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Random variate generation for Laguerre-type exponentially tilted α-stable distributions

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Abstract: Exact sampling methods have been recently developed for generating random variates for exponentially tilted α-stable distributions. In this paper we show how to generate, exactly, random variates for a more general class of tilted α-stable distributions, which is referred to as the class of Laguerre-type exponentially tilted α-stable distributions. Beside the exponentially tilted α-stable distribution, such a class includes also the Erlang tilted α-stable distribution. This is a special case of the so-called gamma tilted α-stable distribution, for which an efficient exact random variate generator is currently not available in the literature. Our result fills this gap.

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1. Introduction

A distribution F is stable if, for any two independent random variables X_1 and X_2, with common distribution F, and for any constants a_1, a_2, there exist $A > 0$ and $B(a_1, a_2)$ such that the random variable $X_3 = A(a_1X_1+a_2X_2+B)$ has again distribution F. The class of stable distributions includes several distributions and, among these, in this paper we focus on the unilateral stable distribution with parameter $\alpha \in (0, 1)$. This is referred to as the α-stable distribution and it is characterized by a Laplace-Stieltjes transform of the form $\exp\{-\lambda^\alpha\}$, for any $\lambda \geq 0$. We refer to Zolotarev [19] for a comprehensive account on the α-stable distribution.

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A method for generating random variates for α-stable distributions was proposed by Kanter [16]. Specifically, let S_α be a random variable distributed according to the α-stable distribution, and let U and E be two independent random variables such that U is distributed as a uniform distribution on $(0, \pi)$ and E is distributed as an exponential distribution with parameter 1. Kanter [16] showed that

$$S_\alpha \stackrel{\mathrm{d}}{=} \left(\frac{1}{E} \left(\frac{(\sin(\alpha U))^\alpha (\sin(1-\alpha)U)^{1-\alpha}}{\sin(U)} \right)^{\frac{1}{1-\alpha}} \right)^{\frac{1-\alpha}{\alpha}}, \tag{1}$$

where $\stackrel{\mathrm{d}}{=}$ denotes the equality in distribution. The distributional identity (1) is known as the Kanter's method for sampling from S_α. A similar random variate generator was proposed by Chambers et al. [5] for the entire class of stable random variables. See also Zolotarev [19] for details. A different sampling method based on the series expansion of the stable density function was developed by Devroye [8].

Let f_α be the density function of the α-stable random variable S_α, which, as we know, is concentrated on the positive real line. A noteworthy generalization of f_α is the exponentially tilted, or Esscher-transformed, α-stable density function. See, e.g., Sato [17] and references therein for details. Specifically, for any positive real number β we say that the density function $f_{\alpha,\beta}$ is the exponential tilting of f_α if

$$f_{\alpha,\beta}(x) = \frac{\exp\{-\beta x\} f_\alpha(x)}{\int_0^{+\infty} \exp\{-\beta x\} f_\alpha(x) dx} = \exp\{\beta^\alpha - \beta x\} f_\alpha(x). \tag{2}$$

We denote by $S_{\alpha,\beta}$ a random variable with density function of the form (2). Exponentially tilted α-stable density functions first appeared in Tweedie [18] and Hougaard [14] and, since then, they have attracted a great interest due to their important role in the theory of importance sampling and in the study and simulation of rare events. See, e.g, the monographs by Bucklew [2] and Bucklew [3] for details.

Random variate generation for exponentially tilted α-stable distributions could be done by a trivial rejection sampling with envelope distribution f_α. However, this approach becomes unacceptable if the tilting parameter β is large. Indeed, the expected number of iterations before halting the rejection sampling is $\exp\{\beta^\alpha\}$. In order to overcome this drawback, Devroye [9] developed an exact sampling method for $S_{\alpha,\beta}$ which is uniformly fast over all choices of α and β. An alternative exact sampling method for $S_{\alpha,\beta}$, the so-called fast rejection sampling, has been recently proposed by Hofert [13]. In principle, the sampling method by Hofert [13] works for any exponentially tilted density function over the positive real line.

