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Remarks on Structure Theorems for ω_1-Saturated Models

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Abstract We give a characterization for those stable theories whose ω_1-saturated models have a “Shelah-style” structure theorem. We use this characterization to prove that if a theory is countable, stable, and 1-based without dop or didip, then its ω_1-saturated models have a structure theorem. Prior to us, this is proved in a paper of Hart, Pillay, and Starchenko (in which they also count the number of models, which we do not do here). Some other remarks are also included.

In this paper we will assume that T is a complete countable stable theory. In order to simplify the notation we use the monster model $\mathbf{M}$, and by a model we mean an elementary submodel of $\mathbf{M}$. So if $A \subseteq B$ are models then $A \prec B$.

We write $\mathcal{A}$, $\mathcal{B}$, and so on for ω_1-saturated (elementary sub-) models (of $\mathbf{M}$) and A, B, and so on for subsets of $\mathbf{M}$. By a, b, and so on we mean finite sequences of elements of $\mathbf{M}$.

We write ω_1-prime for $F^s_{\omega_1}$-prime and $\mathcal{A}[A]$ for ω_1-prime model over $\mathcal{A} \cup A$. By $A \downarrow_B C$ we mean $t(A, C \cup B)$ does not fork over B, by $A \triangleright_B C$ we mean A dominates C over B, and by $t(a, A) \dashv B$ we mean $t(a, A)$ is orthogonal to B.

Let P be a tree without branches of length $> \omega$. Then by t^- we mean the immediate predecessor of t if $t \in P$ is not the root.

Definition 1 (Shelah [4]) We say that (P, f, g) is a decomposition of $\mathcal{A}$ if the following hold.

1. P is a tree without branches of length $> \omega$, $f : (P - \{r\}) \to \mathcal{A}$ and $g : P \to P(\mathcal{A})$, where $r \in P$ is the root of P and $P(\mathcal{A})$ is the power set of $\mathcal{A}$;
2. $g(r)$ is an ω_1-prime model (over $\emptyset$);
3. if t is not the root and $u^- = t$ then $t(f(u), g(t)) \dashv g(t^-)$;
4. for all $t \in P$, $\{f(u) \mid u^- = t\}$ is a maximal independent sequence over $g(t)$ of elements of $\mathcal{A}$ satisfying (3);

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5. if $t = u^-$, then $g(u) = g(t)[f(u)]$.

In order to simplify the notation, we write a_t for $f(t)$, and $\mathcal{A}_t$ for $g(t)$ and say that $\mathcal{A}$ is ω_1-prime over a decomposition (S, f, g) (of some $\mathcal{B}$) if $\mathcal{A}$ is ω_1-prime over $\bigcup\{\mathcal{A}_t | \; t \in P\}$.

The following basic property of a decomposition is frequently used in this paper (for proof, see the proof of XVII 1.6 Claim 2 in Baldwin [1] and add an easy induction). If $S \neq \emptyset$ is a downward closed subset of P, $t \in P$ and $t^{-S} = \{t' \in P | \; t' > t \;\&\; \forall\, t < p \leq t',\; p \notin S\}$, then $\cup\{A_{t'} | \; t' \in t^{-S}\} \downarrow_{\mathcal{A}_t} \cup\{\mathcal{A}_s | \; s \in S\}$.

Definition 2

1. We say that T has the structure property (SP) if the following holds: For all $\mathcal{A}$, if (P, f, g) is a decomposion of $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$ is ω_1-prime over (P, f, g) then $\mathcal{B} = \mathcal{A}$.
2. We say that T has the weak structure property (wSP) if the following holds: For all $\mathcal{A}$, if (P, f, g) is a decomposion of $\mathcal{A}$ then $\mathcal{A}$ is ω_1-prime over (P, f, g).
3. We say that $t(a, \mathcal{A})$, $\mathcal{A}$ ω_1-saturated, is a c-type if for all ω_1-saturated $\mathcal{B} \subseteq \mathcal{A}$ the following holds: if $t(a, \mathcal{A}) \not\dashv \mathcal{B}$ then there is $b \notin \mathcal{B}$ such that $b \downarrow_{\mathcal{B}} \mathcal{A}$ and $a \rhd_{\mathcal{A}} b$.
4. We say that T has the compulsion property (CP) if for all ω_1-saturated $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$, there is $a \in \mathcal{B} - \mathcal{A}$ such that $t(a, \mathcal{A})$ is a c-type.
5. We say that T has the strong compulsion property (sCP) if every nonalgebraic type over an ω_1-saturated model is a c-type.

Clearly SP implies wSP, and sCP implies CP. We will show in fact that all these are equivalent, assuming T has ndop and ndidip.

Lemma 3 wSP *implies* sCP.

Proof: Let $\mathcal{A}$ be ω_1-saturated, we show that $t(a, \mathcal{A})$ is a c-type. Let $\mathcal{B} \subseteq \mathcal{A}$ be ω_1-saturated and $t(a, \mathcal{A}) \not\dashv \mathcal{B}$. Without loss of generality we may assume that $\mathcal{B}$ is ω_1-prime. By $t(a, \mathcal{A}) \not\dashv \mathcal{B}$, we get b such that $b \downarrow_{\mathcal{B}} \mathcal{A}$ and $b \not\downarrow_{\mathcal{A}} a$.

We can choose a decomposition (P, f, g) for $\mathcal{A}$ so that $g(r) = \mathcal{B}$, where r is the root of P. Let $C = \mathcal{A}[a]$ and choose a decomposition (P', f', g') for C extending (P, f, g). Clearly if there is $t \in P' - P$ such that $t^- = r$, we have proved the lemma. So for a contradiction, we assume that for all $t \in P' - P$, $t^- \neq r$.

