

# Functional delta-method for the bootstrap of quasi-Hadamard differentiable functionals

Eric Beutner

*Department of Quantitative Economics, Maastricht University, Netherlands*  
*e-mail: [e.beutner@maastrichtuniversity.nl](mailto:e.beutner@maastrichtuniversity.nl)*

and

Henryk Zähle\*

*Department of Mathematics, Saarland University, Germany*  
*e-mail: [zaehle@math.uni-sb.de](mailto:zaehle@math.uni-sb.de)*

**Abstract:** The functional delta-method provides a convenient tool for deriving the asymptotic distribution of a plug-in estimator of a statistical functional from the asymptotic distribution of the respective empirical process. Moreover, it provides a tool to derive bootstrap consistency for plug-in estimators from bootstrap consistency of empirical processes. It has recently been shown that the range of applications of the functional delta-method for the asymptotic distribution can be considerably enlarged by employing the notion of quasi-Hadamard differentiability. Here we show in a general setting that this enlargement carries over to the bootstrap. That is, for quasi-Hadamard differentiable functionals bootstrap consistency of the plug-in estimator follows from bootstrap consistency of the respective empirical process. This enlargement often requires convergence in distribution of the bootstrapped empirical process w.r.t. a nonuniform sup-norm. The latter is not problematic as will be illustrated by means of examples.

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## 1. Introduction

The bootstrap is a widely used technique to approximate the unknown error distribution of estimators. Since the seminal paper by Efron (1979) many variants of his bootstrap procedure have been introduced in the literature. Furthermore, the bootstrap has quickly been extended to other data than a sample of independent and identically distributed random variables. For general accounts

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on the bootstrap one may refer to Efron and Tibshirani (1994), Shao and Tu (1995), Davison and Hinkely (1997), Lahiri (2003), among others.

For a (tangentially) Hadamard differentiable map  $f$  the functional delta-method leads to the asymptotic distribution of  $a_n(f(\widehat{T}_n) - f(\theta))$  whenever the asymptotic distribution of  $a_n(\widehat{T}_n - \theta)$  is known. Here  $\widehat{T}_n$  is an estimator for a (possibly infinite-dimensional) parameter  $\theta$ , and  $(a_n)$  is a sequence of real numbers tending to infinity such that  $a_n(\widehat{T}_n - \theta)$  has a non-degenerate limiting distribution. This extends to the bootstrap, i.e. bootstrap consistency of  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  follows from bootstrap consistency of  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  for (tangentially) Hadamard differentiable  $f$ ; see, for instance, van der Vaart and Wellner (1996, Theorems 3.9.11 and 3.9.13). Here  $\widehat{T}_n^*$  is a bootstrapped version of  $\widehat{T}_n$  based on some random mechanism. For a recent partial generalization of these results, see also Volgushev and Shao (2014). Parr (1985) established a functional delta-method for the bootstrap of Fréchet differentiable maps  $f$ , and Cuevas and Romo (1997) obtained a corresponding result for the so-called smoothed bootstrap.

A drawback of the classical functional delta-method is its restricted range of applications. For many statistical functionals  $f$  (including classical L-, V- and M-functionals) the condition of (tangential) Hadamard differentiability is simply too strong. For this reason Beutner and Zähle (2010) introduced the notion of quasi-Hadamard differentiability, which is weaker than (tangential) Hadamard differentiability but still strong enough to obtain a generalized version of the classical functional delta-method; see also the Appendix C. Combined with results for weak convergence of empirical processes w.r.t. nonuniform sup-norms the concept of quasi-Hadamard differentiability led to some new weak convergence results for plug-in estimators of statistical functionals based on dependent data; see Beutner and Zähle (2010, 2012), Ahn and Shyamalkumar (2011), Beutner et al. (2012), Krätschmer et al. (2015), and Krätschmer and Zähle (2016). See also Beutner and Zähle (2014) and Buchsteiner (2015) for some recent results on weak convergence of empirical processes w.r.t. nonuniform sup-norms.

In this article, we will show that the notion of quasi-Hadamard differentiability admits even a functional delta-method for the bootstrap. This enlarges the set of functionals  $f$  for which bootstrap consistency of  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  follows immediately from bootstrap consistency of  $a_n(\widehat{T}_n^* - \widehat{T}_n)$ . To illustrate this, let us briefly discuss distortion risk functionals as examples for  $f$  where the parameter  $\theta$  is a distribution function  $F$  on the real line,  $\widehat{T}_n$  represents the empirical distribution function  $\widehat{F}_n$  of  $n$  real-valued random variables with distribution function  $F$ , and  $\widehat{T}_n^*$  corresponds to a bootstrapped version  $\widehat{F}_n^*$  of  $\widehat{F}_n$ .

Given a continuous concave distortion function  $g$ , i.e. a concave function  $g : [0, 1] \rightarrow [0, 1]$  being continuous at 0 and satisfying  $g(0) = 0 = 1 - g(1)$ , the corresponding *distortion risk functional*  $f_g : \mathbf{D}(f_g) \rightarrow \mathbb{R}$  is defined by

$$f_g(F) := \int_{-\infty}^0 g(F(t)) dt - \int_0^{\infty} (1 - g(F(t))) dt, \quad (1)$$

where  $\mathbf{D}(f_g)$  is a suitable subset of the set of all distribution functions  $F$  for

which both integrals on the right-hand side are finite. Note that distortion risk functionals associated with continuous concave distortion functions correspond to coherent distortion risk measures (cf. Example 4.5) which are of special interest in mathematical finance and actuarial mathematics. It was discussed in Beutner and Zähle (2010) and Krättschmer et al. (2015) that these functionals are typically not Hadamard differentiable w.r.t. the usual sup-norm  $\|\cdot\|_\infty$  but only quasi-Hadamard differentiable w.r.t. suitable nonuniform sup-norms  $\|v\|_\phi := \|v\phi\|_\infty$  stronger than  $\|\cdot\|_\infty$  (i.e. with continuous weight functions  $\phi: \mathbb{R} \rightarrow [1, \infty)$  satisfying  $\lim_{|x| \rightarrow \infty} \phi(x) = \infty$ ). The functional delta-method in the form of Corollary 4.2 below then shows that  $a_n(f_g(\widehat{F}_n^*) - f_g(\widehat{F}_n))$  has the same limiting distribution as  $a_n(f_g(\widehat{F}_n) - f_g(F))$  whenever the bootstrapped empirical process  $a_n(\widehat{F}_n^* - \widehat{F}_n)$  converges in distribution to the same limit as the empirical process  $a_n(\widehat{F}_n - F)$ . As “differentiability” can be obtained only for certain nonuniform sup-norms  $\|\cdot\|_\phi$ , the latter convergence in distribution has to be guaranteed for exactly these nonuniform sup-norms  $\|\cdot\|_\phi$ . Fortunately, such results can be easily obtained from Donsker results for appropriate classes of functions; see Sections 5.1–5.2 for examples. So the notion of quasi-Hadamard differentiability together with the functional delta-method based on it provides an interesting field of applications for the bootstrap of Donsker classes. We emphasize that our approach leads in particular to new bootstrap results for empirical distortion risk measures based on  $\beta$ -mixing data; for details and other examples see Section 5.3.

It is worth recalling that the empirical process  $a_n(\widehat{F}_n - F)$ , regarded as a mapping from  $\Omega$  to the nonseparable space of all bounded càdlàg functions equipped with the sup-norm, is not measurable w.r.t. the Borel  $\sigma$ -algebra. This problem was first observed by Chibisov (1965) and carries over to nonuniform sup-norms. There are different ways to deal with this fact; for a respective discussion see, for instance, Section 1.1 in van der Vaart and Wellner (1996). One possibility is to use the concept of weak convergence (or convergence in distribution) in the Hoffmann-Jørgensen sense; see, for instance, van der Vaart and Wellner (1996), Dudley (1999), Lahiri (2003), and Kosorok (2010). Another possibility is to use the open-ball  $\sigma$ -algebra w.r.t. which the empirical process is measurable. Here we work throughout with the open-ball  $\sigma$ -algebra and weak convergence (and convergence in distribution) as defined in Billingsley (1999, Section 6); see also Dudley (1966, 1967), Pollard (1984), and Shorack and Wellner (1986). This implies in particular that we have to take proper care of the measurability of the maps  $a_n(\widehat{T}_n - \theta)$  and  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  for every  $n \in \mathbb{N}$ .

The rest of the article is organized as follows. In Section 2 we briefly explain the setting chosen here and give some definitions that will be used throughout. The main result and its proof are presented in Sections 3 and 6, respectively. Applications of our main result are given in Section 4 and illustrated in Section 5. Additional definitions and results that are needed for our main result are given in the Appendix. The Appendix is organized as follows. In Sections A and B we give some results on weak convergence, convergence in distribution, and convergence in probability for the open-ball  $\sigma$ -algebra which are needed in Sec-

tion C. In Section C we first present an extended Continuous Mapping theorem for convergence in distribution for the open-ball  $\sigma$ -algebra. This complements the extended Continuous Mapping theorems for weak convergence for the Borel  $\sigma$ -algebra and for convergence in distribution in the Hoffmann-Jørgensen sense which are already known from the literature. In the second part of Section C we use the extended Continuous Mapping theorem to prove an extension (compared to Theorem 4.1 in Beutner and Zähle (2010)) of the functional delta-method based on the notion of quasi-Hadamard differentiability. This extension is needed for the proof of our main result, i.e. for the proof of a functional delta-method for the bootstrap. Two results that ensure measurability of maps involved in our approach are given in Section D.

## 2. Basic definitions

In this section we introduce some notation and basic definitions. As mentioned in the introduction, weak convergence and convergence in distribution will always be considered for the open-ball  $\sigma$ -algebra. Borrowed from Billingsley (1999, Section 6) we will use the terminology *weak<sup>o</sup> convergence* (symbolically  $\Rightarrow^o$ ) and *convergence in distribution<sup>o</sup>* (symbolically  $\rightsquigarrow^o$ ). For details see the Appendices A and B. In a separable metric space the notions of weak<sup>o</sup> convergence and convergence in distribution<sup>o</sup> boil down to the conventional notions of weak convergence and convergence in distribution for the Borel  $\sigma$ -algebra. In this case we also use the symbols  $\Rightarrow$  and  $\rightsquigarrow$  instead of  $\Rightarrow^o$  and  $\rightsquigarrow^o$ , respectively.

Let  $\mathbf{V}$  be a vector space and  $\mathbf{E}$  be a subspace of  $\mathbf{V}$ . Let  $\|\cdot\|_{\mathbf{E}}$  be a norm on  $\mathbf{E}$  and  $\mathcal{B}^o$  be the corresponding open-ball  $\sigma$ -algebra on  $\mathbf{E}$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and  $(\widehat{T}_n)$  be a sequence of maps

$$\widehat{T}_n : \Omega \longrightarrow \mathbf{V}.$$

Regard  $\omega \in \Omega$  as a sample drawn from  $\mathbb{P}$ , and  $\widehat{T}_n(\omega)$  as a statistic derived from  $\omega$ . Let  $\theta \in \mathbf{V}$ , and  $(a_n)$  be a sequence of positive real numbers tending to  $\infty$ . Assume that  $a_n(\widehat{T}_n - \theta)$  takes values only in  $\mathbf{E}$  and is  $(\mathcal{F}, \mathcal{B}^o)$ -measurable for every  $n \in \mathbb{N}$ , and that

$$a_n(\widehat{T}_n - \theta) \rightsquigarrow^o \xi \quad \text{in } (\mathbf{E}, \mathcal{B}^o, \|\cdot\|_{\mathbf{E}}) \quad (2)$$

for some  $(\mathbf{E}, \mathcal{B}^o)$ -valued random variable  $\xi$ .

Now, let  $(\Omega', \mathcal{F}', \mathbb{P}')$  be another probability space and set

$$(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}').$$

The probability measure  $\mathbb{P}'$  represents a random experiment that is run independently of the random sample mechanism  $\mathbb{P}$ . In the sequel,  $\widehat{T}_n$  will frequently be regarded as a map defined on the extension  $\overline{\Omega}$  of  $\Omega$ . Let

$$\widehat{T}_n^* : \overline{\Omega} \longrightarrow \mathbf{V}$$

be any map and assume that  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  takes values only in  $\mathbf{E}$  and is  $(\overline{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable for every  $n \in \mathbb{N}$ . Since  $\widehat{T}_n^*(\omega, \omega')$  depends on both the original sample  $\omega$  and the outcome  $\omega'$  of the additional independent random experiment, we may regard  $\widehat{T}_n^*$  as a bootstrapped version of  $\widehat{T}_n$ . For the formula display (3) in the following Definition 2.1, note that the mapping  $\omega' \mapsto a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))$  is  $(\mathcal{F}', \mathcal{B}^\circ)$ -measurable for every fixed  $\omega \in \Omega$ , because  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  is  $(\overline{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable with  $\overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}'$ . That is,  $a_n(\widehat{T}_n^*(\omega, \cdot) - \widehat{T}_n(\omega))$  can be seen as an  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variable on  $(\Omega', \mathcal{F}', \mathbb{P}')$  for every fixed  $\omega \in \Omega$ .

**Definition 2.1 (Bootstrap version almost surely)** *We say that  $(\widehat{T}_n^*)$  is almost surely a bootstrap version of  $(\widehat{T}_n)$  w.r.t. the convergence in (2) if*

$$a_n(\widehat{T}_n^*(\omega, \cdot) - \widehat{T}_n(\omega)) \rightsquigarrow^\circ \xi \quad \text{in } (\mathbf{E}, \mathcal{B}^\circ, \|\cdot\|_{\mathbf{E}}), \quad \mathbb{P}\text{-a.e. } \omega. \quad (3)$$

Next we intend to introduce the notion of bootstrap version in (outer) probability. To this end let the map  $P_n : \overline{\Omega} \times \mathcal{B}^\circ \rightarrow [0, 1]$  be defined by

$$P_n((\omega, \omega'), A) := P_n(\omega, A) := \mathbb{P}' \circ \{a_n(\widehat{T}_n^*(\omega, \cdot) - \widehat{T}_n(\omega))\}^{-1}[A], \quad (\omega, \omega') \in \overline{\Omega}, A \in \mathcal{B}^\circ. \quad (4)$$

It provides a conditional distribution of  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  given  $\Pi$ , where the  $(\overline{\mathcal{F}}, \mathcal{F})$ -measurable map  $\Pi : \overline{\Omega} \rightarrow \Omega$  is defined by

$$\Pi(\omega, \omega') := \omega. \quad (5)$$

This follows from Lemma D.2 (with  $X(\omega, \omega') = g(\omega, \omega') = a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))$  and  $Y = \Pi$ ). Informally,  $\Pi(\omega, \omega')$  specifies that part of the realization  $(\omega, \omega')$  of the extended random mechanism  $\mathbb{P} \otimes \mathbb{P}'$  that represents the ‘‘observed data’’; see also Remark 2.5 below and the discussion preceding it. By definition  $P_n$  is a probability kernel from  $(\overline{\Omega}, \sigma(\Pi))$  to  $(\mathbf{E}, \mathcal{B}^\circ)$ . However, it is directly clear from (4) that  $P_n$  can also be seen as a probability kernel from  $(\Omega, \mathcal{F})$  to  $(\mathbf{E}, \mathcal{B}^\circ)$ .

Let  $d_{\text{BL}}^\circ$  denote the bounded Lipschitz distance (defined in (45) in the Appendix A) on the set  $\mathcal{M}_1^\circ$  of all probability measures on  $(\mathbf{E}, \mathcal{B}^\circ)$ . Note that a sequence  $(\mu_n) \subseteq \mathcal{M}_1^\circ$  converges weakly to some  $\mu_0 \in \mathcal{M}_1^\circ$  which concentrates on a separable set, if and only if  $d_{\text{BL}}^\circ(\mu_n, \mu_0) \rightarrow 0$ ; cf. Theorem A.3. In general the mapping  $\omega \mapsto d_{\text{BL}}^\circ(P_n(\omega, \cdot), \text{law}\{\xi\})$  is not necessarily  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable. For this reason we have to use the outer probability in (6). Recall that the outer probability  $\mathbb{P}^{\text{out}}[S]$  of an arbitrary subset  $S \subseteq \Omega$  is defined to be the infimum of  $\mathbb{P}[\overline{S}]$  over all  $\overline{S} \in \mathcal{F}$  with  $\overline{S} \supseteq S$ .

**Definition 2.2 (Bootstrap version in (outer) probability)** *We say that  $(\widehat{T}_n^*)$  is a bootstrap version in outer probability of  $(\widehat{T}_n)$  w.r.t. the convergence in (2) if*

$$\lim_{n \rightarrow \infty} \mathbb{P}^{\text{out}}[\{\omega \in \Omega : d_{\text{BL}}^\circ(P_n(\omega, \cdot), \text{law}\{\xi\}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0. \quad (6)$$

When  $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$  is separable, we may replace in (6) the outer probability  $\mathbb{P}^{\text{out}}$  by the ordinary probability  $\mathbb{P}$  and we will say that  $(\widehat{T}_n^*)$  is a bootstrap version in probability of  $(\widehat{T}_n)$  w.r.t. the convergence in (2).

The second part of Definition 2.2 can be justified as follows. The assumed separability of  $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$  implies that  $\mathcal{M}_1^\circ$  is just the set  $\mathcal{M}_1$  of all Borel probability measures on  $\mathbf{E}$  and that  $\omega \mapsto P_n(\omega, \cdot)$  can be seen as an  $(\mathcal{F}, \sigma(\mathcal{O}_w))$ -measurable mapping from  $\Omega$  to  $\mathcal{M}_1$  (cf. Lemma D.1); here  $\mathcal{O}_w$  refers to the weak topology on  $\mathcal{M}_1$  (cf. Remark A.1). By the reverse triangle inequality for metrics we also have that the mapping  $\mu \mapsto d_{\text{BL}}(\mu, \text{law}\{\xi\})$  is continuous (recall that  $d_{\text{BL}} := d_{\text{BL}}^\circ$  is a metric when  $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$  is separable) and thus  $(\sigma(\mathcal{O}_w), \mathcal{B}(\mathbb{R}_+))$ -measurable. It follows that the mapping  $\omega \mapsto d_{\text{BL}}(P_n(\omega, \cdot), \text{law}\{\xi\})$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable.

As our interest lies in deriving bootstrap results for functionals  $f$  of  $\widehat{T}_n^*$  from bootstrap results for  $\widehat{T}_n^*$  itself, we introduce some more notation and restate Definition 2.2 for  $f(\widehat{T}_n^*)$ . Let  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  be another normed vector space and assume that  $\|\cdot\|_{\widetilde{\mathbf{E}}}$  is separable. In particular, the open-ball  $\sigma$ -algebra coincides with the Borel  $\sigma$ -algebra  $\widetilde{\mathcal{B}}$  on  $\widetilde{\mathbf{E}}$ . Denote by  $\widetilde{\mathcal{M}}_1$  the set of all probability measures on  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ . Let

$$f : \mathbf{V}_f \longrightarrow \widetilde{\mathbf{E}}$$

be any map defined on some subset  $\mathbf{V}_f \subseteq \mathbf{V}$ . Assume that  $\widehat{T}_n$  and  $\widehat{T}_n^*$  take values only in  $\mathbf{V}_f$  and that  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  is  $(\overline{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable. Moreover let the map  $\widetilde{P}_n : \overline{\Omega} \times \widetilde{\mathcal{B}} \rightarrow [0, 1]$  be defined by

$$\begin{aligned} \widetilde{P}_n((\omega, \omega'), A) &:= \widetilde{P}_n(\omega, A) := \mathbb{P}' \circ \{a_n(f(\widehat{T}_n^*(\omega, \cdot)) - f(\widehat{T}_n(\omega)))\}^{-1}[A], \\ &(\omega, \omega') \in \overline{\Omega}, A \in \widetilde{\mathcal{B}}. \end{aligned} \tag{7}$$

It provides a conditional distribution of  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  given  $\Pi$ , where  $\Pi$  is as in (5). This follows from Lemma D.2 (with  $X(\omega, \omega') = g(\omega, \omega') = a_n(f(\widehat{T}_n^*(\omega, \omega')) - f(\widehat{T}_n(\omega)))$  and  $Y = \Pi$ ). By definition  $\widetilde{P}_n$  is a probability kernel from  $(\overline{\Omega}, \sigma(\Pi_n))$  to  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ . However, it is directly clear from (7) that  $\widetilde{P}_n$  can also be seen as a probability kernel from  $(\Omega, \mathcal{F})$  to  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ . Finally assume that

$$a_n(f(\widehat{T}_n) - f(\theta)) \rightsquigarrow \widetilde{\xi} \quad \text{in } (\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}}, \|\cdot\|_{\widetilde{\mathbf{E}}}) \tag{8}$$

for some  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ -valued random variable  $\widetilde{\xi}$  and let  $\widetilde{d}_{\text{BL}}$  denote the bounded Lipschitz distance on  $\widetilde{\mathcal{M}}_1$  as defined in (45).

