Notre Dame Journal of Formal Logic
Volume 39, Number 3, Summer 1998

Irrevocable Belief Revision in Dynamic Doxastic Logic

KRISTER SEGERBERG

Abstract In this paper we present a new modeling for belief revision that is what we term irrevocable. This modeling is of philosophical interest since it captures some features of suppositional reasoning, and of formal interest since it is closely connected with AGM, yet provides for iterated belief revision. The analysis is couched in terms of dynamic doxastic logic.

1 Introduction There seems to be a need to distinguish actual belief revision from belief revision that is merely hypothetical. Let φ and ψ be two logically incompatible propositions. If an agent, engaged in actual belief revision and with φ among his current beliefs, decides to accept ψ as a new belief, then according to all reasonable theories of belief revision he will at the same time give up his old belief in φ; only if he does will the resulting set of beliefs remain consistent. Suppose, however, that the agent, in conversation with another agent, has agreed to accept φ "for the sake of argument" and that he now agrees also to accept ψ "for the sake of argument." In this case, the resulting set of beliefs is just inconsistent. Since the result is consistent in one case and inconsistent in the other, the two cases must be different.

Ordinary theories of belief change do not seem suited to handle the sort of hypothetical belief change that goes on, for example, in debates where the participants agree, "for the sake of argument," on a certain common ground on which possibilities can be explored and disagreements can be aired. One need not actually believe what one accepts in this way. Nevertheless such acceptance amounts to what may be called a *doxastic commitment*, one that cannot be given up within the perimeter of the debate. Someone who no longer wishes to honor such a commitment may be described as in effect abandoning the debate, perhaps in order to initiate another debate with a different set of doxastic commitments.

Semantically speaking, a modeling of belief revision can be built as follows. Consider a logical space, the points of which represent all the possible states of the world (from some point of view). Any (relevant) proposition about the world may be

September 19, 1997; revised September 15, 1998

identified with a certain subset of the space—with the whole space if the proposition is logically true, with the empty set if it is logically false, otherwise with a subset in between those two extremes. By the same token, a theory may be identified with a subset of the space that is the intersection of a set of propositions. In particular, the set of propositions believed by an agent to be true forms a theory in this sense, the *belief set*. The *belief state* of the agent, however, is something more complicated. The author's suggestion, based on the work of Lewis and Grove, is that belief states can be thought of as *hypertheories*, that is, nonempty sets of theories for which Lindström and Rabinowicz [5] have suggested the term *fallbacks* (see also Segerberg [8]). A doxastic action is a binary relation over the set of hypertheories. A belief change (due to a doxastic action) is a change from one hypertheory (belief state) to another. The intuition is that in order to describe an agent's doxastic state it is not enough to describe his beliefs about the world (the belief set); one must also describe his doxastic dispositions, how he would respond to new information about the world. The fallbacks are theoretical positions with the help of which the agent is able to work out a new belief set if new information forces him to give up his current one.

In this paper we describe a modeling for belief revision (IR) which accommodates the intuitions about hypothetical reasoning described above. In the minimal version presented here it is a rather special modeling that is probably too wasteful in its treatment of old information to be of much practical interest, but it should be possible—and perhaps not so difficult—to combine it with other, more general modelings. Its theoretical interest is that it highlights doxastic commitment, a feature that has received little attention before but which is a component in many cases, for example, in default reasoning. On the technical side it may be noted that IR is closely related to the classic theory of belief revision due to Alchourrón, Gärdenfors, and Makinson (AGM) as supplemented by the semantic representation of Grove (Alchourrón, Gärdenfors, and Makinson [1], Grove [3]). One important difference is that IR specifically provides for iterated belief change. Iteration seems not to have been of great interest to the creators of AGM, which in effect is a "one-shot" theory, and it has proven surprisingly difficult to give an iterative extension of AGM that is natural.

Many ideas in this paper—and in the model theory of AGM type belief revision generally—go back to Lewis's pioneering work [4].

2 Semantics and syntax Let $\mathfrak{A}$ be a Boolean algebra of subsets of a given set U; the elements of $\mathfrak{A}$ are called *propositions* in $\mathfrak{A}$ and the set $U = Univ\,\mathfrak{A}$ the *universe* of $\mathfrak{A}$. We write $Prop\,\mathfrak{A}$ for the set of propositions; thus $\mathfrak{A} = (Prop\,\mathfrak{A}, \cap, \cup, -, U, \varnothing)$. A nonempty subset $T \subseteq U$ is called a *theory (in the semantical sense)* if there is a subset $S \subseteq Prop\,\mathfrak{A}$ such that $T = \bigcap S$.

A hypertheory in $\mathfrak{A}$ is a special kind of subset of $\mathfrak{P}U$; the exact definition will vary from case to case. In this paper we require a *hypertheory* H to be nonempty (NE), to be linearly ordered by inclusion (LIN), and to satisfy the Limit Condition (LIM):

(NE) $H \neq \varnothing$.

(LIN) For all $X, Y \in H$, either $X \subseteq Y$ or $Y \subseteq X$.

(LIM) Suppose that $C = \{X \in H : X \cap P \neq \varnothing\}$, where P is any proposition in $\mathfrak{A}$. If $C \neq \varnothing$ then $\bigcap C \in C$.

A hypertheory every element of which is a theory is said to be *closed*. A hypertheory H is *inconsistent* if $\varnothing \in H$. A proposition P is *inaccessible* to H if $\bigcup H \cap P = \varnothing$.

A *doxastic action* is a binary relation over some set of hypertheories. In this paper all doxastic actions are of the type $*P$ where P is a proposition: *irrevocable revision* by P. We say that a hypertheory H' is obtained from a hypertheory H by irrevocable revision by P if either

(i) P is inaccessible to H and $H' = \{\varnothing\}$, or
(ii) H' consists of all nonempty intersections $Y \cap P$ such that $Y \in H$, plus $\varnothing$ if $\varnothing \in H$.

A *system of hypertheories* in $\mathfrak{A}$ is a structure $\mathfrak{H} = (S, R)$ where S is a set of hypertheories in $\mathfrak{A}$ and $R = \{R^{*P} : P \in \mathit{Prop}\ \mathfrak{A}\ \&\ R^{*P}$ is irrevocable revision by P over $S\}$. Note that if H and H' are hypertheories, then

$$(H, H') \in R^{*P} \text{ iff } H' = \{\varnothing : \forall Y \in H(Y \cap P = \varnothing)\} \cup \{X : X \neq \varnothing\ \&\ \exists Y \in H(X = Y \cap P)\} \cup \{\varnothing : \varnothing \in H\}.$$

We offer the following informal motivation for this conceptual edifice. *Prop* $\mathfrak{A}$ contains the propositions about the world that are in principle expressible (on a certain occasion, in a certain context). A hypertheory represents a possible belief state of a rational agent; the relation R^{*P} models the change the agent's belief state undergoes if he revises his beliefs by the proposition P. Revision is to be understood in the sense of irrevocable revision: once P has been accepted (perhaps "for the sake of argument") it cannot be given up later. Notice that the action of irrevocable revision is always possible to carry out, even though the result may be inconsistent; this is in accord with the intuition that even a rational agent should be able to investigate the logical consequences of any hypothesis.

What is a suitable language in which to discuss these structures? Among several possibilities, for this paper we choose the language of dynamic doxastic logic (DDL). This is a language containing *terms* as well as *formulas*. There are no primitive terms; the primitive formulas are a denumerable set of propositional letters. The operators taking formulas to formulas include a truth-functionally complete set of Boolean operators as well as the unary, *doxastic operators* **B** and **K** and the binary *plausibility operator* $\leq$. The *revision operator* $*$ takes formulas to terms; in fact, in this paper, the only terms are of the form $*\varphi$, where φ is a formula. Finally, the binary *dynamic operator* [ ] may be thought of as operating in two steps: applying it to a term $*\varphi$ results in a unary operator $[*\varphi]$ which can then be applied to a formula to yield a formula. To complete this description of our language we add an important restriction. A formula not containing any non-Boolean operator but built exclusively from propositional letters and Boolean operators is called *purely Boolean*. *Throughout the paper* **B** *and* **K** *and* $\leq$ *and* $*$ *operate only on purely Boolean formulas*. Readers are warned that they will not always be reminded of this restriction.

To guide the informal understanding of our symbolism we offer the following unofficial translations:

$\mathbf{B}\varphi$	the agent believes that φ,
$\mathbf{K}\varphi$	the agent knows that φ (alternatively, has a doxastic

	commitment to φ),
$[*\varphi]\chi$	necessarily, after revision of the agent's beliefs by φ, it is the case that χ,
$\varphi \leq \psi$	φ is at least as plausible (according to the agent's belief) as ψ.

We will also find the following defined operators useful:

$$\begin{aligned}
\mathbf{b}\varphi &=_{\text{df}} \neg\mathbf{B}\neg\varphi,\\
\mathbf{k}\varphi &=_{\text{df}} \neg\mathbf{K}\neg\varphi,\\
\langle *\varphi\rangle\chi &=_{\text{df}} \neg[*\varphi]\neg\chi,\\
\varphi < \psi &=_{\text{df}} (\varphi \leq \psi) \wedge \neg(\psi \leq \varphi).
\end{aligned}$$

The following unofficial readings are suggested:

$\mathbf{b}\varphi$	it is consistent with what the agent believes that φ,
$\mathbf{k}\varphi$	it is consistent with what the agent knows (alternatively, it is consistent with his doxastic commitments) that φ,
$\langle *\varphi\rangle\chi$	possibly, after revision of the agent's beliefs by φ, it is the case that χ,
$\varphi < \psi$	φ is more plausible (according to the agent) than ψ.