1. If $P' = P$, then $\{\mathcal{A}_t | \; t \in P\} \rhd_{\mathcal{B}} C$, which implies $b \downarrow_{\mathcal{B}} C$, a contradiction.
2. Assume $P' \neq P$.

Claim: $t(\cup\{\mathcal{A}_t | \; t \in P' - P\}, \mathcal{A}) \dashv \mathcal{B}$.

Proof of Claim: We show that for all P^* as P' above $t(\cup\{\mathcal{A}_t | \; t \in P^* - P\}, \mathcal{A}) \dashv \mathcal{B}$. Clearly it is enough to prove this for all P^* such that $|P^* - P|$ is finite. We prove this by induction on $n = |P^* - P|$.

$n = 1$: Let $P^* - P = \{t_0\}$. Because $t(\mathcal{A}_{t_0}, \mathcal{A}_{t_0^-}) \dashv \mathcal{B}$, $\cup\{\mathcal{A}_t | \; t \in P\} \rhd_{\mathcal{A}_{t_0^-}} \mathcal{A}$ and $\mathcal{A}_{t_0} \downarrow_{\mathcal{A}_{t_0^-}} \cup\{\mathcal{A}_t | \; t \in P\}$ we get the claim immediately.

$n = m + 1$: Let $P^* = P \cup \{t_0, \ldots, t_m\}$ and $P^- = P \cup \{t_0, \ldots, t_{m-1}\}$. We may assume t_m is a leaf of P^*. Let d be such that $d \downarrow_{\mathcal{B}} \mathcal{A}$. It is enough to show $d \downarrow_{\mathcal{A}} \cup_{i<n} \mathcal{A}_{t_i}$.

Choose $\mathcal{D}$ so that $\mathcal{A} \subseteq \mathcal{D}$ and $\mathcal{D}$ is ω_1-prime over $\cup\{A_t|\ t \in P^-\}$. By induction assumption $d \downarrow_{\mathcal{A}} \cup\{\mathcal{A}_t|\ t \in P^- - P\}$ and because $d \downarrow_{\mathcal{B}} \mathcal{A}$ we get $d \downarrow_{\mathcal{B}} \cup\{\mathcal{A}_t|\ t \in P^-\}$. So $d \downarrow_{\mathcal{B}} \mathcal{D}$. Because $t(\mathcal{A}_{t_m}, \mathcal{A}_{t_m^-}) \dashv \mathcal{B}$, $\cup\{\mathcal{A}_t|\ t \in P^-\} \triangleright_{\mathcal{A}_{t_m^-}} \mathcal{D}$ and $\mathcal{A}_{t_m} \downarrow_{\mathcal{A}_{t_m^-}} \cup\{\mathcal{A}_t|\ t \in P^-\}$ we get $t(\mathcal{A}_{t_m}, \mathcal{D}) \dashv \mathcal{B}$. So $d \downarrow_{\mathcal{D}} \mathcal{A}_{t_m}$ which implies $d \downarrow_{\mathcal{A}} \cup_{i<n}\mathcal{A}_{t_i}$. Hence the Claim.

By the claim $b \downarrow_{\mathcal{B}} \cup\{\mathcal{A}_t|\ t \in P'\}$, and because $\cup\{\mathcal{A}_t|\ t \in P'\} \triangleright_{\mathcal{B}} C$ we get $b \downarrow_{\mathcal{B}} C$, a contradiction. This completes the proof of the lemma.

We say that T has didip if there are $\mathcal{A}_i$, $i \leq \omega$, ω_1-saturated and nonalgebraic $p \in S(\mathcal{A}_\omega)$ such that

1. for all $i < j \leq \omega$, $\mathcal{A}_i \subseteq \mathcal{A}_j$,
2. $\mathcal{A}_\omega$ is ω_1-prime over $\cup_{i<\omega}\mathcal{A}_i$,
3. for all $i < \omega$, $p \dashv \mathcal{A}_i$.

If T does not have didip we say it has ndidip. We write $p \dashv^a B$ for p almost orhogonal to B.

Lemma 4 *If T has* didip, *then there are $\mathcal{A}_i$, $i \leq \omega$, ω_1-saturated such that*

1. *for all $i < j \leq \omega$, $\mathcal{A}_i \subseteq \mathcal{A}_j$,*
2. *$\mathcal{A}_\omega$ is ω_1-prime over $\cup_{i<\omega}\mathcal{A}_i$,*
3. *$\mathcal{A}_\omega$ is not $F^s_{\omega_1}$-minimal over $\cup_{i<\omega}\mathcal{A}_i$ (see [4]).*

Proof: This goes essentially as Lemma X 2.2 in [4]. Let $\mathcal{A}_i$, $i \leq \omega$, and $p \in \mathcal{A}_\omega$ be as in the definition of didip. We show that (3) above holds. By [4], Theorem IV 4.21, it is enough to show that in $\mathcal{A}_\omega$ there is an infinite indiscernible sequence over $\cup_{i<\omega}\mathcal{A}_i$. For this it is enough to show that for all countable $A \subseteq \mathcal{A}_\omega$, $p \restriction (A \cup \bigcup_{i<\omega} \mathcal{A}_i)$ is satisfied in $\mathcal{A}_\omega$. Without loss of generality we may assume that,

(α) p does not fork over A and A a model,

(β) for all $i \leq \omega$, $A \cap \mathcal{A}_i$ is a model,

(γ) for all $i \leq \omega$, $A \downarrow_{A \cap \mathcal{A}_i} \mathcal{A}_i$.

