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No Arbitrage and General Semimartingales

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Abstract: No free lunch with vanishing risk (NFLVR) is known to be equivalent to the existence of an equivalent martingale measure for the price process semimartingale. We give necessary conditions for such a semimartingale to have the property NFLVR. We also extend Novikov's criterion for the stochastic exponential of a local martingale to be a martingale to the general case; that is, the case where the paths need not be continuous.

1. Introduction

The question of whether the absence of arbitrage is equivalent to the existence of an equivalent measure has now been clarified for some time, in the papers of Delbaen and Schachermayer ([5] and [6]). They showed that one has no arbitrage in the sense of *no free lunch with vanishing risk* if and only if there exists an equivalent probability measure rendering the price process a sigma martingale. (In the continuous case, all sigma martingales are local martingales.) Their conditions, known by its acronym NFLVR, imply also that the price process must be a semimartingale, as a consequence of the Bichteler-Dellacherie theorem. Therefore a natural question arises: which semimartingales actually satisfy NFLVR, and thus can be used to model price processes in arbitrage free models? To analyze this, one wants to give conditions on the original semimartingale which imply that it is a sigma martingale after one changes to a risk neutral measure. Once one has the risk neutral measure, checking when a semimartingale is a sigma martingale follows from Proposition 6.35 on page 215 of [10]; what we are concerned with here is giving the conditions on the original semimartingale, before the change to the risk neutral measure. Partial results in this direction have been obtained by E. Strasser (see [28]) in the continuous case, and also by E. Eberlein and J. Jacod (see [9]) in the case of geometric Lévy processes. In the first half of this paper we consider the general situation and obtain primarily necessary conditions for a semimartingale price process to engender a model without arbitrage. Our primary result in this half is Theorem 2.

When dealing with sufficient conditions, some difficult issues arise: how does one find an equivalent sigma martingale measure? Obvious constructions lead to

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*Tom Kurtz has been a leader in the field of stochastic processes his entire career, and without his vision, guidance, and generosity, the field would be much the poorer. The authors wish to dedicate this paper to Tom Kurtz on his sixty-fifth birthday.

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measures which *a priori* could be sub probability measures, and not true probability measures. The Radon Nikodym densities of these measures can often be constructed as stochastic exponentials of local martingales. A classic tool (in the continuous case) used to verify that the exponential of a local martingale is itself a martingale, and not just a supermartingale, is Novikov's theorem. Often Novikov's theorem is insufficient, but it is always appealing due to its simple nature and ease of computation. In the second half of this paper we propose an analog of Novikov's criterion for the general case (that is, the case with jumps). Our results build on the pioneering work of J. Mémin, A.S. Shiryaev, and their co-authors. Our primary result in this half of the paper is Theorem 9.

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2. Necessary Conditions for No Arbitrage

2.1. The Continuous Case

Let $X_t = X_0 + M_t + A_t$, $t \geq 0$ be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Here M represents the continuous local martingale part and A is a process with paths of finite variation on compact time sets, almost surely. We seek necessary and sufficient conditions such that there exists an equivalent probability measure P^* such that X is a P^* sigma martingale. Since X is continuous, and since all continuous sigma martingales are in fact local martingales, we need only concern ourselves with local martingales. Theorem 1 below is essentially already well known. See for example ([27]), Theorem 1, which itself has its roots in ([1]); we include it here for the reader's convenience, and since it illustrates what we are trying to do in Theorem 2.

Theorem 1. *Let $X_t = X_0 + M_t + A_t$, $0 \leq t \leq T$ be a continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. Let $C_t = [X,X]_t = [M,M]_t$, $0 \leq t \leq T$. There exists an equivalent probability measure P^* on $\mathcal{F}_T$ such that X is a P^* sigma martingale only if*

1. *$dA \ll dC$ a.s.;*
2. *If J is such that $A_t = \int_0^t J_s dC_s$ for $0 \leq t \leq T$, then $\int_0^T J_s^2 dC_s < \infty$ a.s.;*

   If in addition one has the condition below, then we have sufficient conditions for there to exist an equivalent probability measure P^* on $\mathcal{F}_T$ such that X is a P^* sigma martingale:
3. *$E\{\mathcal{E}(-J \cdot M)_T\} = 1$, where $\mathcal{E}(U)$ denotes the stochastic exponential of a semimartingale U.*

Proof. Suppose there exists P^* equivalent to P such that X is a P^* local martingale. Let $Z = \frac{dP^*}{dP}$ and let $Z_t = E\{Z|\mathcal{F}_t\}$ for all t, $0 \leq t \leq T$. We then have, by Girsanov's theorem, that the decomposition of X under P^* is given by:

$$X_t = X_0 + \left\{ M_t - \int_0^t \frac{1}{Z_s} d[Z,M]_s \right\} + \left\{ A_t + \int_0^t \frac{1}{Z_s} d[Z,M]_s \right\}. \tag{1}$$

Since X is a P^* local martingale and continuous semimartingales have unique decompositions of the type (1), we conclude that we must have

$$A_t = -\int_0^t \frac{1}{Z_s} d[Z, M]_s \tag{2}$$

and since further, by the Kunita-Watanabe inequality, we have $d[Z, M] \ll d[M, M]$ a.s., we conclude that for some predictable process J that

$$A_t = \int_0^t J_s d[M, M]_s. \tag{3}$$

Since Z is a strictly positive P martingale, we can write it as a solution of an exponential equation. (Note that even though X is assumed to be a continuous semimartingale, that does not imply that Z too must be continuous.)

$$Z_t = 1 + \int_0^t Z_{s-} dY_s, \qquad Z_0 = 1 \tag{4}$$

for a local martingale Y with $Y_0 = 0$. This Y is often called a stochastic logarithm of Z and is given by $Y = \int_0^t (1/Z_{s-}) dZ_s$. A local martingale Y has a decomposition $Y = Y^c + Y^d$ where Y^c is a continuous martingale part of Y and Y^d is a purely discontinuous martingale part of Y. (See [10, P.85]) Since Y^c is continuous, it is locally square integrable local martingale. Therefore we have a unique representation of the form

$$Y_t^c = \int_0^t H_s dM_s + N_t^c \tag{5}$$

where H is a predictable process such that the stochastic integral in (5) exists, and N^c is a continuous local martingale orthogonal to $H \cdot M$ in the sense that $[H \cdot M, N]^c$ is a local martingale. Since the stochastic integral in (5) exists, we have of necessity that $\int_0^t H_s^2 d[M, M]_s < \infty$ a.s. for each t, $0 \le t \le T$. Let's let $N = N_t^c + Y_t^d$. Then

$$[H \cdot M, N] = [H \cdot M, N^c] + [H \cdot M, Y^d] = [H \cdot M, N^c]. \tag{6}$$