**Definition 2.3 (Bootstrap version in probability)** *We say that  $(f(\widehat{T}_n^*))$  is a bootstrap version in probability of  $(f(\widehat{T}_n))$  w.r.t. the convergence in (8) if*

$$\lim_{n \rightarrow \infty} \mathbb{P}[\{\omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \text{law}\{\widetilde{\xi}\}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0. \tag{9}$$

Note that the mapping  $\omega \mapsto \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \text{law}\{\widetilde{\xi}\})$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable. Indeed, one can argue as subsequent to Definition 2.2, because we assumed that  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  is separable.

**Remark 2.4** Note that (9) implies that (9) still holds when the bounded Lipschitz distance  $\widetilde{d}_{\text{BL}}$  is replaced by any other metric on  $\widetilde{\mathcal{M}}_1$  which generates the weak topology. When  $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$  is separable, then the same is true for (6).  $\diamond$

We conclude this section with some comments on the probability kernel  $P_n$  defined in (4). As mentioned above, it is a conditional distribution of  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  given  $\Pi$ , where to some extent  $\Pi(\omega, \omega') = \omega$  can be seen as the “observable” sample. On the other hand, for technical reasons the sample space  $\Omega$  is often so complex so that only a portion  $\Pi_n(\omega)$  of an element  $\omega \in \Omega$  can indeed be “observed”. For instance, when the sample space is an infinite product space, i.e.  $(\Omega, \mathcal{F}) = (S^{\mathbb{N}}, \mathcal{S}^{\otimes \mathbb{N}})$  for some measurable space  $(S, \mathcal{S})$ , then de facto one can only observe a finite-dimensional sample, say the first  $n$  coordinates  $(\omega_1, \dots, \omega_n)$  of the infinite-dimensional sample  $\omega = (\omega_1, \omega_2, \dots) \in S^{\mathbb{N}}$ . In this case it is obviously appealing to interpret  $P_n$  as a conditional distribution of  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  given  $\Pi_n$ , where  $\Pi_n : S^{\mathbb{N}} \times \Omega' \rightarrow S^n$  is given by

$$\Pi_n((\omega_1, \omega_2, \dots), \omega') := (\omega_1, \dots, \omega_n). \tag{10}$$

Under additional mild assumptions this is indeed possible. This follows from the next Remark 2.5 if we take there  $\Pi_n$  as given in (10) and  $(\Omega^{(n)}, \mathcal{F}^{(n)})$  equal to  $(S^n, \mathcal{S}^{\otimes n})$ . Analogously one can regard  $\widetilde{P}_n$  defined in (7) as a conditional distribution of  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  given  $\Pi_n$ .

**Remark 2.5** Let  $(\Omega^{(n)}, \mathcal{F}^{(n)})$  be a measurable space and  $\Pi_n : \overline{\Omega} \rightarrow \Omega^{(n)}$  be an  $(\overline{\mathcal{F}}, \mathcal{F}^{(n)})$ -measurable map for every  $n \in \mathbb{N}$ . Assume that for every  $n \in \mathbb{N}$  the value  $\Pi_n(\omega, \omega')$  depends only on  $\omega$  and that there exist maps  $\tau_n : \Omega^{(n)} \rightarrow \mathbf{V}$  and  $\tau_n^* : \Omega^{(n)} \times \Omega' \rightarrow \mathbf{V}$  such that for all  $\omega \in \Omega$ ,  $\omega' \in \Omega'$

$$\tau_n(\Pi_n(\omega, \omega')) = \widehat{T}_n(\omega) \quad \text{and} \quad \tau_n^*(\Pi_n(\omega, \omega'), \omega') = \widehat{T}_n^*(\omega, \omega') \tag{11}$$

and

$$g_n(\omega^{(n)}, \omega') := a_n(\tau_n^*(\omega^{(n)}, \omega') - \tau_n(\omega^{(n)})), \quad (\omega^{(n)}, \omega') \in \Omega^{(n)} \times \Omega' \tag{12}$$

provides an  $(\mathcal{F}^{(n)} \otimes \mathcal{F}', \mathcal{B}^\circ)$ -measurable map  $g_n : \Omega^{(n)} \times \Omega' \rightarrow \mathbf{E}$ . (This implies in particular that  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  takes values only in  $\mathbf{E}$  and is  $(\overline{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable). Then the map  $P_n : \overline{\Omega} \times \mathcal{B}^\circ \rightarrow [0, 1]$  defined by (4) provides a conditional distribution of  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  given  $\Pi_n$ . This follows again from Lemma D.2 (with  $X(\omega, \omega') = a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))$ ,  $Y = \Pi_n$ , and  $g = g_n$ ).  $\diamond$

### 3. Abstract delta-method for the bootstrap

Theorem 3.1 below establishes an abstract delta-method for the bootstrap for quasi-Hadamard differentiable maps. It uses the notation and definitions introduced in Section 2. More precisely, let  $\mathbf{V}$ ,  $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $(\Omega', \mathcal{F}', \mathbb{P}')$ ,  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ ,  $\widehat{T}_n$ ,  $\widehat{T}_n^*$ ,  $P_n$ ,  $f$ ,  $\mathbf{V}_f$ ,  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$ , and  $\widetilde{P}_n$  be as in Section 2. As before assume that  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  is separable, and that  $\widehat{T}_n$  and  $\widehat{T}_n^*$  take values only in  $\mathbf{V}_f$ .

As already discussed in the introduction, in statistical applications the role of  $\widehat{T}_n$  is often played by the empirical distribution function of  $n$  identically distributed random variables (sample), so that the plug-in estimator  $f(\widehat{T}_n)$  can be

represented as a function of the sample. This special case will be studied in detail in Section 4. Due to the measurability problems discussed in the introduction we work with the open-ball  $\sigma$ -algebra  $\mathcal{B}^\circ$  in our general setting. This is different from the conventional functional delta-method for the bootstrap in the form of van der Vaart and Wellner (1996, Theorems 3.9.11 and 3.9.13) where the measurability problem is overcome by using the concept of convergence in distribution in the Hoffmann-Jørgensen sense. Moreover, compared to the conventional functional delta-method we work with a weaker notion of differentiability, namely with *quasi*-Hadamard differentiability. This kind of differentiability was introduced by Beutner and Zähle (2010) and is recalled in Definition C.3 in the Appendix.

**Theorem 3.1 (Delta-method for the bootstrap)** *Let  $\theta \in \mathbf{V}_f$ . Let  $\mathbf{E}_0 \subseteq \mathbf{E}$  be a separable subspace and assume that  $\mathbf{E}_0 \in \mathcal{B}^\circ$ . Let  $(a_n)$  be a sequence of positive real numbers tending to  $\infty$ , and consider the following conditions:*

- (a)  $a_n(\widehat{T}_n - \theta)$  takes values only in  $\mathbf{E}$ , is  $(\mathcal{F}, \mathcal{B}^\circ)$ -measurable, and satisfies

$$a_n(\widehat{T}_n - \theta) \rightsquigarrow^\circ \xi \quad \text{in } (\mathbf{E}, \mathcal{B}^\circ, \|\cdot\|_{\mathbf{E}}) \quad (13)$$

for some  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variable  $\xi$  on some probability space  $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$  with  $\xi(\Omega_0) \subseteq \mathbf{E}_0$ .

- (b) The map  $f(\widehat{T}_n) : \Omega \rightarrow \widetilde{\mathbf{E}}$  is  $(\mathcal{F}, \widetilde{\mathcal{B}})$ -measurable.  
(c) The map  $f$  is quasi-Hadamard differentiable at  $\theta$  tangentially to  $\mathbf{E}_0(\mathbf{E})$  with quasi-Hadamard derivative  $\dot{f}_\theta$  in the sense of Definition C.3.  
(d) The quasi-Hadamard derivative  $\dot{f}_\theta$  can be extended from  $\mathbf{E}_0$  to  $\mathbf{E}$  such that the extension  $\dot{f}_\theta : \mathbf{E} \rightarrow \widetilde{\mathbf{E}}$  is linear and  $(\mathcal{B}^\circ, \widetilde{\mathcal{B}})$ -measurable. Moreover, the extension  $\dot{f}_\theta : \mathbf{E} \rightarrow \widetilde{\mathbf{E}}$  is continuous at every point of  $\mathbf{E}_0$ .  
(e) The map  $f(\widehat{T}_n^*) : \widetilde{\Omega} \rightarrow \widetilde{\mathbf{E}}$  is  $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable.  
(f)  $a_n(\widehat{T}_n^* - \theta)$  and  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  take values only in  $\mathbf{E}$  and are  $(\widetilde{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable, and  $(\widehat{T}_n^*)$  is almost surely a bootstrap version of  $(\widehat{T}_n)$  w.r.t. the convergence in (13) in the sense of Definition 2.1. The latter means that

$$a_n(\widehat{T}_n^*(\omega, \cdot) - \widehat{T}_n(\omega)) \rightsquigarrow^\circ \xi \quad \text{in } (\mathbf{E}, \mathcal{B}^\circ, \|\cdot\|_{\mathbf{E}}), \quad \mathbb{P}\text{-a.e. } \omega. \quad (14)$$

- (f')  $a_n(\widehat{T}_n^* - \theta)$  and  $a_n(\widehat{T}_n^* - \widehat{T}_n)$  take values only in  $\mathbf{E}$  and are  $(\widetilde{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable, and  $(\widehat{T}_n^*)$  is a bootstrap version in outer probability of  $(\widehat{T}_n)$  w.r.t. the convergence in (13) in the sense of Definition 2.2. The latter means that

$$\lim_{n \rightarrow \infty} \mathbb{P}^{\text{out}}[\{\omega \in \Omega : d_{\text{BL}}^\circ(P_n(\omega, \cdot), \text{law}\{\xi\}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0. \quad (15)$$

Then the following assertions hold:

- (i) If conditions (a)–(c) hold, then  $a_n(f(\widehat{T}_n) - f(\theta))$  and  $\dot{f}_\theta(\xi)$  are respectively  $(\mathcal{F}, \widetilde{\mathcal{B}})$ - and  $(\mathcal{F}_0, \widetilde{\mathcal{B}})$ -measurable, and



$$a_n(f(\widehat{T}_n) - f(\theta)) \rightsquigarrow \dot{f}_\theta(\xi) \quad \text{in } (\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}}, \|\cdot\|_{\widetilde{\mathbf{E}}}). \quad (16)$$

(ii) If conditions (a)–(f) hold, then  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  and  $\dot{f}_\theta(\xi)$  are respectively  $(\overline{\mathcal{F}}, \widetilde{\mathcal{B}})$ - and  $(\mathcal{F}_0, \widetilde{\mathcal{B}})$ -measurable, and  $(f(\widehat{T}_n^*))$  is a bootstrap version in probability of  $(f(\widehat{T}_n))$  w.r.t. the convergence in (16) in the sense of Definition 2.3. The latter means that

$$\lim_{n \rightarrow \infty} \mathbb{P}[\{\omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \text{law}\{\dot{f}_\theta(\xi)\}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0. \quad (17)$$

(iii) Assertion (ii) still holds when assumption (f) is replaced by (f').

Recall that  $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$  was *not* assumed to be separable, so that the mapping  $\omega \mapsto d_{\text{BL}}^{\circ}(P_n(\omega, \cdot), \text{law}\{\xi\})$  is not necessarily  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable. Further note that the Counterexample 1.9.4 in van der Vaart and Wellner (1996) (where  $\mathbb{P}^{\text{out}}[|\xi_n - 0| \geq \delta] = \mathbb{P}[\mathbb{1}_{B_n}^{\text{out}} \geq \delta] = 1$  obviously holds for every  $n \in \mathbb{N}$  and  $\delta \in (0, 1)$ , with  $\xi_n := \mathbb{1}_{B_n}$ ) shows that in general  $\mathbb{P}$ -a.s. convergence of a sequence  $(\xi_n)$  of *non-* $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable functions  $\xi_n : \Omega \rightarrow \mathbb{R}$  does not imply convergence in outer probability of  $(\xi_n)$ . In particular it is not clear to us whether or not condition (f) implies condition (f'). For that reason we consider both conditions separately.

Note that in contrast to the conventional functional delta-method in the form of van der Vaart and Wellner (1996, Theorems 3.9.11 and 3.9.13) condition (a) of Theorem 3.1 does not involve convergence in distribution in the Hoffmann-Jørgensen sense (based on outer integrals) and condition (f) of Theorem 3.1 does not involve the concept of convergence in outer probability. Thus assertion (ii) of Theorem 3.1 shows in particular that a comprehensive version of the functional delta-method for the bootstrap can be stated without using the concepts of outer integrals and outer probabilities. Indeed, (part (ii) of) Theorem 3.1 in the form of (part (ii) of) Corollary 4.2 below (together with Lemmas 5.1 and 5.3) covers plenty of classical plug-in estimators.

#### 4. Application to plug-in estimators of statistical functionals

Let  $\mathbf{D}$  be the space of all càdlàg functions  $v$  on  $\mathbb{R}$  with finite sup-norm  $\|v\|_{\infty} := \sup_{t \in \mathbb{R}} |v(t)|$ , and  $\mathcal{D}$  be the  $\sigma$ -algebra on  $\mathbf{D}$  generated by the one-dimensional coordinate projections  $\pi_t$ ,  $t \in \mathbb{R}$ , given by  $\pi_t(v) := v(t)$ . Let  $\phi : \mathbb{R} \rightarrow [1, \infty)$  be a weight function, i.e. a continuous function being non-increasing on  $(-\infty, 0]$  and non-decreasing on  $[0, \infty)$ . Let  $\mathbf{D}_{\phi}$  be the subspace of  $\mathbf{D}$  consisting of all  $x \in \mathbf{D}$  satisfying  $\|x\|_{\phi} := \|x\phi\|_{\infty} < \infty$  and  $\lim_{|t| \rightarrow \infty} |x(t)| = 0$ . The latter condition automatically holds when  $\lim_{|t| \rightarrow \infty} \phi(t) = \infty$ . Let  $\mathcal{D}_{\phi} := \mathcal{D} \cap \mathbf{D}_{\phi}$  be the trace  $\sigma$ -algebra on  $\mathbf{D}_{\phi}$ . The  $\sigma$ -algebra on  $\mathbf{D}_{\phi}$  generated by the  $\|\cdot\|_{\phi}$ -open balls will be denoted by  $\mathcal{B}_{\phi}^{\circ}$ . The following lemma shows that it coincides with  $\mathcal{D}_{\phi}$ .

**Lemma 4.1**  $\mathcal{D}_{\phi} = \mathcal{B}_{\phi}^{\circ}$ .

**Proof** Without loss of generality we assume  $\lim_{|x| \rightarrow \infty} \phi(x) = \infty$ . We denote by  $B_r(x)$  the  $\|\cdot\|_\phi$ -open ball around  $x \in \mathbf{D}_\phi$  with radius  $r$ , that is,  $B_r(x) := \{y \in \mathbf{D}_\phi : \|x - y\|_\phi < r\}$ . On the one hand, for every  $t \in \mathbb{R}$  and  $a \in \mathbb{R}$  we have

$$\pi_t^{-1}((a/\phi(t), \infty)) = \{x \in \mathbf{D}_\phi : x(t) > a/\phi(t)\} = \bigcup_{n \in \mathbb{N}} B_n(x_n), \quad (18)$$

where  $x_n = x_{n,t,a}$  is defined by  $x_n(s) := (a + (n+1/n)\mathbb{1}_{[t, t+1/n)}(s))/\phi(s)$ . Thus,  $\pi_t^{-1}((b, \infty))$  lies in  $\mathcal{B}_\phi^\circ$  for every  $t \in \mathbb{R}$  and  $b \in \mathbb{R}$ . That is,  $\pi_t$  is  $(\mathcal{B}_\phi^\circ, \mathcal{B}(\mathbb{R}))$ -measurable. Hence,  $\mathcal{D}_\phi \subseteq \mathcal{B}_\phi^\circ$ . On the other hand, any open ball  $B_r(x)$  can be represented as

$$\begin{aligned} B_r(x) &= \bigcap_{t \in \mathbb{Q}} \{y \in \mathbf{D}_\phi : |x(t) - y(t)|\phi(t) < r\} \\ &= \bigcap_{t \in \mathbb{Q}} \pi_t^{-1}((x(t) - r/\phi(t), x(t) + r/\phi(t))), \end{aligned}$$

and so it lies in  $\mathcal{D}_\phi$ . Hence,  $\mathcal{B}_\phi^\circ \subseteq \mathcal{D}_\phi$ .  $\square$

For any given distribution function  $F$  on the real line, let  $\mathbf{C}_{\phi,F} \subset \mathbf{D}_\phi$  be a  $\|\cdot\|_\phi$ -separable subspace and assume  $\mathbf{C}_{\phi,F} \in \mathcal{D}_\phi$ . Moreover let  $f : \mathbf{D}(f) \rightarrow \mathbb{R}$  be a map defined on a set  $\mathbf{D}(f)$  of distribution functions of finite (not necessarily probability) Borel measures on  $\mathbb{R}$ . In particular,  $\mathbf{D}(f) \subset \mathbf{D}$ . In the following,  $\mathbf{D}$ ,  $(\mathbf{D}_\phi, \mathcal{D}_\phi, \|\cdot\|_\phi)$ ,  $\mathbf{C}_{\phi,F}$ ,  $f$ ,  $\mathbf{D}(f)$ , and  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), |\cdot|)$  will play the roles of  $\mathbf{V}$ ,  $(\mathbf{E}, \mathcal{B}^\circ, \|\cdot\|_{\mathbf{E}})$ ,  $\mathbf{E}_0$ ,  $f$ ,  $\mathbf{V}_f$ , and  $(\tilde{\mathbf{E}}, \tilde{\mathcal{B}}, \|\cdot\|_{\tilde{\mathbf{E}}})$ , respectively.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $F \in \mathbf{D}(f)$  be the distribution function of a Borel probability measure on  $\mathbb{R}$ . Let  $(X_i)$  be a sequence of identically distributed real-valued random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with distribution function  $F$ . Let  $\hat{F}_n : \Omega \rightarrow \mathbf{D}$  be the empirical distribution function of  $X_1, \dots, X_n$ , which will play the role of  $\hat{T}_n$ . It is defined by

$$\hat{F}_n := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[X_i, \infty)}. \quad (19)$$

Assume that  $\hat{F}_n$  takes values only in  $\mathbf{D}(f)$ . Let  $(\Omega', \mathcal{F}', \mathbb{P}')$  be another probability space and set  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$ . Moreover let  $\hat{F}_n^* : \bar{\Omega} \rightarrow \mathbf{D}$  be any map; see Section 5 for an illustration. Assume that  $\hat{F}_n^*$  take values only in  $\mathbf{D}(f)$ . In the present setting Theorem 3.1 can be reformulated as follows.

**Corollary 4.2** *Let  $F \in \mathbf{D}(f)$ . Let  $(a_n)$  be a sequence of positive real numbers tending to  $\infty$ , and consider the following conditions:*

(a)  $a_n(\hat{F}_n - F)$  takes values only in  $\mathbf{D}_\phi$  and satisfies

$$a_n(\hat{F}_n - F) \rightsquigarrow^\circ B \quad \text{in } (\mathbf{D}_\phi, \mathcal{D}_\phi, \|\cdot\|_\phi) \quad (20)$$

for some  $(\mathbf{D}_\phi, \mathcal{D}_\phi)$ -valued random variable  $B$  on some probability space  $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$  with  $B(\Omega_0) \subseteq \mathbf{C}_{\phi,F}$ .

- (b) The map  $f(\widehat{F}_n) : \Omega \rightarrow \mathbb{R}$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable.
- (c) The map  $f$  is quasi-Hadamard differentiable at  $F$  tangentially to  $\mathbf{C}_{\phi, F}(\mathbf{D}_\phi)$  with quasi-Hadamard derivative  $\dot{f}_F$  in the sense of Definition C.3.
- (d) The quasi-Hadamard derivative  $\dot{f}_F$  can be extended from  $\mathbf{C}_{\phi, F}$  to  $\mathbf{D}_\phi$  such that the extension  $\dot{f}_F : \mathbf{D}_\phi \rightarrow \mathbb{R}$  is linear and  $(\mathcal{D}_\phi, \mathcal{B}(\mathbb{R}))$ -measurable. Moreover, the extension  $\dot{f}_F : \mathbf{D}_\phi \rightarrow \mathbb{R}$  is continuous at every point of  $\mathbf{C}_{\phi, F}$ .
- (e) The map  $f(\widehat{F}_n^*) : \overline{\Omega} \rightarrow \mathbb{R}$  is  $(\overline{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$ -measurable.
- (f)  $a_n(\widehat{F}_n^* - \widehat{F}_n)$  takes values only in  $\mathbf{D}_\phi$  and is  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable, and  $(\widehat{F}_n^*)$  is almost surely a bootstrap version of  $(\widehat{F}_n)$  w.r.t. the convergence in (20) in the sense of Definition 2.1. The latter means that

$$a_n(\widehat{F}_n^*(\omega, \cdot) - \widehat{F}_n(\omega)) \rightsquigarrow^\circ B \quad \text{in } (\mathbf{D}_\phi, \mathcal{D}_\phi, \|\cdot\|_\phi), \quad \mathbb{P}\text{-a.e. } \omega. \quad (21)$$

- (f')  $a_n(\widehat{F}_n^* - \widehat{F}_n)$  takes values only in  $\mathbf{D}_\phi$  and is  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable, and  $(\widehat{F}_n^*)$  is a bootstrap version in outer probability of  $(\widehat{F}_n)$  w.r.t. the convergence in (20) in the sense of Definition 2.2. The latter means that (with  $P_n$  defined as in (4))

$$\lim_{n \rightarrow \infty} \mathbb{P}^{\text{out}}[\{\omega \in \Omega : d_{\text{BL}}^\circ(P_n(\omega, \cdot), \text{law}\{B\}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0, \quad (22)$$

Then the following assertions hold:

- (i) If conditions (a)–(c) hold, then  $a_n(f(\widehat{F}_n) - f(\theta))$  and  $\dot{f}_F(B)$  are respectively  $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ - and  $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}))$ -measurable, and

$$a_n(f(\widehat{F}_n) - f(F)) \rightsquigarrow \dot{f}_F(B) \quad \text{in } (\mathbb{R}, \mathcal{B}(\mathbb{R})). \quad (23)$$

- (ii) If conditions (a)–(f) hold, then  $a_n(f(\widehat{F}_n^*) - f(\widehat{F}_n))$  and  $\dot{f}_F(B)$  are respectively  $(\overline{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$ - and  $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}))$ -measurable, and  $(f(\widehat{F}_n^*))$  is a bootstrap version in probability of  $(f(\widehat{F}_n))$  w.r.t. the convergence in (23) in the sense of Definition 2.3. The latter means that (with  $\widetilde{P}_n$  defined as in (7))

$$\lim_{n \rightarrow \infty} \mathbb{P}[\{\omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \text{law}\{\dot{f}_F(B)\}) \geq \delta\}] = 0 \quad \text{for all } \delta > 0.$$

- (iii) Assertion (ii) still holds when assumption (f) is replaced by (f').