The informal translations are given only for heuristic purposes; whenever questions of interpretation of the modeling arise, it is to the formal definitions one should turn. In particular, reading **K** for *knowledge* may not be a good idea; better perhaps to regard both **B** and **K** as doxastic operators and take the following slogan to heart: **B** *for belief and* **K** *for kommitment*.

A final comment on the purely-Boolean-formula restriction, which is adopted mostly for technical reasons. A consequence of this restriction is that we are only able to model the agent's beliefs about the world. In extensions of the present modeling one could allow nestings of **B** and **K** as a first step—beliefs about beliefs—and as a second step unrestricted nesting of all operators—beliefs about anything.

3 Truth-value conditions Let $\mathfrak{A}$ be a given algebra of sets. A *valuation* in $\mathfrak{A}$ is a function from the set of propositional letters to *Prop* $\mathfrak{A}$. The structure $(\mathfrak{A}, V)$ is called a *model* (on $\mathfrak{A}$). Let $\mathfrak{M} = (\mathfrak{A}, V)$ be a model. Any purely Boolean formula φ has an *intension* $||\varphi||_{\mathfrak{M}}$ defined in the usual way (we omit the subscript "$\mathfrak{M}$" whenever clarity allows):

$||\pi|| = V(\pi)$, if π is a propositional letter,

$||\varphi \wedge \psi|| = ||\varphi|| \cap ||\psi||$, etc.

Notice that $||\varphi|| \in$ *Prop* $\mathfrak{A}$ for all purely Boolean φ. Let $\mathfrak{H} = (S, R)$ be a system of hypertheories in $\mathfrak{A}$. The *truth*—in symbols $H \models_u \varphi$ (with respect to $\mathfrak{A}$ and $\mathfrak{H}$)—of a formula φ with respect to any hypertheory $H \in S$ and any point $u \in$ *Univ* $\mathfrak{A}$ can be given as follows (we omit the qualification "with respect to $\mathfrak{A}$ and $\mathfrak{H}$" which henceforth is regarded as implicit):

$$
\begin{array}{lll}
H \models_u \varphi & \text{iff} & u \in ||\varphi||, \text{ if } \varphi \text{ is a purely Boolean formula,} \\
H \models_u \varphi \wedge \psi & \text{iff} & H \models_u \varphi \text{ and } H \models_u \psi, \\
H \models_u \neg\varphi & \text{iff} & \text{not } H \models_u \varphi,
\end{array}
$$

and similarly for other Boolean connectives:

$$
\begin{array}{lll}
H \models_u \mathbf{B}\varphi & \text{iff} & \bigcap H \subseteq ||\varphi||; \\
H \models_u \mathbf{K}\varphi & \text{iff} & \bigcup H \subseteq ||\varphi||; \\
H \models_u [*\varphi]\chi & \text{iff} & \text{for all } H' \text{ such that } (H, H') \in R^{*||\varphi||}, H' \models_u \chi; \\
H \models_u \varphi \leq \psi & \text{iff} & \text{for every } X \in H, \text{ if } X \cap ||\varphi|| = \varnothing \text{ then } X \cap ||\psi|| = \varnothing.
\end{array}
$$

Let $\mathfrak{M} = (\mathfrak{A}, V)$ be a model and $\mathfrak{H} = (S, R)$ a system of hypertheories in $\mathfrak{A}$. A set Γ of formulas is said to be *satisfiable* (in $\mathfrak{M}$ with $\mathfrak{H}$) if there is some point $u \in Univ\, \mathfrak{A}$ and some hypertheory $H \in S$ such that, for all formulas $\varphi \in \Gamma$, $H \models_u \varphi$. A formula ψ is *valid* in a class of models with systems of hypertheories if true with respect to all relevant points and hypertheories.

Several observations are in order. First, there are the derived conditions for the defined operators:

$$
\begin{array}{lll}
H \models_u \mathbf{b}\varphi & \text{iff} & \bigcap H \cap ||\varphi|| \neq \varnothing. \\
H \models_u \mathbf{k}\varphi & \text{iff} & \bigcup H \cap ||\varphi|| \neq \varnothing. \\
H \models_u \langle *\varphi \rangle \chi & \text{iff} & \text{there is some } H' \text{ such that } (H, H') \in R^{*||\varphi||} \text{ and} \\
& & H' \models_u \chi; \\
H \models_u \varphi < \psi & \text{iff} & \text{there is some } X \in H \text{ such that } X \cap ||\varphi|| \neq \varnothing \text{ but} \\
& & X \cap ||\psi|| = \varnothing.
\end{array}
$$

Second, notice that in our definition we might have introduced two binary relations $R_H^{\mathbf{B}}$ and $R_H^{\mathbf{K}}$ over $Univ\, \mathfrak{A}$—strictly speaking, ternary, since they depend on the hypertheory H—stipulating that

$$
\begin{aligned}
R_H^{\mathbf{B}} &= \{(u, v) : v \in \textstyle\bigcap H\}, \\
R_H^{\mathbf{K}} &= \{(u, v) : v \in \textstyle\bigcup H\}.
\end{aligned}
$$

If so, we could then have replaced the official truth-conditions for **B** and **K** by the (here unofficial) conditions

$$
\begin{array}{lll}
H \models_u \mathbf{B}\varphi & \text{iff} & \text{for all } v \text{ such that } (u, v) \in R_H^{\mathbf{B}}, H \models_v \varphi, \\
H \models_u \mathbf{K}\varphi & \text{iff} & \text{for all } v \text{ such that } (u, v) \in R_H^{\mathbf{K}}, H \models_v \varphi.
\end{array}
$$

To have done so would have been clumsy, but the observation reveals that **B** and **K** are modal, Kripke/Hintikka type operators (although restricted by our rules for well-formedness).

By contrast—a third remark—we might quite profitably rewrite the rules for the dynamic operators:

$$
H \models_u [*\varphi]\chi \quad \text{iff} \quad H * ||\varphi|| \models_u \chi,
$$

where

$$
\begin{aligned}
H * ||\varphi|| =_{\text{df}} \{\varnothing : \textstyle\bigcup H \cap ||\varphi|| = \varnothing\} \cup \\
\{X \cap ||\varphi|| : X \in H \,\&\, X \cap ||\varphi|| \neq \varnothing\} \cup \{\varnothing : \varnothing \in H\}.
\end{aligned}
$$

Thus $H * ||\varphi||$ is inconsistent if and only if $||\varphi||$ is inaccessible to H or H is inconsistent.

Fourth, notice that H seems to play no role in the definition of $||\varphi||$ and u none in the (official) truth-conditions of the non-Boolean operators. This is because u represents the actual world which (in this modeling) is assumed not to change. All actions in this paper are doxastic and purely doxastic actions do not change the state of the world. “Real” actions do. It is a virtue of the present modeling that it is easy—or at least possible—to extend it by the addition of terms for “real” actions.

4 Heuristic remarks Readers are encouraged to familiarize themselves with the semantics by drawing diagrams. The general picture of a revision is obtained by two diagrams giving the belief state before and after the change, as in Fig. 1. The shaded areas indicate belief sets; notice that the belief sets (but of course not the belief states) delivered by IR are the same as AGM would give (provided, in the latter case, that the whole space is regarded as a fallback).

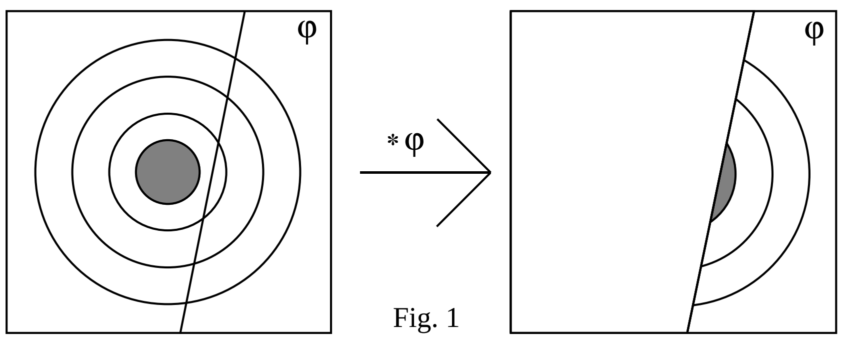

Fig. 1

Notice also that revision by a proposition can change the belief state of an agent even if the proposition is already believed by the agent (Fig. 2). This fact, surprising at first, becomes intelligible as soon as one distinguishes mere belief in a proposition from doxastic commitment to a proposition—one effect of revising by φ is that belief in φ becomes *irrevocable*.

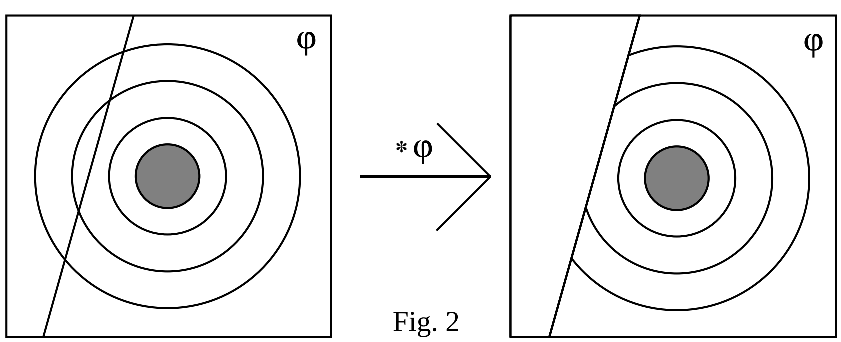

Fig. 2

It is helpful to consider an example given by McGee [6]. Consider California on the eve of the elections of 1980 and the following sentences:

(α) Anderson will win.
(γ) Carter will win.
(ρ) Reagan will win.
(π) A Republican will win.

