), y) \geq \epsilon$ for all $x \in M$. Fix $\varphi(x, y, a)$ as in the above definition. Note that, for all $x, y_1, y_2 \in M$, we have

$$\max\big(\varphi(x, y_1, a), \varphi(x, y_2, a)\big) = 0 \Rightarrow d(y_1, y_2) = 0.$$

Since (M, a) is ω-saturated, there is an increasing continuous function $\alpha\colon [0, 1] \to [0, 1]$ satisfying $\alpha(0) = 0$ and such that, for all $x, y_1, y_2 \in M$,

$$d(y_1, y_2) \leq \alpha\big(\max\big(\varphi(x, y_1, a), \varphi(x, y_2, a)\big)\big).$$

Similarly, since f is injective, there is an increasing continuous $\beta\colon [0, 1] \to [0, 1]$ satisfying $\beta(0) = 0$ and such that, for all $x_1, x_2, y \in M$,

$$d(x_1, x_2) \leq \beta\big(\max\big(\varphi(x_1, y, a), \varphi(x_2, y, a)\big)\big).$$

Consider the following formulas:

$$\begin{aligned}
P(z) &:= \sup_x \inf_y \varphi(x, y, z),\\
Q(z) &:= \sup_{x, y_1, y_2} \big[d(y_1, y_2) \dot{-} \alpha\big(\max\big(\varphi(x, y_1, z), \varphi(x, y_2, z)\big)\big)\big],\\
R(z) &:= \sup_{x_1, x_2, y} \big[d(x_1, x_2) \dot{-} \beta\big(\max\big(\varphi(x_1, y, z), \varphi(x_2, y, z)\big)\big)\big],\\
S(z) &:= \inf_y \sup_x \big(\epsilon \dot{-} \varphi(x, y, z)\big).
\end{aligned}$$

Then

$$M \models \inf_z \max\big(P(z), Q(z), R(z), S(z)\big) = 0.$$

Since M is strongly pseudofinite, there is a finite L-structure A such that

$$A \models \inf_z \max\big(P(z), Q(z), R(z), S(z)\big) = 0.$$

Since A is finite, inf quantifiers are actually realized, and thus there is $a_0 \in A$ such that $\varphi(x, y, a_0)$ defines an injective function $A \to A$ which is not surjective, leading to a contradiction.

Conversely, suppose that f is surjective but not injective. Define $P(z)$ and $Q(z)$ as above. Since f is not injective, there are $b_1, b_2 \in M$ such that $d(b_1, b_2) =: \epsilon > 0$ and $f(b_1) = f(b_2)$. Consequently, there is an increasing continuous $\gamma\colon [0, 1] \to [0, 1]$ satisfying $\gamma(0) = 0$ and so that, for all $w_1, w_2 \in M$,

$$d(w_1, w_2) \leq \gamma\big(\max\big(\varphi(b_1, w_1, a), \varphi(b_2, w_2, a)\big)\big).$$

Consider the following formulas:

$$\begin{aligned}
\Gamma(x_1, x_2, z) &:= \sup_{w_1, w_2} \big[d(w_1, w_2) \dot{-} \gamma\big(\max\big(\varphi(x_1, w_1, z), \varphi(x_2, w_2, z)\big)\big)\big],\\
T(z) &:= \inf_{x_1, x_2} \max\big(\big|d(x_1, x_2) - \epsilon\big|, \Gamma(x_1, x_2, z)\big),\\
U(z) &:= \sup_y \inf_x \varphi(x, y, z).
\end{aligned}$$

Then

$$M \models \inf_z \max\big(P(z), Q(z), T(z), U(z)\big) = 0,$$

leading to a similar contradiction. □

Remark 6.3 The above proof only required that M was *almost* strongly pseudofinite.

Remark 6.4 Similarly, if X is the zero set of a formula in a power of M, and $f: X \to X$ is formula-definable (so there are φ, a such that $d(f(x), y) = \varphi(x, y, a)$ for all $x, y \in X$), then f is injective if and only if f is surjective.