Then

(*) $p \restriction A \dashv^a A \cap \mathcal{A}_i$,

for all $i < \omega$. Let $a \in \mathcal{A}_\omega$ satisfy $p \restriction A$. It is enough to show that $a \downarrow_A \cup_{i<\omega}\mathcal{A}_i$. If not, choose $i < \omega$ and $c \in \mathcal{A}_i$ such that $a \not\downarrow_A c$. By (γ) above $c \downarrow_{A\cap\mathcal{A}_i} A$, which contradicts (*) above.

Definition 5 Let (P, f, g) be a decomposition of $\mathcal{A}$. We say that $(\mathcal{A}_i)_{i\leq\alpha}$ is a generating sequence, if there is an enumeration of P, $P = \{t_i|\ i < \alpha\}$, such that if $t_i < t_j$ then $i < j$ and the following hold.

1. For all $i < \alpha$, $\mathcal{A}_i \subseteq \mathcal{A}$;
2. $\mathcal{A}_0 = \mathcal{A}_{t_0}$ (t_0 must be the root of P);
3. $\mathcal{A}_{i+1} = \mathcal{A}_i[\mathcal{A}_{t_i}]$;
4. if $i \leq \alpha$ is limit then $\mathcal{A}_i$ is ω_1-prime over $\bigcup_{j<i} \mathcal{A}_j$.

Lemma 6 *Let (P, f, g) be a decomposition of $\mathcal{A}$, $(\mathcal{A}_i)_{i\le\alpha}$ a generating sequence, $i^* < \alpha$ and $P_{i^*} = \{t_i |\ i < i^*\}$. Assume $t_0, \dots, t_{n-1} \in P_{i^*}$ are distinct and a_m, $m < n$, are finite sequences from $\bigcup\{A_t |\ t \in t_m^{-P_{i^*}}\}$, where $t_m^{-P_{i^*}} = \{t \in P |\ t > t_m \;\&\; \forall\, t_m < p \le t,\ p \notin P_{i^*}\}$. Then for all $m < n$,*

$$(*) \quad a_m \downarrow_{\mathcal{A}_{t_m}} \mathcal{A}_{i^*} \cup \bigcup\{a_k |\ k < n,\ k \ne m\}.$$

Proof: We prove this by induction on i^*.

$i^* = 1$: Trivial.

$i^* = j + 1$: Let t_m and a_m, $m < n$, be as in the claim. For a contradiction assume (*) is not true.

Claim: $a_m \downarrow_{\mathcal{A}_{t_m}} \mathcal{A}_j \cup \mathcal{A}_{t_j} \cup \bigcup\{a_k |\ k < n,\ k \ne m\}$.

Proof of Claim: Without loss of generality we may assume that $p, r < n$ are such that $t_p = t_j$ and $t_r = t_j^-$. There are three cases.

$m \ne p, r$: By the induction assumption for all sequences a from $\mathcal{A}_{t_j}$, $a_m \downarrow_{\mathcal{A}_{t_m}} \mathcal{A}_j \cup a \cup \bigcup\{a_k |\ k < n,\ k \ne m\}$. So $a_m \downarrow_{\mathcal{A}_{t_m}} \mathcal{A}_j \cup \mathcal{A}_{t_j} \cup \bigcup\{a_k |\ k < n,\ k \ne m\}$.

$m = p$: By the induction assumption for all sequences a from $\mathcal{A}_{t_j}$, $a_m \cup a_r \cup a \downarrow_{\mathcal{A}_{t_r}} \mathcal{A}_j \cup \bigcup\{a_k |\ k < n,\ k \ne m, r\}$. So $a_m \cup a_r \cup \mathcal{A}_{t_j} \downarrow_{\mathcal{A}_{t_r}} \mathcal{A}_j \cup \bigcup\{a_k |\ k < n,\ k \ne m, r\}$ which implies $a_m \downarrow_{\mathcal{A}_{t_m} \cup a_r} \mathcal{A}_j \cup \bigcup\{a_k |\ k < n,\ k \ne m, r\}$. Because $a_m \downarrow_{\mathcal{A}_{t_m}} a_r$ we get the claim.

$m = r$: Essentially symmetric with the case $m = p$.

By the claim there are finite $c \in \mathcal{A}_{j+1} - (\mathcal{A}_j \cup \mathcal{A}_{t_j})$ and countable $C \subseteq \mathcal{A}_j \cup \mathcal{A}_{t_j}$ such that,

1. $t(c, C) \vdash t(c, \mathcal{A}_j \cup \mathcal{A}_{t_j})$,
2. $(a_m)_{m<n} \not\downarrow_C c$.

Let $P' = \{t \in P_{j+1} |\ \exists m < n,\ t \le t_m\}$. Choose countable $D \subseteq \bigcup_{t\in P'} \mathcal{A}_t$ so that,

a. for all $m < n$, $D_{t_m} = D \cap \mathcal{A}_{t_m}$ is a model,
b. for all $m < n$, $a_m \downarrow_{D_{t_m}} C \cup D \cup \bigcup\{a_k |\ k < m\}$.

The existence of such D follows from the claim and the finite character of forking. Finaly we choose $a'_m \in \mathcal{A}_{t_m}$ so that for all $m < n$,

i. $t(a'_m, D_{t_m}) = t(a_m, D_{t_m})$,
ii. $a'_m \downarrow_{D_{t_m}} C \cup D \cup \bigcup\{a'_k |\ k < m\}$.

