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Variance estimator for fractional diffusions with variance and drift depending on time

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Abstract: We propose punctual and functional estimators for the local variance of pseudo-diffusions driven by Gaussian noises. The consistency and asymptotic normality are shown. The proofs are simplified by using the Central Limit Theorem for non-linear functionals belonging to Itô-Wiener's Chaos, of Peccati-Nualart-Tudor. Besides, a simulation study is made to assess the performance of those estimators. This study reveals, through

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various examples, that the estimators give good approximations for the true local variance.

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Contents

1. Introduction

Recently there has been an increasing interest in models of stochastic differential equations (SDE) driven by fractional processes. Examples of such an interest are the recent books of Prakasa Rao (2010) and Mishura (2008). These SDE arise when modeling data obtained from real measurements of some physical, environmental or economic phenomena. For example, we can cite among other areas: fractional models for studying high frequency financial data, tracking of pollutant particles on water surfaces, turbulent fluids and medical imaging. Among these models, the more usual one is the fractional pseudo-diffusion driven by a fractional Brownian motion (fBm), see Nualart and Răşcanu (2002). In the present article we tackle the estimation problems for the following pseudo-diffusion models.

$$\mathrm{d}X(t) = \sigma(t)\,\mathrm{d}Y(t) + \mu(t)\,\mathrm{d}t \qquad t \geq 0,$$

where the functions $\sigma(\cdot)$ and $\mu(\cdot)$ are deterministic or random functions and Y is a Gaussian process whose covariance structure is given below, but that can be considered as a smooth perturbation of a fractional process. The solutions X can be viewed as a generalization of the cited pseudo-diffusion.

We will study three different models.

- ***Fractional pseudo-diffusion with deterministic coefficients (fDdc).*** In this case, the functions σ and μ are real and deterministic.
- ***Generalized fractional Orstein-Uhlenbeck processes (gfOUp).*** Here, σ is a deterministic function and $\mu(t)$ is random defined in the following form. Introducing the function $\widetilde{\mu}(x) = -\lambda x$, we set $\mu(t) = \widetilde{\mu} \circ X(t)$ and more generally we set $\mu(t) = \widetilde{\mu}(X(t))$.
- ***Fractional pseudo-diffusion with random coefficients (fDrc).*** In this last case $\sigma(t) = \sigma(Y(t))$ and $\mu(t) = \mu(Y(t))$.

The goal of this work is to study the estimation of the local variance function $\sigma(\cdot)$, both in a functional framework and also estimating $\sigma(t)$ for each point t belonging to the domain of the function.

It is important to point out that the estimation procedure needs only one trajectory observation and asymptotics are established when the norm of the mesh, where the trajectory is observed, tends to zero. Therefore our study is infilled in nature. As we mentioned, the class of integrators Y includes smooth perturbations of fBm and also stationary processes whose covariance function $r(\cdot)$ behaves as $1-|t|^{2H}$ when $t \to 0$. The estimators are built with the second order increments. In previous works, Berzin, Latour and León (2014) and Berzin and León (2008), we pinpointed the quality of these statistics for estimating the Hurst parameter and some functionals of the local variance. In this framework, punctual estimation of $\sigma(\cdot)$ is studied for the first time, even in the case where the integrator Y is the fBm. Note that the case $H = 1/2$, the true diffusion case, was studied by Soulier (1998) and Genon-Catalot, Laredo and Picard (1992).

For punctual estimation we get consistency and asymptotic normality results similar to those obtained for Brownian diffusions, see Soulier (1998). They are also similar to those obtained for the density estimation for weakly independent samples. The proofs of the asymptotic behavior of the estimators are considerably simplified by using the method of Peccati and Tudor (2005) to obtain central limit theorem for non-linear functionals of Gaussian processes.

In the case of (fDrc) we obtain in Theorem 5.5 a new and astonishing result that can be related to the results on the local time of ordinary diffusions of Florens-Zmirou (1989) and Jacod (2000).

Let us mention that there is an increasing literature about statistics in fractional models. Among others, we can cite the aforementioned two books and the references therein. Also in Tudor and Viens (2007) the authors give a very interesting approach to the estimation of the drift in the model $\mathrm{d}X(t) = \mathrm{d}b_H(t) + \theta\mu(X(t))\,\mathrm{d}t$, where θ is a parameter and b_H is the fBm of parameter H. More recently and more in the mind of the present work, we can cite the article about the parametric estimation in fractional Orsntein-Uhlenbeck processes written by Xiao, Zhang and Xu (2011) where the authors estimate both the diffusion and the drift.

The article has eight sections including this introduction. Section 2 describes the studied models. Section 3 establishes the hypothesis and the notations. Section 4 and 5 contain the results. In Section 6, we report an intensive simulation study to assess our estimators. The obtained results exhibit a very good asymptotic behavior of the estimators and moreover the whole section can serve as a guide to design future simulations of the same type. Sections 7, 8 contain the proofs of the results. An Appendix providing useful tools is also included.

2. Studied models

2.1. Fractional pseudo-diffusions with deterministic coefficients (fDdc)

In the following, σ and μ are real functions; obviously, we assume that σ is positive. Consider the stochastic process X_μ satisfying the pseudo-diffusion model

$$\mathrm{d}X_\mu(t) = \sigma(t)\,\mathrm{d}Y(t) + \mu(t)\,\mathrm{d}t, \quad t \geq 0 \tag{2.1}$$

where Y is a centered Gaussian process satisfying the equation

$$Y(t) = \int_{-\infty}^{+\infty} [\exp(it\lambda) - 1] \sqrt{f(\lambda)}\, \mathrm{d}W(\lambda) \tag{2.2}$$

where W stands for the standard Brownian motion and where the function f can be written as

$$f(\lambda) = \frac{1}{2\pi} |\lambda|^{-2H-1} + G(\lambda), \tag{2.3}$$

with G an even positive integrable function.