It follows that $[H \cdot M, N]$ is also a local martingale and we have a decomposition of Y into two orthogonal components:

$$Y_t = \left(\int_0^t H_s dM_s + N_t^c \right) + Y_t^d = \int_0^t H_s dM_s + \left(N_t^c + Y_t^d \right) = \int_0^t H_s dM_s + N_t. \tag{7}$$

We next apply the Meyer-Girsanov theorem to calculate the decomposition of X under P^*. (Since M is continuous there is no issue about the existence of $d\langle Z, M\rangle_s$.) We get:

$$X_t = X_0 + \left\{ M_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s \right\} + \left\{ A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s \right\}.$$

By the uniqueness of the decomposition, we must have

$$\begin{aligned} A_t &= -\int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s = -\langle M, Y\rangle = -\langle M, H \cdot M + N\rangle \\ &= -\int_0^t H_s d[M, M]_s = \int_0^t J_s dC_s, \end{aligned}$$

and by the definition of C we have $-H = J,\ d[M,M]_s dP$ almost everywhere. Thus $\int_0^t J_s^2 d[M,M]_s < \infty$ for all t, $0 \le t \le T$. This gives the necessity.

For the sufficiency, let us take Z to be the stochastic exponential of $-J \cdot M$. Applying the Meyer-Girsanov theorem we again have

$$X_t = X_0 + \left\{ M_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s \right\} + \left\{ A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s \right\}$$

and by construction we have that $\{A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s\} = 0$. The process Z is a strictly positive local martingale with $Z_0 = 1$, hence it is a positive supermartingale, and it is a martingale as soon as $E\{Z_t\} = 1$ for all t, $0 \le t \le T$. If Z is known to be a martingale on $[0,T]$ then we define P^* by $dP^* = Z_T dP$, and we can conclude that $\{M_t - \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s\}$ is a local martingale under P^*. However the third hypothesis guarantees that Z is a martingale and hence that P^* is a probability measure (and not a sub probability measure), and we have sufficiency. □

Remark 1. *The sufficiency is not as useful in practice as it might seem. The first two conditions should be, in principle, possible to verify, but the third condition is in general not. Depending on the structure of Y, different techniques are available. An obvious one is Novikov's condition, but while easy to state, this too is difficult to verify in practice.*

Remark 2. *If condition (2) of Theorem 1 is satisfied for* **all** ω *(instead of P-a.s.), then condition (3) is automatically satisfied. (See, for example, [20]) This is sufficient but not necessary in general. This difference seems subtle but plays an important role. Essentially this is because a probability measure P^* such that X is P^*-local martingale, if it exists, is not necessarily equivalent to P in general a priori.*

Remark 3. *Condition (1) is often called a structure condition (SC) in the literature. See for example Schweizer [26, page 1538]. Also see Jarrow and Protter (2004) [11] for a constructive example of an arbitrage opportunity when this condition is violated.*

Remark 4. *In an interesting paper, Strasser [28] discusses a similar problem in the case of continuous semimartingales. She focuses on the condition (1), and does not take the approach we take here.*

2.2. General Case

The techniques used in the continuous case break down in the general case (ie, the case with jumps). The reason is that to use formally the same ideas, one would need to use the Meyer-Girsanov theorem, which requires the existence of the process $\langle Z, M\rangle$. When M has continuous paths, such a process always exists, even if Z can have jumps. But if both Z and M have jumps, then the process $\langle Z, M\rangle$ exists if and only if the process $[Z,M]$ is locally integrable, which need not in general be the case. (We mention here that $[Z,M]$ is called locally integrable if there exists a sequence of stopping times $(\tau_n)_{n\ge 1}$ such that $\tau_{n-1} \le \tau_n$ a.s. for each $n \ge 1$, $\lim_{n\to\infty} \tau_n \ge T$ a.s., and $E\{[Z,M]_{\tau_n}\} < \infty$ for each $n \ge 1$). A technique developed to circumvent this kind of technical integrability problem is that of random measures, and in particular the use of the *characteristics* of a semimartingale. We assume the reader is familiar with the basic definitions and theorems concerning the characteristics of a semimartingale. We refer the reader to (for example) [10] for an expository treatment of them.

Let X be an arbitrary semimartingale with characteristics (B, C, ν) on our usual filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Then there exists a predictable process A_t with $A_0 = 0$ such that

$$(8) \qquad \nu(ds, dx) = K_s(\omega, dx) dA_s(\omega), \qquad C_t = \int_0^t c_s dA_s, \qquad B_t = \int_0^t b_s dA_s.$$

Let P^* be another probability measure equivalent to P. Then of course X is a semimartingale under P^*, with characteristics (B^*, C, ν^*). (We write C instead of C^* because it is the same process for any equivalent probability measure.) We then know (see Theorem 3.17 on page 170 of [10]) that the random measure ν^* is absolutely continuous with respect to ν, and that there exists a predictable process (predictable in the extended sense) $Y(s, x)_{s \geq 0, x \in \mathbb{R}}$ such that

$$(9) \qquad \nu^* = Y \cdot \nu.$$

We have the following theorem, which gives necessary conditions for X to have no arbitrage in the Delbaen-Schachermayer sense of "No free lunch with vanishing risk," hereafter abbreviated as **NFLVR**. See Delbaen and Schachermayer [5] or alternatively [12]. One can also consult [13]. In Kabanov's paper [13], conditions are given for a semimartingale to be a sigma martingale; these are also given in [10]. In the theorem below we present conditions on the semimartingale such that it is not necessarily a sigma martingale, but that it is one when viewed under a risk neutral measure, which of course is a different situation. The authors just learned that Karatzas and Kardaras ([15]) have recently obtained similar results, although in a different context.

Theorem 2. *Let X be a P semimartingale with characteristics (B, C, ν). For X to have an equivalent sigma martingale measure and hence satisfy the NFLVR condition, there must exist a predictable process $\beta = (\beta_t)_{t \geq 0}$ and an (extended) predictable process $Y(\cdot, t, x)$ such that following four conditions are satisfied:*

1. $b_t + \beta_t c_t + \int \{x(Y(t, x) - 1_{\{|x| \leq 1\}}) K_t(dx) = 0$; *$P(d\omega) dA_s(\omega)$ almost everywhere;*
2. $\int_0^T \beta_s^2 dC_s < \infty$, *a.s.;*
3. $\Delta A_t > 0$ *implies that* $\int x Y(s, x) K(s, dx) = 0$;
4. $\int |x^2| \wedge |x| Y(t, x) K_t(dx) < \infty$, *$P(d\omega) dA_s(\omega)$ almost everywhere,*

where the predictable process A_t is defined by (8).