**Proof** Corollary 4.2 is a consequence of Theorem 3.1, because the measurability assumption in condition (a) and the first measurability assumption of condition (f) (respectively (f')) of Theorem 3.1 are automatically satisfied in the present setting. Indeed,  $a_n(\widehat{F}_n - F)$  is easily seen to be  $(\mathcal{F}, \mathcal{D}_\phi)$ -measurable, and the sum of two  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable maps is clearly  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable and we assumed here (through (f) (respectively (f'))) that  $a_n(\widehat{F}_n^* - \widehat{F}_n)$  is  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable.  $\square$

Conditions (e)–(f') of Corollary 4.2 will be illustrated in Sections 5.1–5.2. The following examples illustrate conditions (a)–(d) of Corollary 4.2. See also Section 5.3 for specific applications.

**Example 4.3 (for condition (a))** Assume that  $X_1, X_2, \dots$  are i.i.d. with distribution function  $F$ , and let  $\phi$  be a weight function. If  $\int \phi^2 dF < \infty$ , then Theorem 6.2.1 in Shorack and Wellner (1986) shows that

$$\sqrt{n}(\widehat{F}_n - F) \rightsquigarrow^\circ B_F \quad \text{in } (\mathbf{D}_\phi, \mathcal{D}_\phi, \|\cdot\|_\phi),$$

where  $B_F$  is an  $F$ -Brownian bridge, i.e. a centered Gaussian process with covariance function  $\Gamma(t_0, t_1) = F(t_0 \wedge t_1)(1 - F(t_0 \vee t_1))$ . Note that  $B_F$  jumps where  $F$  jumps and that  $\lim_{|t| \rightarrow \infty} B_F(t) = 0$ . Thus,  $B_F$  takes values only in the set  $\mathbf{C}_{\phi, F} \subset \mathbf{D}_\phi$  consisting of all  $x \in \mathbf{D}_\phi$  whose discontinuities are also discontinuities of  $F$ . It was shown in Krätschmer et al. (2015, Corollary B.4) that the set  $\mathbf{C}_{\phi, F}$  is  $\|\cdot\|_\phi$ -separable and contained in  $\mathcal{D}_\phi$ .  $\diamond$

**Example 4.4 (for condition (a))** Let  $\phi$  be any weight function,  $(X_i)$  be strictly stationary and  $\beta$ -mixing with distribution function  $F$ , and assume that  $\mathbb{E}[\phi(X_1)^p] < \infty$  for some  $p > 2$  and that the mixing coefficients satisfy  $\beta_n = o(n^{-p/(p-2)}(\log n)^{2(p-1)/(p-2)})$ . Then

$$\sqrt{n}(\widehat{F}_n - F) \rightsquigarrow^\circ \widetilde{B}_F \quad \text{in } (\mathbf{D}_\phi, \mathcal{D}_\phi, \|\cdot\|_\phi),$$

where  $\widetilde{B}_F$  is a centered Gaussian process with covariance function  $\Gamma(t_0, t_1) = F(t_0 \wedge t_1)(1 - F(t_0 \vee t_1)) + \sum_{i=0}^1 \sum_{k=2}^{\infty} \text{Cov}(\mathbb{1}_{\{X_1 \leq t_i\}}, \mathbb{1}_{\{X_k \leq t_{1-i}\}})$ . The result follows by verifying the assumptions of Theorem 2.1 in Arcones and Yu (1994). We will verify these assumptions in the proof of Theorem 5.4 below. Note that  $\widetilde{B}_F$  jumps where  $F$  jumps and that  $\lim_{|x| \rightarrow \infty} \widetilde{B}_F(x) = 0$ . Thus,  $\widetilde{B}_F$  takes values only in the  $\|\cdot\|_\phi$ -separable and  $\mathcal{D}_\phi$ -measurable set  $\mathbf{C}_{\phi, F}$  introduced in Example 4.3. For illustration, note that many GARCH processes are strictly stationary and  $\beta$ -mixing; see, for instance, Francq and Zakoian (2010, Chapter 3) and Boussama et al. (2011).  $\diamond$

Further examples for condition (a) can be found in Beutner and Zähle (2010, 2012, 2014), Beutner et al. (2012), and Buchsteiner (2015).

**Example 4.5 (for condition (b))** Let  $g$  be a continuous concave distortion function as introduced before (1). For every real-valued random variable  $X$  (on some given atomless probability space) satisfying  $\int_0^\infty g(1 - F_{|X|}(x)) dx < \infty$  the *distortion risk measure* associated with  $g$  is defined by  $\rho_g(X) := f_g(F_X)$  with  $f_g$  as in (1). Here  $F_X$  and  $F_{|X|}$  denote the distribution functions of  $X$  and  $|X|$ , respectively. The set  $\mathcal{X}_g$  of all random variables  $X$  satisfying the above integrability condition provides a linear subspace of  $L^1$ ; this follows from Denneberg (1994, Proposition 9.5) and Föllmer and Schied (2011, Proposition 4.75). It is known that  $\rho_g$  is a law-invariant coherent risk measure; see, for instance, Wang and Dhaene (1998). If specifically  $g(s) = (s/\alpha) \wedge 1$  for any fixed  $\alpha \in (0, 1)$ , then we have  $\mathcal{X}_g = L^1$  and  $\rho_g$  is nothing but the Average Value at Risk at level  $\alpha$ .

The *risk functional*  $f_g : \mathbf{D}(f_g) \rightarrow \mathbb{R}$  corresponding to  $\rho_g$  was already introduced in (1), where  $\mathbf{D}(f_g)$  is the set of all distribution functions of the random variables of  $\mathcal{X}_g$ . Now, the mapping  $\omega \mapsto \widehat{F}_n(\omega, t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[X_i(\omega), \infty)}(t)$

is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable for every  $t \in \mathbb{R}$ . Due to the monotonicity of  $g$  also the mapping  $\omega \mapsto g(\widehat{F}_n(\omega, t))$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable for every  $t \in \mathbb{R}$ . By the right-continuity of the mapping  $t \mapsto g(\widehat{F}_n(\omega, t))$  for every fixed  $\omega \in \Omega$  we obtain in particular that the mapping  $(\omega, t) \mapsto g(\widehat{F}_n(\omega, t))$  is  $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$ -measurable. Fubini's theorem then implies that the mapping  $\omega \mapsto f_g(\widehat{F}_n(\omega, \cdot))$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable. So we have in particular that condition (b) of Corollary 4.2 holds.  $\diamond$

**Example 4.6 (for conditions (c)–(d))** Let  $f_g : \mathbf{D}(f_g) \rightarrow \mathbb{R}$  be as in Example 4.5. Let  $F \in \mathbf{D}(f_g)$  with  $0 < F(\cdot) < 1$ , and  $\phi$  be a weight function satisfying the integrability condition

$$\int_{-\infty}^{\infty} \frac{g(\gamma F(t))}{F(t)\phi(t)} dt < \infty \quad \text{for some } \gamma \in (0, 1). \quad (24)$$

Assume that the set of points  $t \in \mathbb{R}$  for which  $g$  is not differentiable at  $F(t)$  has Lebesgue measure zero. Then Theorem 2.7 in Krättschmer et al. (2015) shows that the functional  $f_g$  is quasi-Hadamard differentiable at  $F$  tangentially to  $\mathbf{C}_{\phi, F}(\mathbf{D}_{\phi})$  with quasi-Hadamard derivative  $\dot{f}_{g;F} : \mathbf{C}_{\phi, F} \rightarrow \mathbb{R}$  given by

$$\dot{f}_{g;F}(x) := \int_{-\infty}^{\infty} g'(F(t)) x(t) dt, \quad x \in \mathbf{C}_{\phi, F},$$

where  $g'$  denotes the right-sided derivative of  $g$  and  $\mathbf{C}_{\phi, F}$  is as in Example 4.3. Recall that  $\mathbf{C}_{\phi, F}$  is  $\|\cdot\|_{\phi}$ -separable and contained in  $\mathcal{D}_{\phi}$ ; cf. Corollary B.4 in Krättschmer et al. (2015). The derivative  $\dot{f}_{g;F}$  can be extended to  $\mathbf{D}_{\phi}$  through

$$\dot{f}_{g;F}(x) := \int_{-\infty}^{\infty} g'(F(t)) x(t) dt, \quad x \in \mathbf{D}_{\phi},$$

and the extension is linear and continuous on  $\mathbf{D}_{\phi}$ . The linearity is obvious and the continuity is ensured by part (ii) of Lemma 4.1 in Krättschmer et al. (2015). Thus, condition (c) of Corollary 4.2 holds. Moreover, using arguments as in Example 4.5, one can easily show that the extension  $\dot{f}_{g;F} : \mathbf{D}_{\phi} \rightarrow \mathbb{R}$  is also  $(\mathcal{D}_{\phi}, \mathcal{B}(\mathbb{R}))$ -measurable. That is, condition (d) of Corollary 4.2 holds too.  $\diamond$

### 5. Bootstrap results for empirical processes

In the following two subsections, we will give examples for bootstrap versions  $(\widehat{T}_n^*)$  of  $(\widehat{T}_n)$  in the sense of Definitions 2.1 and 2.2 in the context of Section 4, i.e. in the case where  $\widehat{T}_n$  is given by an empirical distribution function  $\widehat{F}_n$  of real-valued random variables. As mentioned in the introduction these examples can be combined with the quasi-Hadamard differentiability of statistical functionals to lead to bootstrap consistency for the corresponding plug-in estimators. Examples include empirical distortion risk measures as well as U- and V-statistics which will be discussed in Section 5.3.

### 5.1. I.i.d. observations

We will adopt the notation introduced in Section 4. In particular,  $(X_i)$  will be a sequence of identically distributed real-valued random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with distribution function  $F$ , and  $\widehat{F}_n$  will be given by (19). Let  $(W_{ni})$  be a triangular array of nonnegative real-valued random variables on  $(\Omega', \mathcal{F}', \mathbb{P}')$  such that  $(W_{n1}, \dots, W_{nn})$  is an exchangeable random vector for every  $n \in \mathbb{N}$ , and define the map  $\widehat{F}_n^* : \overline{\Omega} \rightarrow \mathbf{D}$  by

$$\widehat{F}_n^*(\omega, \omega') := \frac{1}{n} \sum_{i=1}^n W_{ni}(\omega') \mathbb{1}_{[X_i(\omega), \infty)}. \quad (25)$$

Note that the sequence  $(X_i)$  and the triangular array  $(W_{ni})$  regarded as families of random variables on the product space  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$  are independent. Of course, we will tacitly assume that  $(\Omega', \mathcal{F}', \mathbb{P}')$  is rich enough to host all of the random variables described in (a)–(b) in Theorem 5.2.

**Lemma 5.1**  $a_n(\widehat{F}_n^* - \widehat{F}_n)$  takes values only in  $\mathbf{D}_\phi$  and is  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable. That is, the first part of condition (f) (respectively (f')) of Corollary 4.2 holds true.

**Proof** First of all note that  $a_n(\widehat{F}_n^*((\omega, \omega'), t) - \widehat{F}_n(\omega, t))$  can be written as

$$a_n \left( \frac{1}{n} \sum_{i=1}^n W_{ni}(\omega') \mathbb{1}_{[X_i(\omega), \infty)}(t) - \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{[X_i(\omega), \infty)}(t) \right) =: \Xi_n((\omega, \omega'), t)$$

for all  $t \in \mathbb{R}$  and  $(\omega, \omega') \in \overline{\Omega}$ . The mapping  $(\omega, \omega') \mapsto \Xi_n((\omega, \omega'), t)$  is  $(\overline{\mathcal{F}}, \mathcal{B}(\mathbb{R}))$ -measurable for every  $t \in \mathbb{R}$ , and the mapping  $t \mapsto \Xi_n((\omega, \omega'), t)$  is right-continuous for every  $(\omega, \omega') \in \overline{\Omega}$ . Thus the mapping  $(\omega, \omega') \mapsto \Xi_n((\omega, \omega'), \cdot)$  from  $\overline{\Omega}$  to  $\mathbf{D}$  is  $(\overline{\mathcal{F}}, \mathcal{D})$ -measurable. Further,  $\Xi_n((\omega, \omega'), \cdot)$  obviously takes values only in  $\mathbf{D}_\phi$  for every  $(\omega, \omega') \in \overline{\Omega}$ . Thus  $\Xi_n$  can indeed be seen as an  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable map from  $\overline{\Omega}$  to  $\mathbf{D}_\phi (\subseteq \mathbf{D})$ .  $\square$

The proof of the following Theorem 5.2 strongly relies on Section 3.6.2 in van der Vaart and Wellner (1996). In fact, the elaborations in Section 3.6.2 in van der Vaart and Wellner (1996) yield slightly stronger results compared to those of Theorem 5.2, because van der Vaart and Wellner work in a more general framework. More precisely, they establish *outer* almost sure bootstrap results for the empirical process w.r.t. convergence in distribution in the *Hoffmann-Jørgensen sense*. The first result on Efron's bootstrap for the empirical process of i.i.d. random variables was given by Bickel and Freedman (1981, Theorem 4.1) for the uniform sup-norm, that is, for  $\phi \equiv 1$ . Gaenssler (1986) extended this result to Vapnik–Červonenkis classes. For a version of Efron's bootstrap in a very general set-up, see also Giné and Zinn (1990, Theorem 2.4).

**Theorem 5.2** Assume that the random variables  $X_1, X_2, \dots$  are i.i.d., their distribution function  $F$  satisfies  $\int \phi^2 dF < \infty$ , and one of the following two settings is met.

- (a) **(Efron’s bootstrap)** The random vector  $(W_{n1}, \dots, W_{nn})$  is multinomially distributed according to the parameters  $n$  and  $p_1 = \dots = p_n = \frac{1}{n}$  for every  $n \in \mathbb{N}$ .
- (b) **(Bayesian bootstrap)**  $W_{ni} = Y_i/\bar{Y}_n$  for every  $n \in \mathbb{N}$  and  $i = 1, \dots, n$ , where  $\bar{Y}_n := \frac{1}{n} \sum_{j=1}^n Y_j$  and  $(Y_j)$  is any sequence of nonnegative i.i.d. random variables on  $(\Omega', \mathcal{F}', \mathbb{P}')$  with distribution  $\mu$  which satisfies the integrability condition  $\int_0^\infty \mu[(x, \infty)]^{1/2} dx < \infty$  and whose standard deviation coincides with its mean and is strictly positive.

Then (condition (a) and) the second part of condition (f) of Corollary 4.2 hold for  $a_n = \sqrt{n}$ ,  $B = B_F$  and  $\hat{F}_n^*$  defined in (25), where  $B_F$  is an  $F$ -Brownian bridge, i.e. a centered Gaussian process with covariance function  $\Gamma(t_0, t_1) = F(t_0 \wedge t_1) \bar{F}(t_0 \vee t_1)$ .

**Proof** The claim of Theorem 5.2 would follow from the second assertion of Theorem 3.6.13 in van der Vaart and Wellner (1996) with  $\mathcal{F} = \mathbb{F}_\phi := \{\phi(x)\mathbb{1}_{(-\infty, x]} : x \in \mathbb{R}\}$  if we could show that the assumptions of Theorem 3.6.13 in van der Vaart and Wellner (1996) are fulfilled in each of the settings (a)–(b). At this point we stress the facts that convergence in distribution in the Hoffmann-Jørgensen sense implies convergence in distribution<sup>o</sup> for the open-ball  $\sigma$ -algebra and that outer almost sure convergence (as defined in part (iii) of Definition 1.9.1 in van der Vaart and Wellner (1996)) implies almost sure convergence (i.e. convergence almost everywhere) in the classical sense. The latter follows from Proposition 1.1 in Dudley (2010).

In Theorem 3.6.13 in van der Vaart and Wellner (1996) it is assumed that the following three assertions hold:

- 1)  $\mathbb{F}_\phi$  is a Donsker class w.r.t.  $\mathbb{P}$ , and  $(t_1, \dots, t_n) \mapsto \sup_{f \in \mathbb{F}_{\phi, \delta}} |\sum_{i=1}^n \lambda_i f(t_i)|$  is a measurable mapping on the completion of  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mathbb{P}_{X_1}^{\otimes n})$  for every  $\delta > 0$ ,  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Here we set  $\mathbb{F}_{\phi, \delta} := \{f_1 - f_2 : f_1, f_2 \in \mathbb{F}_\phi, \rho_{\mathbb{P}}(f_1 - f_2) < \delta\}$  with  $\rho_{\mathbb{P}}(f) := \text{Var}_{\mathbb{P}}[f(X_1)]^{1/2}$ , where  $\text{Var}_{\mathbb{P}}$  refers to the variance w.r.t.  $\mathbb{P}$ .
- 2)  $\mathbb{E}_{\mathbb{P}}^{\text{out}}[\bar{f}(X_1)^2] < \infty$  for the envelope function  $\bar{f}(t) := \sup_{f \in \mathbb{F}_\phi} (f(t) - \mathbb{E}_{\mathbb{P}}[f(X_1)])$ , where  $\mathbb{E}_{\mathbb{P}}^{\text{out}}$  refers to the outer expectation w.r.t.  $\mathbb{P}$ .
- 3)  $(W_{n1}, \dots, W_{nn})$  is an exchangeable nonnegative random vector for every  $n \in \mathbb{N}$ , and the triangular array  $(W_{ni})$  satisfies condition (3.6.8) in van der Vaart and Wellner (1996).

We will now verify 1)–3).

1): The assumption  $\int \phi^2 dF < \infty$  ensures that  $\mathbb{F}_\phi$  is a Donsker class w.r.t.  $\mathbb{P}$ ; cf. Example 4.3. To verify the second part of assertion 1), let  $\delta > 0$  arbitrary but fixed and  $f \in \mathbb{F}_{\phi, \delta}$  with  $\rho_{\mathbb{P}}(f) < \delta$ . Now,  $f$  has the representation  $f =$

$\phi(x_1)\mathbb{1}_{(-\infty, x_1]} - \phi(x_2)\mathbb{1}_{(-\infty, x_2]}$  for some  $x_1, x_2 \in \mathbb{R}$ , and

$$\begin{aligned} \rho_{\mathbb{P}}(f) &= \text{Var}_{\mathbb{P}}[\phi(x_1)\mathbb{1}_{(-\infty, x_1]}(X_1) - \phi(x_2)\mathbb{1}_{(-\infty, x_2]}(X_1)] \\ &= \phi(x_1)^2 F(x_1)(1 - F(x_1)) + \phi(x_2)^2 F(x_2)(1 - F(x_2)) \\ &\quad - \phi(x_1)\phi(x_2)F(x_1 \wedge x_2)(1 - F(x_1 \wedge x_2)) \end{aligned}$$

depends (right) continuously on  $(x_1, x_2)$ . So we can find a sequence  $(g_m)$  in the countable subclass  $\mathbb{G}_{\phi, \delta} := \{g_{q_1, q_2} = \phi(q_1)\mathbb{1}_{(-\infty, q_1]} - \phi(q_2)\mathbb{1}_{(-\infty, q_2]} : q_1, q_2 \in \mathbb{Q}, \rho_{\mathbb{P}}(g_{q_1, q_2}) < \delta\}$  of  $\mathbb{F}_{\phi, \delta}$  such that  $g_m(t) \rightarrow f(t)$  for every  $t \in \mathbb{R}$ . For instance,  $g_m := g_{q_{1,m}, q_{2,m}}$  for any sequences  $(q_{1,m})$  and  $(q_{2,m})$  in  $\mathbb{Q}$  such that  $q_{1,m} \searrow x_1$ ,  $q_{2,m} \searrow x_2$  and  $\rho_{\mathbb{P}}(g_{q_{1,m}, q_{2,m}}) < \delta$ . As discussed in Example 2.3.4 in van der Vaart and Wellner (1996) this implies that the second part of assertion 1) holds.