We tackle the estimation problem of the function σ in model (2.1). An example of such a process Y is given by

$$Y(t) = b_H(t) + Z(t) - Z(0),$$

where both b_H and Z are Gaussian independent processes. We assume that Z is stationary with covariance function given by $\widehat{G}$, the Fourier transform of G, while b_H is a fractional Brownian motion (FBM) with Hurst parameter $0 < H < 1$ and covariance function

$$\mathrm{E}\left[b_H(t) b_H(s)\right] = \frac{1}{2} v_{2H}^2 \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]$$

where

$$v_{2H}^2 = \left[\Gamma(2H+1)\sin(\pi H)\right]^{-1}, \tag{2.4}$$

see for instance Samorodnitsky and Taqqu (1994). Note that b_H is a zero mean process. Using the chaos representation for the FBM (see Hunt (1951)), we can write

$$b_H(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [\exp(i\lambda t) - 1] |\lambda|^{-H-\frac{1}{2}} \, \mathrm{d}W(\lambda), \quad t \geq 0,$$

so the FBM is a particular case of (2.2) with $G \equiv 0$.

With our techniques we can study the case where in model (2.1), Y is a centered stationary Gaussian process defined by

$$Y(t) = \int_{-\infty}^{+\infty} \exp(it\lambda) \sqrt{f(\lambda)}\, \mathrm{d}W(\lambda), \tag{2.5}$$

where $f(\lambda) = f_0(\lambda) + G(\lambda)$, G is as before and f_0 is an integrable even function such that for $0 < t \leq 1$, $\widehat{f_0}(t) = 1 - L(t)t^{2H} = r(t)$, $0 < H < 1$, $\lim_{t \to 0^+} L(t) = C_0 > 0$ and $L(t) - C_0 = O(t^{2H})$ as $t \to 0^+$. We suppose that L is an even non negative function with three continuous derivatives except at the origin where L satisfies

1. $tL'(t) = O(1)$;
2. $t^2 L''(t) = O(1)$, as $t \to 0^+$.

Furthermore, we suppose that $|((L(t)-C_0)t^{2H})'''| \le \boldsymbol{C} t^{4H-3}$, for $t \in]0,1]$. A centered Gaussian stationary process with covariance r given by $r(t) = e^{-|t|^{2H}}$ is an example of such a process. We will refer to this process as the *stationary fractional exponential covariance* process in short, the SFEC process. Finally, in model (2.1) Y could also be defined as

$$Y(t) = Y_1(t) - Y_1(0), \tag{2.6}$$

where process Y_1 plays the role of Y in equation (2.5). We assume that the function σ belongs to C^1 on $[0;\ 1]$ and that function μ is Lipschitz on $[0;\ 1]$. Then, for Y solution of (2.2), (2.5) or (2.6), the solution of (2.1) is

$$X_\mu(t) = X_\mu(0) + Y(t)\sigma(t) - Y(0)\sigma(0) - \int_0^t \sigma'(u)Y(u)\,\mathrm{d}u + \int_0^t \mu(u)\,\mathrm{d}u. \tag{2.7}$$

See Nualart and Răşcanu (2002). When the function μ is such that $\mu \equiv 0$, we note $X_\mu = X_0$.

2.2. Generalized fractional Orstein-Uhlenbeck processes (gfOUp)

In the following σ is positive and belongs to C^1 on $[0;\ 1]$. Our techniques also allow the study of the case where X_λ is a stochastic process satisfying the linear SDE:

$$\mathrm{d}X_\lambda(t) = \sigma(t)\,\mathrm{d}Y(t) - \lambda X_\lambda(t)\,\mathrm{d}t, \quad \lambda > 0 \text{ and } t \ge 0, \tag{2.8}$$

Y satisfying (2.2), (2.5) or (2.6).

Let us note that X_λ is not included in the framework induced by model (2.1). However it is quite close to the latter one if we consider that the drift $\mu(t)$ in (2.1) is random being $-\lambda X(t)$. The solution of (2.8) is given by

$$X_\lambda(t) = e^{-\lambda t}\left(X_\lambda(0) + \int_0^t \sigma(u)e^{\lambda u}\,\mathrm{d}Y(u)\right) \tag{2.9}$$

Note that if $\sigma(u)$ is constant and Y is a FBM ($G \equiv 0$ in equation (2.2)), this process is nothing else than the fractional Orstein-Uhlenbeck process, see Cheridito, Kawaguchi and Mæjima (2003).

We also consider the following model

$$\mathrm{d}X_\lambda(t) = \sigma(t)\,\mathrm{d}b_H(t) + \mu(X_\lambda(t))\,\mathrm{d}t, \quad t \ge 0, \tag{2.10}$$

where b_H is the FBM with parameter $H > \frac{1}{2}$, σ is positive and belongs to C^1 on $[0;\ 1]$ and μ is a Lipschitz function on $\mathbb{R}$. These conditions on H, σ and μ, ensure that there exists a unique process X_λ solution of the SDE (2.10). The solution of (2.10) is given by

$$X_\lambda(t) = X_\lambda(0) + b_H(t)\sigma(t) - \int_0^t \sigma'(u)b_H(u)\,\mathrm{d}u + \int_0^t \mu(X_\lambda(u))\,\mathrm{d}u \tag{2.11}$$

Furthermore X_λ has almost-surely $(H-\delta)$-Hölder continuous trajectories on $[0;\ 1]$. See Nualart and Răşcanu (2002).

Note that in the particular case where $\sigma(u)$ is constant and $\mu(x) = -\lambda x$, this process is also the fractional Orstein-Uhlenbeck process. For this reason (2.9) and (2.11) will be named *generalized fractional Orstein-Uhlenbeck processes*, in short gfOUp and we will note both of them X_λ.