In this paper we show how to generate random variates for a more general class of tilted α-stable distributions. Such a novel class, which includes as a special case the exponentially tilted α-stable distribution, is referred to as the class of Laguerre-type exponentially tilted α-stable distributions. Specifically, for any $n \in \mathbb{N}_0$, $\beta \geq 0$ and $\gamma \leq 0$, a Laguerre-type exponentially tilted α-stable random variable is defined as a positive random variable $S_{\alpha,\beta,n,\gamma}$ with density function

$$f_{\alpha,\beta,n,\gamma}(x) = \frac{L_n^{(\gamma-n)}(\beta x)\exp\{-\beta x\}f_\alpha(x)}{\int_0^{+\infty} L_n^{(\gamma-n)}(\beta y)\exp\{-\beta y\}f_\alpha(y)\mathrm{d}y}, \tag{3}$$

where $L_n^{(\gamma-n)}(\cdot)$ denotes the generalized (or associated) Laguerre polynomial, namely

$$L_n^{(\gamma-n)}(x) = \sum_{i=0}^{n}(-1)^i \binom{\gamma}{n-i}\frac{x^i}{i!}. \tag{4}$$

The polynomial $L_n^{(\gamma-n)}$ is a generalization of the classical Laguerre polynomial, which is recovered under the assumption $\gamma = n$. The reader is referred to the monograph by Ismail [15] for a detailed account on Laguerre and generalized Laguerre polynomials. Since $L_0^{(\gamma)} = 1$, for any $\gamma \leq 0$, the exponentially tilted α-stable density function is recovered from the density function $f_{\alpha,\beta,n,\gamma}$ by setting $n = 0$.

Another noteworthy special case of the density function $f_{\alpha,\beta,n,\gamma}$ is obtained by setting $\gamma = 0$. Under this assumption $L_n^{(-n)}(x) = (-x)^n/n!$ and the Laguerre-type exponentially tilted α-stable density function reduces to a density function which we refer to as the Erlang tilted α-stable density function. Such a density function represents a special case of the so-called gamma tilted α-stable density function

$$g_{\alpha,\beta,\nu}(x) = \frac{x^\nu \exp\{-\beta x\}f_\alpha(x)}{\int_0^{\infty} y^\nu \exp\{-\beta y\}f_\alpha(y)\mathrm{d}y}, \tag{5}$$

for any $\nu > 0$. Precisely, the Erlang tilted α-stable density function $f_{\alpha,\beta,n,0}$ is a gamma tilted α-stable density function with ν restricted to be a nonnegative integer. The gamma tilted α-stable density function was first introduced by Barndorff-Nielsen and Shephard [1] and, according to Remark 5 in Devroye [9], no efficient methods for exact sampling from it are currently available in the literature.

The paper is structured as follows. In Section 2 we show that the random variable $S_{\alpha,\beta,n,\gamma}$ is equal in distribution to the convolution of two independent random variables: a random variable distributed according to an exponentially tilted α-stable distribution, and a random variable distributed according to a suitable compound gamma distribution. Such a result thus provides a direct method for sampling from $S_{\alpha,\beta,n,\gamma}$. Indeed both the random variables involved in our distributional identity can be sampled exactly: the former by exploiting the random variate generators introduced in Devroye [9] and Hofert [13], whereas the latter by means of standard sampling techniques. In Section 3 we provide details on the exact sampling from the random variable $S_{\alpha,\beta,n,\gamma}$, that, in turn, forms the basis for an efficient rejection sampler for generating random variates for the gamma tilted α-stable distribution. Numerical performances of such a rejection sampler are also analyzed by means of a comprehensive simulation study. The R codes used to run the simulations reported in the paper, are available online in a supplemental file [10].

2. Main result

We start by recalling the noncentral generalized factorial coefficients introduced in Chak [4] and Gould and Hopper [11]. See also Charalambides and Koutras [6] for details. For any real number a and any $n \in \mathbb{N}_0$ let $(a)_{(n)} = a(a+1)\dots(a+n-1)$ be the n-th order ascending factorial of a, with the proviso $(a)_{(0)} = 1$. Then, for any $n \in \mathbb{N}_0$, $0 \leq k \leq n$ and any real numbers s and r, the noncentral generalized factorial coefficient $\mathscr{C}(n, k; s, r)$ is the coefficient of the k-th order ascending factorial of t in the expansion of the n-th order ascending factorial of $(st - r)$. Formally,

$$(st - r)_{(n)} = \sum_{k=0}^{n} \mathscr{C}(n, k; s, r)(t)_{(k)}.$$

This definition implies that $\mathscr{C}(0, 0; s, r) = 0$. Also, $\mathscr{C}(n, 0; s, r) = (-r)_{(n)}$ for any $n \in \mathbb{N}$ and $\mathscr{C}(n, k; s, r) = 0$ for any $k > n$. Noncentral generalized factorial coefficients are exploited in the next proposition in order to establish the distributional identity for the Laguerre-type exponentially tilted α-stable random variable $S_{\alpha,\beta,n,\gamma}$.