2): We first of all note that in the present setting the outer expectation  $\mathbb{E}_{\mathbb{P}}^{\text{out}}$  can be replaced by the classical expectation  $\mathbb{E}_{\mathbb{P}}$  w.r.t.  $\mathbb{P}$ . Indeed, the envelope function  $\bar{f}$  can be written as

$$\bar{f}(t) = \sup_{x \in \mathbb{R}} (\mathbb{1}_{(-\infty, x]}(t) - F(x))\phi(x) = \sup_{q \in \mathbb{Q}} (\mathbb{1}_{(-\infty, q]}(t) - F(q))\phi(q)$$

and is thus Borel measurable. So it remains to show  $\mathbb{E}[\bar{f}(X_1)^2] < \infty$ . To this end, we note that the assumption  $\int \phi^2 dF < \infty$  implies

$$M_1 := \sup_{t \leq 0} F(t)^2 \phi(t)^2 < \infty \quad \text{and} \quad M_2 := \sup_{t > 0} (1 - F(t))^2 \phi(t)^2 < \infty.$$

Furthermore, for  $t \leq 0$  we have

$$(\mathbb{1}_{(-\infty, x]}(t) - F(x))^2 \phi(x)^2 = \begin{cases} (1 - F(x))^2 \phi(x)^2 & , \quad t \leq x \\ F(x)^2 \phi(x)^2 & , \quad t > x \end{cases}$$

and so, since the mapping  $x \mapsto (1 - F(x))^2 \phi(x)^2$  is non-increasing on  $[t, 0]$ ,

$$\begin{aligned} \bar{f}(t)^2 &= \sup_{x \in \mathbb{R}} (\mathbb{1}_{(-\infty, x]}(t) - F(x))^2 \phi(x)^2 \\ &\leq \max\{M_1, (1 - F(t))^2 \phi(t)^2, M_2\} =: g(t). \end{aligned}$$

For  $t > 0$  we obtain similarly

$$\bar{f}(t)^2 = \sup_{x \in \mathbb{R}} (\mathbb{1}_{(-\infty, x]}(t) - F(x))^2 \phi(x)^2 \leq \max\{M_1, F(t)^2 \phi(t)^2, M_2\} =: g(t),$$

because the mapping  $x \mapsto (F(x)\phi(x))^2$  is non-decreasing on  $(0, t]$ . Hence, we get  $\mathbb{E}[\bar{f}(X_1)^2] \leq \mathbb{E}[g(X_1)^2] < \infty$  due to our assumption  $\int \phi^2 dF < \infty$ .

3): Examples 3.6.10 and 3.6.12 in van der Vaart and Wellner (1996) show that assertion 3) holds in each of the settings (a)–(b). □



### 5.2. Stationary, $\beta$ -mixing observations

As in Section 5.1, we will adopt the notation introduced in Section 4. In particular,  $(X_i)$  will be a sequence of identically distributed real-valued random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with distribution function  $F$ , and  $\widehat{F}_n$  will be given by (19). Let  $(\ell_n)$  be a sequence of integers such that  $\ell_n \nearrow \infty$  as  $n \rightarrow \infty$ , and  $\ell_n < n$  for all  $n \in \mathbb{N}$ . Set  $k_n := \lfloor n/\ell_n \rfloor$  for all  $n \in \mathbb{N}$ . Let  $(I_{nj})_{n \in \mathbb{N}, 1 \leq j \leq k_n}$  be a triangular array of random variables on  $(\Omega', \mathcal{F}', \mathbb{P}')$  such that  $I_{n1}, \dots, I_{nk_n}$  are i.i.d. according to the uniform distribution on  $\{1, \dots, n\}$  for every  $n \in \mathbb{N}$ . Define the map  $\widehat{F}_n^* : \overline{\Omega} \rightarrow \mathbf{D}$  by

$$\widehat{F}_n^*(\omega, \omega') := \frac{1}{n} \sum_{i=1}^n W_{ni}(\omega') \mathbb{1}_{[X_i(\omega), \infty)} \tag{26}$$

with

$$\begin{aligned} W_{ni}(\omega') := \sum_{j=1}^{k_n} & \left( \mathbb{1}_{\{I_{nj} \leq i \leq (I_{nj} + \ell_n - 1) \wedge n\}}(\omega') \right. \\ & \left. + \mathbb{1}_{\{I_{nj} + \ell_n - 1 > n, 1 \leq i \leq I_{nj} + \ell_n - 1 - n\}}(\omega') \right). \end{aligned} \tag{27}$$

Note that, as before, the sequence  $(X_i)$  and the triangular array  $(W_{ni})$  regarded as families of random variables on the product space  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$  are independent.

At an informal level this means that given a sample  $X_1, \dots, X_n$ , we pick  $k_n$  blocks of length  $\ell_n$  in the (artificially) extended sample  $X_1, \dots, X_n, X_{n+1}, \dots, X_{n+\ell_n-1}$  (with  $X_{n+i} := X_i$ ,  $i = 1, \dots, \ell_n - 1$ ) where the start indices  $I_{n1}, I_{n2}, \dots, I_{nk_n}$  are chosen independently and uniformly in the set of all indices  $\{1, \dots, n\}$ :

$$\begin{aligned} \text{block 1:} & \quad X_{I_{n1}}, X_{I_{n1}+1}, \dots, X_{I_{n1}+\ell_n-1} \\ \text{block 2:} & \quad X_{I_{n2}}, X_{I_{n2}+1}, \dots, X_{I_{n2}+\ell_n-1} \\ & \quad \vdots \\ \text{block } k_n: & \quad X_{I_{nk_n}}, X_{I_{nk_n}+1}, \dots, X_{I_{nk_n}+\ell_n-1} \end{aligned}$$

The bootstrapped empirical distribution function  $\widehat{F}_n^*$  is then defined to be the distribution function of the discrete finite (not necessarily probability) measure with atoms  $X_1, \dots, X_n$  carrying masses  $W_{n1}, \dots, W_{nn}$  respectively, where  $W_{ni}$  specifies the number of blocks which contain  $X_i$ .

**Lemma 5.3**  $a_n(\widehat{F}_n^* - \widehat{F}_n)$  takes values only in  $\mathbf{D}_\phi$  and is  $(\overline{\mathcal{F}}, \mathcal{D}_\phi)$ -measurable. That is, the first part of condition (f) (respectively (f')) of Corollary 4.2 holds true.

**Proof** The proof of Lemma 5.1 with the obvious modifications also applies to Lemma 5.3.  $\square$

The bootstrap method induced by the bootstrapped empirical distribution function  $\widehat{F}_n^*$  defined in (26)–(27) is the so-called *circular bootstrap*; see, for instance, Politis and Romano (1992) and Radulovic (1996). The circular bootstrap is only a slight modification of the moving blocks bootstrap that was independently introduced by Künsch (1989) in the context of the sample mean and by Liu and Singh (1992). Bühlmann (1994, 1995), Naik-Nimbalkar and Rajarshi (1994), and Radulovic (1996) extended Künsch’s approach to empirical processes of strictly stationary, mixing observations. Doukhan et al. (2015) extended Shao’s so-called dependent wild bootstrap for smooth functions of the sample mean (cf. Shao (2010)) to the empirical process of strictly stationary and  $\beta$ -mixing observations. For an application of the delta-method based on the notion of quasi-Hadamard differentiability the most interesting results are those that allow for weight functions  $\phi$  with  $\lim_{|x| \rightarrow \infty} \phi(x) = \infty$ . The following result is derived from Theorem 1 in Radulovic (1996).

**Theorem 5.4 (Circular bootstrap)** *Denote by  $F$  the distribution function of  $X_1$  and assume that the following conditions hold:*

- (a)  $\int \phi^p dF < \infty$  for some  $p > 2$ .
- (b) The sequence of random variables  $(X_i)$  is strictly stationary and  $\beta$ -mixing with mixing coefficients  $(\beta_i)$  satisfying  $\beta_i = \mathcal{O}(i^{-b})$  for some  $b > p/(p-2)$ .
- (c) The block length  $\ell_n$  satisfies  $\ell_n = \mathcal{O}(n^\gamma)$  for some  $\gamma \in (0, \frac{p-2}{2(p-1)})$ .

Then (condition (a) and) the second part of condition (f’) of Corollary 4.2 hold for  $a_n = \sqrt{n}$ ,  $B = \widetilde{B}_F$  and  $\widehat{F}_n^*$  defined in (26), where  $\widetilde{B}_F$  is a centered Gaussian process with covariance function  $\Gamma(t_0, t_1) = F(t_0 \wedge t_1)(1 - F(t_0 \vee t_1)) + \sum_{i=0}^1 \sum_{k=2}^{\infty} \text{Cov}(\mathbb{1}_{\{X_1 \leq t_i\}}, \mathbb{1}_{\{X_k \leq t_{1-i}\}})$ .

A similar result that allows to verify condition (f) of Corollary 4.2 (where in (21) the empirical distribution function  $\widehat{F}_n$  is replaced by the conditional expectation of  $\widehat{F}_n^*$ ) can be found in Bühlmann (1995, Theorem 1).

**Proof of Theorem 5.4** It was shown in Arcones and Yu (1994, Theorem 2.1) that under conditions (a)–(b) of Theorem 5.4 the condition (a) of Corollary 4.2 is satisfied; see also Example 4.4. In the following we will show that under assumption (a) of Theorem 5.4 the following two assumptions of Theorem 1 in Radulovic (1996) are met for the class of functions  $\mathbb{F}_\phi := \{f_x : x \in \mathbb{R}\}$  with  $f_x(\cdot) := \phi(x)\mathbb{1}_{(-\infty, x]}(\cdot)$  for  $x \leq 0$  and  $f_x(\cdot) := -\phi(x)\mathbb{1}_{(x, \infty)}(\cdot)$  for  $x > 0$ :

- 1)  $\mathbb{F}_\phi$  is a VC-subgraph class.
- 2)  $\int \bar{f}^p dF < \infty$  for the envelope function  $\bar{f}(t) := \sup_{x \in \mathbb{R}} |f_x(t)|$ .

The other assumptions of Theorem 1 in Radulovic (1996) are just our assumptions (b) and (c). Then, since we may identify the maps  $x \mapsto \sqrt{n}(\widehat{F}_n(x) - F(x))\phi(x)$  and  $x \mapsto \sqrt{n}(\widehat{F}_n^*(x) - \widehat{F}_n(x))\phi(x)$  with respectively  $f_x \mapsto \sqrt{n}(\int f_x d\widehat{F}_n - \int f_x dF)$  and  $f_x \mapsto \sqrt{n}(\int f_x d\widehat{F}_n^* - \int f_x d\widehat{F}_n)$ , Theorem 1 in Radulovic (1996) implies that condition (f’) of Corollary 4.2 is satisfied too.

Before verifying 1), let us recall the definition of VC-subgraph class; cf., for instance, van der Vaart and Wellner (1996, Section 2.6). First recall that the VC-index of a collection  $\mathcal{C}$  of subsets of a nonempty set  $\mathbf{Y}$  is defined by  $V(\mathcal{C}) := \inf\{n : m^{\mathcal{C}}(n) < 2^n\}$  with the convention  $\inf \emptyset := \infty$ , where

$$m^{\mathcal{C}}(n) := \max_{y_1, \dots, y_n \in \mathbf{Y}} \#\{C \cap \{y_1, \dots, y_n\} : C \in \mathcal{C}\}. \tag{28}$$

A collection  $\mathcal{C}$  is said to be a VC-class if  $V(\mathcal{C}) < \infty$ . A class  $\mathbb{F}$  of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be a VC-subgraph class if the collection  $\mathcal{C}_{\mathbb{F}} := \{\{(x, t) \in \mathbb{R}^2 : t < f(x)\} : f \in \mathbb{F}\}$  is a VC-class of sets in  $\mathbf{Y} := \mathbb{R}^2$ .

1): We will show that  $\mathbb{F}_{\phi}$  is a VC-subgraph class with  $V(\mathcal{C}_{\mathbb{F}_{\phi}}) \leq 3$ . For  $V(\mathcal{C}_{\mathbb{F}_{\phi}}) \leq 3$  it suffices to show that  $m^{\mathcal{C}_{\mathbb{F}_{\phi}}}(3) < 2^3$ . Note that that  $m^{\mathcal{C}_{\mathbb{F}_{\phi}}}(3) < 2^3$  means that for every choice of  $y_1, y_2, y_3 \in \mathbb{R}^2$  there exists at least one of the  $2^3$  subsets of  $\{y_1, y_2, y_3\}$  which cannot be represented as  $C \cap \{y_1, y_2, y_3\}$  for any  $C \in \mathcal{C}_{\mathbb{F}_{\phi}}$ . By way of contradiction assume that there exist  $y_1 = (x_1, t_1)$ ,  $y_2 = (x_2, t_2)$ ,  $y_3 = (x_3, t_3)$  in  $\mathbb{R}^2$  such that every subset of  $\{y_1, y_2, y_3\}$  has the representation  $C \cap \{y_1, y_2, y_3\}$  for some  $C \in \mathcal{C}_{\mathbb{F}_{\phi}}$ . Then, in particular, there exist  $C_{12}, C_{13}, C_{23} \in \mathcal{C}_{\mathbb{F}_{\phi}}$  such that

$$\begin{aligned} C_{12} \cap \{(x_1, t_1), (x_2, t_2), (x_3, t_3)\} &= \{(x_1, t_1), (x_2, t_2)\}, \\ C_{13} \cap \{(x_1, t_1), (x_2, t_2), (x_3, t_3)\} &= \{(x_1, t_1), (x_3, t_3)\}, \\ C_{23} \cap \{(x_1, t_1), (x_2, t_2), (x_3, t_3)\} &= \{(x_2, t_2), (x_3, t_3)\}. \end{aligned} \tag{29}$$

We may and do assume without loss of generality that  $x_1 \leq x_2 \leq x_3$ . Then, if (29) held true, there would exist  $x_{12}, x_{13}, x_{23} \in \mathbb{R}$  such that

$$\begin{aligned} t_1 < f_{x_{12}}(x_1), & \quad t_2 < f_{x_{12}}(x_2), & \quad t_3 \geq f_{x_{12}}(x_3), \\ t_1 < f_{x_{13}}(x_1), & \quad t_2 \geq f_{x_{13}}(x_2), & \quad t_3 < f_{x_{13}}(x_3), \\ t_1 \geq f_{x_{23}}(x_1), & \quad t_2 < f_{x_{23}}(x_2), & \quad t_3 < f_{x_{23}}(x_3). \end{aligned} \tag{30}$$

First assume  $x_{12} \leq 0$ . In this case we have  $f_{x_{12}}(\cdot) = \mathbb{1}_{(-\infty, x_{12}]}\phi(x_{12})$  and thus  $t_3 \geq 0$  (due to  $t_3 \geq f_{x_{12}}(x_3)$ ). But then  $f_{x_{13}}$  and  $f_{x_{23}}$  are also of the form  $f_{x_{13}}(\cdot) = \mathbb{1}_{(-\infty, x_{13}]}\phi(x_{13})$  and  $f_{x_{23}}(\cdot) = \mathbb{1}_{(-\infty, x_{23}]}\phi(x_{23})$ , because  $t_3 < f_{x_{13}}(x_3)$ ,  $t_3 < f_{x_{23}}(x_3)$ , and functions of the form  $f_x(\cdot) = -\mathbb{1}_{(x, \infty)}\phi(x)$  take values only in  $(-\infty, -1] \cup \{0\}$ . From the second and the third line of (30) we can now conclude that  $f_{x_{13}}(x_1) = f_{x_{13}}(x_3)$ ,  $x_3 \leq x_{13}$ , and  $f_{x_{23}}(x_2) = f_{x_{23}}(x_3)$ ,  $x_3 \leq x_{23}$ , respectively. It follows that

$$f_{x_{13}}(x_1) = f_{x_{13}}(x_2) \quad \text{and} \quad f_{x_{23}}(x_1) = f_{x_{23}}(x_2), \tag{31}$$

because  $x_2 \leq x_3$  (which implies  $x_2 \in (-\infty, x_{13}]$ ) and  $x_1 \leq x_2$  (which implies  $x_1 \in (-\infty, x_{23}]$ ). On the other hand, by (30) we obviously have

$$f_{x_{13}}(x_1) > f_{x_{23}}(x_1) \quad \text{and} \quad f_{x_{23}}(x_2) > f_{x_{13}}(x_2). \tag{32}$$

But (31) and (32) contradict each other.

Now assume  $x_{12} > 0$ . This implies that  $f_{x_{12}}$  takes values only in  $(-\infty, -1] \cup \{0\}$ , and therefore  $f_{x_{12}}(x_1) \leq 0$  and  $f_{x_{12}}(x_2) \leq 0$ . It follows that  $t_1 < 0$  and  $t_2 < 0$ . The latter two inequalities imply  $f_{x_{23}}(x_1) < 0$  and  $f_{x_{13}}(x_2) < 0$ , respectively. It follows that  $x_{23} > 0$  and  $x_{13} > 0$ , because otherwise  $f_{x_{23}}$  or  $f_{x_{13}}$  would take values only in  $\{0\} \cup [1, \infty)$ . In particular,  $t_3 < 0$  (since  $t_3 < f_{x_{23}}(x_3)$ ). That is, we have  $t_1, t_2, t_3 < 0$  and  $f_{x_{12}}(\cdot) = -\mathbb{1}_{(x_{12}, \infty)}(\cdot)\phi(x_{12})$ ,  $f_{x_{13}}(\cdot) = -\mathbb{1}_{(x_{13}, \infty)}(\cdot)\phi(x_{13})$ ,  $f_{x_{23}}(\cdot) = -\mathbb{1}_{(x_{23}, \infty)}(\cdot)\phi(x_{23})$ . From the third line of (30) we first conclude that  $x_1 > x_{23}$ , because  $t_1 < 0$  (so that  $t_1 \geq f_{x_{23}}(x_1)$  is only possible if  $x_1 > x_{23}$ ). Then we also have  $x_2 > x_{23}$  and  $x_3 > x_{23}$ , because  $x_3 \geq x_2 \geq x_1$ . This implies  $f_{x_{23}}(x_1) = f_{x_{23}}(x_2) = f_{x_{23}}(x_3)$ , and we conclude from the third line of (30) that  $t_1 > t_2$ . Similarly, from the second line of (30) we obtain  $t_2 > t_3$ . Summarizing we must have

$$0 > t_1 > t_2 > t_3. \quad (33)$$

Recall that we assumed (by way of contradiction) that  $y_1 = (x_1, t_1)$ ,  $y_2 = (x_2, t_2)$ ,  $y_3 = (x_3, t_3)$  are such that every subset of  $\{y_1, y_2, y_3\}$  has the representation  $C \cap \{y_1, y_2, y_3\}$  for some  $C \in \mathcal{C}_{\mathbb{F}_\phi}$ . In particular, there exists a set  $C_{2|1,3} \in \mathcal{C}_{\mathbb{F}_\phi}$  with

$$C_{2|1,3} \cap \{(x_1, t_1), (x_2, t_2), (x_3, t_3)\} = \{(x_2, t_2)\}.$$

That is, there exists some  $x_{2|1,3} \in \mathbb{R}$  such that

$$t_1 \geq f_{x_{2|1,3}}(x_1), \quad t_2 < f_{x_{2|1,3}}(x_2), \quad t_3 \geq f_{x_{2|1,3}}(x_3). \quad (34)$$

Since  $t_1 < 0$ , we must have  $x_{2|1,3} > 0$  (i.e.  $f_{x_{2|1,3}}(\cdot) = -\mathbb{1}_{(x_{2|1,3}, \infty)}(\cdot)\phi(x_{2|1,3})$ ) and  $x_1 > x_{2|1,3}$ . The latter inequality implies in particular  $x_2 > x_{2|1,3}$  and  $x_3 > x_{2|1,3}$ , because  $x_3 \geq x_2 \geq x_1$ . Hence  $f_{x_{2|1,3}}(x_1) = f_{x_{2|1,3}}(x_2) = f_{x_{2|1,3}}(x_3)$ . In view of (34), this gives  $t_2 < t_3$ . But this contradicts (33).

2): The envelope function  $\bar{f}$  is given by  $\bar{f}(t) = \phi(t)$  for  $t \leq 0$  and by  $\bar{f}(t) = \phi(t-) = \phi(t)$  (recall that  $\phi$  is continuous) for  $t > 0$ . Then under assumption (a) the integrability condition 2) holds.  $\square$

### 5.3. Some applications

In this section we discuss two specific examples. First we rigorously treat the case of empirical distortion risk measures. Thereafter we informally discuss bootstrap results for U- and V-statistics.