As the example is given, it would have been rational for a well-informed, rational agent to believe (1) "A Republican will win," and (2) "If a Republican will win, then if Reagan does not win then Anderson will win" but not believe (3) "If Reagan will not win, then Anderson will win." In other words, the argument

$$\begin{array}{rl} & \pi \\ & \underline{\pi \Longrightarrow (\neg\rho \Longrightarrow \alpha)} \\ \therefore & \neg\rho \Longrightarrow \alpha \end{array}$$

fails. McGee offered this example as a case in which modus ponens (with respect to the conditional $\Longrightarrow$) fails. It is interesting that our modeling suits the doxastic version of McGee's argument: also the argument

$$\begin{array}{rl} & \mathbf{B}\pi \\ & \underline{[*\pi][*\neg\rho]\mathbf{B}\alpha} \\ \therefore & [*\neg\rho]\mathbf{B}\alpha \end{array}$$

fails. A formal proof of this claim is given by defining $U = \{0, 1, 2\}$, $V(\alpha) = \{2\}$, $V(\gamma) = \{1\}$, $V(\rho) = \{0\}$, $V(\pi) = \{0, 2\}$, and $H = \{\{0\}, \{0, 1\}, \{0, 1, 2\}\}$. As is readily checked, $(H * ||\pi||) * ||U - \rho|| = \{\{2\}\}$ and $H * ||U - \rho|| = \{\{1\}, \{1, 2\}\}$. Thus with respect to H and the actual state of the world, whether it be $0, 1,$ or 2, $\mathbf{B}\pi$ and $[*\pi][*\neg\rho]\mathbf{B}\alpha$ are true while $[*\neg\rho]\mathbf{B}\alpha$ is false.

5 *An axiom system*

(TF) τ, if τ is a truth-functional tautology.
(MP) If $\vdash \varphi$ and $\vdash \varphi \supset \psi$ then $\vdash \psi$.

If $\circ$ is $\mathbf{B}$ or $[*\theta]$, for any purely Boolean θ:

(01) $\circ(\varphi \supset \psi) \supset (\circ\varphi \supset \circ\psi)$.
(02) If $\vdash \varphi$ then $\vdash \circ\varphi$.

In addition we have the rule

(03) If $\vdash \varphi \equiv \psi$ then $\vdash [*\varphi]\chi \equiv [*\psi]\chi$.

as well as the following axiom schemata:

(#11) $\chi \equiv [*\varphi]\chi$, if χ is purely Boolean.
(#12) $\langle *\varphi \rangle \chi \equiv [*\varphi]\chi$.
(#13) $[*(\varphi \wedge \psi)]\chi \equiv [*\varphi][*\psi]\chi$.

(#21) $\mathbf{b}\varphi \supset ([*\varphi]\mathbf{B}\chi \equiv \mathbf{B}(\varphi \supset \chi))$.
(#22) $\langle *\varphi \rangle \mathbf{b}\psi \supset \langle *\psi \rangle \mathbf{b}\top$.
(#23) $\mathbf{B}\bot \supset [*\varphi]\mathbf{B}\bot$.
(#24) $\langle *\varphi \rangle \mathbf{b}\top \equiv \mathbf{k}\varphi$.

(#25) $[*\varphi]\mathbf{K}\varphi$.
(#26) $\mathbf{K}\varphi \supset \mathbf{B}\varphi$.

(#31) $((\varphi \leq \psi) \wedge (\psi \leq \theta)) \supset (\varphi \leq \theta)$.
(#32) $(\varphi \leq \psi) \vee (\psi \leq \varphi)$.
(#33) $(\varphi \leq \psi) \equiv (\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\varphi)$.
(#34) $(\varphi < \psi) \supset ([*(\varphi \vee \psi)]\mathbf{B}\chi \equiv [*\varphi]\mathbf{B}\chi)$.
(#35) $(\varphi \leq \psi) \supset (\langle *\psi\rangle \mathbf{b}\top \supset \langle *\varphi\rangle \mathbf{b}\top)$.

Thus (#24) is in effect a definition of the operator **K** in terms of the operators **B** and [ ]; we might have refrained from including **K** among the primitive operators and instead introduced it by abbreviation:

$$\mathbf{K}\varphi =_{\mathrm{df}} [*\neg\varphi]\mathbf{B}\bot.$$

Similarly, (#33) may be regarded as a definition of the operator $\leq$; we could have dispensed with it as a primitive operator at the expense of a less transparent axiom system.

It is easy to prove the following soundness result.

Theorem 5.1 *All formal theorems derivable in our axiom system are valid. Consequently, if a formula set is satisfiable, then it is consistent.*

Proof: The first part of the theorem is proved by checking that the axioms are valid and that the rules preserve validity. The second part follows from this and from the fact that our logic is finitary, that is, the rules—(MP), (02), and (03)—have only finitely many premises. □

The axiom system is strong enough to make, not only **B** and $[*\theta]$, for all purely Boolean θ, normal operators, but also **K** and $\leq$. This is an important fact that might be worth proving. In fact it is enough to prove the following.

Lemma 5.2 *The following schemata and rules are derivable in our system:*

(i) $\mathbf{K}(\varphi \wedge \psi) \equiv (\mathbf{K}\varphi \wedge \mathbf{K}\psi)$,
(ii) $\mathbf{K}\top$,
(iii) *if* $\vdash \varphi \equiv \psi$ *then* $\vdash \mathbf{K}\varphi \equiv \mathbf{K}\psi$;
(iv) $((\varphi \vee \psi) \leq \theta) \supset ((\varphi \leq \theta) \vee (\psi \leq \theta))$,
(v) *if* $\vdash \varphi \supset \psi$ *then* $\vdash \psi \leq \varphi$.

Proof: We provide outlines of the formal proofs. By ‘ML’ we mean reasoning depending on (TF), (MP), and (01 – 03). We begin with (iv).

1.	$\neg(((\varphi \vee \psi) \leq \theta) \supset ((\varphi \leq \theta) \vee (\psi \leq \theta)))$	premise
2.	$\neg(\varphi \leq \theta)$	ML: 1
3.	$\neg(\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\varphi)$	ML: (#33), 2
4.	$\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top$	ML: 3
5.	$[*(\varphi \vee \psi)]\mathbf{B}\neg\varphi$	ML: 3
6.	$\neg(\psi \leq \theta)$	ML: 1
7.	$[*(\varphi \vee \psi)]\mathbf{B}\neg\psi$	similarly
8.	$[*(\varphi \vee \psi)]\mathbf{B}\neg(\varphi \vee \psi)$	ML: 5, 7
9.	$[*(\varphi \vee \psi)]\mathbf{B}(\varphi \vee \psi)$	ML: (#25, #26)
10.	$[*(\varphi \vee \psi)]\mathbf{B}\bot$	ML: 8, 9
11.	$\bot$	ML: 4, 10

Since premise 1 leads to contradiction, we have established (iv). Next we turn to (i). Thanks to the result (iv) just proved and (#32) and ML,

$$\vdash (\varphi \leq (\varphi \vee \psi)) \vee (\psi \leq (\varphi \vee \psi)).$$

Hence by (#35)

$$\vdash (\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset \langle *\varphi\rangle \mathbf{b}\top) \vee (\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset \langle *\psi\rangle \mathbf{b}\top),$$
$$\vdash \langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset (\langle *\varphi\rangle \mathbf{b}\top \vee \langle *\varphi\rangle \mathbf{b}\top).$$

By (#24) and ML, this is half of (i). Now the converse:

1.	$\neg((\varphi \vee \psi) \leq \varphi)$	premise
2.	$\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top$	ML: (#33), 1
3.	$[*(\varphi \vee \psi)]\mathbf{B}\neg\varphi$	ML: (#33), 1
4.	$\varphi < (\varphi \vee \psi)$	ML: (#32), 1
5.	$[*\varphi\mathbf{B}\neg\varphi$	ML: (#34), 3, 4
6.	$[*\varphi]\mathbf{B}\varphi$	ML: (#25, #26)
7.	$[*\varphi]\mathbf{B}\bot$	ML: 5, 6
8.	$\langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset \langle *\varphi\rangle \mathbf{b}\top$	ML: (#35), 4
9.	$\langle *\varphi\rangle \mathbf{b}\top$	ML: 2, 8
10.	$\bot$	ML: 7, 9

This deduction shows that $\vdash (\varphi \vee \psi) \leq \varphi$. Hence by (#35), $\vdash \langle *\varphi\rangle \mathbf{b}\top \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\top$. By a symmetrical argument, $\vdash \langle *\psi\rangle \mathbf{b}\top \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\top$. Hence the desired result, $\vdash (\langle *\varphi\rangle \mathbf{b}\top \vee \langle *\psi\rangle \mathbf{b}\top) \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\top$. This ends the proof of (i).

For (ii) we have to prove that $\vdash [*\neg\top]\mathbf{B}\bot$. This follows readily by ML from (#25, #26).