Such a'_m exists because $C \cup D \cup \bigcup\{a'_k |\ k < m\}$ is countable and in $\mathcal{A}_{t_m}$ there is an uncountable D_{t_m}-independent set of sequences a'_m satisfying (i) above.

Then by induction on $0 < m \le n$ we can see that

$$t((a'_k)_{k<m}, C \cup D) = t((a_k)_{k<m}, C \cup D).$$

Especially $t((a'_m)_{m<n}, C) = t((a_m)_{m<n}, C)$ which contradicts (1) and (2) above.

i^* is limit: Let t_m and a_m, $m < n$, be as in the claim and $\mathcal{A}'_{i^*} = \bigcup_{i<i^*} \mathcal{A}_i$. Then by induction assumption and finite character of forking for all $m < n$

$$a_m \downarrow_{\mathcal{A}_{t_m}} \mathcal{A}'_{i^*} \cup \bigcup\{a_k |\ k < n,\ k \ne m\}.$$

So we can prove the claim as in the successor case. This completes the proof of the lemma.

In the proof of Lemma 8 the following deduction is frequently needed. So although trivial we state it as lemma.

Lemma 7 *Assume $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ are ω_1-saturated and a and b are such that $a \downarrow_{\mathcal{B}} \mathcal{C}$, $b \downarrow_{\mathcal{A}} \mathcal{B}$ and $a \triangleright_{\mathcal{B}} b$. Then there is c such that $t(c, \mathcal{B}) = t(b, \mathcal{B})$, $c \downarrow_{\mathcal{A}} \mathcal{C}$ and $a \triangleright_{\mathcal{C}} c$.*

Proof: Just choose c so that $t(c, \mathcal{B} \cup a) = t(b, \mathcal{B} \cup a)$ and $c \downarrow_{\mathcal{B} \cup a} \mathcal{C}$.

Lemma 8 *Assume T has* ndop *and* ndidip. *Then* CP *implies* SP.

Proof: Let $\mathcal{A}$ be ω_1-saturated and (P, f, g) a decomposition of $\mathcal{A}$. Let $(\mathcal{A}_i)_{i \leq \alpha}$ be any generating sequence and let $P = \{t_i | \, i < \alpha\}$ be the enumeration of P given by the definition of generating sequence.

Claim: $\mathcal{A}_\alpha = \mathcal{A}$.

Proof of Claim: Assume not. For all $a \in \mathcal{A} - \mathcal{A}_\alpha$ let i_a be the least ordinal such that $t(a, \mathcal{A}_\alpha) \not\dashv \mathcal{A}_{i_a}$. Choose $a \in \mathcal{A} - \mathcal{A}_\alpha$ so that,

i. for some $l \leq \alpha$ either $t(a, \mathcal{A}_l)$ is a c-type and $a \downarrow_{\mathcal{A}_l} \mathcal{A}_\alpha$ or $t(a, \mathcal{A}_{t_l})$ is a c-type and $a \downarrow_{\mathcal{A}_{t_l}} \mathcal{A}_\alpha$; and
ii. among these a, $i = i_a$ is the least.

By CP such an a exists.

There are two cases: assume first that for some $l < \alpha$ $t(a, \mathcal{A}_{t_l})$ is a c-type and $a \downarrow_{\mathcal{A}_{t_l}} \mathcal{A}_\alpha$. Let $t \leq t_l$ be the least t such that $t(a, \mathcal{A}_{t_l}) \not\dashv \mathcal{A}_t$. Since $t(a, \mathcal{A}_{t_l})$ is a c-type choose b so that

1. $b \downarrow_{\mathcal{A}_t} \mathcal{A}_{t_l}$, and
2. $a \triangleright_{\mathcal{A}_{t_l}} b$.

Then if t^- exists, by (2) $t(b, \mathcal{A}_{t_l}) \dashv \mathcal{A}_{t^-}$ and so by (1), $t(b, \mathcal{A}_t) \dashv \mathcal{A}_{t^-}$.

By Lemma 7 we may choose $b' \in \mathcal{A} - \mathcal{A}_\alpha$ so that $b' \downarrow_{\mathcal{A}_t} \mathcal{A}_\alpha$ and $t(b', \mathcal{A}_l) = t(b, \mathcal{A}_l)$, which contradicts Definition 1.4.

So we may assume that $l \leq \alpha$ is such that $t(a, \mathcal{A}_l)$ is a c-type and $a \downarrow_{\mathcal{A}_l} \mathcal{A}_\alpha$. Let b' be the element given by $t(a, \mathcal{A}_l)$ being a c-type: $b' \downarrow_{\mathcal{A}_i} \mathcal{A}_l$ and $a \triangleright_{\mathcal{A}_l} b'$. By CP we may choose b'' so that $b' \triangleright_{\mathcal{A}_i} b''$ and $t(b'', \mathcal{A}_i)$ is a c-type. By using Lemma 7 twice we find $b \in \mathcal{A} - \mathcal{A}_\alpha$ such that $b \downarrow_{\mathcal{A}_i} \mathcal{A}_\alpha$ and $t(b, \mathcal{A}_i)$ is a c-type.