1) Let  $f_g : \mathbf{D}(f_g) \rightarrow \mathbb{R}$  be the distortion risk functional associated with a continuous concave distortion function as in (1) and Example 4.5, and let  $\phi : \mathbb{R} \rightarrow [1, \infty)$  be any continuous function. Let  $F \in \mathbf{D}(f_g)$  satisfy the integrability condition (24). Let  $(X_i)$  be a strictly stationary sequence of real-valued random variables on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with distribution function  $F$ . Let  $\widehat{F}_n$  be the empirical distribution function of  $X_1, \dots, X_n$  defined by (19). If  $X_1, X_2, \dots$  are independent,  $\int \phi^2 dF < \infty$ , and  $\widehat{F}_n^*$  is as in Theorem 5.2 (on

some extension  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$  of the original probability space), then Corollary 4.2, Example 4.3, Examples 4.5–4.6, and Theorem 5.2 show that  $(f_g(\widehat{F}_n^*))$  is a bootstrap version in probability of  $(f_g(\widehat{F}_n))$ . This bootstrap consistency can also be obtained by results on L-statistics by Helmers et al. (1990) and Gribkova (2002). However, the latter results rely on the independence of  $X_1, X_2, \dots$ . To the best of our knowledge so far there do not exist general results on bootstrap consistency for empirical distortion risk measures associated with continuous concave distortion functions when the data  $X_1, X_2, \dots$  are dependent. On the other hand, our theory admits such results. Indeed, if the sequence  $(X_i)$  is  $\beta$ -mixing with mixing rate as in condition (b) of Theorem 5.4,  $\int \phi^p dF < \infty$  for some  $p > 2$ , and  $\widehat{F}_n^*$  is as in Theorem 5.4, then Corollary 4.2, Example 4.4, Examples 4.5–4.6, and Theorem 5.4 show that  $(f_g(\widehat{F}_n^*))$  is a bootstrap version in probability of  $(f_g(\widehat{F}_n))$ . We emphasize that the results by Lahiri (2003, Chapter 4.4) for  $\alpha$ -mixing data do not cover this bootstrap consistency, because Lahiri assumes Fréchet differentiable for  $f_g$  which fails for continuous concave distortion functions  $g$ .

2) Let  $f_h : \mathbf{D}(f_h) \rightarrow \mathbb{R}$  be the V-functional corresponding to a given Borel measurable function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  (sometimes referred to as kernel) which is defined by

$$f_h(F) := \iint h(x_1, x_2) dF(x_1)dF(x_2), \quad (35)$$

where  $\mathbf{D}(f_h)$  denotes the set of all distribution functions on the real line for which the double integral in (35) exists. It was shown in Theorem 4.1 in Beutner and Zähle (2012) that subject to some regularity conditions on  $h$  and  $F$  the V-functional  $f_h$  is quasi-Hadamard differentiable at  $F$  w.r.t. a suitable nonuniform sup-norm. Similar as in Example 4.6 it can be shown that condition (d) of Corollary 4.2 holds for the quasi-Hadamard derivative of  $f_h$ . Then again, if  $(X_i)$  is a stationary  $\beta$ -mixing sequence of random variables with distribution function  $F$  and mixing rate as in condition (b) of Theorem 5.4,  $\int \phi^p dF < \infty$  for some  $p > 2$ , and  $\widehat{F}_n^*$  is as in Theorem 5.4, Corollary 4.2 shows that  $(f_h(\widehat{F}_n^*))$  is a bootstrap version in probability of  $(f_h(\widehat{F}_n))$ . Other approaches to show bootstrap consistency for non-degenerate U- and V-statistics can be found, for example, in Arcones and Giné (1992), Janssen (1994), and Dehling and Wendler (2010) (yet another approach was exemplified for the variance by Dudley (1990)); see also Bücher and Kojadinovic (2015) who use results of Dehling and Wendler (2010). Among other things Dehling and Wendler (2010, Theorem 2.1) also establish bootstrap consistency for non-degenerate U- and V-statistics for  $\beta$ -mixing sequences. Whereas their approach requires an additional integrability condition on  $(X_1, X_k)$ , our approach (based on Corollary 4.2 that we just outlined) requires stronger regularity conditions on the kernel  $h$ . Looking at condition (b) in Theorem 5.4 and the condition on the mixing coefficient in Dehling and Wendler (2010, Theorem 2.1), it seems that both approaches impose the same condition on the mixing coefficient. Thus, the approach based on Corollary 4.2 may supplement the results in Dehling and Wendler (2010).

### 6. Proof of Theorem 3.1

We start with a convention and a general remark. We will equip the product space  $\overline{\mathbf{E}} := \mathbf{E} \times \mathbf{E}$  with the metric  $\overline{d}((x_1, x_2), (y_1, y_2)) := \max\{\|x_1 - y_1\|_{\mathbf{E}}; \|x_2 - y_2\|_{\mathbf{E}}\}$ , and denote the corresponding open-ball  $\sigma$ -algebra on  $\overline{\mathbf{E}}$  by  $\overline{\mathcal{B}}^\circ$ . Note that  $\overline{\mathcal{B}}^\circ \subseteq \mathcal{B}^\circ \otimes \mathcal{B}^\circ$ , because any  $\overline{d}$ -open ball in  $\overline{\mathbf{E}}$  is the product of two  $\|\cdot\|_{\mathbf{E}}$ -open balls in  $\mathbf{E}$ . Analogously the product space  $\widetilde{\mathbf{E}} := \widetilde{\mathbf{E}} \times \widetilde{\mathbf{E}}$  will be equipped with the metric  $\widetilde{d}((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)) := \max\{\|\tilde{x}_1 - \tilde{y}_1\|_{\widetilde{\mathbf{E}}}; \|\tilde{x}_2 - \tilde{y}_2\|_{\widetilde{\mathbf{E}}}\}$ . By the separability of  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  the corresponding Borel  $\sigma$ -algebra  $\widetilde{\mathcal{B}}$  coincides with the product  $\sigma$ -algebra  $\widetilde{\mathcal{B}} \otimes \widetilde{\mathcal{B}}$ ; cf. Dudley (2002, Proposition 4.1.7). So the couple  $(\xi_1, \xi_2)$  is an  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ -valued random variable when  $\xi_1$  and  $\xi_2$  are  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ -valued random variables. In particular,  $h(\xi_1, \xi_2)$  is an  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ -valued random variable when  $h : \widetilde{\mathbf{E}} \rightarrow \widetilde{\mathbf{E}}$  is continuous. Since the addition and the multiplication by constants in normed vector spaces are continuous, we have in particular that a linear combination of two  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ -valued random variables is again an  $(\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}})$ -valued random variable. This fact will be used frequently in the sequel without further mentioning.

(i): By assumption (b) we have that  $f(\widehat{T}_n)$  is  $(\mathcal{F}, \widetilde{\mathcal{B}})$ -measurable. This implies that  $a_n(f(\widehat{T}_n) - f(\theta))$  is  $(\mathcal{F}, \widetilde{\mathcal{B}})$ -measurable for every  $n \in \mathbb{N}$ , because we assumed that  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  is separable. Now, assertion (i) directly follows from the functional delta-method in the form of Theorem C.4.

(ii): Recall that  $\widehat{T}_n$  will frequently be seen as a map defined on the extension  $\overline{\Omega}$  of  $\Omega$ . From the above we therefore have that  $f(\widehat{T}_n)$  is  $(\overline{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable. Moreover,  $f(\widehat{T}_n^*)$  is  $(\overline{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable due assumption (e). In particular, the map  $a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n))$  is  $(\overline{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable, because we assumed that  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  is separable. By assumptions (a) and (d) we also have that the map  $\dot{f}_\theta(\xi)$  is  $(\mathcal{F}_0, \widetilde{\mathcal{B}})$ -measurable, and by assumptions (d) and (f) we have that the map  $\dot{f}_\theta(a_n(\widehat{T}_n^* - \widehat{T}_n))$  is  $(\overline{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable.

To verify (17), we will adapt the arguments of Section 3.9.3 in van der Vaart and Wellner (1996). First note that  $\widetilde{Q}_n$  defined by

$$\widetilde{Q}_n(\omega, \widetilde{A}) := \mathbb{P}' \circ \left\{ \dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \cdot) - \widehat{T}_n(\omega))) \right\}^{-1}[\widetilde{A}], \quad \omega \in \Omega, \widetilde{A} \in \widetilde{\mathcal{B}}$$

provides a conditional distribution of  $\dot{f}_\theta(a_n(\widehat{T}_n^* - \widehat{T}_n))$  given  $\Pi$ . This follows from Lemma D.2 (with  $X(\omega, \omega') = g(\omega, \omega') = \dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega)))$  and  $Y = \Pi$ ). Now, let  $\delta > 0$  be arbitrary but fixed. For (17) it suffices to show that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \left\{ \omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \widetilde{Q}_n(\omega, \cdot)) \geq \frac{\delta}{2} \right\} \right] = 0 \quad (36)$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \left\{ \omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{Q}_n(\omega, \cdot), \text{law}\{\dot{f}_\theta(\xi)\}) \geq \frac{\delta}{2} \right\} \right] = 0. \quad (37)$$

Note that the mappings  $\omega \mapsto \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \widetilde{Q}_n(\omega, \cdot))$  and  $\omega \mapsto \widetilde{d}_{\text{BL}}(\widetilde{Q}_n(\omega, \cdot), \text{law}\{\dot{f}_\theta(\xi)\})$  are  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable, because  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  was assumed to be

separable. For the latter map one can argue as subsequent to Definition 2.2. For the former map one can argue in the same way, noting that  $(\widetilde{\mathcal{M}}_1, \widetilde{d}_{\text{BL}})$  is separable (cf. Remark A.2 and Theorem A.4) and that the metric distance of two random variables in a separable metric space is also measurable (cf. Klenke (2014, Lemma 6.1)). In particular, the events in (36) and (37) are  $\mathcal{F}$ -measurable.

We first show (37). By (14) in assumption (f), the Continuous Mapping theorem in the form of Billingsley (1999, Theorem 6.4) (along with  $\mathbb{P}_0 \circ \xi^{-1}[\mathbf{E}_0] = 1$  and the continuity of  $f_\theta$ ), and the implication (a) $\Rightarrow$ (g) in the Portmanteau theorem A.3, we have

$$\lim_{n \rightarrow \infty} \widetilde{d}_{\text{BL}}(\widetilde{Q}_n(\omega, \cdot), \text{law}\{\dot{f}_\theta(\xi)\}) = 0 \quad \mathbb{P}\text{-a.e. } \omega.$$

Since almost sure convergence of real-valued random variables implies convergence in probability, we arrive at (37).

To verify (36), we set

$$\eta_n(\omega, \omega') := a_n(f(\widehat{T}_n^*(\omega, \omega')) - f(\widehat{T}_n(\omega))) - \dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega)))$$

and

$$\eta_{n, \widetilde{h}}(\omega, \omega') := \widetilde{h}\left(a_n(f(\widehat{T}_n^*(\omega, \omega')) - f(\widehat{T}_n(\omega)))\right) - \widetilde{h}\left(\dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega)))\right)$$

for every  $\widetilde{h} \in \widetilde{\text{BL}}_1$  with  $\widetilde{\text{BL}}_1$  as defined before (45). We then obtain

$$\begin{aligned} & \mathbb{P}\left[\left\{\omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \widetilde{Q}_n(\omega, \cdot)) \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \widetilde{h}(\widetilde{x}) \widetilde{P}_n(\omega, d\widetilde{x}) - \int \widetilde{h}(\widetilde{x}) \widetilde{Q}_n(\omega, d\widetilde{x}) \right| \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \widetilde{h}\left(a_n(f(\widehat{T}_n^*(\omega, \omega')) - f(\widehat{T}_n(\omega)))\right) \mathbb{P}'[d\omega'] \right. \right. \\ &\quad \left. \left. - \int \widetilde{h}\left(\dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega)))\right) \mathbb{P}'[d\omega'] \right| \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \eta_{n, \widetilde{h}}(\omega, \omega') \mathbb{P}'[d\omega'] \right| \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}^{\text{out}}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \eta_{n, \widetilde{h}}(\omega, \omega') \mathbb{P}'[d\omega'] \right| \geq \frac{\delta}{2}\right\}\right] \\ &\leq \mathbb{P}^{\text{out}}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \int |\eta_{n, \widetilde{h}}(\omega, \omega')| \mathbb{P}'[d\omega'] \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}^{\text{out}}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left( \int |\eta_{n, \widetilde{h}}(\omega, \omega')| \mathbb{1}_{\{|\eta_{n, \widetilde{h}}| < \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega'] \right. \right. \right. \\ &\quad \left. \left. + \int |\eta_{n, \widetilde{h}}(\omega, \omega')| \mathbb{1}_{\{|\eta_{n, \widetilde{h}}| \geq \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega'] \right) \geq \frac{\delta}{2}\right\}\right] \\ &\leq \mathbb{P}^{\text{out}}\left[\left\{\omega \in \Omega : \frac{\delta}{4} + \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \int |\eta_{n, \widetilde{h}}(\omega, \omega')| \mathbb{1}_{\{|\eta_{n, \widetilde{h}}| \geq \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega'] \geq \frac{\delta}{2}\right\}\right] \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{P}^{\text{out}} \left[ \left\{ \omega \in \Omega : \sup_{\tilde{h} \in \widetilde{\text{BL}}_1} \int 2 \mathbb{1}_{\{|\eta_n, \tilde{h}| \geq \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega'] \geq \frac{\delta}{4} \right\} \right] \\
&\leq \mathbb{P}^{\text{out}} \left[ \left\{ \omega \in \Omega : 2 \int \mathbb{1}_{\{\|\eta_n\|_{\widetilde{\mathbf{E}}} \geq \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega'] \geq \frac{\delta}{4} \right\} \right], \tag{38}
\end{aligned}$$

where the second last and the last step are ensured by  $\|\tilde{h}\|_\infty \leq 1$  and the Lipschitz continuity of  $h$  (with Lipschitz constant 1), respectively. We have seen above that the maps  $a_n(f(\widehat{T}_n) - f(\widehat{T}_n^*))$  and  $\dot{f}_\theta(a_n(\widehat{T}_n^* - \widehat{T}_n))$  are  $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable. Since  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  is separable, we can conclude that the map  $\eta_n$  is  $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{B}})$ -measurable. Since the map  $\|\cdot\|_{\widetilde{\mathbf{E}}} : \widetilde{\mathbf{E}} \rightarrow \mathbb{R}_+$  is continuous and thus  $(\widetilde{\mathcal{B}}, \mathcal{B}(\mathbb{R}_+))$ -measurable, we obtain that the map  $\|\eta_n\|_{\widetilde{\mathbf{E}}} : \widetilde{\Omega} \rightarrow \mathbb{R}_+$  is  $(\widetilde{\mathcal{F}}, \mathcal{B}(\mathbb{R}_+))$ -measurable. By Fubini's theorem we can conclude that the mapping

$$\omega \mapsto \int \mathbb{1}_{\{\|\eta_n\|_{\widetilde{\mathbf{E}}} \geq \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega']$$

is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable. Therefore, we may replace the outer probability  $\mathbb{P}^{\text{out}}$  by the ordinary probability  $\mathbb{P}$  in the last line of (38). So we obtain

$$\begin{aligned}
&\mathbb{P} \left[ \left\{ \omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{P}_n(\omega, \cdot), \widetilde{Q}_n(\omega, \cdot)) \geq \frac{\delta}{2} \right\} \right] \\
&\leq \mathbb{P} \left[ \left\{ \omega \in \Omega : 2 \int \mathbb{1}_{\{\|\eta_n\|_{\widetilde{\mathbf{E}}} \geq \delta/4\}}(\omega, \omega') \mathbb{P}'[d\omega'] \geq \frac{\delta}{4} \right\} \right] \\
&= \mathbb{P} \left[ \left\{ \omega \in \Omega : \mathbb{P}' \left[ \left\{ \omega' \in \Omega' : \|\eta_n(\omega, \omega')\|_{\widetilde{\mathbf{E}}} \geq \frac{\delta}{4} \right\} \right] \geq \frac{\delta}{8} \right\} \right] \\
&\leq \frac{8}{\delta} \int \mathbb{P}' \left[ \left\{ \omega' \in \Omega' : \|\eta_n(\omega, \omega')\|_{\widetilde{\mathbf{E}}} \geq \frac{\delta}{4} \right\} \right] \mathbb{P}[d\omega] \\
&= \frac{8}{\delta} \overline{\mathbb{P}} \left[ \left\{ (\omega, \omega') \in \widetilde{\Omega} : \|\eta_n(\omega, \omega')\|_{\widetilde{\mathbf{E}}} \geq \frac{\delta}{4} \right\} \right] \\
&= \frac{8}{\delta} \overline{\mathbb{P}} \left[ \left\| a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n)) - \dot{f}_\theta(a_n(\widehat{T}_n^* - \widehat{T}_n)) \right\|_{\widetilde{\mathbf{E}}} \geq \frac{\delta}{4} \right],
\end{aligned}$$

where for the third and the fourth step we used respectively Markov's inequality and the representation of the product measure  $\overline{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$  as given in Bauer (2001, Formula (23.3)). Thus, it remains to show that

$$a_n(f(\widehat{T}_n^*) - f(\widehat{T}_n)) - \dot{f}_\theta(a_n(\widehat{T}_n^* - \widehat{T}_n)) \xrightarrow{\mathbb{P}} 0_{\widetilde{\mathbf{E}}} \quad \text{in } (\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}}) \text{ w.r.t. } \overline{\mathbb{P}}, \tag{39}$$

where  $\xrightarrow{\mathbb{P}}$  refers to convergence in probability and  $0_{\widetilde{\mathbf{E}}}$  denotes the null in  $\widetilde{\mathbf{E}}$ . To prove (39), we note that by assumption (b) we have that  $a_n(\widehat{T}_n - \theta)$  converges in distribution<sup>o</sup> to some separable random variable,  $\xi$ . So we may apply part (ii) of Theorem C.4 to obtain

$$a_n(f(\widehat{T}_n) - f(\theta)) - \dot{f}_\theta(a_n(\widehat{T}_n - \theta)) \xrightarrow{\mathbb{P}} 0_{\widetilde{\mathbf{E}}} \quad \text{in } (\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}}) \text{ w.r.t. } \overline{\mathbb{P}}, \tag{40}$$

where condition (g) of Theorem C.4 holds since  $(\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}})$  was assumed to be separable (cf. the discussion at the beginning of the proof). Further, in the following we will show that  $a_n(\widehat{T}_n^* - \theta)$  converges in distribution<sup>o</sup> to some separable



random variable too. So we may apply part (ii) of Theorem C.4 once more to obtain

$$a_n(f(\widehat{T}_n^*) - f(\theta)) - \dot{f}_\theta(a_n(\widehat{T}_n^* - \theta)) \xrightarrow{\mathbb{P}} 0_{\mathbf{E}} \quad \text{in } (\widetilde{\mathbf{E}}, \|\cdot\|_{\widetilde{\mathbf{E}}}) \text{ w.r.t. } \widetilde{\mathbb{P}}. \quad (41)$$

Now, (40), (41) and the linearity of  $\dot{f}_\theta$  imply (39).

It remains to show that  $a_n(\widehat{T}_n^* - \theta)$  converges in distribution $^\circ$  to some separable random variable. For this it suffices to show that  $(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n))$  converges in distribution $^\circ$  to  $(\xi_1, \xi_2)$  in  $(\overline{\mathbf{E}}, \overline{\mathcal{B}}^\circ, \bar{d})$ , where  $(\xi_1, \xi_2)$  is an  $(\overline{\mathbf{E}}, \overline{\mathcal{B}}^\circ)$ -valued random variable (on some probability space) which takes values only in  $\overline{\mathbf{E}}_0 := \mathbf{E}_0 \times \mathbf{E}_0$ . In fact, the extended Continuous Mapping theorem C.1 applied to the functions  $h_n : \overline{\mathbf{E}} \rightarrow \mathbf{E}$  and  $h_0 : \overline{\mathbf{E}}_0 \rightarrow \mathbf{E}_0 \subseteq \mathbf{E}$  given by respectively  $h_n(x, y) := x + y$  and  $h_0(x, y) := x + y$  then implies that  $a_n(\widehat{T}_n^* - \theta) = a_n(\widehat{T}_n^* - \widehat{T}_n) + a_n(\widehat{T}_n - \theta)$  converges in distribution $^\circ$  to the separable random variable  $\xi_1 + \xi_2$ . For the application of the extended Continuous Mapping theorem note that  $h_n(a_n(\widehat{T}_n^* - \widehat{T}_n), a_n(\widehat{T}_n - \theta)) = a_n(\widehat{T}_n^* - \theta)$  is  $(\overline{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable by the first part of condition (g) and that the map  $h_0 : \overline{\mathbf{E}}_0 \rightarrow \mathbf{E}$  is continuous and  $(\overline{\mathcal{B}}_0, \mathcal{B}^\circ)$ -measurable for  $\overline{\mathcal{B}}_0 := \overline{\mathcal{B}}^\circ \cap \overline{\mathbf{E}}_0 = \overline{\mathcal{B}} \cap \overline{\mathbf{E}}_0$ . For the latter measurability take into account that  $\overline{\mathbf{E}}_0$  is separable w.r.t.  $\bar{d}$  and argue as at the beginning of the proof. Also note that  $(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n))$  can be seen as an  $(\overline{\mathbf{E}}, \overline{\mathcal{B}}^\circ)$ -valued random variable, because it is obviously  $(\mathcal{F}, \mathcal{B}^\circ \otimes \mathcal{B}^\circ)$ -measurable and  $\overline{\mathcal{B}}^\circ \subseteq \mathcal{B}^\circ \otimes \mathcal{B}^\circ$ .