(iii)—congruentiality for **K**—follows readily from congruentiality for **B** and [ ] (that is, (02 – 03)).

Finally we turn to (v):

1.	$\vdash \varphi \supset \psi$	premise
2.	$\vdash (\varphi \vee \psi) \equiv \psi$	ML: 1
3.	$\vdash \langle *\psi\rangle \mathbf{b}\top \supset \langle *\psi\rangle \mathbf{b}\psi$	ML: (#25, #26)
4.	$\vdash \langle *(\varphi \vee \psi)\rangle \mathbf{b}\top \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\psi$	ML: 2, 3
5.	$\vdash \psi \leq \varphi$	ML: (#33), 4

□

We end the section with some technical results.

Lemma 5.3 *The following theorem schemata are derivable in our axiom system:*

- *(a)* $\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset \langle *\varphi\rangle \mathbf{b}\top$.
- *(b)* $\langle *\varphi\rangle \mathbf{b}\top \supset \langle *(\varphi \vee \psi)\rangle \mathbf{b}\top$.
- *(c)* $\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset [*\varphi]\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top$.
- *(d)* $\langle *\varphi\rangle \mathbf{b}\theta \equiv \langle *\varphi\rangle \mathbf{b}(\varphi \wedge \theta)$.
- *(e)* $\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset (\langle *\varphi\rangle(\theta \leq \psi) \equiv ((\varphi \wedge \theta) \leq (\varphi \wedge \psi)))$.

Proof: (a) and (b) follow from Lemma 5.2. For (c), use (#12, #13), for (d) use (#23, #25, #26). For (e) we sketch a syntactic proof.

1.	$\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top$	premise
2.	$\langle *\psi\rangle \mathbf{b}\top$	Lemma 5.3(a): 1
3.	$\langle *(\theta \vee \psi)\rangle \mathbf{b}\top$	Lemma 5.3(b): 2
4.	$\theta \leq \psi \equiv \langle *(\theta \vee \psi)\rangle \mathbf{b}\theta$	ML: (#33), 3

This argument shows that $\vdash \langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset (\theta \leq \psi \equiv \langle *(\theta \vee \psi)\rangle \mathbf{b}\theta)$. Hence

$$\vdash [*\varphi]\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset [*\varphi](\theta \leq \psi \equiv \langle *(\theta \vee \psi)\rangle \mathbf{b}\theta).$$

By Lemma 5.3(c) and more ML

$$\vdash \langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset (\langle *\varphi\rangle(\theta \leq \psi) \equiv \langle *\varphi\rangle\langle *(\theta \vee \psi)\rangle \mathbf{b}\theta).$$

By (#13)

$$\vdash \langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset (\langle *\varphi\rangle(\theta \leq \psi) \equiv \langle *((\varphi \wedge \theta) \vee (\varphi \wedge \psi))\rangle \mathbf{b}\theta).$$

Hence with the help of Lemma 5.3(d),

$$\vdash \langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset (\langle *\varphi\rangle(\theta \leq \psi) \equiv \langle *((\varphi \wedge \theta) \vee (\varphi \wedge \psi))\rangle \mathbf{b}(\varphi \wedge \theta)).$$

By Lemma 5.3(b) and ML,

$$\vdash \langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \supset \langle *((\varphi \wedge \theta) \vee (\varphi \wedge \psi))\rangle \mathbf{b}\top.$$

The desired result follows by (#33) and ML.[1] □

6 *World states, belief sets, hypertheories* A *small Lindenbaum* set is a maximal consistent set of purely Boolean formulas, a *big Lindenbaum set* a maximal consistent set of any formulas. (Consistency here is with respect to the axiom system in the preceding section.) If in the sequel we speak of Lindenbaum sets without specifying small or big, it is big that we have in mind.

The set of small Lindenbaum sets is denoted by U. If φ is a purely Boolean formula, then we write $|\varphi|$ for the set $\{u \in U : \varphi \in u\}$. Let Σ be a big Lindenbaum set. Already in [8] we have the following definitions.

Definition 6.1 $wst\ \Sigma = \{\chi \in \Sigma : \chi \text{ is purely Boolean}\}$.

Definition 6.2 $bst\ \Sigma = \{u \in U : \forall\chi(\mathbf{B}\chi \in \Sigma \Longrightarrow \chi \in u)\}$.

Definition 6.3 $\Sigma^{*\varphi} = \{\chi : [*\varphi]\chi \in \Sigma\}$.

Lemma 6.4 *Suppose that Σ and Θ are big Lindenbaum sets.*

(i) *bst* $\Sigma \subseteq |\varphi|$ *if and only if* $\mathbf{B}\varphi \in \Sigma$.
(ii) *bst* $\Sigma \subseteq$ *bst* Θ *if and only if, for all* χ, $\mathbf{B}\chi \in \Theta$ *only if* $\mathbf{B}\chi \in \Sigma$.
(iii) $\Sigma^{*\varphi}$ *is a big Lindenbaum set.*
(iv) *wst* $\Sigma^{*\varphi} =$ *wst* Σ.

Proof: (In accordance with our general policy, it is assumed that φ and χ are purely Boolean formulas.) The proofs are similar to the proofs of similar claims in [8]. Axiom schema (#12) is used for (iii), (#11) for (iv). □

Notice also that $\Sigma^{*\perp}$ is a big Lindenbaum set even though *bst* $\Sigma^{*\perp} = \varnothing$. *Throughout the remainder of this section let Σ be a fixed, given big Lindenbaum set.*

Definition 6.5 $f(\psi, \Sigma) = \bigcup\{bst\ \Sigma^{*\theta} : \theta \leq \psi \in \Sigma)\}$.

We shall say that $X \subseteq U$ is a *basic fallback* (with respect to Σ) if, for some ψ, $X = f(\psi, \Sigma)$. A basic fallback $f(\psi, \Sigma)$ is *proper* if $\langle *\psi\rangle\mathbf{b}\top \in \Sigma$, otherwise *improper*. A necessary and sufficient condition for $f(\psi, \Sigma)$ to be proper is that $\mathbf{k}\psi \in \Sigma$ or, equivalently, $\psi < \perp \in \Sigma$.

Lemma 6.6 *If* $\mathbf{B}\perp \in \Sigma$ *then* $f(\psi, \Sigma) = \varnothing$.

Proof: Suppose that $u \in f(\psi, \Sigma)$. Then there is some formula θ such that $\theta \leq \psi \in \Sigma$ and $u \in bst\ \Sigma^{*\theta}$. If $\mathbf{B}\perp \in \Sigma$ then $[*\theta]\mathbf{B}\perp \in \Sigma$ by (#23). Hence $\mathbf{B}\perp \in \Sigma^{*\theta}$ and so $\perp \in u$, which is absurd. □

Lemma 6.7 *If* $\varphi \leq \psi \in \Sigma$ *then* $f(\varphi, \Sigma) \subseteq f(\psi, \Sigma)$.

Proof: By (#31). □

Lemma 6.8 *Assume that* $\langle *(\varphi \wedge \psi)\rangle\mathbf{b}\top \in \Sigma$. *Then* $f(\psi, \Sigma^{*\varphi}) = f(\varphi \wedge \psi, \Sigma) \cap |\varphi|$.

Proof: Assume that $\langle *(\varphi \wedge \psi)\rangle\mathbf{b}\top \in \Sigma$. Note that

$$f(\psi, \Sigma^{*\varphi}) = \bigcup\{bst\ \Sigma^{*(\varphi\wedge\theta)} : \theta \leq \psi \in \Sigma^{*\varphi}\}.$$

First suppose that $u \in f(\psi, \Sigma^{*\varphi})$. Then there is some θ such that $\theta \leq \psi \in \Sigma^{*\varphi}$ and $u \in bst\ \Sigma^{*(\varphi\wedge\theta)}$. Hence $[*\varphi](\theta \leq \psi) \in \Sigma$. Therefore $\langle *\varphi\rangle(\theta \leq \psi) \in \Sigma$ by (#12). By Lemma 5.3(e) then $\varphi \wedge \theta \leq \varphi \wedge \psi \in \Sigma$. Hence $bst\ \Sigma^{*(\varphi\wedge\theta)} \subseteq f(\varphi \wedge \psi, \Sigma)$. That $bst\ \Sigma^{*(\varphi\wedge\theta)} \subseteq |\varphi|$ is obvious. Consequently, $u \in f(\varphi \wedge \psi, \Sigma) \cap |\varphi|$.