Proof. Our primary tool will be the Jacod-Mémin version of a Girsanov theorem with characteristics (see Theorem 3.24 on page 172 of [10]). Let P^* be an equivalent sigma martingale measure. Let (B^*, C^*, ν^*) be the characteristics of X under P^*. Then there exist c_s, b_s^*, K_s^* such that

$$(10) \qquad C_t^* = c_s dA_s,\ B_t^* = \int_0^t b_s^* dA_s;\ \nu^*(ds, dx) = dA_s K_s^*(\omega, dx).$$

Note that in the above, we write c and not c^*, and also A and not A^*, since under our hypothesis, we can take $A^* = A$. In addition the process C does not change under an equivalent change of measure. We next invoke Proposition 6.35 on page 215 of [10] to conclude that X is a P^* sigma martingale if and only if

1. $b_t^* + \int x 1_{\{|x|>1\}} K_t^*(dx) = 0$, $P(d\omega)dA_s(\omega)$, almost everywhere;
2. When $\Delta A_t > 0$ then $\int x K_t^*(dx) = 0$; and
3. $\int |x^2| \wedge |x| K_t^*(dx) < \infty$, $P(d\omega)dA_s(\omega)$, almost everywhere.

We wish to interpret these three conditions in terms of the original characteristics under P. We know from the continuous case that for C we need a new predictable process coming from the density process Z of $\frac{dP^*}{dP}$, which we denote β, with the property that $\int_0^t \beta_s^2 dC_s = \int_0^t \beta_s^2 c_s dA_s < \infty$ a.s. We also use a key fact that ν^* must be absolutely continuous with respect to ν and there must exist $Y = Y(s,x)$, predictable in the extended sense, such that

$$\nu^* = Y \cdot \nu. \tag{11}$$

This is proved in Theorem 3.17 on page 170 of [10]. (We remark that both β and Y derive from the P martingale Z where $Z_T = \frac{dP^*}{P}$, with β coming from the continuous martingale part of Z, and Y coming from the 'purely discontinuous' part of Z.) Moreover since for any bounded U we have

$$\begin{aligned}\int_0^t \int U(\omega,s,x)\nu^*(dx,ds) &= \int_0^t \int U(\omega,s,x) dA_s K_s^*(dx) \\ &= \int_0^t \int U(\omega,s,x) Y(\omega,s,x) dA_s K_s(dx)\end{aligned}$$

we can conclude that $K^* = Y \cdot K$,

Now we need only to re-express the three conditions in (2.2) to conclude that we must have:

1. $b_t + \beta_t c_t + \int \{x(Y(t,x) - 1_{\{|x|\le 1\}}) K_t(dx) = 0$; $P(d\omega)dA_s(\omega)$ almost everywhere;
2. $\int_0^T \beta_s^2 dC_s < \infty$, a.s.;
3. $\Delta A_t > 0$ implies that $\int x K_t^*(dx) = \int x Y(s,x) K(s,dx) = 0$;
4. $\int |x^2| \wedge |x| K_t^*(dx) = \int |x^2| \wedge |x| Y(t,x) K_t(dx) < \infty$, $P(d\omega)dA_s(\omega)$ almost everywhere.

□

Corollary 3. *Let X be a semimartingale as in Theorem 2. Suppose in addition that $\mathbb{F}$ is a quasi-left continuous filtration. If X is a P^* sigma martingale, then we must have the following three conditions satisfied:*

1. $b_t + \beta_t c_t + \int \{x(Y(t,x) - 1_{\{|x|\le 1\}}) K_t(dx) = 0$; *$P(d\omega)dA_s(\omega)$ almost everywhere;*
2. $\int_0^T \beta_s^2 dC_s < \infty$, *a.s.;*
3. $\int |x^2| \wedge |x| Y(t,x) K_t(dx) < \infty$, *$P(d\omega)dA_s(\omega)$ almost everywhere.*

Proof. We are able to remove the condition on the jumps of A because if $\mathbb{F}$ is quasi-left continuous, then A does not jump, it being increasing and predictably measurable. □

Corollary 4. *Let X be a semimartingale as in Theorem 2. If X is a P^* local martingale, then we must have the following three conditions satisfied:*

1. $b_t + \beta_t c_t + \int \{x(Y(t,x) - 1_{\{|x|\le 1\}}) K_t(dx) = 0$; *$P(d\omega)dA_s(\omega)$ almost everywhere;*
2. $\int_0^T \beta_s^2 dC_s < \infty$, *a.s.;*

3. $\Delta A_t > 0$ *implies that* $\int xY(s,x)K(s,dx) = 0$*;*

and if the filtration $\mathbb{F}$ *is quasi-left continuous, we must have the following two conditions satisfied:*

1. $b_t + \beta_t c_t + \int \{x(Y(t,x) - 1_{\{|x|\leq 1\}})K_t(dx) = 0$*;* $P(d\omega)dA_s(\omega)$ *almost everywhere;*
2. $\int_0^T \beta_s^2 dC_s < \infty$*, a.s.*

Proof. This follows from Theorem 2 and Proposition 6.35 on page 215 of [10]. That quasi-left continuity of $\mathbb{F}$ implies we can drop the condition on the jumps of A is a trivial consequence of A not having jumps when the filtration is quasi-left continuous. □

Remark 5. *Comparing Theorem 1 and Theorem 2 illustrates how incompleteness of the market corresponding to the price process* X *can arise in two different ways. Theorem 1 shows that (in the continuous case) the choice of the orthogonal martingale* N *is essentially arbitrary, and each such choice potentially leads to a different equivalent probability measure rendering* X *a local martingale. Theorem 2 shows that in the general case (the case where jumps are present) incompleteness can still arise for the same reasons as in the continuous case, but also because of the jumps, through the choice of* Y*. Indeed, we are free to change* Y *appropriately at the cost of changing* b*. Only if* K *reduces to a point mass is it then possible to have uniqueness of* P^* *(and hence market completeness), and then of course only if* $C = 0$*.*

Remark 6. *For the special case where* X *is a geometric Lévy process, Eberlein and Jacod [9] give a necessary and sufficient condition for the existence of an equivalent martingale measure.*

We can derive a structure condition for the general case, with an additional hypothesis involving integrability.