To show that  $(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n))$  converges in distribution $^\circ$  to some separable random element  $(\xi_1, \xi_2)$ , we will adapt some of the arguments of the proof of Theorem 2.2 in Kosorok (2008) where weak convergence is understood in the Hoffmann-Jørgensen sense. Let  $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2) := (\overline{\mathbf{E}}, \mathcal{B}^\circ \otimes \mathcal{B}^\circ, (\mathbb{P}_0 \circ \xi^{-1}) \otimes (\mathbb{P}_0 \circ \xi^{-1}))$  (with  $\xi$  and  $\mathbb{P}_0$  as in condition (b)) and  $\xi_i$  be the  $i$ -th coordinate projection on  $\Omega_1 \times \Omega_2 = \overline{\mathbf{E}}$ ,  $i = 1, 2$ . Then  $(\xi_1, \xi_2)$  can be seen as an  $(\overline{\mathbf{E}}, \overline{\mathcal{B}}^\circ)$ -valued random variable on  $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ , because by  $\overline{\mathcal{B}}^\circ \subseteq \mathcal{B}^\circ \otimes \mathcal{B}^\circ$  it is clearly  $(\mathcal{B}^\circ \otimes \mathcal{B}^\circ, \overline{\mathcal{B}}^\circ)$ -measurable. In view of the implication (f) $\Rightarrow$ (a) in the Portmanteau theorem A.3, for the convergence in distribution $^\circ$  of the pair  $(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n))$  to the random variable  $(\xi_1, \xi_2)$  it suffices to show that

$$\int \bar{h}(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n)) d(\mathbb{P} \otimes \mathbb{P}') \longrightarrow \int \bar{h}(\xi_1, \xi_2) d(\mathbb{P}_1 \otimes \mathbb{P}_2)$$

for every  $\bar{h} \in \overline{\text{BL}}_1^\circ$ , where  $\overline{\text{BL}}_1^\circ$  denotes the set of all real-valued functions on  $\overline{\mathbf{E}} = \mathbf{E} \times \mathbf{E}$  that are  $(\overline{\mathcal{B}}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable, bounded by 1 and Lipschitz continuous with Lipschitz constant 1 (as defined before (45)). So, let  $\bar{h} \in \overline{\text{BL}}_1^\circ$ . We have

$$\begin{aligned} & \left| \int \bar{h}(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n)) d(\mathbb{P} \otimes \mathbb{P}') - \int \bar{h}(\xi_1, \xi_2) d(\mathbb{P}_1 \otimes \mathbb{P}_2) \right| \\ & \leq \left| \int \bar{h}(a_n(\widehat{T}_n - \theta), a_n(\widehat{T}_n^* - \widehat{T}_n)) d(\mathbb{P} \otimes \mathbb{P}') \right. \\ & \quad \left. - \int \bar{h}(a_n(\widehat{T}_n - \theta), \xi_2) d(\mathbb{P} \otimes \mathbb{P}_2) \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int \bar{h}(a_n(\widehat{T}_n - \theta), \xi_2) d(\mathbb{P} \otimes \mathbb{P}_2) - \int \bar{h}(\xi_1, \xi_2) d(\mathbb{P}_1 \otimes \mathbb{P}_2) \right| \\
& =: S_1(n) + S_2(n).
\end{aligned}$$

For every  $x_2 \in \mathbf{E}$ , define the function  $h_{x_2} : \mathbf{E} \rightarrow \mathbb{R}$  by  $h_{x_2}(x_1) := \bar{h}(x_1, x_2)$  and note that  $h_{x_2}$  is bounded, continuous, and  $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable. The latter measurability means that  $h_{x_2}^{-1}(B) = (\bar{h}^{-1}(B))_{x_2} := \{x_1 \in \mathbf{E} : (x_1, x_2) \in \bar{h}^{-1}(B)\}$  lies in  $\mathcal{B}^\circ$  for every  $B \in \mathcal{B}(\mathbb{R})$ . By the  $(\bar{\mathcal{B}}^\circ, \mathcal{B}(\mathbb{R}))$ -measurability of  $\bar{h}$  the set  $\bar{h}^{-1}(B)$  lies in  $\bar{\mathcal{B}}^\circ$ . By Lemma 23.1 of Bauer (2001) the set  $A_{x_2} := \{x_1 \in \mathbf{E} : (x_1, x_2) \in A\}$  lies in  $\mathcal{B}^\circ$  for every  $A \in \mathcal{B}^\circ \otimes \mathcal{B}^\circ$ . Thus, in view of  $\bar{\mathcal{B}}^\circ \subseteq \mathcal{B}^\circ \otimes \mathcal{B}^\circ$ , it follows that the functions  $h_{x_2}$  is indeed  $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable. Now, with the help of Fubini's theorem we obtain

$$\begin{aligned}
& S_2(n) \\
& \leq \int \left| \int \bar{h}(a_n(\widehat{T}_n(\omega) - \theta), \xi_2(\omega_2)) \mathbb{P}[d\omega] \right. \\
& \quad \left. - \int \bar{h}(\xi_1(\omega_1), \xi_2(\omega_2)) \mathbb{P}_1(d\omega_1) \right| \mathbb{P}_2[d\omega_2] \\
& = \int \left| \int h_{\xi_2(\omega_2)}(a_n(\widehat{T}_n(\omega) - \theta)) \mathbb{P}[d\omega] - \int h_{\xi_2(\omega_2)}(\xi_1(\omega_1)) \mathbb{P}_1(d\omega_1) \right| \mathbb{P}_2[d\omega_2].
\end{aligned}$$

In view of assumption (a), the integrand of the outer integral converges to 0 for every  $\omega_2$ . So, since  $\|h_{x_2}(\cdot)\|_\infty \leq 1$  for every  $x_2 \in \mathbf{E}$ , the Dominated Convergence theorem implies that the summand  $S_2(n)$  converges to 0. For every  $x_1 \in \mathbf{E}$ , define the function  $h_{x_1} : \mathbf{E} \rightarrow \mathbb{R}$  by  $h_{x_1}(x_2) := \bar{h}(x_1, x_2)$  and note that  $h_{x_1} \in \text{BL}_1^\circ$  for every  $x_1 \in \mathbf{E}$  (for the measurability of  $h_{x_1}$  one can argue as for  $h_{x_2}$  above). With the help of Fubini's theorem we obtain

$$\begin{aligned}
S_1(n) & \leq \int \left| \int \bar{h}(a_n(\widehat{T}_n(\omega) - \theta), a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))) \mathbb{P}'[d\omega'] \right. \\
& \quad \left. - \int \bar{h}(a_n(\widehat{T}_n(\omega) - \theta), \xi_2(\omega_2)) \mathbb{P}_2[d\omega_2] \right| \mathbb{P}[d\omega] \\
& = \int \left| \int h_{a_n(\widehat{T}_n(\omega) - \theta)}(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))) \mathbb{P}'[d\omega'] \right. \\
& \quad \left. - \int h_{a_n(\widehat{T}_n(\omega) - \theta)}(\xi_2(\omega_2)) \mathbb{P}_2[d\omega_2] \right| \mathbb{P}[d\omega] \\
& \leq \int \sup_{m \in \mathbb{N}} \left| \int h_{a_m(\widehat{T}_m(\omega) - \theta)}(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))) \mathbb{P}'[d\omega'] \right. \\
& \quad \left. - \int h_{a_m(\widehat{T}_m(\omega) - \theta)}(\xi_2(\omega_2)) \mathbb{P}_2[d\omega_2] \right| \mathbb{P}[d\omega]. \quad (42)
\end{aligned}$$

The integrand of the outer integral is bounded above by  $d_{\text{BL}}^\circ(P_n(\omega, \cdot), \text{law}\{\xi_2\})$ . So it follows by the second part of assumption (f) and the implication (a) $\Rightarrow$ (g) in the Portmanteau theorem A.3 that the integrand of the outer integral converges to 0 for  $\mathbb{P}$ -a.e.  $\omega$ . In view of  $\|h_{a_m(\widehat{T}_m(\omega) - \theta)}(\cdot)\|_\infty \leq 1$  for every  $m \in \mathbb{N}$ , the

Dominated Convergence theorem implies that the summand  $S_1(n)$  converges to 0 too. This completes the proof of part (ii).

(iii): One can proceed as for the proof of part (ii). It again suffices to show (36) and (37). The proof of (36) can be transferred nearly verbatim. The convergence of the upper bound in (42) to zero was justified by the classical Dominated Convergence theorem. This time one has to use slightly different arguments. The upper bound in (42) is bounded above by

$$\int^{\text{out}} \sup_{h \in \text{BL}_1^c} \left| \int h(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))) \mathbb{P}'[d\omega'] - \int h(\xi_2(\omega_2)) \mathbb{P}_2[d\omega_2] \right| \mathbb{P}[d\omega],$$

which equals

$$\int^{\text{out}} d_{\text{BL}}^c(P_n(\omega, \cdot), \text{law}\{\xi\}) \mathbb{P}[d\omega].$$

Here  $\int^{\text{out}}$  refers to the outer integral (outer expectation). By (15) in assumption (f'), the integrand of the latter integral converges to 0 in outer probability. Lemma 3.3.4 in Dudley (1999) then implies

$$\limsup_{n \rightarrow \infty} \int^{\text{out}} d_{\text{BL}}^c(P_n(\omega, \cdot), \text{law}\{\xi\}) \mathbb{P}[d\omega] \leq 0.$$

It follows that the summand  $S_1(n)$  again converges to 0. This gives (36).

It remains to show that (37) can also be derived from assumption (f'). We have

$$\begin{aligned} & \mathbb{P}\left[\left\{\omega \in \Omega : \widetilde{d}_{\text{BL}}(\widetilde{Q}_n(\omega, \cdot), \text{law}\{\dot{f}_\theta(\xi)\}) \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \widetilde{h}(\widetilde{x}) \widetilde{Q}_n(\omega, d\widetilde{x}) - \int \widetilde{h}(\widetilde{x}) \text{law}\{\dot{f}_\theta(\xi)\}[d\widetilde{x}] \right| \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \widetilde{h}\left(\dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega)))\right) \mathbb{P}'[d\omega'] \right. \right. \\ &\quad \left. \left. - \int \widetilde{h}(\dot{f}_\theta(\xi(\omega_0))) \mathbb{P}_0[d\omega_0] \right| \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \widetilde{h} \circ \dot{f}_\theta(a_n(\widehat{T}_n^*(\omega, \omega') - \widehat{T}_n(\omega))) \mathbb{P}'[d\omega'] \right. \right. \\ &\quad \left. \left. - \int \widetilde{h} \circ \dot{f}_\theta(\xi(\omega_0)) \mathbb{P}_0[d\omega_0] \right| \geq \frac{\delta}{2}\right\}\right] \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \sup_{\widetilde{h} \in \widetilde{\text{BL}}_1} \left| \int \widetilde{h} \circ \dot{f}_\theta(x) P_n(\omega, dx) \right. \right. \right. \\ &\quad \left. \left. - \int \widetilde{h} \circ \dot{f}_\theta(x) \text{law}\{\xi\}[dx] \right| \geq \frac{\delta}{2}\right\}\right] \\ &\leq \mathbb{P}^{\text{out}}\left[\left\{\omega \in \Omega : \sup_{h \in \text{BL}_1^c} \left| \int h(x) P_n(\omega, dx) \right. \right. \right. \end{aligned} \tag{43}$$

$$\begin{aligned}
& - \int h(x) \text{law}\{\xi\}[dx] \Big| \geq \frac{\delta}{2(L_{f,\theta} \vee 1)} \Big\} \\
= & \mathbb{P}^{\text{out}} \left[ \left\{ \omega \in \Omega : d_{\text{BL}}^{\circ}(P_n(\omega, \cdot), \text{law}\{\xi\}) \geq \frac{\delta}{2(L_{f,\theta} \vee 1)} \right\} \right], \tag{44}
\end{aligned}$$

where  $L_{f,\theta} > 0$  denotes the Lipschitz constant of the linear and continuous (thus Lipschitz continuous) map  $\dot{f}_{\theta}$ . The last line in (44) converges to 0 as  $n \rightarrow \infty$  by (15) in assumption (f'). This gives (37), and the proof is complete.  $\square$

### Appendix A: Weak topology and weak convergence for the open-ball $\sigma$ -algebra

Let  $(\mathbf{E}, d)$  be a metric space and  $\mathcal{B}^{\circ}$  be the  $\sigma$ -algebra on  $\mathbf{E}$  generated by the open balls  $B_r(x) := \{y \in \mathbf{E} : d(x, y) < r\}$ ,  $x \in \mathbf{E}$ ,  $r > 0$ . We will refer to  $\mathcal{B}^{\circ}$  as *open-ball  $\sigma$ -algebra*. If  $(\mathbf{E}, d)$  is separable, then  $\mathcal{B}^{\circ}$  coincides with the Borel  $\sigma$ -algebra  $\mathcal{B}$ . If  $(\mathbf{E}, d)$  is not separable, then  $\mathcal{B}^{\circ}$  might be strictly smaller than  $\mathcal{B}$  and thus a continuous real-valued function on  $\mathbf{E}$  is not necessarily  $(\mathcal{B}^{\circ}, \mathcal{B}(\mathbb{R}))$ -measurable. Let  $C_{\text{b}}^{\circ}$  be the set of all bounded, continuous and  $(\mathcal{B}^{\circ}, \mathcal{B}(\mathbb{R}))$ -measurable real-valued functions on  $\mathbf{E}$ , and  $\mathcal{M}_1^{\circ}$  be the set of all probability measures on  $(\mathbf{E}, \mathcal{B}^{\circ})$ . For every  $f \in C_{\text{b}}^{\circ}$  we consider the mapping

$$\pi_f : \mathcal{M}_1^{\circ} \longrightarrow \mathbb{R}, \quad \mu \longmapsto \int f d\mu.$$

The *weak $^{\circ}$  topology*  $\mathcal{O}_{\text{w}}^{\circ}$  on  $\mathcal{M}_1^{\circ}$  is defined to be the topology  $\mathcal{O}(\mathbb{F})$  generated by the class of functions  $\mathbb{F} := \{\pi_f : f \in C_{\text{b}}^{\circ}\}$ . That is,  $\mathcal{O}_{\text{w}}^{\circ} := \mathcal{O}(\mathcal{S}_{\mathbb{F}}) := \bigcap_{\mathcal{O}} \text{topology on } \mathcal{M}_1^{\circ} \text{ with } \mathcal{O} \supseteq \mathcal{S}_{\mathbb{F}}$  for the system  $\mathcal{S}_{\mathbb{F}} := \{\pi_f^{-1}(G') : f \in C_{\text{b}}^{\circ}, G' \in \mathcal{O}_{\mathbb{R}}\}$ , where  $\mathcal{O}_{\mathbb{R}}$  is the usual topology of open sets in  $\mathbb{R}$ . In other words, the weak $^{\circ}$  topology is the coarsest topology on  $\mathcal{M}_1^{\circ}$  w.r.t. which each of the maps  $\pi_f$ ,  $f \in C_{\text{b}}^{\circ}$ , is continuous. A sequence  $(\mu_n)$  in  $\mathcal{M}_1^{\circ}$  converges to some  $\mu_0 \in \mathcal{M}_1^{\circ}$  in the weak $^{\circ}$  topology  $\mathcal{O}_{\text{w}}^{\circ}$  if and only if

$$\int f d\mu_n \longrightarrow \int f d\mu_0 \quad \text{for all } f \in C_{\text{b}}^{\circ};$$

see, for instance, Lemma 2.52 in Aliprantis and Border (2006) (take into account that every sequence is a net). In this case, we also say that  $(\mu_n)$  *converges weak $^{\circ}$ ly* to  $\mu_0$  and write  $\mu_n \Rightarrow^{\circ} \mu_0$ . It is worth mentioning that two probability measures  $\mu_0, \nu_0 \in \mathcal{M}_1^{\circ}$  coincide if  $\mu_0[\mathbf{E}_0] = \nu_0[\mathbf{E}_0] = 1$  for some separable  $\mathbf{E}_0 \in \mathcal{B}^{\circ}$  and  $\int f d\mu_0 = \int f d\nu_0$  for all uniformly continuous  $f \in C_{\text{b}}^{\circ}$ ; see, for instance, Billingsley (1999, Theorem 6.2).

**Remark A.1** Recall that  $\mathcal{B}^{\circ} = \mathcal{B}$  when  $(\mathbf{E}, d)$  is separable. In this case we suppress the superscript  $^{\circ}$  and write simply  $C_{\text{b}}$ ,  $\mathcal{M}_1$ , weak,  $\mathcal{O}_{\text{w}}$ , and  $\Rightarrow$  instead of  $C_{\text{b}}^{\circ}$ ,  $\mathcal{M}_1^{\circ}$ , weak $^{\circ}$ ,  $\mathcal{O}_{\text{w}}^{\circ}$ , and  $\Rightarrow^{\circ}$ , respectively.  $\diamond$

Denote by  $\text{BL}_1^\circ$  the set of all  $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable functions  $f : \mathbf{E} \rightarrow \mathbb{R}$  satisfying  $|f(x) - f(y)| \leq d(x, y)$  for all  $x, y \in \mathbf{E}$  and  $\sup_{x \in \mathbf{E}} |f(x)| \leq 1$ . Note that  $\text{BL}_1^\circ$  is contained in the set of all uniformly continuous functions in  $C_b^\circ$ . The bounded Lipschitz distance on  $\mathcal{M}_1^\circ$  is defined by

$$d_{\text{BL}}^\circ(\mu, \nu) := \sup_{f \in \text{BL}_1^\circ} \left| \int f d\mu - \int f d\nu \right|. \tag{45}$$

It is easily seen that the mapping  $d_{\text{BL}}^\circ : \mathcal{M}_1^\circ \times \mathcal{M}_1^\circ \rightarrow \mathbb{R}_+$  satisfies the axioms of a pseudo-metric on  $\mathcal{M}_1^\circ$ , i.e. that it is symmetric and satisfies  $d_{\text{BL}}^\circ(\mu, \mu) = 0$  as well as the triangle inequality.

**Remark A.2** If  $(\mathbf{E}, d)$  is separable, then we again suppress the superscript  $^\circ$  and write simply  $\text{BL}_1$  and  $d_{\text{BL}}$  instead of  $\text{BL}_1^\circ$  and  $d_{\text{BL}}^\circ$ , respectively. In this case the bounded Lipschitz distance  $d_{\text{BL}}$  provides even a metric on  $\mathcal{M}_1$ , because  $\text{BL}_1$  is separating in  $\mathcal{M}_1$ ; the latter follows from the proof of Theorem 1.2 in Billingsley (1999).  $\diamond$

**Theorem A.3 (Portmanteau theorem)** *Let  $\mu_n \in \mathcal{M}_1^\circ$ ,  $n \in \mathbb{N}_0$ , and assume that  $\mu_0[\mathbf{E}_0] = 1$  for some separable  $\mathbf{E}_0 \in \mathcal{B}^\circ$ . Then the following conditions are equivalent:*

- (a)  $\mu_n \Rightarrow^\circ \mu_0$ .
- (b)  $\int f d\mu_n \rightarrow \int f d\mu_0$  for all uniformly continuous  $f \in C_b^\circ$ .
- (c)  $\limsup_{n \rightarrow \infty} \mu_n[F] \leq \mu_0[F]$  for all closed  $F \in \mathcal{B}^\circ$ .
- (d)  $\liminf_{n \rightarrow \infty} \mu_n[G] \geq \mu_0[G]$  for all open  $G \in \mathcal{B}^\circ$ .
- (e)  $\mu_n[A] \rightarrow \mu_0[A]$  for every  $A \in \mathcal{B}^\circ$  for which  $\mathcal{B}^\circ$  contains an open set  $G$  and a closed set  $F$  such that  $G \subseteq A \subseteq F$  and  $\mu_0[F \setminus G] = 0$ .
- (f)  $\int f d\mu_n \rightarrow \int f d\mu_0$  for all  $f \in \text{BL}_1^\circ$ .
- (g)  $d_{\text{BL}}^\circ(\mu_n, \mu_0) \rightarrow 0$ .

**Proof** The equivalence of the conditions (a), (b), (c), (d), and (e) is known from Theorem 6.3 of Billingsley (1999), and the implications (b) $\Rightarrow$ (f) is trivial. The arguments in the proof of (b) $\Rightarrow$ (c) in Theorem 6.3 of Billingsley (1999) also prove the implication (f) $\Rightarrow$ (c). Indeed, the function  $f$  defined in (6.1) in Billingsley (1999) is bounded by 1 and Lipschitz continuous with Lipschitz constant  $\varepsilon^{-1}$ ,  $\varepsilon f$  is an element of  $\text{BL}_1^\circ$  for  $\varepsilon \in (0, 1]$ , and  $\int f d\mu_n \rightarrow \int f d\mu$  if and only if  $\int \varepsilon f d\mu_n \rightarrow \int \varepsilon f d\mu$ . Finally, the equivalence of (a) and (g) was discussed in Example IV.3.22 of Pollard (1984).  $\square$

The following Theorem A.4 is a special case of Theorem 15.12 in Aliprantis and Border (2006). Recall that a topological space is separable if it contains a countable dense subset; a subset is dense in a topological space if its closure coincides with the whole space.