Example 6.5 Let $\mathbb{S}^1 = \{z \in \mathbb{C}: |z| = 1\}$ have the metric which is half the one induced by the canonical metric in $\mathbb{C}$ (so it has values in $[0, 1]$), and consider the ternary relation $P(u, v, w) := d(uv, w)$, where the usual product in $\mathbb{C}$ is used. Consider also $f: \mathbb{S}^1 \to \mathbb{S}^1$, $f(z) = z^2$, which is surjective, but not injective. Then the relational structure $(\mathbb{S}^1, P)$ (in the minimal adequate language with the right modulus of uniformity for P) is compact, hence pseudofinite. Because of the total categoricity of compact models, $(\mathbb{S}^1, P)$ is saturated. Also, f is formula-definable in $(\mathbb{S}^1, P)$: we have $d(f(z), w) = P(z, z, w)$. Therefore, $(\mathbb{S}^1, P)$ is not strongly pseudofinite.

Example 6.6 Let $[0, 1]$ have the usual metric, and consider $f: [0, 1] \to [0, 1]$, $f(x) = x/2$, which is injective, but not surjective. Similarly to the previous example, one obtains a compact, pseudofinite, but not strongly pseudofinite structure.

Question 6.7 Is there a natural property of endofunctions in continuous logic corresponding to injectivity-surjectivity, and which holds in pseudofinite structures?

Note that the function $f: [0, 1] \to [0, 1]$ defined thus: $f(x) = 2x$ if $0 \le x \le \frac{1}{2}$ and $f(x) = 1$ otherwise, is surjective, yet $f^{-1}(1)$ is large by several metric and topological standards.

In general, suppose that $P: M^n \to [0, 1]$ is a definable predicate in M (over some countable parameter set). Let L_P be the language obtained by adding a predicate symbol for P, together with its own modulus of uniform continuity, and let (M, P) be the natural expansion of M to an L_P-structure. Given an L_P-formula $\psi(y)$ without parameters and an L-formula $\varphi(x, a)$ with parameters a, where x is an n-tuple, one naturally gets an $L(a)$-formula ψ_φ by replacing every occurrence of $P(t)$ in ψ with $\varphi(t, a)$ (for any occurring tuple t of terms).

Lemma 6.8 *If M is ω_1-saturated, then (M, P) is ω_1-saturated.*

Proof Suppose that $\{\psi^i(y) = 0: i \in I\}$ is a finitely satisfiable collection of L_P-conditions in countably many parameters. Replace $\psi^i(y) = 0$ by $\psi^i_{\varphi^i_n}(y) \le \frac{1}{n}$, where φ^i_n is an L-formula approximating P well enough, so as to obtain a finitely satisfiable collection of L-conditions in countably many parameters. Then use ω_1-saturation of M. □

Corollary 6.9 *Suppose that $f: M \to M$ is a definable function in an ω_1-saturated structure M, let $P(x, y) = d(f(x), y)$, and suppose further that (M, P) is almost strongly pseudofinite. Then f is injective if and only if it is surjective.*

Such a result naturally poses the following question.

Question 6.10 If M is strongly pseudofinite, is (M, P) almost strongly pseudofinite?

We can settle the corresponding question for pseudofinite structures.

Lemma 6.11 *Given any L_P-formula $\psi(y)$ and $\epsilon > 0$, there are parameters a in M and an $L(a)$-formula $\varphi(x, a)$ such that $|\psi^{(M,P)}(b) - \psi^M_\varphi(b)| \le \epsilon$ for every $b \in M$.*

Proof If ψ is actually an L-formula, then there is nothing to do. Otherwise, if $\psi(y)$ is atomic, then it is $P(t_1(y),\dots,t_n(y))$, where $t_1,\dots,t_n$ are L-terms; now choose $\varphi(x,a)$ such that $|P^M(x)-\varphi^M(x,a)|\le\epsilon$ for all $x\in M^n$, so $\varphi(t_1(y),\dots,t_n(y),a)$ is the desired formula. Proceeding by induction, connectives are handled as usual and the case of quantifiers is also immediate. □