Theorem 5. *Let* X *be a special semimartingale with characteristics* (B, C, ν)*. Then* X *has a canonical decomposition* $X = X_0 + M + A$*. Assume* $(x^2 \wedge x) * \nu_t < \infty$*. If there exists* P^* *such that* X *is* P^* *local martingale, then*

$$dA_t \ll d\left[C_t + (x^2 \wedge x) * \nu_t\right]. \tag{12}$$

In particular if X *is locally square integrable then* $\langle M, M\rangle$ *exists and*

$$dA_t \ll d\langle M, M\rangle_t \quad a.s. \tag{13}$$

Proof. Suppose an equivalent local martingale local measure P^* exists. Let (B, C, ν) and (B^*, C, ν^*) be characteristics of X under P and P^* with truncation function $h(x) = 1_{\{|x|\leq 1\}}$. Let μ be a jump measure of X. Since X is P^*-local martingale, $(x1_{\{|x|>1\}}) * \mu$ is P^*-locally integrable and X has a representation:

$$X_t = X_0 + X_t^{c*} + x * (\mu - \nu^*)_t. \tag{14}$$

Since $P \ll P^*$ by hypothesis, applying Girsanov's theorem (Theorem 3.24 in page 172 of [10]), there exists a predictable process β' and $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ measurable non-negative function Y' such that

$$B = B^* + \beta' \cdot [X^c, X^c] + 1_{\{|x|<1\}}(Y - 1) * d\nu^* \tag{15}$$

$$\nu = Y' \cdot \nu^* \tag{16}$$

where $|1_{|x|\geq 1}(Y-1)| * \nu^*_t < \infty$ P^*-a.s. Under P, since we assume X is a special semimartingale, $x1_{\{|x|>1\}} * \mu$ is P-locally integrable and

$$X_t = X_0 + X^c_t + x*(\mu-\nu)_t + x*(\nu-\nu^*) + \int_0^t \beta'_s dC_s = X_0 + M_t + A_t. \tag{17}$$

By the uniqueness of Doob–Meyer decomposition, we have

$$A_t = x(1-Y')*\nu_t + \int_0^t \beta'_s dC_s. \tag{18}$$

Then clearly $A_t \ll d((x^2\wedge x)*\nu_t + C_t)$. Finally suppose in addition that X is locally square integrable. Then $[M,M]$ is locally integrable and $\langle M,M\rangle = C + x^2*\nu$ exists. It is clear from (18) that $dA_t \ll d\langle M,M\rangle_t$, a.s. □

Remark 7. *The case when X is locally bounded (and hence X is a special semimartingale such that M is automatically a locally square integrable local martingale) is shown by Delbaen and Schachermayer [7, Theorem 3.5]. Theorem 5 extends their result to the case when X is not necessarily locally bounded. In addition, Theorem 5 does not depend on the notion of admissibility.*

Remark 8. *The structure condition has a clear economic interpretation. On the set E such that $\int_E d\langle M,M\rangle = 0$, M is constant and $P(\int_E\int_{\mathbb{R}} |x|\mu(dx,dt) = 0) = 1$ where μ is a jump measure of X. Therefore any trading strategy supported on E is risk-free in the sense that any movement of X comes from the predictable component A and hence we can construct a trading strategy which takes advantage of the information of an infinitesimal future. Indeed it is easy to construct such a trading strategy to exploit an arbitrage opportunity if $dA \not\ll d\langle M,M\rangle$: Consider a price process X on a finite time horizon $[0,T]$. Without loss of generality, we assume that A is an increasing predictable process. Suppose there exists a set $E \in \mathcal{B}(\mathbb{R}_+)$ such that $\int_E d\langle M,M\rangle_s = 0$ but $P(\int_E dA_s > 0) > \eta$ for some $\eta > 0$. Let $A^c_t = A_t - \sum_{0\leq s\leq t}\triangle A_s$. Let h_t be a predictable process defined by*

$$h_t = A^c_t 1_{E\cap\{t:\triangle A_t=0\}} + sgn(\triangle A_t)1_{\{\triangle A_t\neq 0\}}. \tag{19}$$

The following equation is well defined:

$$\begin{aligned}\int_0^T h_s dX_s &= \int_E h_s dM_s + \int_{E\cap\{\triangle A_t=0\}} A^c_s dA^c_s + \sum_{s\leq T}|\triangle A_s| \\ &= \frac{1}{2}\int_E d((A^c_s)^2) + \sum_{s\leq T}|\triangle A_s|.\end{aligned} \tag{20}$$

Therefore $P(\int_0^T h_s dX_s \geq 0) = 1$ and $P(\int_0^T h_s dX_s > 0) > \eta > 0$. Since $(h\cdot X)_t \geq 0$ for all $t\in[0,T]$, h is a 0-admissible trading strategy and hence this is an arbitrage opportunity.

3. Stochastic exponential of local martingales

3.1. Definition and notations

One of the key components of the sufficient conditions for no arbitrage is that a martingale density Z be a true martingale. However it is not easy to verify this directly in general. The literature is rich on this topic especially for the case when a martingale density Z is continuous. For example Novikov [22], Kazamaki [16], [17], [18], Cherney & Shiryaev [4] studied this question for the continuous case and derived several sufficient conditions in terms of integrability conditions. Mémin [21], Lépingle and Mémin [19], and Kallsen and Shiryaev [14] studied the same question in a general (non-continuous) setting.

The purpose of this section is to show that a formula similar to the famous Novikov condition works in a general setting. More precisely we want to show that a Novikov-type condition $E[\exp\{c\langle M,M\rangle\}]<\infty$ for some c is sufficient to show that $\mathcal{E}(M)$ is a martingale. This condition belongs to the *predictable type* introduced by Revuz and Yor [24].

It should be noted that a Novikov-type condition is often difficult (even in the continuous case) to apply directly. Therefore a common use of this type of condition occurs together with a localization argument. We illustrate this with Example 14.

Let $M=\{M_t\}_{t\geq 0}$ be a càdlàg local martingale vanishing at 0 on a given filtered probability space $(\Omega,\mathcal{F}_t,\mathbb{F},P)$. A process $X=\{X_t\}_{t\geq 0}$ defined by

$$X_t=\exp\left\{M_t-\frac{1}{2}[M,M]^c_t\right\}\prod_{s\leq t}(1+\triangle M_s)\exp(-\triangle M_s) \tag{21}$$

is called a *Doléans exponential* or the stochastic exponential of M, and it is denoted by $\mathcal{E}(M)_t$. X_t is also given as a solution of the stochastic differential equation

$$dX_t=X_{t-}dM_t,\qquad X_0=1. \tag{22}$$

Since X_- is left continuous and therefore locally bounded, it follows that $\mathcal{E}(M)$ is a local martingale in all cases. When $\triangle M_t>-1$ for all t a.s., it is a positive local martingale. By Fatou's lemma, $\mathcal{E}(M)$ is also a positive supermartingale. Throughout this paper, we assume that $\triangle M_t>-1$.

3.2. Results

3.2.1. Lévy process and additive martingales

We start with a basic result shown by Lépingle and Mémin [19].