1. i is not a limit with $cf(i) = \omega$. This is because otherwise by ndidip $t(b, \mathcal{A}_i) \not\dashv \mathcal{A}_j$ for some $j < i$ and so $t(b, \mathcal{A}_\alpha) \not\dashv \mathcal{A}_j$, which contradicts the choice of a.
2. i is not a limit with $cf(i) > \omega$. We see this as above, use $\kappa(T) \leq \omega_1$ instead of ndidip.
3. i is not a successor. Assume it is, $i = j + 1$. Then $\mathcal{A}_i$ is ω_1-prime over $\mathcal{A}_j \cup \mathcal{A}_{t_j}$ and by Lemma 6 $\mathcal{A}_j \downarrow_{\mathcal{A}_{t_j^-}} \mathcal{A}_{t_j}$. By the choice of a, $t(b, \mathcal{A}_i) \dashv \mathcal{A}_j$. So by ndop $t(b, \mathcal{A}_i) \not\dashv \mathcal{A}_{t_j}$. Because $t(b, \mathcal{A}_i)$ is a c-type we choose c' so that $c' \downarrow_{\mathcal{A}_{t_j}} \mathcal{A}_i$ and $b \triangleright_{\mathcal{A}_i} c'$. By Lemma 7 we can find $c \in \mathcal{A} - \mathcal{A}_\alpha$ such that $t(c, \mathcal{A}_i) = t(c', \mathcal{A}_i)$ and $c \downarrow_{\mathcal{A}_i} \mathcal{A}_\alpha$. If t_j is not the root, then by the choice of a, $t(c, \mathcal{A}_\alpha) \dashv \mathcal{A}_j$ and since $\mathcal{A}_{t_j^-} \subseteq \mathcal{A}_j$, $c \downarrow_{\mathcal{A}_{t_j}} \mathcal{A}_\alpha$ and $\mathcal{A}_{t_j}$ is ω_1-saturated, we get $t(c, \mathcal{A}_{t_j}) \dashv \mathcal{A}_{t_j^-}$. But this contradicts Definition 1.4.

By (1), (2), and (3), $i = 0$, which, together with the fact that $t(a, \mathcal{A}_l)$ is a c-type and Lemma 7, contradicts Definition 1.4. Hence the Claim.

Let $\mathcal{B} \subseteq \mathcal{A}$ be ω_1-prime over $\bigcup\{\mathcal{A}_t | \ t \in P\}$. We want to show that $\mathcal{B} = \mathcal{A}$. For this we choose a generating sequence $(\mathcal{A}_i)_{i \leq \alpha}$, so that $\mathcal{A}_i \subseteq \mathcal{B}$ for all $i \leq \alpha$. By the claim above $\mathcal{A}_\alpha = \mathcal{A}$ and so $\mathcal{B} = \mathcal{A}$. This completes the proof of the lemma.

Corollary 9 *Assume T has* ndop *and* ndidip*. Then SP,* wSP*,* CP *and* sCP *are equivalent.*

We want to point out that this is not a main gap for ω_1-saturated models. For main gap we should prove a proper nonstructure theorem for theories without CP.

Notice that, instead of choosing i in the proof of Lemma 8 the way we did, the author could not choose $t \in P$ so that $t(a, \mathcal{B}) \not\dashv \mathcal{A}_t$, because he does not know how to prove the existence of such t from ndop and ndidip alone. (Of course, SP implies the existence of such t.) This is also the reason why the author thinks that it is not completely trivial that wSP implies SP (under ndop and ndidip).

Lemma 10 *(i) If T has* dop *then it does not have* SP*. (ii) If T has* didip *then it does not have* SP.

Proof: Since these are similar, we prove only (i).

For a contradiction we assume T has SP. Let ω_1-saturated $\mathcal{A}_i$, $i < 4$, and nonalgebraic $p \in S(\mathcal{A}_3)$ be as in the definition of dop, i.e.,

i. $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,
ii. $\mathcal{A}_3$ ω_1-prime over $\mathcal{A}_1 \cup \mathcal{A}_2$,
iii. $p \dashv \mathcal{A}_i$ for $i \in \{1, 2\}$.

Without loss of generality we may assume $\mathcal{A}_0$ is ω_1-prime over $\emptyset$. Let a realize p and $\mathcal{B} = \mathcal{A}_3[a]$. Let $(P_1, \ f_1, g_1)$ and $(P_2, \ f_2, g_2)$ be decompositions of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We may assume $P_1 \cap P_2 = \{r\}$, where r is the root of both P_1 and P_2 and $g_1(r) = g_2(r) = \mathcal{A}_0$.

Claim: $P_1 \cup P_2$ is a decomposition of $\mathcal{A}_3$.

Proof of Claim: By (i) above, $P_1 \cup P_2$ satisfies (1)–(3) and (5) in the definition of decomposition. In order to show (4) it is enough to prove that $\mathcal{A}_3$ is ω_1-prime over $P_1 \cup P_2$.

Let $\mathcal{A}$ be ω_1-prime over $P_1 \cup P_2$. Then $P_1 \cup P_2$ is a decomposition of $\mathcal{A}$. By (i) and (ii) above we may assume that $\mathcal{A}_3 \subseteq \mathcal{A}$. On the other hand we can choose $\mathcal{A}' \subseteq \mathcal{A}_3$ which is ω_1-prime over $P_1 \cup P_2$. By SP $\mathcal{A}' = \mathcal{A}$. Hence the Claim.

Now $t(a, \mathcal{A}_3) \dashv g_1(t)$ for all $t \in P_1$ and $t(a, \mathcal{A}_3) \dashv g_2(t)$ for all $t \in P_2$, which implies that $P_1 \cup P_2$ is a decomposition of $\mathcal{B}$, a contradiction with SP.

Lemma 11 *If T^{eq} is 1-based then T^{eq} has* sCP.

Proof: The following property of 1-based theories is used in this proof: in M^{eq},

$$A \downarrow_{acl(A) \cap acl(B)} B$$

for all A and B (see Bouscaren and Hrushovski [2]).