**Theorem A.4** *The topological space  $(\mathcal{M}_1, \mathcal{O}_w)$  is metrizable and separable if  $(\mathbf{E}, d)$  is separable.*

The bounded Lipschitz distance  $d_{\text{BL}}$  provides a metric on  $\mathcal{M}_1$  when  $(\mathbf{E}, d)$  is separable; cf. Remark A.2. Also recall that the topology generated by a metric consists of all  $d$ -open subsets of the underlying space. As a consequence of Theorem A.4 and the Portmanteau theorem A.3 we obtain the following well known result.

**Corollary A.5** *If  $(\mathbf{E}, d)$  is separable, then the bounded Lipschitz distance  $d_{\text{BL}}$  generates the weak topology  $\mathcal{O}_w$  on  $\mathcal{M}_1$ .*

**Proof** First, two topologies  $\mathcal{O}$  and  $\mathcal{O}'$  on a nonempty set coincide if and only if the identity is a homeomorphism w.r.t.  $\mathcal{O}$  and  $\mathcal{O}'$ . Second, a topology is first countable if it is metrizable; cf. Aliprantis and Border (2006, p. 27). Thus it follows by the second part of Theorem 2.40 in Aliprantis and Border (2006) that two metrizable topologies coincide if and only if convergence of any sequence in  $\mathcal{O}$  implies convergence of the sequence in  $\mathcal{O}'$  and vice versa. By Theorem A.4 the topology  $\mathcal{O}_w$  is metrizable, and the topology  $\mathcal{O}(d_{\text{BL}})$  generated by the metric  $d_{\text{BL}}$  is metrizable anyway. Thus the equivalence of (a) and (g) in Theorem A.3 implies  $\mathcal{O}_w = \mathcal{O}(d_{\text{BL}})$ , i.e. the metric  $d_{\text{BL}}$  indeed generates the weak topology  $\mathcal{O}_w$ .  $\square$

## Appendix B: Convergence in distribution and convergence in probability for the open-ball $\sigma$ -algebra

Let  $(\mathbf{E}, d)$  be a metric space and  $\mathcal{B}^\circ$  the open-ball  $\sigma$ -algebra on  $\mathbf{E}$ . A sequence  $(X_n)$  of  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variables is said to *converge in distribution*<sup>o</sup> to an  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variable  $X_0$  if the sequence  $(\text{law}\{X_n\})$  weak<sup>o</sup>ly converges to  $\text{law}\{X_0\}$ . In this case, we write  $X_n \rightsquigarrow^\circ X_0$ . In the case where the random variables  $X_n$ ,  $n \in \mathbb{N}_0$ , are all defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  the sequence  $(X_n)$  is said to *converge in probability*<sup>o</sup> to  $X_0$  if the mappings  $\omega \mapsto d(X_n(\omega), X_0(\omega))$ ,  $n \in \mathbb{N}$ , are  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable and satisfy

$$\lim_{n \rightarrow \infty} \mathbb{P}[d(X_n, X_0) \geq \varepsilon] = 0 \quad \text{for all } \varepsilon > 0. \quad (46)$$

In this case, we write  $X_n \rightarrow^{\text{P}, \circ} X_0$ . As usual, by  $\mathbb{P}$ -almost sure convergence of the sequence  $(X_n)$  to  $X_0$ , abbreviated by  $X_n \rightarrow X_0$   $\mathbb{P}$ -a.s., we will mean that there exists a set  $N \in \mathcal{F}$  with that  $\mathbb{P}[N] = 0$  and  $d(X_n(\omega), X_0(\omega)) \rightarrow 0$  for all  $\omega \in \Omega \setminus N$ .

**Proposition B.1** *Let  $X_n$ ,  $n \in \mathbb{N}_0$ , be  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variables on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and assume that the mappings  $\omega \mapsto d(X_n(\omega), X_0(\omega))$ ,  $n \in \mathbb{N}$ , are  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable. Then  $X_n \rightarrow X_0$   $\mathbb{P}$ -a.s. implies  $X_n \rightarrow^{\text{P}, \circ} X_0$ .*

**Proof** By assumption the variables  $d(X_n, X_0)$ ,  $n \in \mathbb{N}$ , are  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable, and therefore the variable  $\limsup_{n \rightarrow \infty} d(X_n, X_0)$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}_+))$ -measurable. Since  $X_n \rightarrow X_0$   $\mathbb{P}$ -a.s., we obtain  $\mathbb{P}[\limsup_{n \rightarrow \infty} d(X_n, X_0) = 0] = 1$ . This implies

$$\mathbb{P}\left[\limsup_{n \rightarrow \infty} \{d(X_n, X_0) \geq \varepsilon\}\right] \leq \mathbb{P}\left[\limsup_{n \rightarrow \infty} d(X_n, X_0) \geq \varepsilon\right] = 0 \quad \text{for all } \varepsilon > 0$$

which together with the reverse of Fatou's lemma gives  $\limsup_{n \rightarrow \infty} \mathbb{P}[d(X_n, X_0) \geq \varepsilon] = 0$  for every  $\varepsilon > 0$ .  $\square$

When  $X_0$  takes almost surely values in a separable measurable set, then convergence in probability $^\circ$  implies convergence in distribution $^\circ$  of  $X_n$  to  $X_0$ :

**Proposition B.2** *Let  $X_n$ ,  $n \in \mathbb{N}_0$ , be  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variables on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and assume that  $\mathbb{P}[X_0 \in \mathbf{E}_0] = 1$  for some separable  $\mathbf{E}_0 \in \mathcal{B}^\circ$ . Then  $X_n \rightarrow^{\mathbb{P}, \circ} X_0$  implies  $X_n \rightsquigarrow^\circ X_0$ .*

**Proof** For any  $f \in \text{BL}_1^\circ$  we have  $|\int f d\mathbb{P}_{X_n} - \int f d\mathbb{P}_{X_0}| \leq 2\mathbb{P}[d(X_n, X_0) \geq \varepsilon/2] + \varepsilon/2$  for all  $\varepsilon > 0$ , i.e.  $\int f d\mathbb{P}_{X_n} \rightarrow \int f d\mathbb{P}_{X_0}$ . The claim then follows by the implication (f) $\Rightarrow$ (a) in the Portmanteau theorem A.3.  $\square$

The following lemma implies that the measurability condition in the definition of convergence in probability $^\circ$  is automatically satisfied when  $X_0$  is constant, i.e. when  $X_0(\cdot) = x$  for some  $x \in \mathbf{E}$ .

**Lemma B.3** *For every  $x \in \mathbf{E}$ , the mapping  $y \mapsto d(x, y)$  is continuous and  $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurable.*

**Proof** The continuity is obvious, and the  $(\mathcal{B}^\circ, \mathcal{B}(\mathbb{R}))$ -measurability follows by

$$\{d(x, \cdot) < a\} = \{y \in \mathbf{E} : d(x, y) < a\} = B_a(x) \in \mathcal{B}^\circ \quad \text{for every } a > 0$$

and  $\{d(x, \cdot) < a\} = \emptyset \in \mathcal{B}^\circ$  for every  $a \leq 0$ .  $\square$

For constant  $X_0$  we also have that convergence in probability $^\circ$  of  $X_n$  to  $X_0$  is equivalent to convergence in distribution $^\circ$  of  $X_n$  to  $X_0$ :

**Proposition B.4** *Let  $X_n$ ,  $n \in \mathbb{N}$ , be  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variables on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and  $x_0 \in \mathbf{E}$  be a constant. Then:*

- (i)  $X_n \rightarrow x_0$   $\mathbb{P}$ -a.s. implies  $X_n \rightarrow^{\mathbb{P}, \circ} x_0$ .
- (ii)  $X_n \rightarrow^{\mathbb{P}, \circ} x_0$  if and only if  $X_n \rightsquigarrow^\circ x_0$ .

**Proof** Part (i) follows from Proposition B.1 and Lemma B.3. To prove part (ii), first assume  $X_n \rightsquigarrow^\circ x_0$ . Set  $f(x) := \min\{d(x, x_0); 1\}$ ,  $x \in \mathbf{E}$ , and note that  $f \in C_b^\circ$ . By Markov's inequality and Lemma B.3 we obtain

$$\mathbb{P}[d(X_n, x_0) \geq \varepsilon] \leq \frac{1}{\varepsilon} \int f(X_n(\omega)) \mathbb{P}[d\omega] \longrightarrow \frac{1}{\varepsilon} \int f(x_0) \mathbb{P}[d\omega] = 0, \quad n \rightarrow \infty$$

for every  $\varepsilon > 0$ . That is,  $X_n \rightarrow^\circ x_0$ . The other direction in part (ii) follows from Proposition B.2, because the set  $\{x_0\} = \bigcap_{n \in \mathbb{N}} B_{1/n}(x_0)$  is separable and lies in  $\mathcal{B}^\circ$ .  $\square$

Recall that  $\mathcal{B}^\circ = \mathcal{B}$  when  $(\mathbf{E}, d)$  is separable. In this case we suppress the superscript  $^\circ$  and write simply  $\rightsquigarrow$ ,  $\rightarrow^{\mathbb{P}}$ , convergence in distribution, and convergence in probability instead of  $\rightsquigarrow^\circ$ ,  $\rightarrow^{\mathbb{P}, \circ}$ , convergence in distribution $^\circ$ , and convergence in probability $^\circ$ , respectively.

### Appendix C: An extended Continuous Mapping theorem and a delta-method for the open-ball $\sigma$ -algebra

As mentioned in the introduction, Theorem 3.1 is based on a generalization of Theorem 4.1 in Beutner and Zähle (2010), which in turn is a generalization of the classical functional delta-method in the form of Theorem 3 of Gill (1989). The proof of the generalization of Theorem 4.1 in Beutner and Zähle (2010) is based on the extended Continuous Mapping theorem C.1 below. An extended Continuous Mapping theorem for convergence in distribution for the Borel  $\sigma$ -algebra can be found in Kallenberg (2002, Theorem 4.27). A corresponding result for convergence in distribution in the Hoffmann-Jørgensen is given, for example, in van der Vaart and Wellner (1996, Theorem 1.11.1). However, we could not find a version of this result for convergence in distribution $^\circ$  for the open-ball  $\sigma$ -algebra. So we include a proof for Theorem C.1. Note that Theorem C.1 is a generalization of the “ordinary” Continuous Mapping theorem for convergence in distribution $^\circ$  for the open-ball  $\sigma$ -algebra as given by Billingsley (1999, Theorem 6.4). Let  $(\mathbf{E}, d)$  and  $(\tilde{\mathbf{E}}, d_{\tilde{\mathbf{E}}})$  be metric spaces and  $\mathcal{B}^\circ$  and  $\tilde{\mathcal{B}}^\circ$  be the open-ball  $\sigma$ -algebras on  $\mathbf{E}$  and  $\tilde{\mathbf{E}}$ , respectively.

**Theorem C.1 (Extended CMT for random variables)** *Let  $\mathbf{E}_n \subseteq \mathbf{E}$  and  $\xi_n$  be an  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variable on some probability space  $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$  such that  $\xi_n(\Omega_n) \subseteq \mathbf{E}_n$ ,  $n \in \mathbb{N}$ . Let  $\xi_0$  be an  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variable on some probability space  $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$  such that  $\xi_0(\Omega_0) \subseteq \mathbf{E}_0$  for some separable  $\mathbf{E}_0 \in \mathcal{B}^\circ$ . Let  $h_n : \mathbf{E}_n \rightarrow \tilde{\mathbf{E}}$  be a map such that the map  $h_n(\xi_n) : \Omega_n \rightarrow \tilde{\mathbf{E}}$  is  $(\mathcal{F}_n, \tilde{\mathcal{B}}^\circ)$ -measurable,  $n \in \mathbb{N}$ . Let  $h_0 : \mathbf{E}_0 \rightarrow \tilde{\mathbf{E}}$  be a  $(\mathcal{B}_0^\circ, \tilde{\mathcal{B}}^\circ)$ -measurable map, where  $\mathcal{B}_0^\circ := \mathcal{B}^\circ \cap \mathbf{E}_0 (\subseteq \mathcal{B}^\circ)$ . Moreover, assume that the following two assertions hold:*

- (a)  $\xi_n \rightsquigarrow^\circ \xi_0$ .
- (b) For every  $x_n \in \mathbf{E}_n$ ,  $n \in \mathbb{N}_0$ , we have  $\tilde{d}(h_n(x_n), h_0(x_0)) \rightarrow 0$  when  $d(x_n, x_0) \rightarrow 0$ .

Then  $h_n(\xi_n) \rightsquigarrow^\circ h_0(\xi_0)$ .

**Remark C.2** Note that we do *not* assume in Theorem C.1 that the maps  $h_n$ ,  $n \in \mathbb{N}$ , are  $(\mathcal{B}^\circ, \tilde{\mathcal{B}}^\circ)$ -measurable. This implies that for  $n \in \mathbb{N}$  the law  $\mathbb{P}_n \circ (h_n(\xi_n))^{-1}$  of  $h_n(\xi_n)$  can not necessarily be represented as the image law of  $\xi_n$ 's law  $\mathbb{P}_n \circ \xi_n^{-1}$  w.r.t.  $h_n$ .  $\diamond$

**Proof of Theorem C.1** According to the implication (d) $\Rightarrow$ (a) in the Portmanteau theorem A.3, it suffices to show that  $\liminf_{n \rightarrow \infty} \mathbb{P}_n \circ h_n(\xi_n)^{-1}[\tilde{G}] \geq$



$\mathbb{P}_0 \circ h_0(\xi_0)^{-1}[\tilde{G}]$  for every open set  $\tilde{G} \in \tilde{\mathcal{B}}^\circ$ . So, let  $\tilde{G} \in \tilde{\mathcal{B}}^\circ$  be open. First we note that

$$h_0^{-1}(\tilde{G}) \cap \mathbf{E}_0 \subseteq \bigcup_{m=1}^{\infty} \left( \left\{ \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}} \cap \mathbf{E}_0 \right), \quad (47)$$

where the superscript  $\text{int}$  refers to the interior of a set. Indeed: For every  $x_0 \in h_0^{-1}(\tilde{G}) \cap \mathbf{E}_0$  there exists an  $m \in \mathbb{N}$  and a neighborhood  $U$  of  $x_0$  such that  $h_k(x) \in \tilde{G}$  for all  $k \geq m$  and  $x \in U$ . Otherwise we could find for every  $m \in \mathbb{N}$  some  $k_m \geq m$  and  $x_m \in B_{1/m}(x_0)$  such that  $h_{k_m}(x_m) \notin \tilde{G}$ . But then we had  $d(x_m, x_0) \rightarrow 0$  and  $\tilde{d}(h_{k_m}(x_m), h_0(x_0)) \not\rightarrow 0$  (take into account that  $h_0(x_0) \in \tilde{G}$  and  $\tilde{G}$  is open), which contradicts assumption (b). Hence  $U \subseteq \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G})$  and thus  $x_0 \in \left\{ \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$ . In particular,  $h_0^{-1}(\tilde{G}) \cap \mathbf{E}_0 \subseteq \bigcup_{m=1}^{\infty} \left\{ \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$ . Now (47) is obvious.

Further, for every  $m \in \mathbb{N}$  we can find a union  $G_m$  of countably many open balls such that

$$\left\{ \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}} \cap \mathbf{E}_0 \subseteq G_m \subseteq \left\{ \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}, \quad (48)$$

and we may and do assume  $G_1 \subseteq G_2 \subseteq \dots$ . To prove this one can proceed by an induction on  $m$ . First let  $m = 1$ . For every  $x \in \left\{ \bigcap_{k=1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$  we can find an open ball  $B_{r_x}(x)$  around  $x$  which is contained in  $\left\{ \bigcap_{k=1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$ , because the latter set is open. The system which consists of the open balls  $B_{r_x}(x)$ ,  $x \in \left\{ \bigcap_{k=1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$ , provides an open cover of  $\left\{ \bigcap_{k=1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}} \cap \mathbf{E}_0$ . Since the latter set is separable (recall that  $\mathbf{E}_0$  was assumed to be separable), Lindelöf's theorem ensures that there is a countable subcover. The set  $G_1$  can now be defined as the union of the elements of this subcover. Next assume that  $G_1, \dots, G_M$  are unions of countably many open balls such that  $G_1 \subseteq \dots \subseteq G_M$  and (48) holds for  $m = 1, \dots, M$ . For every  $x \in \left\{ \bigcap_{k=M+1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$  we can find an open ball  $B_{r_x}(x)$  around  $x$  which is contained in  $\left\{ \bigcap_{k=M+1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$ , because the latter set is open. The system which consists of the open balls  $B_{r_x}(x)$ ,  $x \in \left\{ \bigcap_{k=M+1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}} \setminus \left\{ \bigcap_{k=M}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}}$  and of the countably many open balls which unify to  $G_M$  provides an open cover of  $\left\{ \bigcap_{k=M+1}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}} \cap \mathbf{E}_0$ . Since the latter set is separable, Lindelöf's theorem ensures that there is a countable subcover. Without loss of generality we may and do assume that the countably many open balls which unify to  $G_M$  belong to this countable subcover. Defining  $G_{M+1}$  as the union of the elements of this subcover we obtain  $G_M \subseteq G_{M+1}$  and (48) for  $m = M + 1$ .

As countable unions of open balls the sets  $G_m$ ,  $m \in \mathbb{N}$ , are open and lie in  $\mathcal{B}^\circ$ . Then, using (47), the first " $\subseteq$ " in (48), and the inclusions  $G_1 \subseteq G_2 \subseteq \dots$  (along with the continuity from below of  $\mathbb{P}_0 \circ \xi_0^{-1}$ ),

$$\begin{aligned} \mathbb{P}_0 \circ h_0(\xi_0)^{-1}[\tilde{G}] &= \mathbb{P}_0 \circ \xi_0^{-1}[h_0^{-1}(\tilde{G})] \\ &= \mathbb{P}_0 \circ \xi_0^{-1}[h_0^{-1}(\tilde{G}) \cap \mathbf{E}_0] \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{P}_0^{\text{out}} \left[ \xi_0 \in \bigcup_{m=1}^{\infty} \left( \left\{ \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right\}^{\text{int}} \cap \mathbf{E}_0 \right) \right] \\
&\leq \mathbb{P}_0 \left[ \xi_0 \in \bigcup_{m=1}^{\infty} G_m \right] \\
&\leq \mathbb{P}_0 \circ \xi_0^{-1} \left[ \bigcup_{m=1}^{\infty} G_m \right] \\
&= \sup_{m \in \mathbb{N}} \mathbb{P}_0 \circ \xi_0^{-1} [G_m] \\
&\leq \sup_{m \in \mathbb{N}} \liminf_{n \rightarrow \infty} \mathbb{P}_n \circ \xi_n^{-1} [G_m], \tag{49}
\end{aligned}$$

where the last step follows from assumption (a) and the implication (a) $\Rightarrow$ (d) in the Portmanteau theorem A.3. Now, (49) and the second “ $\subseteq$ ” in (48) yield

$$\begin{aligned}
\mathbb{P}_0 \circ h_0(\xi_0)^{-1} [\tilde{G}] &\leq \sup_{m \in \mathbb{N}} \liminf_{n \rightarrow \infty} \mathbb{P}_n^{\text{out}} \left[ \xi_n \in \bigcap_{k=m}^{\infty} h_k^{-1}(\tilde{G}) \right] \\
&\leq \liminf_{n \rightarrow \infty} \mathbb{P}_n [\xi_n \in h_n^{-1}(\tilde{G})] \\
&= \liminf_{n \rightarrow \infty} \mathbb{P}_n \circ h_n(\xi_n)^{-1} [\tilde{G}].
\end{aligned}$$

This completes the proof.  $\square$

Before giving the generalization of Theorem 4.1 in Beutner and Zähle (2010) we recall the definition of quasi-Hadamard differentiability. For this let  $\mathbf{V}$  and  $\tilde{\mathbf{E}}$  be vector spaces, and  $\mathbf{E} \subseteq \mathbf{V}$  be a subspace of  $\mathbf{V}$ . Let  $\|\cdot\|_{\mathbf{E}}$  and  $\|\cdot\|_{\tilde{\mathbf{E}}}$  be norms on  $\mathbf{E}$  and  $\tilde{\mathbf{E}}$ , respectively.