Conversely, suppose that $u \in f(\varphi \wedge \psi, \Sigma) \cap |\varphi|$. Then there is some θ such that $\theta \leq (\varphi \wedge \psi) \in \Sigma$ and $u \in bst\ \Sigma^{*\theta}$ and $u \in |\varphi|$. It follows that $[*\theta]\mathbf{b}\varphi \in \Sigma$, so by (#12, #25, #26) and modal logic $\langle *\theta\rangle\mathbf{b}(\varphi \wedge \theta) \in \Sigma$. Using the fact that $(\varphi \wedge \theta) \vee \theta$ is tautologically equivalent to θ, we infer with the help of (#33) that $(\varphi \wedge \theta) \leq \theta \in \Sigma$. Consequently, $(\varphi \wedge \theta) \leq (\varphi \wedge \psi) \in \Sigma$ by (#31). By Lemma 5.3(e), then, $[*\varphi](\theta \leq \psi) \in \Sigma$, whence $\theta \leq \psi \in \Sigma^{*\varphi}$. Thus in order to be able to conclude that $u \in f(\psi, \Sigma^{*\varphi})$, all we now need to do is to prove that $u \in f(\psi, \Sigma^{*(\varphi\wedge\theta)})$. Let χ be any formula such that $\mathbf{B}\chi \in \Sigma^{*(\varphi\wedge\theta)}$; it will be enough to show that $\chi \in u$. Note that $[*(\varphi \wedge \theta)]\mathbf{B}\chi \in \Sigma$ and so, by (#13) and modal logic, $[*\varphi][*\theta]\mathbf{B}\chi \in \Sigma$. By (#21) and modal logic,

$$[*\theta]\mathbf{b}\varphi \supset ([*\theta][*\varphi]\mathbf{B}\chi \equiv [*\theta]\mathbf{B}(\varphi \supset \chi))$$

is a theorem. We already saw that $[*\theta]\mathbf{b}\varphi \in \Sigma$, hence $[*\theta]\mathbf{B}(\varphi \supset \chi) \in \Sigma$. Consequently, $\mathbf{B}(\varphi \supset \chi) \in \Sigma^{*\theta}$. Since $u \in \mathit{bst}\ \Sigma^{*\theta}$, it follows that $\varphi \supset \chi \in u$. The fact that $u \in |\varphi|$ implies that $\varphi \in u$. Hence $\chi \in u$, as we wanted. □

If $\langle *\psi\rangle\mathbf{b}\top \in \Sigma$, we refer to $f(\psi, \Sigma)$ as the *smallest ψ-fallback* (with respect to Σ), that is, the smallest basic fallback to intersect $|\psi|$. To justify this terminology there is the following result.

Lemma 6.9 *$f(\psi, \Sigma) \cap |\varphi| \neq \varnothing$ implies that $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$ and that $f(\varphi, \Sigma) \subseteq f(\psi, \Sigma)$.*

Proof: Assume that $f(\psi, \Sigma) \cap |\varphi| \neq \varnothing$. Evidently there is some $u \in f(\psi, \Sigma) \cap |\varphi|$. Hence there is some θ such that $\theta \leq \psi \in \Sigma$ and $u \in \mathit{bst}\ \Sigma^{*\theta}$. Since $\varphi \in u$ it is clear that $\langle *\theta\rangle\mathbf{b}\varphi \in \Sigma$. Then $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$ by (#22). This proves one part of the lemma.

Let us now address the remaining part. Suppose by absurdity that $\varphi \leq \theta \notin \Sigma$. Then $[*(\theta \vee \varphi)]\mathbf{B}\neg\varphi \in \Sigma$ thanks to (#33). Furthermore, $\theta < \varphi \in \Sigma$ by (#31, #32), hence $[*\theta]\mathbf{B}\neg\varphi \in \Sigma$ by (#34). This contradicts the fact, noted above, that $\langle *\theta\rangle\mathbf{b}\varphi \in \Sigma$. Therefore $\varphi \leq \theta \in \Sigma$. Since $\theta \leq \psi \in \Sigma$, (#31) ψ yields $\varphi \leq \psi \in \Sigma$. Consequently $f(\varphi, \Sigma) \subseteq f(\psi, \Sigma)$ by Lemma 6.7. □

Lemma 6.10 *Suppose that $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$. Then $\varphi \leq \psi \in \Sigma$ if and only if, for all θ, if $f(\theta, \Sigma) \cap |\varphi| = \varnothing$ then $f(\theta, \Sigma) \cap |\psi| = \varnothing$.*

Proof: Assume that $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$. First suppose that $\varphi \leq \psi \in \Sigma$. Let θ be any formula such that $f(\theta, \Sigma) \cap |\psi| \neq \varnothing$. Then $f(\psi, \Sigma) \subseteq f(\theta, \Sigma)$ by Lemma 6.9. But $\mathit{bst}\ \Sigma^{*\varphi} \subseteq f(\psi, \Sigma)$, and so a fortiori $\mathit{bst}\ \Sigma^{*\varphi} \subseteq f(\theta, \Sigma)$. Moreover, $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$ implies that $\mathit{bst}\ \Sigma^{*\varphi} \neq \varnothing$. Hence $f(\theta, \Sigma) \cap |\varphi| \neq \varnothing$, as we wanted to show.

Conversely, assume that $\varphi \leq \psi \notin \Sigma$. Then $\psi < \varphi \in \Sigma$, and so by (#35) $\langle *\psi\rangle\mathbf{b}\top \in \Sigma$. Then $f(\psi, \Sigma) \cap |\psi| \neq \varnothing$; thus it will be enough to prove that $f(\psi, \Sigma) \cap |\varphi| = \varnothing$. Suppose there is some element $u \in f(\psi, \Sigma) \cap |\varphi|$. Then there exists some formula θ such that $u \in \mathit{bst}\ \Sigma^{*\theta}$ and $\theta \leq \psi \in \Sigma$ and $\langle *\theta\rangle\mathbf{b}\varphi \in \Sigma$. If $\varphi \leq \theta \notin \Sigma$, then by the same argument as in the proof of Lemma 6.9 $[*\theta]\mathbf{B}\neg\varphi \in \Sigma$, which is impossible; consequently, $\varphi \leq \theta \in \Sigma$. Then by (#31) $\varphi \leq \psi \in \Sigma$ in contradiction with our hypothesis. Hence $f(\psi, \Sigma) \cap |\varphi| = \varnothing$. □

We define the *canonical hypertheory induced by* Σ, in symbols $\mathit{hth}\ \Sigma$, as the closure under arbitrary union of the set of basic fallbacks. Thus if $\mathbf{B}\bot \in \Sigma$, then by Lemma 6.6 $\mathit{hth}\ \Sigma = \{\varnothing\}$. But if $\mathbf{b}\top \in \Sigma$—the nontrivial case—then $\mathit{hth}\ \Sigma$ is the closure under arbitrary union of the set $\{f(\psi, \Sigma) : \langle *\psi\rangle\mathbf{b}\top \in \Sigma\}$. In the latter case, $\mathit{hth}\ \Sigma$ is the smallest set S of subsets of U such that

(i) if $\langle *\psi\rangle\mathbf{b}\top \in \Sigma$ then $f(\psi, \Sigma) \in S$,
(ii) if $T \subseteq S$ and $T \neq \varnothing$, then $\bigcup T \in S$.

An element of $\mathit{hth}\ \Sigma$ that is the union of a set of basic fallbacks but is not itself a basic fallback is called a *limit fallback*.

Lemma 6.11 *$hth\ \Sigma$ is a hypertheory.*

Proof: If $\mathbf{B}\bot \in \Sigma$ then $\varnothing \in hth\ \Sigma$. If $\mathbf{b}\top \in \Sigma$ then $\langle * \top\rangle\mathbf{b}\top \in \Sigma$ by (#21), and so $f(\top, \Sigma) \in hth\ \Sigma$. Hence (NE) is satisfied. (LIN) holds thanks to (#31, #32). (LIM) follows from Lemma 6.9. □

Lemma 6.12 $\bigcup hth\ \Sigma = \bigcup\{bst\ \Sigma^{*\theta} : \langle *\theta\rangle\mathbf{b}\top \in \Sigma\}$.

Proof: First suppose that $u \in \bigcup hth\ \Sigma$. Then there is some ψ such that $\langle *\psi\rangle\mathbf{b}\top \in \Sigma$ and $u \in f(\psi, \Sigma)$. Consequently there is some φ such that $\varphi \leq \psi \in \Sigma$ and $u \in bst\Sigma^{*\varphi}$. By (#35), $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$. Thus $u \in \{\bigcup bst\ \Sigma^{*\theta} : \langle *\theta\rangle\mathbf{b}\top \in \Sigma\}$.

Conversely, suppose that $u \in bst\ \Sigma^{*\theta}$ for some formula θ such that $\langle *\theta\rangle\mathbf{b}\top \in \Sigma$. By (#32), $\theta \leq \theta \in \Sigma$, so $bst\ \Sigma^{*\theta} \subseteq f(\theta, \Sigma)$. But $f(\theta, \Sigma) \subseteq \bigcup hth\ \Sigma$, hence $u \in \bigcup hth\ \Sigma$. □

Lemma 6.13 $(\bigcup hth\ \Sigma) \cap |\varphi| \neq \varnothing$ *if and only if* $\mathbf{k}\varphi \in \Sigma$.

Proof: First suppose that $(\bigcup hth\ \Sigma) \cap |\varphi| \neq \varnothing$. Then, by Lemma 6.12, there is some element $u \in |\varphi|$ and some formula θ such that $u \in bst\ \Sigma^{*\theta}$ and $\langle *\theta\rangle\mathbf{b}\top \in \Sigma$. Since $\varphi \in u$ we have $\mathbf{b}\varphi \in \Sigma^{*\theta}$, whence $[*\theta]\mathbf{b}\varphi \in \Sigma$. Hence $\langle *\theta\rangle\mathbf{b}\varphi \in \Sigma$ by (#12). By (#22), $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$, therefore $\mathbf{k}\varphi \in \Sigma$ by (#24).

Conversely, suppose that $\mathbf{k}\varphi \in \Sigma$. Then $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$ by (#24). By Lemma 6.12 therefore $bst\ \Sigma^{*\varphi} \subseteq \bigcup hth\ \Sigma$. By (#25, #26), $bst\ \Sigma^{*\varphi} \subseteq |\varphi|$. Take any $u \in bst\ \Sigma^{*\varphi}$ (according to Lindenbaum, such elements exist!). Evidently, $u \in (\bigcup hth\ \Sigma) \cap |\varphi|$. □

Lemma 6.14 $hth\ (\Sigma^{*\varphi}) = (hth\ \Sigma) * |\varphi|$.