Theorem 6 (Lépingle and Mémin)**.** *Let M be a local martingale. If the compensator C of the process*

$$\langle M^c,M^c\rangle_t+\sum_{s\leq t}\triangle M_s^2 1_{\{|\triangle M_s|\leq 1\}}+\sum_{s\leq t}\triangle M_s 1_{\{|\triangle M_s|>1\}} \tag{23}$$

is bounded, then $E[\sqrt{[\mathcal{E}(M),\mathcal{E}(M)]_t}]<\infty$. In particular, $\mathcal{E}(M)_t$ is a martingale.

Proof. See Lépingle and Mémin [19]. □

Although the requirement of boundedness looks strong, it is enough to show the following well known fact:

Corollary 7. *Let M be a Lévy martingale. Then $\mathcal{E}(M)_\cdot$ is a martingale.*

Proof. Fix $T > 0$ and let $M_t = M_t^T$. Then by the Lévy decomposition theorem,

$$M_t = W_t + \int_{|x|<1} x\,(N(\cdot,[0,t],dx) - t\nu(dx)) + \left(\sum_{0<s\le t} \triangle M_s 1_{\{|\triangle M_s|\ge 1\}} - \alpha t\right)$$

where W is a Brownian motion, N is a Poisson random measure with mean measure $dt\ \nu(dx)$, $\int 1 \wedge x^2\nu(dx) < \infty$ and $\alpha t = E(\sum_{0<s\le t} \triangle M_s 1_{\{|\triangle M_s|\ge 1\}})$. Then the compensator of $[W,W]_t$, $\sum_{0<u\le t} \left|\triangle M_u^1\right|$, $\sum_{0<u\le\cdot} \left(\triangle M_u^2\right)^2$ are bounded by T, $T\int_{|x|<1} x^2\nu(dx)$ and αT respectively. Therefore the preceding theorem implies that $M_\cdot^T$ is a uniformly integrable martingale for all $T \in \mathbb{R}_+$. Thus M is a martingale. □

3.2.2. A general result

Theorem 8. *Let M be a martingale such that $\triangle M > -1$. If the process*

$$A_t = \frac{1}{2}\langle M^c, M^c\rangle_t + \sum_{s\le t\wedge T} \{(1+\triangle M_s)\ln(1+\triangle M_s) - \triangle M_s\} \tag{24}$$

has compensator $B = (B_t)_{t\ge 0}$ which satisfies

$$E\left[e^{B_\infty}\right] < \infty, \tag{25}$$

then $\mathcal{E}(M)$ is a uniformly integrable martingale and $\mathcal{E}(M)_\infty > 0$ a.s.

Proof. See Lépingle and Mémin [19, III.1]. Note that $\langle M^c, M^c\rangle_t = [M,M]_t^c$, which is the way it is denoted in [23]. We have changed the notation here for consistency within Theorem 9, which follows. □

For a local martingale M, we let M^d denote its 'purely discontinuous' part; that is, $M = M^c + M^d$, where M^c is a local martingale with continuous paths, and $[M^c, M^d]$ is also a (possibly zero) local martingale. (See, e.g., [23, page 193] for the case of a square integrable martingale M, and [8] for the more general case of a local martingale.)

Theorem 9. *Let M be a locally square integrable martingale such that $\triangle M > -1$. If*

$$E\left[e^{\frac{1}{2}\langle M^c,M^c\rangle_T + \left\langle M^d,M^d\right\rangle_T}\right] < \infty, \tag{26}$$

where M^c and M^d are continuous and purely discontinuous martingale parts of M, then $\mathcal{E}(M)$ is a martingale on $[0,T]$, where T can be ∞.

Proof. Let $f(x) = (1+x)\ln(1+x) - x$, $g(x) = x^2$. For $x > -1$, $g(x) \ge f(x)$. It follows that

$$L_t \triangleq \sum_{s\le t} \{g(\triangle M_s) - f(\triangle M_s)\} \ge 0 \tag{27}$$

is a locally integrable increasing process and there exists a compensator $\tilde{L}_t$ of L_t by the Doob–Meyer decomposition. Since $\tilde{L}_t = \widetilde{\sum_{s\le t} g(\triangle M_s)} - \widetilde{\sum_{s\le t} f(\triangle M_s)} \ge 0$,

$$\widetilde{\sum_{s\le t} f(\triangle M_s)} \le \widetilde{\sum_{s\le t} g(\triangle M_s)} = \widetilde{\sum_{s\le t} \triangle[M,M]_s} = \left\langle M^d, M^d\right\rangle_t.$$

Thus $E\left[e^{\frac{1}{2}\langle M^c,M^c\rangle_T+\langle M^d,M^d\rangle_T}\right]<\infty$ implies the conditions of Theorem 8 and $\mathcal{E}(M)$ is a martingale on $[0,T]$. □

A natural question is whether we can improve the constant multiplying $\langle M^d,M^d\rangle$. The next example shows that the answer is negative. Namely $E\left(e^{(1-\varepsilon)\langle M,M\rangle_\infty}\right)<\infty$ for any $\varepsilon>0$ is not sufficient in general.

Example 10 ($\alpha<1$ is not sufficient)**.** *Let $(N_t)_{t\geq 0}$ be a standard Poisson process. Define $T^b=\inf\{s:N_s-(1-b)s=1\}$ for $b\in(0,1)$. Let $U_n=\inf_s\{s:N_s=n\}$. Then $P(T^b=\infty)=0$ since $\{T^b=\infty\}=\bigcap_n\left\{U_n\geq\frac{n-1}{1-b}\right\}$ and $\frac{U_n}{n}\to 1$ almost surely. Then*

$$N_{T^b}-(1-b)T^b=1 \qquad a.s. \tag{28}$$

The moment generating function of $N_t-(1-b)t$ exists and for any $\lambda\in\mathbb{R}$

$$E\left[\exp\left\{-\lambda\left[N_t-(1-b)t\right]\right\}\right]=e^{\lambda(1-b)t}E\left[\exp\left(-\lambda N_t\right)\right]=\exp\left\{tf(\lambda)\right\} \tag{29}$$

where

$$f(\lambda)=e^{-\lambda}+\lambda(1-b)-1 \tag{30}$$

and $Z_t:=\exp\{-\lambda\{N_t-(1-b)t\}-tf(\lambda)\}$ is a martingale. Since Z_t is non-negative, by Doob's supermartingale inequality, $E(Z_{T^b})\leq E(Z_0)=1$. From (28), we obtain

$$E\left[\exp\left\{-\lambda\{N_{T^b}-(1-b)T^b\}-T^bf(\lambda)\right\}\right]=E\left[\exp\left\{-\lambda-T^bf(\lambda)\right\}\right]\leq 1$$

$$E\left[e^{-T^bf(\lambda)}\right]\leq e^{\lambda}. \tag{31}$$

Now define $M_t=-a(N_t-t)^{T^b}$ where $a\in(0,1)$. M_t is martingale and $\mathcal{E}(M)_t=\exp\{N_{t\wedge T^b}\ln(1-a)+a(t\wedge T^b)\}$.