**Definition C.3 (Quasi-Hadamard differentiability)** *Let  $H : \mathbf{V}_H \rightarrow \tilde{\mathbf{E}}$  be a map defined on some  $\mathbf{V}_H \subseteq \mathbf{V}$ , and  $\mathbf{E}_0$  be a subset of  $\mathbf{E}$ . Then  $H$  is said to be quasi-Hadamard differentiable at  $x \in \mathbf{V}_H$  tangentially to  $\mathbf{E}_0 \langle \mathbf{E} \rangle$  if there is some continuous map  $\dot{H}_x : \mathbf{E}_0 \rightarrow \tilde{\mathbf{E}}$  such that*

$$\lim_{n \rightarrow \infty} \left\| \dot{H}_x(x_0) - \frac{H(x + \varepsilon_n x_n) - H(x)}{\varepsilon_n} \right\|_{\tilde{\mathbf{E}}} = 0 \tag{50}$$

holds for each triplet  $(x_0, (x_n), (\varepsilon_n))$ , with  $x_0 \in \mathbf{E}_0$ ,  $(x_n) \subseteq \mathbf{E}$  satisfying  $\|x_n - x_0\|_{\mathbf{E}} \rightarrow 0$  as well as  $(x + \varepsilon_n x_n) \subseteq \mathbf{V}_H$ , and  $(\varepsilon_n) \subset (0, \infty)$  satisfying  $\varepsilon_n \rightarrow 0$ . In this case the map  $\dot{H}_x$  is called quasi-Hadamard derivative of  $H$  at  $x$  tangentially to  $\mathbf{E}_0 \langle \mathbf{E} \rangle$ .

Recall that  $\tilde{\mathbf{E}}$  is a vector space equipped with a norm  $\|\cdot\|_{\tilde{\mathbf{E}}}$ , and let  $0_{\tilde{\mathbf{E}}}$  denote the null in  $\tilde{\mathbf{E}}$ . Set  $\tilde{\tilde{\mathbf{E}}} := \tilde{\mathbf{E}} \times \tilde{\mathbf{E}}$  and let  $\tilde{\tilde{\mathcal{B}}}^\circ$  be the  $\sigma$ -algebra on  $\tilde{\tilde{\mathbf{E}}}$  generated by the open balls w.r.t. the metric  $\tilde{d}((\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)) := \max\{\|\tilde{x}_1 - \tilde{y}_1\|_{\tilde{\mathbf{E}}}; \|\tilde{x}_2 - \tilde{y}_2\|_{\tilde{\mathbf{E}}}\}$ . Recall that  $\tilde{\tilde{\mathcal{B}}}^\circ \subseteq \tilde{\mathcal{B}}^\circ \otimes \tilde{\mathcal{B}}^\circ$ , because any  $\tilde{d}$ -open ball in  $\tilde{\tilde{\mathbf{E}}}$  is the product of two

$\|\cdot\|_{\tilde{\mathbf{E}}}$ -open balls in  $\tilde{\mathbf{E}}$ . Let  $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$  be a probability space and  $X_n : \Omega_n \rightarrow \mathbf{E}$  be any map,  $n \in \mathbb{N}$ . Recall that  $\rightsquigarrow^\circ$  and  $\rightarrow^\circ$  refer to convergence in distribution<sup>o</sup> and convergence in probability<sup>p,o</sup>, respectively.

**Theorem C.4 (Delta-method)** *Let  $H : \mathbf{V}_H \rightarrow \tilde{\mathbf{E}}$  be a map defined on some  $\mathbf{V}_H \subseteq \mathbf{E}$ , and  $x \in \mathbf{V}_H$ . Let  $\mathbf{E}_0 \in \mathcal{B}^\circ$  be some  $\|\cdot\|_{\mathbf{E}}$ -separable subset of  $\mathbf{E}$ . Let  $(a_n)$  be a sequence of positive real numbers tending to  $\infty$ , and consider the following conditions:*

- (a)  $X_n$  takes values only in  $\mathbf{V}_H$ .
- (b)  $a_n(X_n - x)$  takes values only in  $\mathbf{E}$ , is  $(\mathcal{F}_n, \mathcal{B}^\circ)$ -measurable and satisfies

$$a_n(X_n - x) \rightsquigarrow^\circ X_0 \quad \text{in } (\mathbf{E}, \mathcal{B}^\circ, \|\cdot\|_{\mathbf{E}}) \quad (51)$$

for some  $(\mathbf{E}, \mathcal{B}^\circ)$ -valued random variable  $X_0$  on some probability space  $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$  with  $X_0(\Omega_0) \subseteq \mathbf{E}_0$ .

- (c)  $a_n(H(X_n) - H(x))$  is  $(\mathcal{F}_n, \tilde{\mathcal{B}}^\circ)$ -measurable.
- (d) The map  $H$  is quasi-Hadamard differentiable at  $x$  tangentially to  $\mathbf{E}_0(\mathbf{E})$  with quasi-Hadamard derivative  $\dot{H}_x : \mathbf{E}_0 \rightarrow \tilde{\mathbf{E}}$ .
- (e)  $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n) = (\Omega, \mathcal{F}, \mathbb{P})$  for all  $n \in \mathbb{N}$ .
- (f) The quasi-Hadamard derivative  $\dot{H}_x$  can be extended to  $\mathbf{E}$  such that the extension  $\dot{H}_x : \mathbf{E} \rightarrow \tilde{\mathbf{E}}$  is continuous at every point of  $\mathbf{E}_0$  and  $(\mathcal{B}^\circ, \tilde{\mathcal{B}}^\circ)$ -measurable.
- (g) The map  $h : \tilde{\mathbf{E}} \rightarrow \tilde{\mathbf{E}}$  defined by  $h(\tilde{x}_1, \tilde{x}_2) := \tilde{x}_1 - \tilde{x}_2$  is  $(\tilde{\mathcal{B}}^\circ, \tilde{\mathcal{B}}^\circ)$ -measurable.

Then the following two assertions hold:

- (i) If conditions (a)–(d) hold true, then  $\dot{H}_x(X_0)$  is  $(\mathcal{F}_0, \tilde{\mathcal{B}}^\circ)$ -measurable and

$$a_n(H(X_n) - H(x)) \rightsquigarrow^\circ \dot{H}_x(X_0) \quad \text{in } (\tilde{\mathbf{E}}, \tilde{\mathcal{B}}^\circ, \|\cdot\|_{\tilde{\mathbf{E}}}).$$

- (ii) If conditions (a)–(g) hold true, then

$$a_n(H(X_n) - H(x)) - \dot{H}_x(a_n(X_n - x)) \xrightarrow{\text{p},\circ} 0_{\tilde{\mathbf{E}}} \quad \text{in } (\tilde{\mathbf{E}}, \|\cdot\|_{\tilde{\mathbf{E}}}). \quad (52)$$

**Remark C.5** It is apparent from the following proof that for part (i) of Theorem C.4 it is not necessary to assume (as in Definition C.3) that the quasi-Hadamard derivative  $\dot{H}_x$  is continuous. It would suffice to require in Definition C.3 that the map  $\dot{H}_x$  is  $(\mathcal{B}_0^\circ, \tilde{\mathcal{B}}^\circ)$ -measurable for the trace  $\sigma$ -algebra  $\mathcal{B}_0^\circ := \mathcal{B}^\circ \cap \mathbf{E}_0$  ( $\subseteq \mathcal{B}^\circ$ ).  $\diamond$

**Proof of Theorem C.4** For the proof of part (i) we adapt the arguments in the proof of Theorem 3.9.4 in van der Vaart and Wellner (1996), which then allow for an easy proof of part (ii).

(i): For every  $n \in \mathbb{N}$ , let  $\mathbf{E}_n := \{x_n \in \mathbf{E} : a_n^{-1}x_n + x \in \mathbf{V}_H\}$  and define the map  $h_n : \mathbf{E}_n \rightarrow \tilde{\mathbf{E}}$  by

$$h_n(x_n) := \frac{H(x + a_n^{-1}x_n) - H(x)}{a_n^{-1}}.$$

Moreover, define the map  $h_0 : \mathbf{E}_0 \rightarrow \widetilde{\mathbf{E}}$  by

$$h_0(x_0) := \dot{H}_x(x_0).$$

Now, the claim would follow by the extended Continuous Mapping theorem C.1 applied to the functions  $h_n$ ,  $n \in \mathbb{N}_0$ , and the random variables  $\xi_n := a_n(X_n - x)$ ,  $n \in \mathbb{N}$ , and  $\xi_0 := X_0$  if we can show that the assumptions of Theorem C.1 are satisfied. First,  $\xi_n(\Omega_n) \subseteq \mathbf{E}_n$  and  $\xi_0(\Omega_0) \subseteq \mathbf{E}_0$  clearly hold. Second, by assumption (c) we have that  $h_n(\xi_n) = a_n(H(X_n) - H(x))$  is  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ)$ -measurable. Third, the map  $h_0$  is continuous by assumption (on the quasi-Hadamard derivative). Thus  $h_0$  is  $(\mathcal{B}_0^\circ, \widetilde{\mathcal{B}}^\circ)$ -measurable, because the trace  $\sigma$ -algebra  $\mathcal{B}_0^\circ := \mathcal{B}^\circ \cap \mathbf{E}_0$  coincides with the Borel  $\sigma$ -algebra on  $\mathbf{E}_0$  (recall that  $\mathbf{E}_0$  is separable). In particular,  $\dot{H}_x(X_0)$  is  $(\mathcal{F}_0, \widetilde{\mathcal{B}}^\circ)$ -measurable. Fourth, condition (a) of Theorem C.1 holds by assumption (b). Fifth, condition (b) of Theorem C.1 is ensured by assumption (d) (note that (d) implies (50)).

(ii): For every  $n \in \mathbb{N}$ , let  $\mathbf{E}_n$  and  $h_n$  be as above and define the map  $\bar{h}_n : \mathbf{E}_n \rightarrow \widetilde{\mathbf{E}}$  by

$$\bar{h}_n(x_n) := (h_n(x_n), \dot{H}_x(x_n)).$$

Moreover, define the map  $\bar{h}_0 : \mathbf{E}_0 \rightarrow \widetilde{\mathbf{E}}$  by

$$\bar{h}_0(x_0) := (h_0(x_0), \dot{H}_x(x_0)) = (\dot{H}_x(x_0), \dot{H}_x(x_0)).$$

We will first show that

$$\bar{h}_n(a_n(X_n - x)) \rightsquigarrow^\circ \bar{h}_0(X_0) \quad \text{in } (\widetilde{\mathbf{E}}, \widetilde{\mathcal{B}}^\circ, \widetilde{d}). \quad (53)$$

For (53) it suffices to show that the assumption of the extended Continuous Mapping theorem C.1 applied to the functions  $\bar{h}_n$  and  $\xi_n$  (as defined above) are satisfied. The claim then follows by Theorem C.1. First, we have already observed that  $\xi_n(\Omega_n) \subseteq \mathbf{E}_n$  and  $\xi_0(\Omega_0) \subseteq \mathbf{E}_0$ . Second, we have seen in the proof of part (i) that  $h_n(\xi_n)$  is  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ)$ -measurable,  $n \in \mathbb{N}$ . By assumption (f) the extended map  $\dot{H}_x : \mathbf{E} \rightarrow \widetilde{\mathbf{E}}$  is  $(\mathcal{B}^\circ, \widetilde{\mathcal{B}}^\circ)$ -measurable, which implies that  $\dot{H}_x(\xi_n)$  is  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ)$ -measurable. Thus,  $\bar{h}_n(\xi_n) = (h_n(\xi_n), \dot{H}_x(\xi_n))$  is  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ \otimes \widetilde{\mathcal{B}}^\circ)$ -measurable (to see this note that, in view of  $\widetilde{\mathcal{B}}^\circ \otimes \widetilde{\mathcal{B}}^\circ = \sigma(\pi_1, \pi_2)$  for the coordinate projections  $\pi_1, \pi_2$  on  $\widetilde{\mathbf{E}} = \widetilde{E} \times \widetilde{E}$ , Theorem 7.4 of Bauer (2001) shows that the map  $(h_n(\xi_n), \dot{H}_x(\xi_n))$  is  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ \otimes \widetilde{\mathcal{B}}^\circ)$ -measurable if and only if the maps  $h_n(\xi_n) = \pi_1 \circ (h_n(\xi_n), \dot{H}_x(\xi_n))$  and  $\dot{H}_x(\xi_n) = \pi_2 \circ (h_n(\xi_n), \dot{H}_x(\xi_n))$  are  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ)$ -measurable). In particular, the map  $\bar{h}_n(\xi_n) = (h_n(\xi_n), \dot{H}_x(\xi_n))$  is  $(\mathcal{F}_n, \widetilde{\mathcal{B}}^\circ)$ -measurable,  $n \in \mathbb{N}$ . Third, we have seen in the proof of part (i) that the map  $h_0 = \dot{H}_x$  is  $(\mathcal{B}_0^\circ, \widetilde{\mathcal{B}}^\circ)$ -measurable. Thus the map  $\bar{h}_0$  is  $(\mathcal{B}_0^\circ, \widetilde{\mathcal{B}}^\circ \otimes \widetilde{\mathcal{B}}^\circ)$ -measurable (one can argue as above) and in particular  $(\mathcal{B}_0^\circ, \widetilde{\mathcal{B}}^\circ)$ -measurable. Fourth, condition (a) of Theorem C.1 holds by assumption (b). Fifth, condition (b) of Theorem C.1 is ensured by assumption (d) and the continuity of the extended map  $\dot{H}_x$  at every point of  $\mathbf{E}_0$  (recall assumption (f)). Hence, (53) holds.

By assumption (g) and the ordinary Continuous Mapping theorem (cf. Billingsley (1999, Theorem 6.4)) applied to (53) and the map  $h : \widetilde{\mathbf{E}} \rightarrow \widetilde{\mathbf{E}}$ ,  $(\widetilde{x}_1, \widetilde{x}_2) \mapsto \widetilde{x}_1 - \widetilde{x}_2$ , we now have

$$h_n(a_n(X_n - x)) - \dot{H}_x(a_n(X_n - x)) \rightsquigarrow^\circ \dot{H}_x(X_0) - \dot{H}_x(X_0),$$

i.e.

$$a_n(H(X_n) - H(x)) - \dot{H}_x(a_n(X_n - x)) \rightsquigarrow^\circ 0_{\widetilde{\mathbf{E}}}.$$

By Proposition B.4 we can conclude (52).  $\square$

### Appendix D: Probability kernels and conditional distributions

Let  $(\Omega, \mathcal{F})$  be a measurable space. Let  $(\mathbf{E}, d)$  be a metric space and  $\mathcal{B}^\circ$  be the open-ball  $\sigma$ -algebra on  $\mathbf{E}$ . A map  $P : \Omega \times \mathcal{B}^\circ \rightarrow [0, 1]$  is said to be a *probability kernel* from  $(\Omega, \mathcal{F})$  to  $(\mathbf{E}, \mathcal{B}^\circ)$  if  $P(\cdot, A)$  is  $(\mathcal{F}, \mathcal{B}([0, 1]))$ -measurable for every  $A \in \mathcal{B}^\circ$ , and  $P(\omega, \cdot)$  is a probability measure on  $(\mathbf{E}, \mathcal{B}^\circ)$  for every  $\omega \in \Omega$ . Of course, we may regard  $P$  as a map from  $\Omega$  to  $\mathcal{M}_1^\circ$ . Recall that  $\mathcal{M}_1^\circ = \mathcal{M}_1$  when  $(\mathbf{E}, d)$  is separable. If in this case the set  $\mathcal{M}_1$  is equipped with the weak topology  $\mathcal{O}_w$ , then a probability kernel can be regarded as an  $\mathcal{M}_1$ -valued random variable (w.r.t. any probability measure on  $(\Omega, \mathcal{F})$ ):

**Lemma D.1** *Let  $(\mathbf{E}, d)$  be separable and  $P$  be a probability kernel from  $(\Omega, \mathcal{F})$  to  $(\mathbf{E}, \mathcal{B})$ . Then the mapping  $\omega \mapsto P(\omega, \bullet)$  is  $(\mathcal{F}, \sigma(\mathcal{O}_w))$ -measurable.*

**Proof** Since  $(\mathbf{E}, d)$  was assumed to be separable, the proof of the implication (4) $\Rightarrow$ (1) in Theorem 19.7 in Aliprantis and Border (2006) shows that  $\sigma(\mathcal{O}_w)$  equals the  $\sigma$ -algebra generated by the system  $\{\pi_f^{-1}(A) : f \in C_b, A \subseteq \mathbb{R} \text{ open}\}$ . So it suffices to show that the set

$$P(\cdot, \bullet)^{-1}(\pi_f^{-1}(A)) = \pi_f(P(\cdot, \bullet))^{-1}(A) = \left( \int f(x) P(\cdot, dx) \right)^{-1}(A)$$

is contained in  $\mathcal{F}$  for every open  $A \subseteq \mathbb{R}$  and  $f \in C_b$ . But this follows from the well known fact (see e.g. Lemma 1.41 in Kallenberg (2002)) that the mapping  $\omega \mapsto \int f(x) P(\omega, dx)$  is  $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable for every  $f \in C_b$ . This finishes the proof.  $\square$

Now, let  $(\Omega', \mathcal{F}')$  and  $(\mathbf{D}, \mathcal{D})$  be further measurable spaces. Let  $\mathbb{P}$  and  $\mathbb{P}'$  be probability measures on respectively  $\Omega$  and  $\Omega'$ , and set  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$ . Let  $Y : \Omega \rightarrow \mathbf{D}$  be an  $(\mathcal{F}, \mathcal{D})$ -measurable map and  $X : \overline{\Omega} \rightarrow \mathbf{E}$  be an  $(\overline{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable map. Note that  $Y$  can also be regarded as a  $(\mathbf{D}, \mathcal{D})$ -valued random variable on  $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ , and we are doing that in Lemma D.2. The following lemma shows that under an additional assumption, the conditional distribution of  $X$  given  $Y$  can be specified explicitly.

**Lemma D.2** Assume that  $X(\omega, \omega') = g(Y(\omega), \omega')$  holds for all  $(\omega, \omega') \in \overline{\Omega}$  and some  $(\mathcal{D} \otimes \mathcal{F}', \mathcal{B}^\circ)$ -measurable map  $g : \mathbf{D} \times \Omega' \rightarrow \mathbf{E}$ . Then the map  $P : \overline{\Omega} \times \mathcal{B}^\circ \rightarrow [0, 1]$  defined by

$$P((\omega, \omega'), A) := P(\omega, A) := \mathbb{P}' \circ X(\omega, \cdot)^{-1}[A], \quad (\omega, \omega') \in \overline{\Omega}, A \in \mathcal{B}^\circ$$

provides a conditional distribution of  $X$  given  $Y$ .

**Proof** First,  $P$  provides a probability kernel from  $(\overline{\Omega}, \overline{\sigma}(Y))$  to  $(\mathbf{E}, \mathcal{B}^\circ)$ . Indeed: The mapping  $\tilde{\omega}' \mapsto X(\omega, \tilde{\omega}')$  is  $(\mathcal{F}', \mathcal{B}^\circ)$ -measurable for every fixed  $\omega \in \Omega$ , because  $X$  is  $(\overline{\mathcal{F}}, \mathcal{B}^\circ)$ -measurable. So it immediately follows that the mapping  $A' \mapsto P(\omega, A')$  is a probability measure on  $(\mathbf{E}, \mathcal{B}^\circ)$  for every  $\omega \in \Omega$ . Further, the mapping  $(\omega, \tilde{\omega}') \mapsto (Y(\omega), \tilde{\omega}')$  is clearly  $(\sigma(Y) \otimes \mathcal{F}', \mathcal{D} \otimes \mathcal{F}')$ -measurable, which implies that the mapping  $(\omega, \tilde{\omega}') \mapsto X(\omega, \tilde{\omega}') = g(Y(\omega), \tilde{\omega}')$  is  $(\sigma(Y) \otimes \mathcal{F}', \mathcal{B}^\circ)$ -measurable. By Tonelli's part of Fubini's theorem it follows that the mapping  $\omega \mapsto \int \mathbb{1}_A(X(\omega, \tilde{\omega}')) \mathbb{P}'[d\tilde{\omega}'] = P(\omega, A)$  is  $(\sigma(Y), \mathcal{B}([0, 1]))$ -measurable for every  $A \in \mathcal{B}^\circ$ . In particular, the mapping  $(\omega, \omega') \mapsto P((\omega, \omega'), A) = P(\omega, A)$  is  $(\overline{\sigma}(Y), \mathcal{B}([0, 1]))$ -measurable for every  $A \in \mathcal{B}^\circ$ .

Second, by Fubini's theorem we obtain for every  $B \in \mathcal{D}$  and  $A \in \mathcal{B}^\circ$ ,

$$\begin{aligned} & \int_{\{Y \in B\}} P((\omega, \omega'), A) \overline{\mathbb{P}}[d(\omega, \omega')] \\ &= \int_{\{Y \in B\}} \mathbb{P}' \circ X(\omega, \cdot)^{-1}[A] \overline{\mathbb{P}}[d(\omega, \omega')] \\ &= \int_{\{Y \in B\}} \mathbb{P}' \circ X(\omega, \cdot)^{-1}[A] \mathbb{P}[d\omega] \\ &= \iint \mathbb{1}_{\{Y \in B\}}(\omega) \mathbb{1}_{\{X(\omega, \cdot) \in A\}}(\omega') \mathbb{P}'[d\omega'] \mathbb{P}[d\omega] \\ &= \int \mathbb{1}_{\{Y \in B\}}(\omega) \mathbb{1}_{\{X(\omega, \cdot) \in A\}}(\omega') \overline{\mathbb{P}}[d(\omega, \omega')] \\ &= \overline{\mathbb{P}}[\{Y \in B\} \cap \{X \in A\}]. \end{aligned}$$

This completes the proof. □

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