Proposition 1. *Let $G_{a,b}$ be a gamma random variable with shape parameter a and rate parameter b. Then,*

$$S_{\alpha,\beta,n,\gamma} \stackrel{d}{=} S_{\alpha,\beta} + G_{X-\alpha Y,\beta} \tag{6}$$

where

$$\mathbb{P}[X = x, Y = y] = \frac{(-1)^x \binom{\gamma}{n-x} \frac{1}{x!} \beta^{\alpha y} \mathscr{C}(x, y; \alpha, 0)}{\sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^i \binom{\gamma}{n-i} \frac{1}{i!} \beta^{\alpha j} \mathscr{C}(i, j; \alpha, 0)}, \tag{7}$$

for any $x \in \{0, 1, \dots, n\}$ and $y \in \{0, 1, \dots, x\}$. Finally, the random variables $S_{\alpha,\beta}$ and $G_{X-Y\alpha,\beta}$ are independent.

Proof. For any integer $n \geq 1$ and any integer $1 \leq k \leq n$ let $\mathcal{D}_{n,k}$ be the set of all vector of integers $(n_1, \dots, n_k)$ such that $n_i \geq 1$ for $i = 1, \dots, k$ and $\sum_{1 \leq i \leq k} n_i = n$. By combining Theorem 2.15 with Equation 2.56 in Charalambides [7], we can write

$$\alpha^k \sum_{s=k}^{n} \binom{n}{s} (-\gamma)_{(n-s)} \frac{s!}{k!} \sum_{(s_1,\dots,s_k) \in \mathcal{D}_{s,k}} \prod_{i=1}^{k} \frac{(1-\alpha)_{(s_i - 1)}}{s_i!} = \mathscr{C}(n, k; \alpha, \gamma).$$

Note that $(s!/k!) \sum_{(s_1,\dots,s_k) \in \mathcal{D}_{s,k}} \prod_{1 \leq i \leq k} (1-\alpha)_{(s_i-1)}/s_i!$ is the k-th partial Bell polynomial $B_{s,k}(w_\bullet)$ with $w_\bullet = (w_i)_{i \geq 1}$ such that $w_i = (1-\alpha)_{(i-1)}$ for $i \geq 1$. Consider the Bell polynomial $B_s(v_\bullet, w_\bullet)$ with $v_\bullet = (v_i)_{i \geq 1}$ such that $v_i = \alpha^i \beta^{\alpha i}$ for $i \geq 1$. Then

$$\sum_{s=0}^{n} \binom{n}{s} (-\gamma)_{(n-s)} B_s(\alpha^\bullet \beta^{\alpha \bullet}, (1-\alpha)_{(\bullet - 1)}) = \sum_{k=0}^{n} \beta^{\alpha k} \mathscr{C}(n, k; \alpha, \gamma). \tag{8}$$

Finally, let us consider the exponential generating function $w_\alpha(\xi)$ associated with $(1-\alpha)_{(\bullet-1)}$, i.e.,

$$w_\alpha(\xi) = \sum_{i=1}^{\infty} (1-\alpha)_{(i-1)} \frac{\xi^i}{i!} = \frac{1-(1-\xi)^\alpha}{\alpha}. \tag{9}$$

According to the well-known Faà di Bruno formula, for any $n \geq 0$ the left-hand side of (8) corresponds to the coefficient of $\xi^n/n!$ in the expansion of the term $(1-\xi)^\gamma \exp\{\beta^\alpha \alpha w_\alpha(\xi)\}$. See Charalambides [7] for details. We can write, then, the identity

$$\begin{aligned} &(1-\xi)^\gamma \exp\{\beta^\alpha \alpha w_\alpha(\xi)\} \\ &\quad = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} (-\gamma)_{(n-s)} B_s(\alpha^\bullet \beta^{\alpha \bullet}, (1-\alpha)_{(\bullet-1)}) \frac{\xi^n}{n!}. \end{aligned} \tag{10}$$

At this stage, we can combine the identity (8), (9) and (10). Such a combination leads to

$$(1-\xi)^\gamma \exp\{\beta^\alpha\} \exp\{-(\beta(1-\xi))^\alpha\} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta^{\alpha k} \mathscr{C}(n,k;\alpha,\gamma) \frac{\xi^n}{n!}. \tag{11}$$