$$\begin{aligned} E\left[\mathcal{E}(M)_{T^b}\right]&=E\left[\exp\{N_{T^b}\ln(1-a)+aT^b\}\right]\\ &=E\left[\exp\left\{\left\{1+(1-b)T^b\right\}\ln(1-a)+aT^b\right\}\right]\\ &=(1-a)E\left[\exp\left\{T^b\left\{(1-b)\ln(1-a)+a\right\}\right\}\right].\end{aligned} \tag{32}$$

Let $\lambda^*=\ln\left(\frac{1}{1-a}\frac{1+b}{2}\right)$. Then by (30), $-f(\lambda^*)=-\frac{1-a}{(1+b)/2}+(1-b)\ln\left(\frac{1-a}{(1+b)/2}\right)+1$. Next define

$$k(a,b)=-f(\lambda^*)-\{(1-b)\ln(1-a)+a\}.$$

Simplifying terms,

$$k(a,b)=(1-b)\ln\left(\frac{1}{(1+b)/2}\right)-(1-a)\left(\frac{1}{(1+b)/2}-1\right). \tag{33}$$

Let $g(b)=1-\frac{(1-b)\ln\{2/(1+b)\}}{2/(1+b)-1}$. $k(a,b)>0$ if $a>g(b)$. Observe that on $\{b:0<b<1\}$, $g(b)$ is an increasing function of b and $1-\ln 2<g(b)<1$. Thus for every $b\in(0,1)$, there exists $a^*\in(0,1)$ such that for all $a\geq a^*$, $k(a,b)>0$. Fix b and choose a so that $k(a,b)>0$. Then by (31),

$$\begin{aligned} E\left[\mathcal{E}(M)_{T^b}\right]&=(1-a)E\left[\exp\left\{T^b\left\{(1-b)\ln(1-a)+a\right\}\right\}\right]\\ &\leq(1-a)E\left[\exp\left\{-T^bf(\lambda^*)\right\}\right]\\ &\leq(1-a)e^{\lambda^*}=\frac{1+b}{2}<1.\end{aligned} \tag{34}$$

$\mathcal{E}(M)$ is not a uniformly integrable martingale. $\langle M,M\rangle_t = a^2(t\wedge T_b)$ since a stopped predictable process is still predictable process. Finally, define $h(b)$ by

$$h(b) = b + (1-b)\ln(1-b) = -f(-\ln(1-b)) = max_\lambda\{-f(\lambda)\}.$$

For all $b : 0 < b < 1$, $h(b) < g(b)^2$. However for every $\varepsilon > 0$, there exists $b^*_\varepsilon < 1$ such that $b > b^*_\varepsilon$ implies $h(b) > (1-\varepsilon)g(b)^2$. Fix $\varepsilon > 0$. Let's choose $b' \in (b^*_\varepsilon, 1)$ and $a' \in \left(g(b'), \sqrt{h(b')/(1-\varepsilon)}\right) \subset (0,1)$ so that

$$\begin{cases} a' & > g(b') \\ h(b') & > (1-\varepsilon)a'^2. \end{cases} \tag{35}$$

By (31),

$$E\left[e^{(1-\varepsilon)\langle M,M\rangle_\infty}\right] = E\left[e^{(1-\varepsilon)a'^2 T_{b'}}\right] \le E\left[e^{(1-\varepsilon)h(b')T_{b'}}\right] < \frac{1}{1-b'} < \infty. \tag{36}$$

As seen in this example, jumps with size close to -1 prohibit the improvement of the condition. The following corollary shows that if $\triangle M$ is bounded away from -1, we can improve the results with a constant α. In particular, if $\triangle M > 0$, then $\alpha = 1/2$.

Corollary 11. *Fix $\varepsilon \in (0,1]$. Let M be a locally square integrable martingale such that $\triangle M > -1+\varepsilon$. Then there exists $\alpha(\varepsilon) \in [1/2, 1]$ such that*

$$E\left[e^{\frac{1}{2}\langle M^c,M^c\rangle_T + \alpha(\varepsilon)\langle M^d,M^d\rangle_T}\right] < \infty, \tag{37}$$

implies that $\mathcal{E}(M)_t$ is martingale on $[0,T]$, where T can be ∞.

Proof. Let $f(x) = (1+x)\ln(1+x) - x$, Then there exists $\alpha(\varepsilon) = \inf\{a : ax^2 - f(x) \ge 0 \text{ on } x > -1+\varepsilon\}$ such that $\alpha(\varepsilon) \in [1/2, 1]$. Especially when $\varepsilon = 1$, we can take $\alpha(\varepsilon) = 1/2$ and $\alpha(\varepsilon)$ is a decreasing function of ε. Let $g(x) = \alpha(\varepsilon)x^2$ For $x > 0$, $g(x) \ge f(x)$. It follows that

$$L_t \triangleq \sum_{s\le t}\{g(\triangle M_s) - f(\triangle M_s)\} \ge 0$$

is a locally integrable increasing process and there exists a compensator $\tilde{L}_t$ of L_t by the Doob–Meyer decomposition. Since $\tilde{L}_t = \sum_{s\le t}\widetilde{g(\triangle M_s)} - \sum_{s\le t}\widetilde{f(\triangle M_s)} \ge 0$

$$\sum_{s\le t}\widetilde{f(\triangle M_s)} \le \sum_{s\le t}\widetilde{g(\triangle M_s)} = \sum_{s\le t}\widetilde{\triangle[M,M]_s} = \langle M^d, M^d\rangle_t.$$

Thus $E\left[\exp\{1/2\langle M^c,M^c\rangle_t T + \alpha(\varepsilon)\langle M^d,M^d\rangle_T\}\right] < \infty$ implies the condition of theorem 8 and hence $\mathcal{E}(M)$ is martingale on $[0,T]$. □

Remark 9. *This integrability approach provides sufficient but not necessary conditions. While it is possible to derive a sequence of sufficient conditions converging in some sense to a necessary and sufficient condition, those stronger conditions become more difficult to verify at the same time. For details on this issue, see Kallsen and Shiryaev [14].*

In the continuous framework, it is well known that the Novikov condition is not optimal. The symmetric nature of a quadratic variation processes requires that if a *continuous martingale M satisfies Novikov's condition, $-M$ has to satisfy Novikov's condition as well. This implies that Novikov's condition is not applicable to identify a class of martingales M such that $\mathcal{E}(M)$ is a martingale but $\mathcal{E}(-M)$ is not. More generally, if there exists a predictable process h_s such that $\int_s h_s dM_s$ is a continuous local martingale satisfying Novikov's condition, then for all integrable predictable $g_s \in L(M)$ such that $\int |g_s|^2 d[M,M]_s = \int h_s^2 d[M,M]_s$, $\mathcal{E}(\int g_s dM_s)$ is a uniformly integrable martingale. See, for example, Stroock [29]. (The authors thank Marc Yor for calling this reference to their attention.)*

Some of these examples can be dealt with using a stronger condition derived in an integrability approach, such as Kazamaki's condition. But other examples requires totally different approaches. See for example Lipster and Shiryaev [20], Cheridito, Filipovic, and Yor [3].