Let $\mathcal{B} \subseteq \mathcal{A}$ be ω_1-saturated models and $t(a, \mathcal{A}) \not\dashv \mathcal{B}$. Choose c so that $c \downarrow_{\mathcal{B}} \mathcal{A}$ and $c \not\downarrow_{\mathcal{A}} a$. Let $C = \mathcal{B}[c]$ and $\mathcal{D} = \mathcal{A}[a]$. Because T is one-based and $c \not\downarrow_{\mathcal{A}} a$, $C \cap \mathcal{D} \not\subseteq \mathcal{B}$. Choose $b \in (C \cap \mathcal{D}) - \mathcal{B}$. Then $a \triangleright_{\mathcal{A}} b$ and because $c \triangleright_{\mathcal{B}} b$, $b \downarrow_{\mathcal{B}} \mathcal{A}$.

Corollary 12 ([3]) *Assume T^{eq} is 1-based. Then T^{eq} has* SP *iff it has* ndop *and* ndidip.

Proof: $\Leftarrow$: Lemma 11 and Corollary 9.
$\Rightarrow$: Lemma 10.

Notice that in Hart, Pillay, and Starchenko [3] the structure theorem is stronger than the one needed for SP. Shelah [4] proved SP for countable superstable theories without dop by using regular types more or less the way we use c-types. We notice the following.

Lemma 13 *If for all ω_1-saturated models $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$, there is $a \in \mathcal{B} - \mathcal{A}$ such that $t(a, \mathcal{A})$ has weight one, then T has* CP.

Proof: It is enough to show that under the assumptions of the lemma every weight one type over ω_1-saturated model is a c-type. So assume $\mathcal{B} \subseteq \mathcal{A}$ are ω_1-saturated, the weight of $t(a, \mathcal{A})$ is one, and $t(a, \mathcal{A}) \not\dashv \mathcal{B}$. Choose b so that $b \downarrow_{\mathcal{B}} \mathcal{A}$ and $b \not\downarrow_{\mathcal{A}} a$. Choose $\overline{c} = (c_i)_{i<\alpha}$, $\alpha \leq \omega$, a maximal independent sequence over $\mathcal{B}$ such that for all $i < \alpha$ the weight of $t(c_i, \mathcal{B})$ is one and $b \not\downarrow_{\mathcal{B}} c_i$. The assumption of the lemma implies that $\alpha > 0$. Choose $\overline{d} = (d_i)_{i<\alpha}$ so that $t(\overline{d} \frown b, \mathcal{B}) = t(\overline{c} \frown b, \mathcal{B})$ and $\overline{d} \downarrow_{\mathcal{B} \cup b} \mathcal{A}$.

Because $\mathcal{B}$ is ω_1-saturated, $\overline{d}$ is a maximal independent sequence over $\mathcal{A}$ such that for all $i < \alpha$ the weight of $t(d_i, \mathcal{B})$ is one and $b \not\downarrow_{\mathcal{B}} d_i$. So $a \not\downarrow_{\mathcal{A}} \overline{d}$. Choose $j \leq \alpha$ least such that $a \not\downarrow_{\mathcal{A}} (d_i)_{i \leq j}$. Let $A \subseteq \mathcal{A}$ be a countable model so that $a \cup \overline{d} \downarrow_A \mathcal{A}$. Let $(e_i)_{i<j}$ be chosen from $\mathcal{A}$ so that $t((e_i)_{i<j}, A) = t((d_i)_{i<j}, A)$. Finaly let b be such that $t(b \frown (e_i)_{i<j} \frown a, A) = t(d_j \frown (d_i)_{i<j} \frown a, A)$ and $b \downarrow_{A \cup a \cup (e_i)_{i<j}} \mathcal{A}$.

Then $b \downarrow_A \mathcal{A}$ and so $t(b, \mathcal{A}) = t(d_j, \mathcal{A})$, especially $b \downarrow_{\mathcal{B}} \mathcal{A}$ and the weight of $t(b, \mathcal{A})$ is one. Also $a \not\downarrow_{\mathcal{A}} b$ which implies $a \triangleright_{\mathcal{A}} b$.

If in Lemma 13 we replace "weight one" by "regular," then the lemma is already proved in [4]. That proof would also work in this case, but we gave a bit different proof for a change.

There are examples (unpublished as far as the author knows) which show that the assumption of Lemma 13 need not hold for theories having SP. Nevertheless we prove the following lemma which may in some cases be helpful for proving CP.

Lemma 14 *Assume $\mathcal{A} \subseteq \mathcal{B}$ are ω_1-saturated models such that $\mathcal{B} \neq \mathcal{A}$ and for all $b \in \mathcal{B} - \mathcal{A}$ the weight of $t(b, \mathcal{A})$ is not one. Then for all $b \in \mathcal{B} - \mathcal{A}$ there are a countable model $A \subseteq \mathcal{A}$ and $a_i \in \mathcal{B}$, $i < \omega$, such that $(a_i)_{i<\omega}$ is independent over A, for all $i < \omega$, $b \not\downarrow_A a_i$ and $b \downarrow_A \mathcal{A}$. Especially then the weight of $t(b, \mathcal{A})$ is infinite.*

Proof: So let $b \in \mathcal{B} - \mathcal{A}$. By induction on $n < \omega$ we choose countable models $A_n \prec \mathcal{A}$, $a_i^n \in \mathcal{B}$, $i < n+1$ and $c^n \in \mathcal{B} - \mathcal{A}$ such that,

i. a_i^n, $i < n+1$ together with c^n is independent over A_n;
ii. $b \not\downarrow_{A_n} c^n$ and $b \not\downarrow_{A_n} a_i^n$ for all $i < n+1$;
iii. for all $n < n'$, $t(b \frown a_0^n \frown \ldots \frown a_n^n, A_n) = t(b \frown a_0^{n'} \frown \ldots \frown a_n^{n'}, A_n)$; and
iv. $b \downarrow_{A_0} \mathcal{A}$ and for all $n < n'$, $A_n \subseteq A_{n'}$.