2.3. Fractional pseudo-diffusions with random coefficients

Y as solution of (2.2), (2.5) or (2.6) has zero quadratic variations for $H > \frac{1}{2}$. See Lemma 7.1 page 962. So if we suppose that σ is positive and belongs to C^1 on $\mathbb{R}$ and μ is a continuous function on $\mathbb{R}$, we can consider the following pseudo-diffusion model in the sense of pathwise integrals, say for $t \geq 0$ and $H > \frac{1}{2}$,

$$\mathrm{d}X_r(t) = \sigma(Y(t))\,\mathrm{d}Y(t) + \mu(Y(t))\,\mathrm{d}t. \tag{2.12}$$

Here the variance and the drift are both random. The solution X_r of (2.12) is given by

$$X_r(t) = X_r(0) + \int_0^{Y(t)} \sigma(u)\,\mathrm{d}u + \int_0^t \mu(Y(u))\,\mathrm{d}u, \tag{2.13}$$

see Lin (1995).

3. Hypothesis and notations

In the following, $\mathbb{N}^*$ is the set $\mathbb{N}^* = \{x \in \mathbb{Z} : x > 0\}$.

For $n \in \mathbb{N}^* - \{1\}$, let $\Delta_n b_H$ the second order increments of the process b_H, defined as $\Delta_n b_H(i)$,

$$\Delta_n b_H(i) = \frac{n^H}{\sigma_{2H}} \delta_n b_H(i),\ i = 0, 1, \ldots, n-2,$$

where δ_n is given by

$$\delta_n b_H(i) = \left[b_H(\tfrac{i+2}{n}) - 2b_H(\tfrac{i+1}{n}) + b_H(\tfrac{i}{n})\right],$$

with $\sigma_{2H}^2 = v_{2H}^2(4 - 2^{2H})$.

Thus the process $\Delta_n b_H$ is a centered stationary Gaussian process with variance 1. Its covariance function is given by $\rho_H(i-j)$ for $i, j = 0, 1, \ldots, n-2$, where $\rho_H(x)$, for $x \in \mathbb{R}$, is

$$\rho_H(x) = \frac{1}{2(4 - 2^{2H})} \Big[-6\,|x|^{2H} + 4\,|x+1|^{2H} - |x+2|^{2H} - |x-2|^{2H} + 4\,|x-1|^{2H}\Big]. \tag{3.1}$$

Also if U_n is a random variable with support $\{0, 1, \ldots, n-2\}$, we define the random variable U_n^* on $[\,0;\ 1\,[$ by

$$U_n^*(u) = U_n(i), \quad \text{if } u \in \left[\, \tfrac{i}{n-1};\ \tfrac{i+1}{n-1} \,\right[.$$

We use Hermite polynomials, denoted by H_p. They satisfy

$$\exp(tx - t^2/2) = \sum_{p=0}^{+\infty} H_p(x) t^p / p!$$

and give an orthogonal system for the standard Gaussian measure $\phi(x)\,\mathrm{d}x$. If $h \in L^2(\phi(x)\,\mathrm{d}x)$ then,

$$h(x) = \sum_{p=0}^{+\infty} h_p H_p(x) \quad \text{and} \quad \|h\|_{2,\phi}^2 = \sum_{p=0}^{+\infty} h_p^2 p!.$$

Mehler's formula (see Breuer and Major (1983)) leads to a simple form to compute the covariance between two L^2 functions of Gaussian random variables. In fact, if $k \in L^2(\phi(x)\,\mathrm{d}x)$ and is written as $k(x) = \sum_{p=0}^{+\infty} k_p H_p(x)$ and if (X, Y) is a centered Gaussian random vector with correlation ρ and unit variance then

$$\mathrm{E}[h(X)k(Y)] = \sum_{p=0}^{+\infty} h_p k_p p! \rho^p. \tag{3.2}$$

Let g be a function in $L^2(\phi(x)\,\mathrm{d}x)$ such that

$$g(x) = \sum_{p=1}^{+\infty} g_p H_p(x), \text{ with } \|g\|_{2,\phi}^2 = \sum_{p=1}^{+\infty} g_p^2 p! < +\infty$$

and let $A_g = \{p : p \geq 2 \text{ and } g_p \neq 0\}$. We suppose that $A_g \neq \emptyset$. We define the Hermite rank of g as the smallest p such that the coefficient g_p is different from zero.

We shall write

$$\sigma_g^2 = \sum_{p=1}^{+\infty} g_p^2 p! \sum_{r=-\infty}^{+\infty} \rho_H^p(r) \qquad (\text{resp. } \widetilde{\sigma}_g^2 = \sum_{p=1}^{+\infty} g_p^2 p! \int_{-\infty}^{+\infty} \rho_H^p(x)\,\mathrm{d}x). \tag{3.3}$$

Note that since $\sum_{r=-\infty}^{+\infty} \rho_H(r) = 0$ (resp. $\int_{-\infty}^{+\infty} \rho_H(x)\,\mathrm{d}x = 0$), then if $A_g = \emptyset$, in such a case $\sigma_g^2 = \widetilde{\sigma}_g^2 = 0$.

Let K be a C^2 density function with a compact support in $[\,-1;\ 1\,]$ such that $\int_{-1}^{1} uK(u)\,\mathrm{d}u = 0$.

Furthermore, let

$$\chi^2 = \frac{1}{2} \int_{-1}^{+1} u^2 K(u)\,\mathrm{d}u, \tag{3.4}$$

and

$$\kappa^2 = \int_{-1}^{+1} K^2(u)\,\mathrm{d}u. \tag{3.5}$$

We assume that G defined in (2.3) for models (2.1), (2.8) and (2.12) is an even function such that $\int_{-\infty}^{+\infty} |\lambda|^3 G(\lambda)\,\mathrm{d}\lambda < +\infty$.