On the left-hand side of (11), the term $\exp\{-(\beta(1-\xi))^\alpha\}$ can be read as the Laplace transform of a positive α-stable random variable. If f_α is the density function of a positive α-stable random variable, then we can write the left-hand side of (11) as

$$\begin{aligned} &(1-\xi)^\gamma \exp\{\beta^\alpha\} \int_0^{+\infty} \exp\{-y(\beta(1-\xi))\} f_\alpha(y) \mathrm{d}y \\ &\quad = (1-\xi)^\gamma \exp\{\beta^\alpha\} \int_0^{+\infty} \exp\{y\xi\beta\} \exp\{-\beta y\} f_\alpha(y) \mathrm{d}y \\ &\quad = \exp\{\beta^\alpha\} \int_0^{+\infty} \exp\{-\beta y\} \sum_{s=0}^{\infty} y^s \beta^s \sum_{n=s}^{\infty} \binom{n}{s} (-\gamma)_{(n-s)} \frac{\xi^n}{n!} f_\alpha(y) \mathrm{d}y \\ &\quad = \exp\{\beta^\alpha\} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} (-\gamma)_{(n-s)} \beta^s \int_0^{+\infty} y^s \exp\{-\beta y\} f_\alpha(y) \mathrm{d}y \frac{\xi^n}{n!} \\ &\quad = \exp\{\beta^\alpha\} \sum_{n=0}^{\infty} (-n)_{(n)} \int_0^{+\infty} L_n^{(\gamma-n)}(\beta y) \exp\{-\beta y\} f_\alpha(y) \mathrm{d}y \frac{\xi^n}{n!} \end{aligned} \tag{12}$$

where the last equality is obtained by standard algebraic manipulations and by a direct application of the identity (4). See Equation 8.970.1 in Gradshteyn and Ryzhik [12] for details. Then, by combining (11) with (12) we obtain the identity

$$\int_0^{+\infty} L_n^{(\gamma-n)}(\beta y) \exp\{-\beta y\} f_\alpha(y) \mathrm{d}y = \frac{\exp\{-\beta^\alpha\}}{(-n)_{(n)}} \sum_{k=0}^{n} \beta^{\alpha k} \mathscr{C}(n,k;\alpha,\gamma). \tag{13}$$

The proof is completed by computing the Laplace transform of $S_{\alpha,\beta,n,\gamma}$. Indeed, by means of (13),

$$\begin{aligned}&\mathbb{E}[\exp\{-\lambda S_{\alpha,\beta,n,\gamma}\}]\\&\quad=\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^i\binom{\gamma}{n-i}\frac{1}{i!}\beta^{\alpha j}\mathscr{C}(i,j;\alpha,0)}{\sum_{l=0}^{n}\sum_{r=0}^{l}(-1)^l\binom{\gamma}{n-l}\frac{1}{l!}\beta^{\alpha r}\mathscr{C}(l,r;\alpha,0)}\\&\qquad\times\exp\{\beta^\alpha-(\lambda+\beta)^\alpha\}\left(\frac{\beta}{\lambda+\beta}\right)^{i-\alpha j}\end{aligned}$$

where $(\beta/(\lambda+\beta))^{i-\alpha j}$ is the Laplace transform of $G_{i-\alpha j,\beta}$ and $\exp\{\beta^\alpha-(\lambda+\beta)^\alpha\}$ is the Laplace transform of the exponentially tilted α-stable random variable $S_{\alpha,\beta}$. Finally, the random variable $S_{\alpha,\beta}$ is independent of the random variable $G_{i-\alpha j,\beta}$. □

3. Exact sampling from $S_{\alpha,\beta,n,\gamma}$

Proposition 1 shows that the problem of generating random variates for a Laguerre-type exponentially tilted α-stable distribution reduces to the problem of sampling from an exponentially tilted α-stable random variable and from a random variable distributed according to a compound gamma distribution. While the former can be sampled by exploiting the methods in Devroye [9] and Hofert [13], in this section we focus on sampling from the compound gamma distribution. If $n \geq 1$, we can exploit Proposition 1 and first simulate from a discrete random vector (X,Y) taking values in $\{(x,y) : x \in \{0,1,\dots,n\}, y \in \{0,1,\dots,x\}\}$, with probability mass function (7). To this end, a simple approach is to compute the non-central generalized factorial coefficients, and hence the probabilities in (7), explicitly. This can be easily achieved by using the following recursion

$$\mathscr{C}(n,k;\alpha,0)=\alpha\mathscr{C}(n-1,k-1;\alpha,0)+(n-1-k\alpha)\mathscr{C}(n-1,k;\alpha,0),$$

for any $n \geq 1$ and $k \leq n$ and with $\mathscr{C}(1,1;\alpha,0)=\alpha$. Note that the algorithm for sampling (X,Y) from (7), scales quadratically with the parameter n due to the computation of the non-central generalized factorial coefficients. However, these can be cheaply precomputed and stored for practical values of n and, in simulation studies we ran, constitute a negligible part of the computational cost. Finally, given $(X,Y)=(x,y)$, we simulate independently the exponentially tilted α-stable variate $S_{\alpha,\beta}$ and the gamma variate $G_{x-\alpha y,\beta}$, and return their sum. The devised random sampler for Laguerre-type exponentially tilted α-stable distributions is exact and therefore allows for independent and identically distributed samples to be generated.