Proof: It follows from a remark at the end of Section 3 that

$$\begin{aligned} hth\ \Sigma * |\varphi| = \{\varnothing : \bigcup hth\ \Sigma \cap |\varphi| = \varnothing\} \cup \\ \{X \cap |\varphi| : X \in hth\ \Sigma\ \&\ X \cap |\varphi| \neq \varnothing\} \cup \{\varnothing : \varnothing \in hth\ \Sigma\}. \end{aligned}$$

There are two main cases, the first one of which is when $[*\varphi]\mathbf{B}\bot \in \Sigma$.

Case 1: In this case, $\mathbf{B}\bot \in \Sigma^{*\varphi}$. Hence by definition, $hth\ (\Sigma^{*\varphi}) = \{\varnothing\}$. Furthermore, if $f(\psi, \Sigma) \cap |\varphi| \neq \varnothing$ for some fallback $f(\psi, \Sigma) \in hth\ \Sigma$, then by Lemma 6.8 $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$, which is absurd; consequently, $(hth\ \Sigma) * |\varphi| = \{\varnothing\}$.

Case 2: The other main case is when $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$. Note that $\mathbf{b}\top \in \Sigma$ by (#23). First we prove inclusion from left to right. There are two subcases.

Subcase 1: First the basic case. Suppose that $X \in hth\ (\Sigma^{*\varphi})$ and that there is some ψ such that $\langle *\psi\rangle\mathbf{b}\top \in \Sigma^{*\varphi}$ and $X = f(\psi, \Sigma^{*\varphi})$. Assume, for contradiction, that $[*(\varphi \vee \psi)]\mathbf{B}\neg\varphi \in \Sigma^{*\varphi}$. Then $[*\varphi][*(\varphi \vee \psi)]\mathbf{B}\neg\varphi \in \Sigma$. Hence $[*\varphi]\mathbf{B}\neg\varphi \in \Sigma$ by (#13) and (03). By (#25, #26) and modal logic then $[*\varphi]\mathbf{B}\bot \in \Sigma$, a result that contradicts the assumption that $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$. This argument shows that $\langle *(\varphi \vee \psi)\rangle\mathbf{b}\varphi \in \Sigma^{*\varphi}$. It follows from (#33) that $\varphi \leq \psi \in \Sigma^{*\varphi}$. But then $bst\ \Sigma^{*\varphi} \subseteq f(\psi, \Sigma^{*\varphi})$. Since the fact that $\langle *\varphi\rangle\mathbf{b}\top \in \Sigma$ implies that $bst\ \Sigma^{*\varphi} \neq \varnothing$, this shows that $X \neq \varnothing$. Now, let $Y = f(\varphi \wedge \psi, \Sigma)$. With the help of (#12, #13) and the fact that $\langle *\psi\rangle\mathbf{b}\top \in \Sigma^{*\varphi}$ it is readily shown that $\langle *(\varphi \wedge \psi)\rangle \mathbf{b}\top \in \Sigma$; hence $Y \in hth\ \Sigma$. By Lemma 6.8, $X = Y \cap |\varphi|$. Hence $X \in (hth\ \Sigma) * |\varphi|$.

Subcase 2: Now the limiting case. Suppose that $X \in hth\ (\Sigma^{*\varphi})$ and that $X = \bigcup\{X_i : i \in I\}$, where I is some nonempty index set and each X_i is a basic fallback in $\Sigma^{*\varphi}$. By the argument presented in the preceding paragraph, for each $i \in I$ there is some $Y_i \in hth\ \Sigma$ such that $X_i = Y_i \cap |\varphi|$. But since $hth\ \Sigma$ is closed under union, $Y = \bigcup\{Y_i : i \in I\}$ is a fallback in $hth\ \Sigma$. Moreover, $X = Y \cap |\varphi|$. Hence $X \in (hth\ \Sigma) * |\varphi|$. This ends the proof of inclusion from left to right.

For the converse direction—inclusion from right to left—suppose that $X \in (hth\ \Sigma) * |\varphi|$. Then $X \neq \varnothing$. Moreover, there is some $Y \in hth\ \Sigma$ such that $X = Y \cap |\varphi|$. Let Z denote the set

$$\bigcup\{ f(\varphi \wedge \theta, \Sigma) : bst\ \Sigma^{*(\varphi\wedge\theta)} \cap X \neq \varnothing\}.$$

We claim that $X = Z \cap |\varphi|$. First suppose that $u \in X$. Note that $u \in |\varphi|$. Furthermore, $u \in Y$. Hence there are some θ and τ such that $f(\tau, \Sigma) \subseteq Y$ and $u \in bst\ \Sigma^{*\theta}$ and $\theta \leq \tau \in \Sigma$. This implies that $[*\theta]\mathbf{b}\varphi \in \Sigma$, whence $[*\theta]\mathbf{b}(\varphi \wedge \theta) \in \Sigma$ by (#25, #26), and so on. Hence $\langle *\theta\rangle \mathbf{b}(\varphi \wedge \theta) \in \Sigma$ by (#12) and so $\langle *(\varphi \wedge \theta)\rangle \mathbf{b}\top \in \Sigma$ by (#22). By (#32), $\theta \leq \theta \in \Sigma^{*\varphi}$, whence $bst\ \Sigma^{*\theta} \subseteq f(\theta, \Sigma^{*\varphi})$. Lemma 6.8 applies, yielding $u \in f(\varphi \wedge \theta, \Sigma) \cap |\varphi|$. The conclusion is that $u \in Z$. Conversely, suppose that $u \in Z \cap |\varphi|$. Then there is some θ such that $u \in f(\varphi \wedge \theta, \Sigma)$ and $bst\ \Sigma^{*(\varphi\wedge\theta)} \cap X \neq \varnothing$. Note that $Y \cap |\varphi \wedge \theta| \neq \varnothing$. Hence $f(\varphi \wedge \theta, \Sigma) \subseteq Y$ since by Lemma 6.9 $f(\varphi \wedge \theta, \Sigma)$ is the smallest fallback intersecting $\varphi \wedge \theta$. Therefore $u \in Y$. We also have $u \in |\varphi|$. Hence $u \in X$. This ends the proof of the claim that $X = Z \cap |\varphi|$.

By distribution, therefore,

$$X = \bigcup\{ f(\varphi \wedge \theta, \Sigma) \cap |\varphi| : bst\ \Sigma^{*(\varphi\wedge\theta)} \cap X \neq \varnothing\}.$$

But whenever θ is such that $bst\ \Sigma^{*(\varphi\wedge\theta)} \cap X \neq \varnothing$ we have $\langle *(\varphi \wedge \theta)\rangle \mathbf{b}\top \in \Sigma$. Hence by Lemma 6.8,

$$X = \bigcup\{ f(\theta, \Sigma^{*\varphi}) : bst\ \Sigma^{*(\varphi\wedge\theta)} \cap X \neq \varnothing\}.$$

Suppose that $bst\ \Sigma^{*(\varphi\wedge\theta)} \cap X \neq \varnothing$, for some θ. Then $\langle *(\varphi \wedge \theta)\rangle \mathbf{b}\varphi \in \Sigma$. By (#12, #13), $[*\varphi]\langle *\theta\rangle \mathbf{b}\top \in \Sigma$ and so $\langle *\theta\rangle \mathbf{b}\top \in \Sigma^{*\varphi}$, implying that $f(\theta, \Sigma^{*\varphi}) \in hth\ \Sigma^{*\varphi}$. The fact that canonical hypertheories are closed under arbitrary union then yields $X \in hth\ \Sigma^{*\varphi}$. $\square$

7 *The completeness proof* As before, U is the set of small Lindenbaum sets. Let $\mathfrak{A}$ be the Boolean algebra generated by the set-theoretical operations intersection, union and complement by the set of elements $\{|\varphi| : \varphi$ is a propositional letter$\}$. Define a valuation V by the requirement that, for each propositional letter π, $V(\pi) = \{u \in U : \pi \in u\}$. The valuation V is extended in the usual way to a valuation V' of all purely Boolean formulas. We adopt the convention of writing $||\varphi||$ for $V'(\varphi)$, when φ is purely Boolean. By a classical argument, $||\varphi|| = |\varphi|$; we refer to this fact by the term *tautological completeness*.

Lemma 7.1 *Let Σ be any big Lindenbaum set. Then*

$$hth\ \Sigma \models_{wst\ \Sigma} \chi \text{ iff } \chi \in \Sigma.$$

Proof: The proof is by induction on the complexity of χ. The basic step holds by definition. The Boolean steps are trivial. The case of **B** is as in modal logic. The cases of **K**, [ ], and $\leq$ are dealt with with the help of Lemmas 6.13, 6.14, and 6.10, respectively.

As an example we give the case of [ ]. Assume that χ is of the form $[*\varphi]\theta$; the induction hypothesis is that the theorem holds for θ.