Proposition 6.12 *If M is pseudofinite, then (M,P) is almost pseudofinite.*

Proof Suppose that σ is an L_P-sentence such that $\sigma^{(M,P)}=0$. Given $\epsilon>0$, let $\varphi(x,a)$ be such that $(M,a)\models\sigma_\varphi\le\frac{\epsilon}{2}$. Thus, $M\models\inf_y\sigma_\varphi(y)\le\frac{\epsilon}{2}$. Assuming that M is pseudofinite, we have that $A\models\inf_y\sigma_\varphi(y)\le\epsilon$ for some finite L-structure A. Let $b\in A$ be such that $(A,b)\models\sigma_\varphi(b)\le\epsilon$, then make A into an almost L_P-structure by interpreting $P(x)$ as $\varphi(x,b)$, so $(A,P^A)\models\sigma\le\epsilon$. □

Appendix: Compact Structures

As mentioned in the Introduction, compact structures have no proper ultrapowers; indeed, since any sequence from a compact space has a unique ultralimit, the diagonal embedding $M\to M^{\mathcal{U}}$ of a compact structure M into any of its ultrapowers is surjective. From this it follows that M is totally categorical, that is, if $N\equiv M$, then $N\cong M$. To see this, we use the Keisler–Shelah theorem for continuous logic: if $N\equiv M$, then $N^{\mathcal{U}}\cong M^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$, hence $N^{\mathcal{U}}$ is compact. By Łoś's theorem, we see that N is compact, whence we have $N\cong N^{\mathcal{U}}\cong M^{\mathcal{U}}\cong M$.

Here, we give a more elementary proof that compact structures are totally categorical.

Theorem *Suppose that L is a countable signature and that M is a compact L-structure. Then for any L-structure N, if $N\equiv M$, then $N\cong M$.*

Proof Let $N\equiv M$. Without loss of generality, we may assume that N is ω_1-saturated. Indeed, if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, then $N^{\mathcal{U}}\equiv M$ and $N^{\mathcal{U}}$ is ω_1-saturated. If further $N^{\mathcal{U}}\cong M$, then again $N\cong M$ by Łoś's theorem.

For ease of notation, we work with one unary predicate symbol P and one unary function symbol F. The proof below extends immediately to finite languages. For a countably infinite language, one needs to replace single conditions by partial types.

For $m\ge 1$, let $\{a_1^m,\dots,a_{n(m)}^m\}$ be a finite $\frac{1}{m}$-net for M. Then for each $i\in\{1,\dots,n(m)\}$, let $r(i,m):=P^M(a_i^m)$, and fix $j(i,m)\in\{1,\dots,n(m)\}$ such that

$$d\big(F(a_i^m),a_{j(i,m)}^m\big)\le\frac{1}{m}.$$

For $i,j\in\{1,\dots,n(m)\}$, set $s(i,j,m):=d(a_i^m,a_j^m)$. Consider the following formulas:

$$\psi_m(x_1,\dots,x_{n(m)}):=\sup_x\Big(\min_{1\le i\le n(m)}\Big(d(x,x_i)\dot{-}\frac{1}{m}\Big)\Big),$$

$$\chi_m(x_1,\dots,x_{n(m)})$$

$$:=\max_{1\le i\le n(m)}\Big(\max\Big(\big|P(x_i)-r(i,m)\big|,d\big(F(x_i),x_{j(i,m)}\big)\dot{-}\frac{1}{m}\Big)\Big),$$

$$\tau_m(x_1,\dots,x_{n(m)}):=\max_{1\le i,j\le n(m)}\big|d(x_i,x_j)-s(i,j,m)\big|,$$

$$\varphi_m(x_1,\dots,x_{n(m)}):=\max(\psi_m,\chi_m,\tau_m).$$

Since $N \equiv M$ and N is ω_1-saturated, we have that there exists, for each $m \geq 1$, $b_1^m, \ldots, b_{n(m)}^m \in N$ such that $\varphi_m^N(b_1^m, \ldots, b_{n(m)}^m) = 0$. Set $A_m := \{a_1^m, \ldots, a_{n(m)}^m\}$ and $B_m := \{b_1^m, \ldots, b_{n(m)}^m\}$. It remains to observe that $A := \bigcup_m A_m$ is dense in M and that $B := \bigcup_m B_m$ is dense in N. $\square$

Our proof shows that, unlike finite structures in finite languages in classical logic, compact structures in a finite language are not finitely axiomatizable, but rather, countably axiomatizable.