The case corresponding to $\gamma=0$ is particularly interesting as it represents the starting point of the random variate generator for the gamma tilted α-stable distribution that we describe in the next section. In this case, if $n \geq 1$, the distribution of X is degenerate on n, while Y take values in $\{1,\dots,n\}$ with probabilities

$$\mathbb{P}[Y = y] = \frac{\beta^{\alpha y}\mathscr{C}(n, y; \alpha, 0)}{\sum_{i=1}^{n} \beta^{\alpha i}\mathscr{C}(n, i; \alpha, 0)}.$$

Finally, we recall that if $n = 0$, the problem of sampling from $S_{\alpha,\beta,n,0}$ boils down to the problem of sampling from $S_{\alpha,\beta}$. Indeed, in such a case $\mathbb{P}[X = 0, Y = 0] = 1$ and $G_{0,\beta} = 0$ almost surely. Accordingly, the right-hand side of (6) reduces to $S_{\alpha,\beta}$.

3.1. Exact random variates for gamma tilted α-stable distributions

We show how the proposed strategy for sampling from $S_{\alpha,\beta,n,0}$ can be used to devise an exact random variate generator for gamma tilted α-stable random variables. The performance of this simulation algorithm will be investigated by means of a comprehensive simulation study in Section 3.2. The rationale of our methodology consists in reducing the problem of sampling from density $g_{\alpha,\beta,\nu}$ in (5), to that one of sampling from $f_{\alpha,\beta,n,0}$ in (3). When $\nu = n \geq 0$ the two densities coincide, while, when $\nu > 0$ is non-integral, we can use a simple rejection sampling procedure. Specifically, we set $n = \lfloor \nu \rfloor$ and observe that, for any $\beta' \in (0, \beta)$,

$$\frac{g_{\alpha,\beta,\nu}(x)}{f_{\alpha,\beta',n,0}(x)} = \frac{Z_{\alpha,\beta',n}}{Z_{\alpha,\beta,\nu}} x^{\nu-n} \exp\{-(\beta - \beta')x\}, \tag{14}$$

where

$$Z_{\alpha,b,c} = \int_0^\infty y^c \exp\{-by\} f_\alpha(y)\mathrm{d}y.$$

As $\nu > n$ and $\beta > \beta'$, the ratio in (14) is bounded above, with the maximum value

$$\frac{Z_{\alpha,\beta',n}}{Z_{\alpha,\beta,\nu}} M_{\beta,\beta',\nu,n} \tag{15}$$

at $x' = (\nu - n)/(\beta - \beta')$, where $M_{\beta,\beta',\nu,n} = ((\nu - n)/(\beta - \beta'))^{\nu-n} \exp\{-(\nu - n)\}$. Then, we can use the distribution of $S_{\alpha,\beta',n,0}$ as the proposal, and accept a draw x with probability

$$\frac{x^{\nu-n} \exp\{-(\beta - \beta')x\}}{M_{\beta,\beta',\nu,n}}.$$

The expected number of proposals required to draw one sample coincides with (15). By means of (13), we observe that (15), as a function of the parameter β', is proportional to

$$h(\beta') = \frac{\exp\{-\beta'^{\alpha}\}}{(-n)_{(n)}} (\beta - \beta')^{n-\nu} \sum_{k=0}^{n} \beta'^{\alpha k - n} \mathscr{C}(n, k; \alpha, 0). \tag{16}$$

In order to maximize the acceptance rate, β' can be set in such a way to minimize the function $h(\beta')$. Unfortunately, we do not have an analytic solution for this optimization problem that, anyway, can be easily solved numerically. In the simulation studies presented in the next section, we consider an alternative approach. Such an approach consists in choosing the heuristic value $\beta^* = \beta(n+1)/$