1.	$hth\ \Sigma \models_{wst\ \Sigma} [*\varphi]\theta$	iff	(by the truth-definition)
2.	$(hth\ \Sigma) * \|\|\varphi\|\| \models_{wst\ \Sigma} \theta$	iff	(by tautological completeness)
3.	$(hth\ \Sigma) * \|\varphi\| \models_{wst\ \Sigma} \theta$	iff	(by Lemma 6.14)
4.	$hth\ (\Sigma^{*\varphi}) \models_{wst\ \Sigma} \theta$	iff	(by Lemma 6.4(iv))
5.	$hth\ (\Sigma^{*\varphi}) \models_{wst\ \Sigma^{*\varphi}} \theta$	iff	(by the induction hypothesis)
6.	$\theta \in \Sigma^{*\varphi}$	iff	(by definition)
7.	$[*\varphi]\theta \in \Sigma.$		

□

From Lemma 7.1 follows a completeness result that is a converse to the soundness result stated in Theorem 5.1.

Theorem 7.2 *If a set of formulas is consistent in our axiom system for IR, then it is satisfiable. In particular, a formula true with respect to all points and hypertheories is derivable in our axiom system for IR.*

8 Closed hypertheories From a philosophical point of view, the completeness result just achieved is not enough. The elements of a hypertheory—the fallbacks—represent positions on which the agent might fall back if his beliefs are challenged; if he modifies his belief state, it is from those elements that (with the help of set-theoretical operations) he molds his new belief state. Therefore one would expect the fallbacks—those theoretical positions—to be theories (in the semantical sense). But the fallbacks of a canonical hypertheory, although unions of theories, are not necessarily theories. In this concluding section we will improve upon that state of affairs.

Let φ range over the set of purely Boolean formulas and Σ over the set of sets of purely Boolean formulas. As before, U is the set of small Lindenbaum sets. Retaining the definition

$$|\varphi| \quad = \quad \{u \in U : \varphi \in u\}$$

we also define

$$|\Sigma| \quad = \quad \{u \in U : \Sigma \subseteq u\}.$$

These notations are consistent if one accepts that $|\{\varphi\}| = |\varphi|$. If $X \subseteq U$ we define

$$\begin{aligned} th\ X &= \{\varphi : X \subseteq |\varphi|\}, \\ \mathbf{C}X &= \bigcap\{|\varphi| : X \subseteq |\varphi|\}. \end{aligned}$$

We say that $\mathbf{C}X$ is the *closure* of X, and that X is *closed* if $X = \mathbf{C}X$. Note that $\bigcap X = \{\varphi : \forall u \in X(\varphi \in u)\} = \{\varphi : \forall u \in X(u \in |\varphi|)\} = th\ X$. The following list of well-known regularities is longer than really needed in this paper; it is included anyway in order to emphasize the interplay between syntax and model theory (or, more precisely, between syntax and the theory of the canonical model).

(a) $X \subseteq \mathbf{C}X$,
(b) $\mathbf{CC}X = \mathbf{C}X$,
(c) $\mathbf{C}|\varphi| = |\varphi|$,
(d) $\mathbf{C}\varnothing = \varnothing$,
(e) $\mathbf{C}|\Sigma| = |\Sigma|$,
(f) $\mathbf{C}(X \cap Y) = \mathbf{C}X \cap \mathbf{C}Y$,
(g) $\mathbf{C}(X \cup Y) = \mathbf{C}X \cup \mathbf{C}Y$,
(h) $\bigcap_{i \in I} \mathbf{C}X_i = \mathbf{C} \bigcap_{i \in I} X_i$,
(i) $\bigcup_{i \in I} \mathbf{C}X_i \subseteq \mathbf{C} \bigcup_{i \in I} X_i$,
(j) if $X \subseteq Y$ then $\mathbf{C}X \subseteq \mathbf{C}Y$,
(k) if $X \subseteq Y$ then $th\ X \supseteq\ th\ Y$,
(l) if Σ tautologically implies φ, then $|\Sigma| \subseteq |\varphi|$,
(m) if $|\Sigma| \subseteq |\varphi|$ then (by compactness) Σ tautologically implies φ,
(n) $\Sigma \subseteq th\ |\Sigma|$,
(o) $\Sigma = th\ |\Sigma|$ (thanks to compactness), if Σ contains all tautologies and is closed under modus ponens,
(p) $X \subseteq |th\ X|$,
(q) $X = |th\ X|$, if X is closed,
(r) $th\ \mathbf{C}X = th\ X$.
(s) $|th\ X| = \mathbf{C}X$.

Lemma 8.1 *If $X \cap |\varphi| = \varnothing$, then $\mathbf{C}X \cap |\varphi| = \varnothing$.*

Proof: Suppose that $X \cap |\varphi| = \varnothing$. Then $\mathbf{C}(X \cap |\varphi|) = \mathbf{C}\varnothing$. Furthermore, with the help of observations (c), (d), and (f), $\mathbf{C}(X \cap |\varphi|) = \mathbf{C}X \cap \mathbf{C}|\varphi| = \mathbf{C}X \cap |\varphi|$ and $\mathbf{C}\varnothing = \varnothing$. □

Definition 8.2 $chth\ \Sigma = \{\mathbf{C}X : X \in hth\ \Sigma\}$.

Thus each element of $chth\ \Sigma$ is a theory, as we wanted. Notice that if $hth\ \Sigma = \{\varnothing\}$, then also $chth\ \Sigma = \{\varnothing\}$.

Obviously what we have called theories in the semantical sense are the same as the closed sets. Thus, according to the definition in Section 2, a hypertheory is closed if all its fallbacks are closed. For philosophical reasons, it is in closed hypertheories that our interest lies. We now wish to establish the following results.

Lemma 8.3 *$chth\ \Sigma$ is a hypertheory.*

Proof: (NE) and (LIN) hold as before. We check the limit assumption (LIM) by verifying that $\mathbf{C}\,f(\varphi, \Sigma)$ is the smallest fallback in $chth\ \Sigma$ to intersect $|\varphi|$. Suppose that $\mathbf{C}\,f(\psi, \Sigma) \cap |\varphi| \neq \varnothing$. Then $f(\psi, \Sigma) \cap |\varphi| \neq \varnothing$ by Lemma 8.1. Hence $f(\varphi, \Sigma) \subseteq f(\psi, \Sigma)$, whence $\mathbf{C}\,f(\varphi, \Sigma) \subseteq \mathbf{C}\,f(\psi, \Sigma)$ by observation (j). □

Lemma 8.4 *$chth\ (\Sigma^{*\varphi}) = (chth\ \Sigma) * |\varphi|$, if $\langle *\varphi \rangle \mathbf{b}\top \in \Sigma$.*

Proof: Assume that $\langle * \varphi \rangle \mathbf{b} \top \in \Sigma$. First suppose that $Z \in chth\ (\Sigma^{*\varphi})$. Then there is some $X \in hth\ (\Sigma^{*\varphi})$ such that $Z = \mathbf{C}X$. It follows from the assumption by Lemma 6.14 that $X \in (hth\ \Sigma) * |\varphi|$. Hence $X \neq \varnothing$ and there is some $Y \in hth\ \Sigma$ such that $X = Y \cap |\varphi|$, for some $Y \in hth\ \Sigma$. Note that $\mathbf{C}X = \mathbf{C}(Y \cap |\varphi|) = \mathbf{C}Y \cap |\varphi|$. Evidently, $\mathbf{C}Y \in chth\ \Sigma$. Moreover, $\mathbf{C}X \neq \varnothing$. Consequently, $Z \in (chth\ \Sigma) * |\varphi|$.

Conversely, suppose that $Z \in (chth\ \Sigma) * |\varphi|$. Then $Z \neq \varnothing$ and there is some $W \in chth\ \Sigma$ such that $Z = W \cap |\varphi|$. Evidently there is some $Y \in hth\ \Sigma$ such that $W = \mathbf{C}Y$. Let $X = Y \cap |\varphi|$. By Lemma 8.1, $X \neq \varnothing$. Hence $X \in hth\ (\Sigma^{*\varphi})$ and so $\mathbf{C}X \in chth\ (\Sigma^{*\varphi})$. Note that $\mathbf{C}X = \mathbf{C}Y \cap |\varphi| = W \cap |\varphi| = Z$. Thus $Z \in chth\ (\Sigma^{*\varphi})$, as we wanted. □

Lemma 8.5 *For all formulas, purely Boolean or not,* $chth\ \Sigma \models_{wst\ \Sigma} \varphi$ *if and only if* $\varphi \in \Sigma$.

Proof: By induction on φ. Remember that $\mathbf{K}$ and $\leq$ could have been introduced as abbreviations. Thus in order to check the inductive step of the induction it is really enough to check the cases when φ is $\mathbf{B}\psi$, for some purely Boolean ψ, or $[*\psi]\chi$, for some purely Boolean ψ and arbitrary χ. We confine ourselves to the latter as the former easily follows from Lemma 8.1.

First assume that $\bigcup chth\ \Sigma \cap |\psi| = \varnothing$. By observation (a) above, $\bigcup hth\ \Sigma \subseteq \bigcup chth\ \Sigma$. Hence $\bigcup hth\ \Sigma \cap |\psi| = \varnothing$. This case is trivial.