Clearly if we can do this we have proved the lemma.

$n = 0$: Because the weight of $t(b, \mathcal{A})$ is ≥ 2 and $\mathcal{A}$ is ω_1-saturated, we can choose countable model $A_0 \subseteq \mathcal{A}$ such that $b \downarrow_{A_0} \mathcal{A}$ and $t(b, A_0)$ has pre-weight ≥ 2. Then also a_0^0 and c^0 can be found easily.

$n > 0$: Assume we have found these elements for n. We show how to choose them for $n+1$.

Let $g = b \frown c^n \frown a_0^n \frown \ldots \frown a_n^n$. If there is d such that $c^n \not\downarrow_{A_n} d$, $d \downarrow_{A_n \cup c^n} a_0^n \frown \ldots \frown a_n^n$ and $b \downarrow_{A_n} d$ we let D be countable such that $d \subseteq D$, $A_n \cup D$ is a model and $D \downarrow_{A_n \cup d} g$. Then we choose $D' \subset \mathcal{A}$ such that $t(D', A_n) = t(D, A_n)$. Then $t(b \frown D', A_n) = t(b \frown D, A_n)$. Finally we choose c'^n and $a_i'^n$, $i < n+1$, so that $t(c'^n \frown a_0'^n \frown \ldots \frown a_n'^n \frown D', A_n \cup b) = t(c^n \frown a_0^n \frown \ldots \frown a_n^n \frown D, A_n \cup b)$. Now $c^n \downarrow_{A_n} a_0^n \frown \ldots \frown a_n^n$ and $d \downarrow_{A_n \cup c^n} a_0^n \frown \ldots \frown a_n^n$ and so $d \frown c^n \downarrow_{A_n} a_0^n \frown \ldots \frown a_n^n$. This implies easily that a_i^n, $i < n+1$ together with c^n is independent over $A_n \cup d$. Because $g \downarrow_{A_n \cup d} D$ it is easy to see that (i) – (iv) remain true if we replace A_n with $A_n \cup D$ everywhere. And so the same is true for D', b, c'^n and $a_i'^n$, $i < n+1$.

Because $\kappa(T) \leq \omega_1$ we can repeat this at most countably many times; limits are no problem because of the finite character of forking. So we can find A_n', c_n' and $a_i'^n$, $i < n+1$, such that (i) – (iv) remain true and if d is such that $c'^n \not\downarrow_{A_n'} d$ and $a_0'^n \frown \ldots \frown a_n'^n \downarrow_{A_n' \cup c'^n} d$ then $b \not\downarrow_{A_n'} d$.

We want to choose A_{n+1} so that in addition to the above, $t(c'^n, A_{n+1})$ has pre-weight ≥ 2. There are two cases.

Case 1: $c'^n \not\downarrow_{A_n'} \mathcal{A}$: We let $A_{n+1} = A_n$. Then $t(c'^n, A_{n+1})$ has pre-weight ≥ 2 because there is $d \subseteq \mathcal{A}$ such that $c'^n \not\downarrow_{A_{n+1}} d$ and on the other hand $c'^n \not\downarrow_{A_{n+1}} b$ and $b \downarrow_{A_n} d$.

Case 2: $c'^n \downarrow_{A_n'} \mathcal{A}$: Because $t(c'^n, \mathcal{A})$ has weight ≥ 2 there is countable D such that $A_n' \cup D$ is a model, $c'^n \downarrow_{A_n'} D$ and $t(c'^n, A_n' \cup D)$ has pre-weight ≥ 2. We choose D so that

$$D \downarrow_{A_n' \cup c'^n} b \frown a_0'^n \frown \ldots \frown a_n'^n$$

i.e.

$$D \downarrow_{A_n'} b \frown c'^n \frown a_0'^n \frown \ldots \frown a_n'^n.$$

Because $c'^n \frown a_0'^n \frown \ldots \frown a_n'^n$ is finite and $t(c'^n, A_n')$ is stationary we can find such D from $\mathcal{A}$.

Let $A_{n+1} = A_n' \cup D$. Clearly (i) – (iv) are true for these elements and $t(c'^n, A_{n+1})$ has pre-weight ≥ 2. We let $a_i^{n+1} = a_i'^n$ for $i < n+1$. Let d be such that $d \not\downarrow_{A_{n+1}} c'^n$ and $d \downarrow_{A_{n+1} \cup c'^n} a_0^{n+1} \frown \ldots \frown a_n^{n+1}$. We want to show that $b \not\downarrow_{A_{n+1}} d$. For a contradiction assume not. Then there is $d' \subseteq A_{n+1} - A_n'$ such that $c'^n \not\downarrow_{A_n'} d \frown d'$. But then $b \downarrow_{A_n'} d \frown d'$, and

$$(*) \qquad d \frown d' \downarrow_{A_n' \cup c'^n} a_0^{n+1} \frown \ldots \frown a_n^{n+1},$$

($c'^n \frown a_0^{n+1} \frown \ldots \frown a_n^{n+1} \downarrow_{A_n'} A_{n+1}$ and so $a_0^{n+1} \frown \ldots \frown a_n^{n+1} \downarrow_{A_n' \cup c'^n} A_{n+1} \cup c'^n$, which implies (*)), a contradiction.