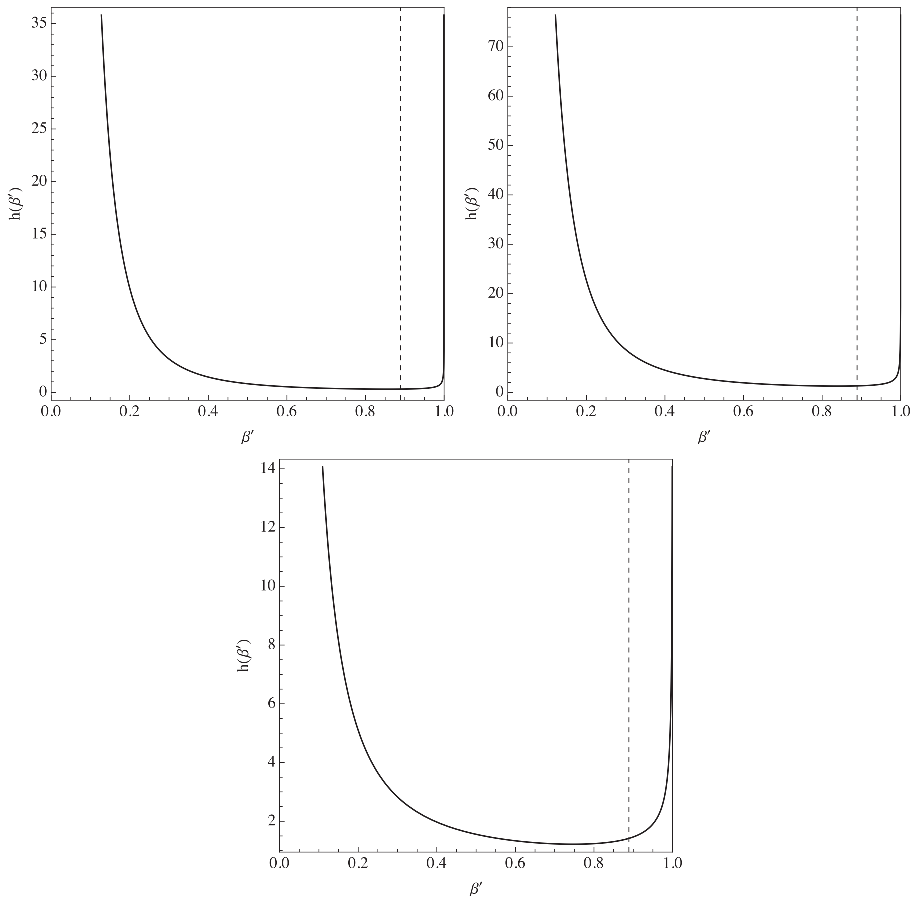

FIG 1. *Plot of the function $h(\beta')$ in (16), proportional to the expected number of proposals needed to draw one sample, for $\beta = 1$, $\nu = 3.5$ and $\alpha = 0.1$ (top), $\alpha = 0.5$ (center) and $\alpha = 0.9$ (bottom). The dashed vertical line corresponds to $\beta^* = 0.889$.*

$(\nu + 1)$ as a proxy for the value β' that minimizes (16). Such value arises by approximating with a constant function the stable density in the integral defining $Z_{\alpha,b,c}$ and, thus, minimizing the resulting approximation of (15). By visually investigating the plot of the function $h(\beta')$ for several combinations of values for α, β and ν, we could observe that the choice of β^* in general works well as the corresponding expected number of iterations is always close to the minimum. This fact can be appreciated in Figure 1, for $\beta = 1$, $\nu = 3.5$ and $\alpha \in \{0.1, 0.5, 0.9\}$.

3.2. Simulation study

We conclude by analyzing the proposed sampler for the gamma tilted α-stable distribution, by means of a simulation study. The goal of our analysis is two-fold:

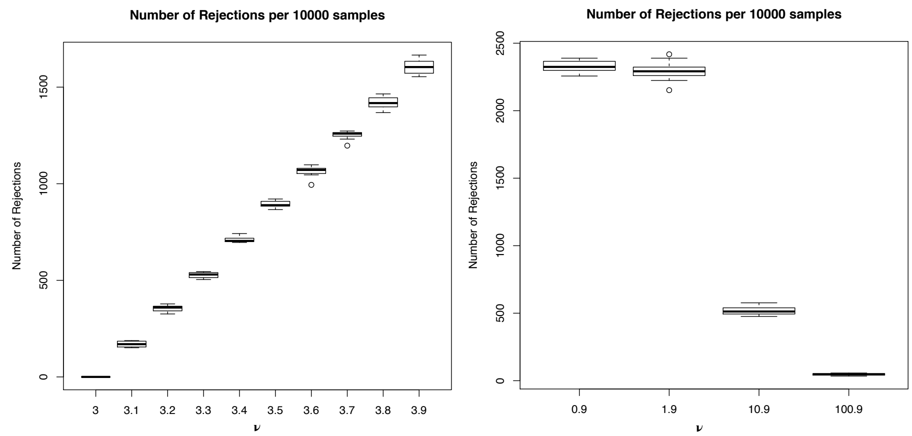

Fig 2. *Number of rejected proposals per 10000 samples. Parameters are* $\alpha = .5$, $\beta = 5$ *and* $\nu = 3.0, \ldots, 3.9$ *in increments of* 0.1 *(left panel) and* $\nu = 0.9, 1.9, 10.9, 100.9$ *(right panel). The number of rejections is small in most scenarios.*