Therefore suppose that $\bigcup chth\ \Sigma \cap |\psi| \neq \varnothing$. Then there is some $X \in hth\ \Sigma$ such that $\mathbf{C}X \cap |\psi| \neq \varnothing$. By Lemma 8.1, $X \cap |\psi| \neq \varnothing$. Hence $\bigcup hth\ \Sigma \cap |\psi| \neq \varnothing$. Note that $\langle * \psi \rangle \mathbf{b} \top \in \Sigma$. Then

1.	$chth\ \Sigma \models_{wst\ \Sigma} [*\psi]\chi$	iff	(by the truth definition)
2.	$chth\ \Sigma * |\psi| \models_{wst\ \Sigma} \chi$	iff	(by Lemma 8.4)
3.	$chth\ (\Sigma^{*\psi}) \models_{wst\ \Sigma} \chi$	iff	(by Lemma 6.4(iv))
4.	$chth\ (\Sigma^{*\psi}) \models_{wst\ \Sigma^{*\psi}} \chi$	iff	(by the induction hypothesis)
5.	$\chi \in \Sigma^{*\psi}$	iff	(by definition)
6.	$[*\psi]\chi \in \Sigma$.		

□

From this lemma follows the second completeness result of this paper:

Theorem 8.6 *If a set of formulas is consistent in our axiom system for IR, then it is satisfiable in a model with a system of closed hypertheories. In particular, in the class of models with systems of closed hypertheories, a formula true with respect to all points and hypertheories is derivable in our axiom system for IR.*

The second part of Theorem 8.6 can be strengthened even further: it is enough to consider finite models. Suppose that φ is a particular formula. Then φ contains only a finite number of propositional letters—say n, where n is a nonnegative integer. Let $\mathcal{L}_\varphi$ be the object language obtained by restricting the normal object language $\mathcal{L}$ to the propositional letters occurring in φ The constructions and arguments in Section 6 go through as before with $\mathcal{L}_\varphi$ taking the place of $\mathcal{L}$. There are some differences, of course; one is that the cardinality of the set U_φ of small Lindenbaum sets in $\mathcal{L}_\varphi$ is 2^n, hence finite. (Notice that this fact trivializes condition (LIM).) The proof of Lemma 7.1 also goes through. Hence we have:

Theorem 8.7 *A formula true in all finite models with respect to all points and all systems of (closed) hypertheories is derivable in our axiom system for IR.*

Corollary 8.8 *IR is decidable.*

Proof: We now have both a proof procedure and a disproof procedure. □

The last result is of theoretical interest only as the complexity of the decision problem is impractical.

9 Concluding remarks This paper was read by several referees who, in addition to suggesting many improvements, made a number of interesting comments. This section is in response to the latter.

First, it should be said that the interest in the paper is mainly formal. It seems obvious to the author that the proposed modeling has some application, in particular to a common variety of hypothetical reasoning; but not much effort was expended in arguing for this view. Rather, the interest is in describing a modeling with formal features that make it worth investigating. Modal logic may be looked upon as a storehouse of systems that can be used to model certain concepts in which philosophers are interested; for most of those concepts, there are many candidates. In the view of the author, the situation in belief revision is similar: we need not just one modeling, but many. It is probably hopeless to look for the logic of belief revision—if there is uniqueness, at least it is not obvious from the outset. Thus IR has not been launched in order to replace AGM.

As stated above, AGM is really a “one-shot” theory whereas IR is iterative, so in some ways it is impossible to compare the two. In the “one-shot” perspective, the difference between AGM and IR is not great and stems mainly from the slightly different conceptions of hypertheory: in AGM but not in IR, hypertheories are replete in the sense that the universe of a model is always a fallback (that is, H is a hypertheory in an algebra with universe U only if $U \in H$). Another difference is that in AGM any hypertheory becomes consistent upon revision by a consistent proposition, whereas in IR inconsistent hypertheories remain inconsistent after any revision (cf. (#23)). This has to do with a feature of IR that might be called “the persistence of commitment”: $[*\varphi][*\psi]\mathbf{K}\varphi$ and $\mathbf{K}\varphi \supset [*\psi]\mathbf{K}\varphi$ are valid schemata.

But even though it is possible to compare AGM and IR in some ways, it is perhaps more fruitful to focus on the doxastic actions that they model. In AGM we find one kind of revision, in IR another; let them be denoted by $*$ and $\overset{*}{*}$, respectively. There is an obvious way in which Grove’s modeling in [3] of AGM (“one-shot”) revision can be adapted to nonreplete hypertheories: in an algebra with universe U, if P is any proposition and H any hypertheory such that $\bigcup H \cap P = \varnothing$, then just stipulate that revision of H by P yields the new belief set $\varnothing$. Consider the obvious modeling in which both $*$ and $\overset{*}{*}$ are represented. The fact that the two different kinds of action—AGM revision and “one-shot” IR revision—lead to the same beliefs in “normal” cases is brought out by the fact that the schema $[*\varphi]\mathbf{B}\chi \equiv [\overset{*}{*}\varphi]\mathbf{B}\chi$ is valid in this modeling, whether or not attention is restricted to replete hypertheories.

To gain further perspective, let us compare another modeling with the idea of irrevocable revision. LR is a system of belief revision described in [9] that differs from AGM in two respects: hypertheories need not be replete and need not be nested (that is, the condition (LIN) is not imposed; cf. [5]), and there is provision for iterated revision. Let $*$ denote revision according to LR and R^{*P} the relation that models the change the agent's belief state undergoes if he revises his beliefs by a proposition P (cf. Section 2). According to LR, R^{*P} is defined as the set of all pairs (H, H') of hypertheories H and H' for which there exists a set Z such that

1. Z is minimal in the set $\{X \in H : X \cap P \neq \varnothing\}$,
2. $H' = \{X : Z \subseteq X\} \cup \{X \cap P : Z \subseteq X\}$.

(Thus belief revision in LR is not functional, in contrast with both revision in AGM and irrevocable revision in IR.) Now let $\overset{*}{*}$ denote irrevocable revision in this system, and let $R^{\overset{*}{*}P}$ denote the relation modeling the change the agent's belief state undergoes if he revises his beliefs by a proposition P in the irrevocable manner. Then it is natural to define $R^{\overset{*}{*}P}$ as the set of all pairs (H, H') of hypertheories such that either $\bigcup H \cap P = \varnothing$ and $H' = \varnothing$, or else there is a set Z such that

3. Z is minimal in the set $\{X \in H : X \cap P \neq \varnothing\}$,
4. $H' = \{X \cap P : Z \subseteq X\}$.

The example suggests that the idea of irrevocable belief revision may be combined with many modelings of "ordinary" belief revision.[2]

Acknowledgments The author is grateful to John Cantwell for helping to sort out a number of confusions and to the anonymous referees.

NOTES

1. One referee found Lemma 5.3(e) "very interesting" and went on to make the following comment: "It states, for the principal case, where a revision of $\varphi \wedge \psi$ is feasible, a definition of how to revise plausibility relations. My conjecture is that the biconditional $\langle *\varphi \rangle (\theta \leq \psi) \equiv ((\varphi \wedge \theta) \leq (\varphi \wedge \psi))$ fully characterizes the author's method of irrevocable revision (save perhaps for the limiting case in which $\neg \langle *(\varphi \wedge \psi) \rangle \mathbf{b} \top$) obtains. Given the equivalence of Grove plausibilities and Gärdenfors-Makinson entrenchments, it turns out that the method was briefly discussed (but finally rejected) by Rott in a paper published in 1991." Evidently, the paper referred to here is [7].

2. The idea underlying the modeling of IR (as distinct from the idea of exploring it with the help of dynamic deontic logic) arose in conversation between Horacio Arló Costa and the author. At one time we had hoped to write a joint paper; a joint abstract was presented by title at the Scandinavian Logic Symposium in Uppsala 1997 but unfortunately never published. The author gratefully remembers many discussions with Arló Costa. In particular, it was Arló Costa who first observed that the McGee example can be handled by IR; see [2].

REFERENCES

[1] Alchourrón, C., P. Gärdenfors, and D. Makinson, "On the logic of theory change," *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 510–30. Zbl 0578.03011 MR 87c:03020 1

[2] Arló Costa, H., "Iterated epistemic conditionals," pp. 14–31 in *The Parikh Project: Seven Papers in Honour of Rohit*, Uppsala Prints and Preprints in Philosophy 18, Uppsala, 1996. 9

[3] Grove, A., "Two modelings for theory change," *Journal of Philosophical Logic*, vol. 17 (1988), pp. 157–70. 1, 9

[4] Lewis, D., *Counterfactuals*, Blackwell, Oxford, 1973. Zbl 01700109 MR 54:9979 1

[5] Lindström, S., and W. Rabinowicz, "Epistemic entrenchment with incomparabilities and relational belief revision," pp. 93–126 in *The Logic of Theory Change*, edited by A. Fuhrmann and M. Morreau, Springer-Verlag, Berlin, 1991. Zbl 0925.03128 MR 1 096 792 1, 9

[6] McGee, V., "A counterexample to modus ponens," *Journal of Philosophy*, vol. 82 (1985), pp. 462–71. MR 87c:03034 4

[7] Rott, H., "Two methods of constructing contractions and revisions of knowledge systems," *Journal of Philosophical Logic*, vol. 20 (1991), pp. 49–73. Zbl 0723.03009 MR 92g:68136 9

[8] Segerberg, K., "Belief revision from the point of view of doxastic logic," *Bulletin of the IGPL*, vol. 3 (1995), pp. 534–53. Zbl 0844.03015 MR 96d:03041 1, 6, 6

[9] Segerberg, K., "Proposal for a theory of belief revision along the lines of Lindström and Rabinowicz," *Fundamenta Informaticae*, vol. 32 (1997), pp. 183–91. Zbl 0890.03011 MR 99f:03032 9

Department of Philosophy
University of Uppsala
Drottninggatan 4
753 10 Uppsala
SWEDEN
email: krister.segerberg@filosofi.uu.se