Notations Throughout the paper, we use the following notations:

- $\boldsymbol{C}$ stands for a generic constant and when such a constant depends on a trajectory ω, we write $\boldsymbol{C}(\omega)$. The value of this generic constant may change during a proof.
- The random variable N denotes a standard Gaussian random variable.
- For a function h, h', h'' and h''' will be the first, second and third derivatives of h.
- For a function h, $h^{(k)}$ will be the k^{th} derivative of h, $k \in \mathbb{N}$.
- $\xrightarrow{\text{Law}}$ indicates the convergence in law while $\xrightarrow{\text{Stable}}$ if for the stable convergence. See Podolskij and Vetter (2010) for the definion of stable convergence.

4. Functional estimation results for $\boldsymbol{\sigma}$

In this section, we proceed to the functional estimation of σ in the four models described in Section 2, by equations (2.1), (2.8), (2.10) and (2.12).

4.1. For the fDdc

We suppose that in model (2.1) the function μ is Lipschitz on $[0;\ 1]$ while the function σ belongs to C^1 on $[0;\ 1]$ and is strictly positive on this interval.

4.1.1. Almost sure convergence of the increments of X_μ

For X_μ solution of model (2.1), with Y solution of (2.2), (2.5) or (2.6), we observe X_μ on a grid $\{\frac{i}{n}, i = 0, 1, \ldots, n\}$, $n \in \mathbb{N}^* - \{1\}$, from which we can compute the second order increments, denoted by $\Delta_n X_\mu(i)$,

$$\Delta_n X_\mu(i) = a(n)\delta_n X_\mu(i),\ i = 0, 1, \ldots, n-2, \tag{4.1}$$

where δ_n is given by

$$\delta_n X_\mu(i) = \left[X_\mu(\tfrac{i+2}{n}) - 2X_\mu(\tfrac{i+1}{n}) + X_\mu(\tfrac{i}{n})\right], \tag{4.2}$$

and where

$$a^2(n) = \begin{cases} \dfrac{n^{2H}}{\sigma_{2H}^2}, & \text{for model (2.2);} \\ \left\{2\left[3 - 4r(\tfrac{1}{n}) + r(\tfrac{2}{n})\right]\right\}^{-1}, & \text{for models (2.5) and (2.6);} \end{cases} \tag{4.3}$$

with $\sigma_{2H}^2 = v_{2H}^2(4 - 2^{2H})$.

The process $\Delta_n X_\mu$ is Gaussian with mean $F_{n,\mu}(i)$ and variance $\sigma_n^2(i)$

$$F_{n,\mu}(i) = a(n)\left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \mu(u)\,\mathrm{d}u - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \mu(u)\,\mathrm{d}u\right) \tag{4.4}$$

and

$$\sigma_n^2(i) = \mathrm{Var}[\Delta_n X_\mu(i)] = \mathrm{E}[\Delta_n^2 X_0(i)]. \tag{4.5}$$

Theorem 4.1. *Almost surely for all continuous function h on $[\,0;\ 1\,]$ and for all real $k \geq 1$,*

$$\frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\frac{|\Delta_n X_\mu(i)|^k}{\mathrm{E}[|N|^k]} \underset{n\to+\infty}{\longrightarrow} \int_0^1 h(u)\sigma^k(u)\,\mathrm{d}u.$$

Remark 4.1. When $\mu \equiv 0$, the convergence holds for all real $k > 0$.

Remark 4.2. In Theorem 3.33 of Berzin, Latour and León (2014), the last convergence is obtained for a fBm.

Let us give an outline of the proof of Theorem 4.1. We need Lemma 4.1.

Lemma 4.1. *For any interval $[\,a;\ b\,] \subseteq [\,0;\ 1\,]$ and for all $k \in \mathbb{N}^*$ we almost surely have,*

$$\int_a^b ((\Delta_n X_0)^*)^k(u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} \left(\int_a^b \sigma^k(u)\,\mathrm{d}u\right)\mathrm{E}[N^k].$$

This lemma implies that if we consider $([\,a;\ b\,], \mathcal{B}_{[\,a;\ b\,]}, \frac{\lambda}{b-a})$ as a probability space, λ being the Lebesgue measure and $\mathcal{B}_{[\,a;\ b\,]}$, the Borel sets of $[\,a;\ b\,]$, then for all interval $[\,a;\ b\,] \subseteq [\,0;\ 1\,]$, almost surely $(\Delta_n X_0)^*$ weakly converges to $\sigma(U)\otimes N$, where U is an uniform random variable on $[\,a;\ b\,]$ independent of N.

Thus, by a density argument, we get that almost surely for all continuous function h on $[\,0;\ 1\,]$ and for all real $k > 0$,

$$\int_0^1 h(u)\left|(\Delta_n X_0)^*\right|^k(u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} \left(\int_a^b h(u)\sigma^k(u)\,\mathrm{d}u\right)\mathrm{E}[|N|^k].$$

Finally using the uniform continuity of h on $[\,0;\ 1\,]$ and the last result of convergence we obtain Remark 4.1.

To complete the proof of Theorem 4.1, we use the two following lemmas proven in the Appendix A.1.

Lemma 4.2. *For $i = 0, 1, \ldots, n-2$,*

$$\Delta_n X_\mu(i) = \Delta_n X_0(i) + F_{n,\mu}(i),$$

where $F_{n,\mu}(i)$ is defined by (4.4)

$$|F_{n,\mu}(i)| \leq \boldsymbol{C}\frac{1}{n^{2-H}}.$$

This lemma implies Lemma 4.3. Remark 4.1 with Lemma 4.3 also imply Theorem 4.1.

Lemma 4.3. *Almost surely for all continuous function h on $[0;\ 1]$ and for all real $k \geq 1$,*

$$A_n(h) = \frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\left\{|\Delta_n X_\mu(i)|^k - |\Delta_n X_0(i)|^k\right\} = o(1/\sqrt{n}).$$

4.1.2. Convergence in law for the weighted sum of a generalized variation of the increments of the fDdc

The following notation is used.