References

[1] Ax, J., "The elementary theory of finite fields," *Annals of Mathematics (2)*, vol. 88 (1968), pp. 239–71. Zbl 0195.05701. MR 0229613. 493

[2] Ben Yaacov, I., "On theories of random variables," *Israel Journal of Mathematics*, vol. 194 (2013), pp. 957–1012. MR 3047098. DOI 10.1007/s11856-012-0155-4. 502

[3] Ben Yaacov, I., A. Berenstein, C. W. Henson, and A. Usvyatsov, "Model theory for metric structures," pp. 315–427 in *Model Theory with Applications to Algebra and Analysis, Vol. 2*, vol. 350 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 2008. MR 2436146. DOI 10.1017/CBO9780511735219.011. 493, 494, 495, 497, 498, 500, 501

[4] Ben Yaacov, I., and H. J. Keisler, "Randomizations of models as metric structures," *Confluentes Mathematici*, vol. 1 (2009), pp. 197–223. MR 2561997. DOI 10.1142/S1793744209000080. 501, 502

[5] Ben Yaacov, I., and A. Usvyatsov, "Continuous first order logic and local stability," *Transactions of the American Mathematical Society*, vol. 362 (2010), pp. 5213–59. Zbl 1200.03024. MR 2657678. DOI 10.1090/S0002-9947-10-04837-3. 493, 494

[6] Ben Yaacov, I., A. Usvyatsov, and M. Zadka, "Generic automorphism of a Hilbert space," preprint, http://ptmat.fc.ul.pt/~alexus/papers.html (accessed May 26, 2015). 500

[7] Berenstein, A., and A. Villaveces, "Hilbert spaces with generic predicates," preprint, arXiv:0704.1633v1 [math.LO]. 500

[8] Carlisle, S., "Model theory of real-trees and their isometries," Ph.D. dissertation, University of Illinois at Urbana-Champaign, 2009. MR 2714159. 502

[9] Chang, C. C., and H. J. Keisler, *Model Theory*, 3rd ed., vol. 73 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1990. MR 1059055. 503

[10] Ealy, C., and I. Goldbring, "Thorn-Forking in continuous logic," *Journal of Symbolic Logic*, vol. 77 (2012), pp. 63–93. Zbl 1257.03059. MR 2951630. DOI 10.2178/jsl/1327068692. 502

[11] Farah, I., B. Hart, and D. Sherman, "Model theory of operator algebras, III: Elementary equivalence and II_1 factors," *Bulletin of the London Mathematical Society*, vol. 46 (2014), pp. 609–28. MR 3210717. DOI 10.1112/blms/bdu012. 493

[12] Felgner, U., "Pseudo-endliche Gruppen," *Jbuch. Kurt Gödel Ges.* vol. 1990 (1991), pp. 94–108. MR 1143712. 493

[13] Henson, C. W., and J. Iovino, "Ultraproducts in analysis," pp. 1–110 in *Analysis and Logic (Mons, 1997)*, vol. 262 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 2002. MR 1967834. 495

[14] Hrushovski, E., "Pseudo-finite fields and related structures," pp. 151–212 in *Model Theory and Applications*, vol. 11 of *Quaderni di Matematica*, Aracne, Rome, 2002. MR 2159717. 493

[15] Moore, L. C., Jr., "Approximately finite-dimensional Banach spaces," *Journal of Functional Analysis*, vol. 42 (1981), pp. 1–11. MR 0620577. DOI 10.1016/0022-1236(81)90045-8. 493

[16] Usvyatsov, A., "Generic separable metric structures," *Topology and Its Applications*, vol. 155 (2008), pp. 1607–1617. MR 2435152. DOI 10.1016/j.topol.2007.08.023. 502

[17] Väänänen, J., "Pseudo-finite model theory," *Matemática Contemporânea*, vol. 24 (2003), pp. 169–83. MR 2007169. 493

Acknowledgment

Goldbring's work was partially supported by National Science Foundation grant DMS-1007144.

Goldbring
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60607-7045
USA
isaac@math.uic.edu
http://homepages.math.uic.edu/~isaac

Lopes
Centro de Matemática, Computação e Cognição
Universidade Federal do ABC
Santo André, São Paulo 09210-580
Brazil
vinicius@ufabc.edu.br
http://hostel.ufabc.edu.br/~vinicius