Despite these examples showing its limitations, a Novikov-type condition is the kind of condition that we could hope to verify in a practical setting. This is due to the fact that the condition is given in terms of an increasing process and the quadratic variation of $\mathcal{L}og(Z)$, where $\mathcal{L}og(\cdot)$ denotes a stochastic logarithm.

3.3. Examples and applications

The following example shows that when the stochastic exponential comes from a driving Lévy martingale, then the condition in Theorem 9 becomes easier to compute. (We could phrase this as "Let M be a Lévy local martingale ..." but a Lévy process which is a local martingale is *a fortiori* a martingale, so it is not any more general, and indeed misleading, to state this example for Lévy local martingales.)

Example 12. *Let M be a Lévy martingale with Lévy triplet (B, C, ν). Let $h \in L(M)$ be a predictable process such that $X = \int h_s dM_s$ is locally square integrable and $\triangle X_t = h_t \triangle M_t > -1$. Then $[X^c, X^c]_t = \int_0^t h_s^2 C ds$, and*

$$\langle X^d, X^d \rangle = \int_0^t h_s^2 d\langle M^d, M^d \rangle_s = \int_0^t h_s^2 \left(\int_{\mathbb{R}} x^2 d\nu(dx) \right) ds. \tag{38}$$

Let $K = \int_{\mathbb{R}} x^2 \nu(dx)$. Then

$$E\left[\exp\left\{ \left(\frac{1}{2} C + K \right) \int_0^T h_s^2 ds \right\} \right] < \infty \tag{39}$$

and hence $\mathcal{E}(X)_t$ is a martingale. In particular, if $K < \infty$ and $|h_t|$ is uniformly bounded, $\mathcal{E}(X)_t$ is a martingale.

The following auxiliary lemma is useful in some situations. It allows us to relax the constants that appear in the exponent (e.g. 1 in (26), $\alpha(\varepsilon)$ in Corollary 11 and 1/2 C+K in (39) etc).

Lemma 13. *Let M be a local martingale with $M_0 = 0$. If there exists $k \in \mathbb{N}_+$ and a partition of $[0, T]$, $0 = \tau_0 < \tau_1 < \ldots < \tau_k = T$ where each τ_i is a stopping time such that*

$$E\left[\exp\left[\langle M, M \rangle_{\tau_{m+1}} - \langle M, M \rangle_{\tau_m} \right] \right] < \infty, \qquad \forall\, 1 \le m \le k-1, \tag{40}$$

then, $\mathcal{E}(M)$ is a martingale.

Proof. Fix m and let $M'_t = M_{t\wedge\tau_{m+1}} - M_{t\wedge\tau_m}$. That is, $M'_t = 0$ if $t < \tau_m$, $M'_t = M_t - M_{\tau_m}$ if $t \in [\tau_m, \tau_{m+1})$, and $M'_t = M_{\tau_{m+1}} - M_{\tau_m}$ for $t \in [\tau_{m+1}, \infty)$. Since M is a local martingale, M' is also a local martingale. By hypothesis and the previous lemma, $\mathcal{E}(M')_t$ is a martingale.

$$E\left[\mathcal{E}(M')_t\right] = 1, \qquad \forall t \in [0, t]. \tag{41}$$

Let's rewrite M' by $M^{(m)}$. Then for $0 \le m \le k-2$

$$\begin{aligned}\mathcal{E}\left(M_\cdot^{(m)}\right)\mathcal{E}\left(M_\cdot^{(m+1)}\right) &= \mathcal{E}\left(M_\cdot^{(m)} + M_\cdot^{(m+1)} - \left[M_\cdot^{(m)}, M_\cdot^{(m+1)}\right]\right)\\ &= \mathcal{E}\left(M_\cdot^{(m)} + M_\cdot^{(m+1)}\right),\end{aligned} \tag{42}$$

and $\mathcal{E}(M_T) = \mathcal{E}(\sum_{n=0}^{k-1} M^{(n)})_T$.

$$\begin{aligned}E[\mathcal{E}(M)_T] = E[E[\mathcal{E}(M)_T|\mathcal{F}_{\tau_{k-1}}]] &= E\left[\sum_{n=0}^{k-2} M_t^{(n)} E\left[M_t^{(k-1)}|\mathcal{F}_{\tau_{k-1}}\right]\right]\\ &= E\left[\mathcal{E}(M_{\tau_{k-1}})\right].\end{aligned} \tag{43}$$

Then by induction, $E\left[\mathcal{E}(M)_T\right] = E\left[\mathcal{E}(M)_0\right] = 1$. Since $\mathcal{E}(M)$ is a local martingale, this implies that it is a martingale. □

We can apply Lemma 13 to refine Example 12.

Example 14. *In Example 12, if there exists an increasing sequence of stopping times* $0 = \tau_0 < \tau_1 \cdots < \tau_k = T$ *such that*

$$E\left[\exp\left\{\left(\frac{1}{2}C + K\right)\int_{\tau_i}^{\tau_{i+1}} h_s^2 ds\right\}\right] < \infty, \forall i \in \{1, \ldots, k-1\} \tag{44}$$

then $\mathcal{E}(X)$ *is a martingale on* $[0, T]$.

Another application of Lemma 13 yields an extension of Beneš theorem [2]:

Theorem 15. *Let* $Z_t = \mathcal{E}(h \cdot M)_t$ *where* M *is a Lévy martingale,* $h \in L(M)$ *be a predictable adapted process such that* $h \cdot M$ *is properly defined and* $Z_t \ge 0$ *for all* $t \in [0, T]$. *Suppose there exists* $a > 0$ *such that* $E[e^{a|M_T|}] < \infty$. *If there exists a constant* K *such that*

$$|h_t(\omega)| \le K(1 + \sup_{u\le t}|M_u(\omega)|^{1/2}) \qquad a.s. \tag{45}$$

for all $t \in [0, T]$ *then* $\mathcal{E}(h \cdot M)$ *is a martingale on* $[0, T]$.