For $n \in \mathbb{N}^*$, $0 \leq t \leq 1$, for any function g described in the notations and for a continuous function h on $[0;\ 1]$, we define $S^{(n)}_{g,h}(t)$ as the weighted sum of a generalized variation of the increments of the fDdc. More precisely, for X_μ solution of model (2.1) for $\mu \equiv 0$, with Y solution of (2.2), (2.5) or (2.6), we define

$$S^{(n)}_{g,h}(t) = \frac{1}{\sqrt{n}}\sum_{i=0}^{\lfloor nt\rfloor -2} h\left(\tfrac{i}{n}\right) g\left(\tfrac{\Delta_n X_0(i)}{\sigma_n(i)}\right), \tag{4.6}$$

with $S^{(n)}_{g,h}(t) = 0$ if $\lfloor nt\rfloor \leq 1$ and where $\lfloor x\rfloor$ denotes the integer part of the positive real number x.

In the aim of giving the rate of convergence in Theorem 4.1, which is $\sqrt{n}$ as proved in Corollary 4.1, we demonstrate Lemma 4.4 in the Appendix A.1. Indeed since $\sigma_n(i)$ defined by (4.5) is such that $\sigma_n(i) \simeq \sigma(\frac{i}{n})$ (see Lemma 7.2, page 963), using Lemma 4.3 we establish Lemma 4.4.

Lemma 4.4. *For all function h belonging to C^1 on $[0;\ 1]$ and for all real $k \geq 1$,*

$$\begin{aligned}
B_n(h) &= \sqrt{n}\left[\frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\frac{|\Delta_n X_\mu(i)|^k}{\mathrm{E}[|N|^k]} - \int_0^1 h(u)\sigma^k(u)\,\mathrm{d}u\right]\\
&= \frac{\sqrt{n}}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\sigma^k(\tfrac{i}{n})\left(\frac{1}{\mathrm{E}[|N|^k]}\left|\frac{\Delta_n X_0(i)}{\sigma_n(i)}\right|^k - 1\right) + o_{a.s.}(1)\\
&= \frac{n}{n-1} S^{(n)}_{g_k,h\sigma^k}(1) + o_{a.s.}(1),
\end{aligned}$$

where g_k is defined by (4.7).

Remark 4.3. The latter result still holds for all real $k > 0$ when $\mu \equiv 0$.

At this step, to obtain the convergence rate in Theorem 4.1, it is sufficient to prove convergence in law for $S^{(n)}_{g_k,h\sigma^k}(1)$ and more generally for $S^{(n)}_{g,h}(1)$, for any function g described in the notations and for h a continuous function on $[0;\ 1]$.

Theorem 4.2.

$$S^{(n)}_{g,h}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(0; \sigma_g^2\int_0^1 h^2(u)\,\mathrm{d}u\right),$$

where σ_g^2 is defined by (3.3).

Remark 4.4. If g has a finite Hermite expansion, then for n large enough

$$\mathrm{E}[S_{g,h}^{(n)}(1)]^4 \leq \boldsymbol{C}.$$

More generally, for all $a \in [\,0;\ 1\,]$, we get $\mathrm{E}[S_{g,h}^{(n)}(a)]^4 \leq \boldsymbol{C}$, for n large enough.

Remark 4.5. Note that last convergence is obtained in Lemma 5.10 of Berzin, Latour and León (2014) for a fBm b_H and $h \equiv 1$.

As a corollary of Theorem 4.2, by Lemma 4.4, we get the convergence rate in Theorem 4.1.

Corollary 4.1. *If the function h belongs to C^1 on $[\,0;\ 1\,]$, then for all real $k \geq 1$*

$$\sqrt{n}\left[\frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\frac{|\Delta_n X_\mu(i)|^k}{\mathrm{E}[|N|^k]} - \int_0^1 h(u)\sigma^k(u)\,\mathrm{d}u\right]$$
$$\xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(0;\sigma_{g_k}^2\int_0^1 h^2(u)\sigma^{2k}(u)\,\mathrm{d}u\right),$$

$$g_k(x) = \frac{|x|^k}{\mathrm{E}[|N|^k]} - 1 = \sum_{p=1}^{\infty} g_{2p,k} H_{2p}(x), \tag{4.7}$$

with

$$g_{2p,k} \quad = \quad \frac{1}{(2p)!}\prod_{i=0}^{p-1}(k-2i)\,. \tag{4.8}$$

Remark 4.6. This result still holds when $\mu \equiv 0$ for all real $k > 0$.

4.2. For the two gfOUp

We assume that in models (2.8) and (2.10) the function σ belongs to C^2 on $[\,0;\ 1\,]$. Furthermore in model (2.10), we suppose that the function μ is Lipschitz on $\mathbb{R}$.

4.2.1. Almost sure convergence of the increments of X_λ

We define $\Delta_n X_\lambda$ as (4.1), Section 4.1.1.

Theorem 4.3. *Almost surely for all continuous function h on $[\,0;\ 1\,]$ and for all real $k \geq 1$,*

$$\frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\frac{|\Delta_n X_\lambda(i)|^k}{\mathrm{E}[|N|^k]} \underset{n\to+\infty}{\longrightarrow} \int_0^1 h(u)\sigma^k(u)\,\mathrm{d}u.$$

Let us give a sketch of the proof of Theorem 4.3. In this aim, we use an approximation via the process Y or b_H based on the two following lemmas.

Lemma 4.5. *For $i = 0, 1, \ldots, n-2$,*

$$\Delta_n X_\lambda(i) = \sigma(\tfrac{i}{n})\Delta_n Y^\star(i) + \varepsilon_n(i),$$

with, for any $\delta > 0$,

$$|\varepsilon_n(i)| \leq \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{(1-\delta)},$$

where $Y^\star$ stands for Y (resp. b_H) if X_λ is solution of (2.8) (resp. (2.10)) and $\Delta_n Y^\star(i)$ is defined by (4.1), Section 4.1.1.