on the one side we aim at investigating the efficiency of the sampler in terms of empirical acceptance rate; on the other side we want to assess the goodness of the choice of β^* as a proxy for the value that minimizes the expected number of rejections.

In our first set of simulations, we study the number of rejected proposals required to draw 10000 samples from a random variable distributed according to the gamma tilted α-stable distribution. The results of our analysis are illustrated in Figure 2, where we have set $\alpha = 0.5$, $\beta = 5$, and varying values for ν. Overall, the number of rejected proposals is small, increasing for larger discrepancy between ν and n. In general, the smaller ν, α and β are, the smaller is the acceptance rate. The number of rejected proposals is anyway manageable even in the worst cases we considered: for example (not shown) when $\alpha = 0.1$, $\beta = 0.1$ and $\nu = 0.9$, approximately 9 proposals are required on average per each sample.

In Figure 3 we explore how the computation time is affected by the choice of β' and made comparisons with a naïve rejection sampler in which the proposal distribution is simply f_α. We see that our proposed method works much better than the naïve approach, and that our heuristic choice of β^* works well. Note that for small values of β and ν, the optimal value of β' is actually smaller than β^* (see, e.g. top left panel): although a better choice would slightly improve both the run time and the number of rejected proposals, our simulations suggest that β^* represents a good compromise as, while not requiring numerical optimizations, it leads to a number of rejections that is close to the empirical minimum. Note that these findings are in line with what previously observed in Figure 1.

In Figure 4, we compare the run time of our sampler and the run time of the naïve rejection sampler over a larger range of values of α, β and ν. We see