So we can choose c^{n+1} and a_{n+1}^{n+1} so that

1. $c'^n \not\downarrow_{A_{n+1}} c^{n+1}$;
2. $c'^n \not\downarrow_{A_{n+1}} a_{n+1}^{n+1}$;
3. $c^{n+1} \downarrow_{A_{n+1}} a_{n+1}^{n+1}$; and
4. $c^{n+1} \frown a_{n+1}^{n+1} \downarrow_{A_{n+1} \cup c'^n} b \frown a_0^{n+1} \frown \ldots \frown a_n^{n+1}$.

Then $b \not\downarrow_{A_{n+1}} c^{n+1}$, $b \not\downarrow_{A_{n+1}} a^{n+1}_{n+1}$ and $c^{n+1} \frown a^{n+1}_{n+1} \downarrow_{A_{n+1}} a^{n+1}_0 \frown \ldots \frown a^{n+1}_n$, which implies that c^{n+1} together with a^{n+1}_i, $i < n+2$, is independent over A_{n+1}.

The especially part follows now immediately from the following easy remark: if q is a nonforking extension of p then the weight of q is not smaller than the pre-weight of p.

We finish this paper by giving another definition of c-types.

Definition 15 $t(a, C)$ is an f-type if the following holds: if $\mathcal{B} \subseteq C$ and $t(a, C) \not\dashv \mathcal{B}$ then there are $\mathcal{A}$, $\mathcal{A}'$, C', a' and c such that,

1. $\mathcal{A} = C[a]$, $\mathcal{A}' = C'[a']$;
2. there is an automorphism f that takes $\mathcal{A}$ to $\mathcal{A}'$, C to C', $f(a) = a'$ and fixes $\mathcal{B} \cup c$ pointwise;
3. $C' \downarrow_{\mathcal{B}} C$, $\mathcal{A}' \downarrow_{C'} C$;
4. $c \not\downarrow_C \mathcal{A}$;
5. $c \downarrow_{\mathcal{A}} \mathcal{A}'$, $c \downarrow_{\mathcal{A}'} \mathcal{A}$.

Lemma 16 *If $t(a, C)$ is a c-type then it is an f-type.*

Proof: Let $\mathcal{B} \subseteq C$ be such that $t(a, C) \not\dashv \mathcal{B}$. Because $t(a, C)$ is a c-type there is c' such that $c' \downarrow_{\mathcal{B}} C$ and $a \rhd_C c'$. We choose $\mathcal{A} = C[a]$ and $c \in \mathcal{A}$ so that $t(c, C) = t(c', C)$. We choose ω_1-saturated models $C' \subseteq \mathcal{A}'$ so that $t(\mathcal{A}', \mathcal{B} \cup c) = t(\mathcal{A}, \mathcal{B} \cup c)$ and $\mathcal{A}' \downarrow_{\mathcal{B} \cup c} \mathcal{A}$. Then it is easy to see that these are as wanted.

Lemma 17 *In $\mathbf{M}^{eq}$, if $t(a, C)$ is an f-type then it is a c-type.*

Proof: Let $\mathcal{B} \subseteq C$ be such that $t(a, C) \not\dashv \mathcal{B}$. Choose $\mathcal{A}$, $\mathcal{A}'$, C', a' and c as in the definition of f-type. Because $c \not\downarrow_C \mathcal{A}$ there are $d \in \mathcal{A}$ and $\varphi(x, y)$ so that $\varphi(d, c)$ forks over C. We choose $\psi(x, y)$ and $e \in \mathcal{A}$ so that $\psi(x, e)$ defines $\varphi(\mathcal{A}, c)$. As in the definition of canonical base we choose equivalence relation E as follows: $E(v, w)$ iff $\forall x(\psi(x, v) \leftrightarrow \psi(x, w))$. Let $b = e/E \notin \mathcal{B}$. Let f be the automorphism given by (2) in the definition of f-type. Let $b' = f(e)/E$.

Because of (5) in the definition of f-type, $b = b'$. So $a \rhd_C b$ and because $a' \rhd_{C'} b$, by the second part of (3) in the definition of f-type $b \downarrow_{C'} C$ and so by the first part of (3), $b \downarrow_{\mathcal{B}} C$.

By using Lemma 17 we can give another proof for Lemma 11. Let C be ω_1-saturated. It is enough to show that $t(a, C)$ is an f-type. So let $\mathcal{B} \subseteq C$ be ω_1-saturated and $t(a, C) \not\dashv \mathcal{B}$. Choose b so that $b \downarrow_{\mathcal{B}} C$ and $b \not\downarrow_C a$. Let $\mathcal{A} = C[a]$. We choose sets $\mathcal{A}_i$, $i < \omega$, so that $\mathcal{A}_0 = \mathcal{A}$ and $\mathrm{stp}(\mathcal{A}_i, \mathcal{B} \cup b) = \mathrm{stp}(\mathcal{A}, \mathcal{B} \cup b)$ and $\mathcal{A}_i \downarrow_{\mathcal{B} \cup b} \cup\{\mathcal{A}_j |\ j < i\}$. Then this sequence is indiscernible but not free over $\mathcal{B}$. Choose c from $\cup\{\mathcal{A}_j |\ 1 < j < \omega\}$ so that $c \not\downarrow_C \mathcal{A}$. If we let $\mathcal{A}' = \mathcal{A}_1$ then these are as in the definition of f-type; (5) follows from the following characterization of 1-based theories: the average type of the sequence is based on any member of the sequence.

Conjecture 18 *If T^{eq} is a complete countable stable theory then it has* CP.

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