Lemma 4.5 leads us to Lemma 4.6.

Lemma 4.6. *Almost surely for all continuous function h on $[0;\ 1]$ and for all real $k \geq 1$,*

$$C_n(h) = \frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\left\{|\Delta_n X_\lambda(i)|^k - \sigma^k(\tfrac{i}{n})\,|\Delta_n Y^\star(i)|^k\right\} = o\left(\tfrac{1}{\sqrt{n}}\right).$$

Theorem 4.3 then follows from Remark 4.1 applied in model (2.1) to $X_0 = Y^\star$, replacing the function h by $\sigma^k h$.

4.2.2. Rate of convergence for the increments $\Delta_n X_\lambda$

Lemma 4.6 and Remark 4.6 applied in model (2.1) to $X_0 = Y^\star$, replacing function h by $h\sigma^k$ lead to the following Theorem.

Theorem 4.4. *If the function h belongs to C^1 on $[0;\ 1]$, then for all real $k \geq 1$*

$$\sqrt{n}\left[\frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\frac{|\Delta_n X_\lambda(i)|^k}{\mathrm{E}[|N|^k]} - \int_0^1 h(u)\sigma^k(u)\,\mathrm{d}u\right] \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(0; \sigma^2_{g_k}\int_0^1 h^2(u)\sigma^{2k}(u)\,\mathrm{d}u\right),$$

where the function g_k is defined by (4.7).

4.3. For the fDrc

We assume that in model (2.12) the function μ is locally Lipschitz on $\mathbb{R}$ and that the function σ belongs to C^1 on $\mathbb{R}$.

4.3.1. Almost sure convergence of the increments of X_r

We define $\Delta_n X_r$ as in (4.1), Section 4.1.1.

Theorem 4.5. *Almost surely for all continuous function h on $\mathbb{R}$ and for all real $k \geq 1$,*

$$\frac{1}{n-1}\sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\frac{|\Delta_n X_r(i)|^k}{\mathrm{E}[|N|^k]} \underset{n\to+\infty}{\longrightarrow} \int_0^1 h(Y(u))\sigma^k(Y(u))\,\mathrm{d}u.$$

Let us give a sketch of the proof of Theorem 4.5. In this aim, as for Lemma 4.5 and 4.6, we use an approximation via the process Y based on the two following lemmas.

Lemma 4.7. *For $i = 0, 1, \ldots, n-2$,*

$$\Delta_n X_r(i) = \sigma(Y(\tfrac{i}{n}))\Delta_n Y(i) + \varepsilon_n(i),$$

with, for any $\delta > 0$,

$$|\varepsilon_n(i)| \leq \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{(H-\delta)},$$

where $\Delta_n Y(i)$ is defined by (4.1), Section 4.1.1.

Lemma 4.7 leads us to Lemma 4.8.

Lemma 4.8. *Almost surely for all continuous function h on $\mathbb{R}$ and for all real $k \geq 1$,*

$$D_n(h) = \frac{1}{n-1}\sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\left\{|\Delta_n X_r(i)|^k - \sigma^k(Y(\tfrac{i}{n}))\,|\Delta_n Y(i)|^k\right\} = o\left(\tfrac{1}{\sqrt{n}}\right).$$

Theorem 4.5 then follows from Remark 4.1 applied in model (2.1) to $X_0 = Y$, replacing function h by $(h \circ Y \cdot \sigma^k \circ Y)$.

4.3.2. Rate of convergence for the increments $\Delta_n X_r$

We will make the additional hypothesis that σ belongs to C^2 on $\mathbb{R}$, bounded away from zero and that for all $x \in \mathbb{R}$, $|\sigma''(x)| \leq P(|x|)$, where P is a polynomial.

The following notation is used.

We will denote by $\sigma^2_{n,1}$ the variance of $\Delta_n Y(i)$ (that does not depend on i), that is

$$\sigma^2_{n,1} = \mathrm{Var}[\Delta_n Y(i)] = \mathrm{E}[\Delta_n^2 Y(i)]. \tag{4.9}$$

$\sigma^2_{n,1}$ is nothing but $\sigma^2_n(i)$ defined by (4.5) in case where $X_0 = Y$.

Mixing techniques of this paper and of Berzin, Latour and León (2014), we can establish the rate of convergence in Theorem 4.5 through the following theorem.

Theorem 4.6. *If the function h belongs to C^2 on $\mathbb{R}$, and $|h''(x)| \leq P(|x|)$, where P is a polynomial, then for all real $k \geq 1$*

$$E_n(h) = \sqrt{n}\left[\frac{1}{n-1}\sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\frac{|\Delta_n X_r(i)|^k}{\mathrm{E}[|N|^k]} - \int_0^1 h(Y(u))\sigma^k(Y(u))\,\mathrm{d}u\right]$$
$$\xrightarrow[n\to\infty]{\text{Stable}} \sigma_{g_k}\int_0^1 h(Y(u))\sigma^k(Y(u))\,\mathrm{d}W_{\perp Y}(u)$$

where $W_{\perp Y}$ is a Brownian motion independent of Y.

We give here an outline of the proof of Theorem 4.6.

First let us remark that almost surely for all function h belonging to C^1 one has

$$\int_0^1 h(Y(u))\,\mathrm{d}u - \frac{1}{n-1}\sum_{i=0}^{n-1} h(Y(\tfrac{i}{n})) = o\left(\tfrac{1}{\sqrt{n}}\right).$$

Thus using Lemma 4.8, last equality replacing function h by $h\sigma^k$ and using the fact that $\sigma_{n,1}$ is such that $\sigma_{n,1} \simeq 1$ (see Lemma 7.2, page 963), Theorem 4.6 will ensue from the following theorem.