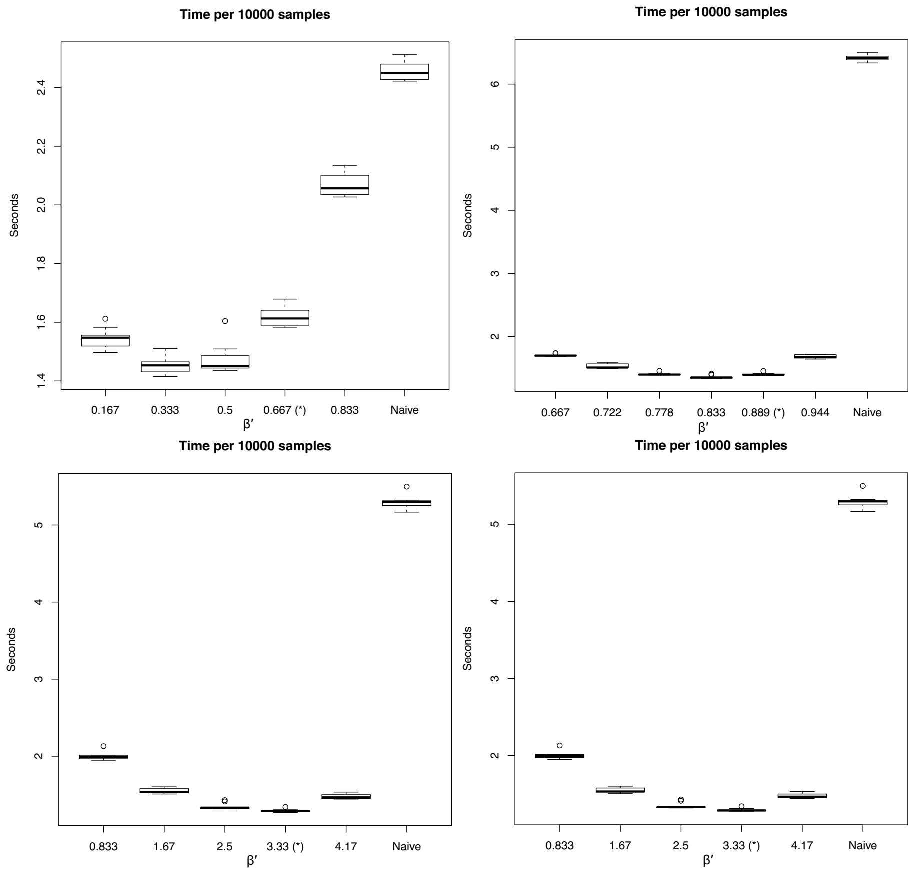

FIG 3. *Computation time required to draw 10000 samples, for varying values of β'. Parameters are $\alpha = .5$, $\beta = 1$ (top panels), $\beta = 5$ (bottom panels), $\nu = 0.5$ (left panels) and $\nu = 3.5$ (right panels). The heuristic value of $\beta^* = \beta(n+1)/(\nu+1)$, indicated by (*) in labels, works reasonably well. The computation time is not very sensitive to the exact β' value, and our proposed method is significantly faster than the naïve rejection method.*

that our proposed sampler is significantly more efficient than the naïve sampler. Similar observations follow by comparing the theoretical expected number of proposals needed to draw one sample with the two approaches, that is (15) for our sampler, and

$$\left(\frac{\nu}{\beta \mathrm{e}}\right)^{\nu} \frac{1}{Z_{\alpha,\beta,\nu}}$$

for the naïve rejection sampler. For example, if $\alpha = 0.5$, $\nu = 1.5$ and $\beta = 9$ (cfr. second panel of Figure 4), then the expected number of proposals is 1.06 with our sampler, and 8.53 with the naïve rejection sampler. Note that the difference becomes even more evident for larger values of ν or β: for example, if $\alpha = 0.5$, $\nu = 1.5$ and $\beta = 1000$, then, with our sampler, about 25 proposals are needed,

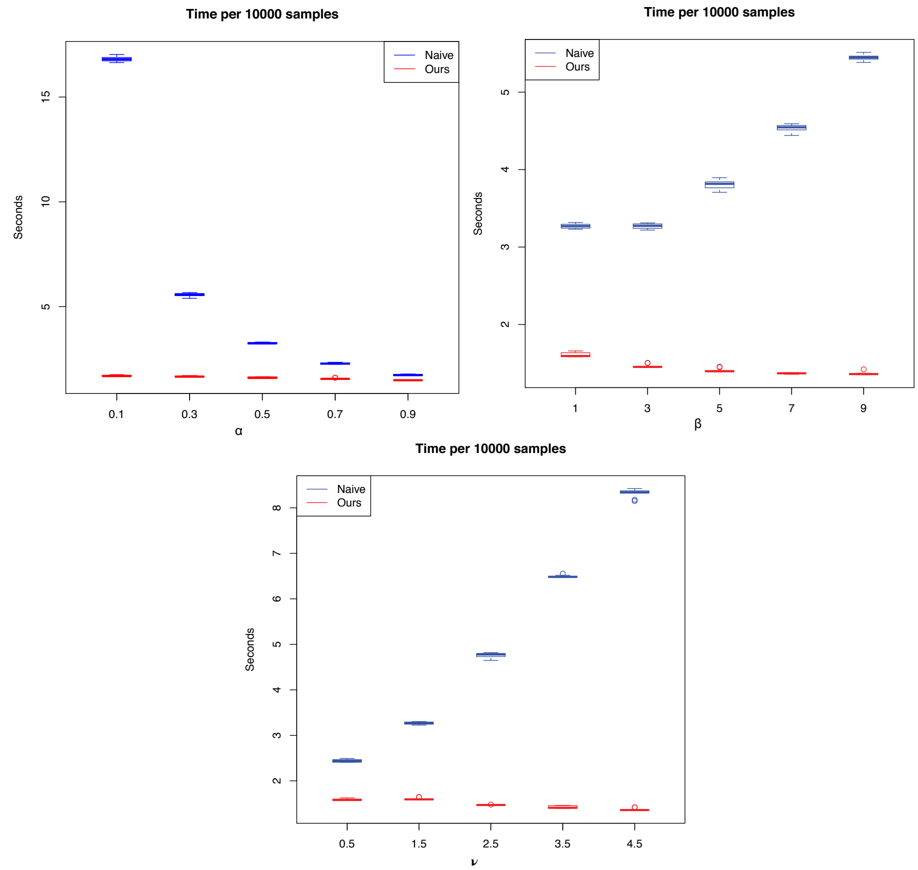

Fig 4. *Computation time required to draw 10000 samples, for varying values of parameters α, β and ν. Default parameters are $\alpha = .5$, $\beta = 1$ and $\nu = 1.5$, with α varying in first panel, β in second panel, and ν in third panel. Our proposed method is significantly faster than the naïve rejection method.*

whereas the naïve approach becomes useless as it requires 10^{17} proposals per sample.

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Supplementary Material

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