Proof. Let $\alpha = \sup_{u\le T}|M_u|^{1/2}$. Then

$$\begin{aligned}\langle h\cdot M, h\cdot M\rangle_{t_{m+1}} - \langle h\cdot M, h\cdot M\rangle_{t_m} &= \int_{t_m}^{t_{m+1}} h_u^2 d\langle M, M\rangle_u = c\int_{t_m}^{t_{m+1}} h_u^2 du\\ &\le cK^2(t_{m+1} - t_m)(1+\alpha)^2 \le 2cK^2(t_{m+1} - t_m)(1+\alpha^2).\end{aligned} \tag{46}$$

Therefore

$$E\left[\exp\left(\langle h\cdot M, h\cdot M\rangle_{t_{m+1}} - \langle h\cdot M, h\cdot M\rangle_{t_m}\right)\right] \le c_1 E\left[\exp(c_2\alpha^2)\right] \tag{47}$$

where $c_1 = cK^2, c_2 = 2cK^2(t_{m+1} - t_m)$. By the hypothesis that $e^{a|M_t|} \in L_1$ for some $a \in \mathbb{R}_+$ and Jensen's inequality, $e^{a|M_t|}$ is a submartingale. By Doob's maximal inequality,

$$E\left[\exp(c_2\alpha^2)\right] \leq 4E\left[e^{c_2|M_T|}\right] < \infty, \tag{48}$$

since we can take c_2 arbitrarily small by refining the partition. Therefore by the previous lemma, $\mathcal{E}(h \cdot M)$ is a martingale. □

Remark 10. *The original Beneš theorem requires $e^{a(M_T)^2} \in L_1$ for some a, instead of $e^{a|M_T|} \in L_1$, while it assumes further that*

$$|h_t(\omega)| \leq K(1 + \sup_{u \leq t} |M_u(\omega)|) \qquad a.s. \tag{49}$$

is a sufficient condition for h (note that there is no square root in the original condition). With a proof similar in spirit to Theorem 15, we can show that this still holds even if M is not continuous. However the condition $e^{a|M_t|} \in L_1$ is satisfied only when $\nu(dx) = 0$, which of course implies that M is a Brownian motion as Lemma 16 indicates.

The following lemma is well known, but hard to find explicitly stated.

Lemma 16. *Let X be a Lévy process such that its Lévy measure $\nu(dx) \not\equiv 0$. If $p > 1$, then $\exp[a|X_t|^p] \notin L_1$ for all $t > 0$, $a > 0$.*

Proof. Suppose there exists ϵ such that $\nu([\epsilon, \infty)) \not\equiv 0$. Let $X_t^{(2)} = \sum_{s \leq t} 1_{[\epsilon,\infty)}(\triangle X_s)$ and $X_t^{(1)} = X_t - X_t^{(2)}$. Then $X^{(1)}$ and $X^{(2)}$ are independent. $P(X_t^{(1)} > 0) > 0$. If $P(X_t^{(2)} = 0) = 1$ then

$$E\left[\exp\{a|X_t|^p\}\right] = E\left[\exp\{a|X_t^{(1)}|^p\}\right] \geq E\left[\exp\{a|\epsilon\nu([\epsilon, \infty))|^p\}\right] = \infty. \tag{50}$$

Consider the case $P(X_t^{(2)} = 0) \neq 1$. For $a, b \in \mathbb{R}$, $|a + b|^p \geq |a| 1_{\{a,b<0\} \cup \{a,b>0\}}$. Therefore

$$\begin{aligned} E[\exp\{a|X_t|^p\}] &\geq E[\exp\{a|X_t^{(1)}|^p\} 1_{\{X_t^{(1)}, X_t^{(2)} < 0\} \cup \{X_t^{(1)}, X_t^{(2)} > 0\}}] \\ &\geq \min[P(X_t^{(2)} > 0), P(X_t^{(2)} < 0)] E\left[\exp\{a|X_t^{(1)}|^p\}\right] \\ &\geq \min[P(X_t^{(2)} > 0), P(X_t^{(2)} < 0)] E\left[\exp\{a|\varepsilon\nu_T([\varepsilon, \infty))\}|^p\right]. \end{aligned} \tag{51}$$

The case that X has jumps with negative size only is similar. □

Lemma 17. *Let $Z_t = \mathcal{E}(h \cdot M)_t$ where M is a Lévy martingale, $h \in L(M)$ is a predictable process such that $h \cdot M$ is well defined and $Z_t \geq 0$. Suppose there exists constants K and p such that*

$$|h_s(\omega)| \leq K(1 + ([M, M]_T)^p), \tag{52}$$

and $E[e^{a[M,M]_T^{2p}}] < \infty$ for some $a \in \mathbb{R}_+$, almost surely for all $s \in [0, T]$. Then $\mathcal{E}(M)$ is a martingale on $[0, T]$.

Proof. Let $\alpha = [M, M]^p_T$. Then

$$(53)\quad \begin{aligned} \langle h \cdot M, h \cdot M\rangle_{t_{m+1}} - \langle h \cdot M, h \cdot M\rangle_{t_m} &= \int_{t_m}^{t_{m+1}} h_u^2 d\langle M, M\rangle_u = c \int_{t_m}^{t_{m+1}} h_u^2 du \\ &\le cK^2(t_{m+1} - t_m)(1+\alpha)^2 \le 2cK^2(t_{m+1} - t_m)(1+\alpha^2) \end{aligned}$$

and the hypothesis that

$$(54)\quad E\left[e^{c_2 [M,M]_T^{2p}}\right] < \infty$$

implies the desired result. □

Example 18. *Let M be a compensated compound Poisson process and let h be a predictable process satisfying*

$$(55)\quad |h_t(\omega)| \le K(1 + \sum_{u \le t} \triangle |M_u|^2).$$

Since $[M, M]_T = \sum_{s \le T} (\triangle M_s)^2$,
(56)

$$E\left[\exp\{c[M, M]_T\}\right] = E\left[E\left(\exp\left\{c \sum_{j=1}^{N_t} \xi_j^2\right\} |\sigma(N_t)\right)\right] = E\left[\left(E\left(e^{c\xi_1^2}\right)\right)^{N_t}\right] < \infty,$$

provided $e^{c\xi_1^2} \in L_1$ for some $c > 0$. For example if ξ is bounded or $\xi = |\chi|$ where χ is normally distributed, then this condition is satisfied. In this case, $\mathcal{E}(h \cdot M)$ is a martingale.

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