Theorem 4.7. *If the function h belongs to C^2 on $\mathbb{R}$, and $|h''(x)| \leq P(|x|)$, where P is a polynomial, then for all real $k > 0$*

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))g_k\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) \xrightarrow[n\to\infty]{\text{Stable}} \sigma_{g_k}\int_0^1 h(Y(u))\,\mathrm{d}W_{\perp Y}(u).$$

Here $W_{\perp Y}$ is still a Brownian motion independent of Y.

Remark 4.7. Let g be a general function with four moments with respect to the standard Gaussian measure, even, or odd, with a Hermite rank greater than or equal to one and such that $A_g \neq \emptyset$ (for the definition of A_g, see Section 3). It can be proved that, under the same hypotheses on H and h that

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\, g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) \xrightarrow[n\to\infty]{\text{Stable}} \sigma_g\int_0^1 h(Y(u))\,\mathrm{d}W_{\perp Y}(u).$$

Furthermore if h belongs to C^4 on $\mathbb{R}$ and $\left|h^{(4)}(x)\right| \leq P(|x|)$, this result is still valid under the weaker hypothesis that $H > \frac{1}{4}$ and under the supplementary condition in the case where g is odd that g has a Hermite rank greater than or equal to three.

Remark 4.8. This theorem and Remark 4.7 have been shown in Theorem 3.34 of Berzin, Latour and León (2014) for the fractional Brownian motion b_H instead of Y.

5. Punctual estimation results for σ

We consider the punctual estimation of σ in the context of the four models (2.1), (2.8), (2.10) and (2.12) described in Section 2. More precisely, we fix $t_1, t_2, \ldots, t_m$, $m \in \mathbb{N}^*$. For $j = 1, \ldots, m$, we suppose that, $0 < t_j < 1$, are all distinct points. Our aim is to estimate $\sigma(t_j)$, for $j = 1, \ldots, m$, in models (2.1), (2.8) and (2.10) and to estimate $\sigma(Y(t_j))$ in model (2.12).

5.1. For the fDdc

We assume that in model (2.1) the function μ is Lipschitz on $[\,0;\ 1\,]$ and that the function σ belongs to C^2 on $[\,0;\ 1\,]$, and is strictly positive on this interval. We make the additional hypothesis that in model (2.1), the function G is such that $\int_{-\infty}^{+\infty} \lambda^4 G(\lambda)\,\mathrm{d}\lambda < +\infty$ (see (2.3)).

5.1.1. Bias and variance

The idea consists in using Theorem 4.1 when the function h tends to the Dirac function at a fixed point t, $0 < t < 1$.

The following notation is used. For $m \in \mathbb{N}^*$, let $t_1, t_2, \ldots, t_m$, m fixed points in $]\,0;\ 1\,[$. For $j = 1, \ldots, m$, $n \in \mathbb{N}^*$, $n \geq n_0$, $i = 0, 1, \ldots, n-1$ and $0 < h \leq h_0$, let

$$a_n^{(i)}(t_j) = t_j + h\left(1 - \tfrac{2i}{n}\right). \tag{5.1}$$

Also, for X_μ solution of model (2.1), with Y solution of (2.2), (2.5) or (2.6), let

$$\Delta_n^{(i)} X_\mu(t) = a(n)\delta_n^{(i)} X_\mu(t), \tag{5.2}$$

where $a(n)$ is defined by (4.3), $\delta_n^{(i)}$ is given by

$$\delta_n^{(i)} X_\mu(t) = \left[X_\mu(a_n^{(i)}(t) + \tfrac{1}{n}) - 2X_\mu(a_n^{(i)}(t)) + X_\mu(a_n^{(i)}(t) - \tfrac{1}{n})\right], \tag{5.3}$$

where $h = h(n) \to 0$ as $n \to +\infty$ is the smoothing parameter and the dependence of h on n is implicit throughout the paper. We suppose that the sequence $nh \to +\infty$ as $n \to +\infty$.

The process $\Delta_n^{(i)} X_\mu(t)$ is Gaussian with mean $F_{n,\mu}^{(i)}(t)$ and variance $(\sigma_n^{(i)}(t))^2$ where

$$F_{n,\mu}^{(i)}(t) = a(n)\left(\int_{a_n^{(i)}(t)}^{a_n^{(i)}(t)+\frac{1}{n}} \mu(u)\,\mathrm{d}u - \int_{a_n^{(i)}(t)-\frac{1}{n}}^{a_n^{(i)}(t)} \mu(u)\,\mathrm{d}u\right) \tag{5.4}$$

and

$$(\sigma_n^{(i)}(t))^2 = \mathrm{Var}[\Delta_n^{(i)} X_\mu(t)] = \mathrm{E}[\Delta_n^{(i)} X_0(t)]^2. \tag{5.5}$$

As in Berzin-Joseph, León and Ortega (2001), for real $k > 0$, for fixed $t = t_j$, $j = 1, \ldots, m$, we consider $\widehat{\alpha}_{n,\mu}(t)$ the non-parametric kernel density estimate of the parameter

$$\alpha(t) = \sigma^k(t), \tag{5.6}$$

given by

$$\widehat{\alpha}_{n,\mu}(t) = \frac{2}{n}\sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n})\frac{\left|\Delta_n^{(i)} X_\mu(t)\right|^k}{\mathrm{E}[|N|^k]}, \tag{5.7}$$

where we recall that K is a C^2 density function with a compact support in $[-1;\ 1]$ such that $\int_{-1}^{1} uK(u)\,\mathrm{d}u = 0$.

When function $\mu \equiv 0$, we note $\widehat{\alpha}_{n,\mu}(t) = \widehat{\alpha}_{n,0}(t)$.

Remark 5.1. In this section we work with $\Delta_n^{(i)} X_\mu(t)$ (see 5.2), instead of

$$a(n)\left[X_\mu(a_n^{(i)}(t) + \tfrac{2}{n}) - 2X_\mu(a_n^{(i)}(t) + \tfrac{1}{n}) + X_\mu(a_n^{(i)}(t))\right], \tag{5.8}$$

as it could be done in the spirit of Section 4. Note that working with (5.8) remains possible. Explanation of this change is postponed after Lemma 8.1 proof in the Appendix A.2.

Theorem 5.1 establishes the consistency of $\widehat{\alpha}_{n,\mu}(t)$ as an estimator of $\alpha(t)$.

Theorem 5.1. *For all real $k \geq 1$, $\widehat{\alpha}_{n,\mu}(t)$ is an L^2 consistent estimator of $\alpha(t)$.*

The idea of the proof exploits the following equality

$$\widehat{\alpha}_{n,0}(t) - \alpha(t) = \frac{2}{n} \sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n}) g_k\Big(\tfrac{\Delta_n^{(i)} X_0(t)}{\sigma_n^{(i)}(t)}\Big) \left(\sigma_n^{(i)}(t)\right)^k + \left(\frac{2}{n} \sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n}) (\sigma_n^{(i)}(t))^k \right) - \alpha(t), \quad (5.9)$$

where $\sigma_n^{(i)}(t)$ and function g_k are respectively defined by equalities (5.5) and (4.7).

Since g_k is a centered function of $L^2(\phi(x)\,\mathrm{d}x)$, by Mehler's formula (see (3.2)) the expectation of the first term is zero and then

$$\mathrm{E}\left[\widehat{\alpha}_{n,0}(t) - \alpha(t)\right]^2 = \mathrm{E}\left[\frac{2}{n} \sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n}) g_k\Big(\tfrac{\Delta_n^{(i)} X_0(t)}{\sigma_n^{(i)}(t)}\Big) \left(\sigma_n^{(i)}(t)\right)^k \right]^2 + \left(\mathrm{E}[\widehat{\alpha}_{n,0}(t) - \alpha(t)]\right)^2.$$

Lemma 5.1. *For all real $k > 0$,*

$$\frac{1}{h^2}\mathrm{E}[\widehat{\alpha}_{n,0}(t) - \alpha(t)] \underset{n\to+\infty}{\longrightarrow} \alpha''(t)\chi^2,$$

where χ^2 is defined by (3.4).

Then, we prove that for all real $k \geq 1$, $\mathrm{E}[\widehat{\alpha}_{n,\mu}(t) - \alpha(t)] = \mathrm{E}[\widehat{\alpha}_{n,0}(t) - \alpha(t)] + o(h^2)$. So Lemma 5.1 remains true for $\widehat{\alpha}_{n,\mu}(t)$ and for all real $k \geq 1$.

The following notation is used.

Let us define, for $0 < t < 1$ and X_μ solution of model (2.1) for $\mu \equiv 0$ and for any function g described in the notations

$$S_n^{(g)}(t) = 2\sqrt{\frac{h}{n}} \sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n}) g\Big(\tfrac{\Delta_n^{(i)} X_0(t)}{\sigma_n^{(i)}(t)}\Big).$$

To establish the equilibrium between variance and bias, we need the convergence rate of the variance to zero, which is nh.

In this aim, we will prove, on the one hand, that since $\sigma_n^{(i)}(t) \simeq \sigma(t)$ (see Lemma 8.1, page 980), we have that for all real $k > 0$, $\sqrt{nh}(\widehat{\alpha}_{n,0}(t) - \mathrm{E}[\widehat{\alpha}_{n,0}(t)])$ is equivalent in L^2 to $S_n^{(g_k)}(t) \cdot \alpha(t)$.

On the other hand, we prove the following lemma.

Lemma 5.2. *For a general function g described in the notations*

$$\mathrm{E}[S_n^{(g)}(t)]^2 \underset{n\to+\infty}{\longrightarrow} \widetilde{\sigma}_g^2 \kappa^2,$$

where $\widetilde{\sigma}_g^2$ and κ^2 are respectively defined by (3.3) and (3.5).

We obtain then the following corollary.

Corollary 5.1. *For all real $k > 0$,*

$$\mathrm{E}[\sqrt{nh}(\widehat{\alpha}_{n,0}(t) - \mathrm{E}[\widehat{\alpha}_{n,0}(t)])]^2 \underset{n\to+\infty}{\longrightarrow} \widetilde{\sigma}^2_{g_k}\kappa^2\alpha^2(t).$$

Then we prove that for all real $k \geq 1$, $\sqrt{nh}(\widehat{\alpha}_{n,\mu}(t) - \mathrm{E}[\widehat{\alpha}_{n,\mu}(t)])$ is equivalent in L^2 to $\sqrt{nh}(\widehat{\alpha}_{n,0}(t) - \mathrm{E}[\widehat{\alpha}_{n,0}(t)])$. So that Corollary 5.1 still holds for $\widehat{\alpha}_{n,\mu}(t)$ and for all real $k \geq 1$.

That yields Theorem 5.1.

5.1.2. Central Limit Theorem

Lemma 5.2 allows to enunciate a theorem similar to Theorem 4.2.

Theorem 5.2. *For any function g as described in Section 3,*

$$\left(S_n^{(g)}(t_1), S_n^{(g)}(t_2), \ldots, S_n^{(g)}(t_m)\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(\mathbf{0}_m; \widetilde{\sigma}^2_g\kappa^2\boldsymbol{I}_m),$$

where $\boldsymbol{I}_m$ is the identity $(m \times m)$-matrix and $\mathbf{0}_m$ is the null column vector of length m.

From this theorem we then get the following corollary.

Corollary 5.2. *For all real $k \geq 1$,*

$$\left(\sqrt{nh}\left(\widehat{\alpha}_{n,\mu}(t_1) - \mathrm{E}[\widehat{\alpha}_{n,\mu}(t_1)]\right), \ldots, \sqrt{nh}\left(\widehat{\alpha}_{n,\mu}(t_m) - \mathrm{E}[\widehat{\alpha}_{n,\mu}(t_m)]\right)\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(\mathbf{0}_m; \widetilde{\sigma}^2_{g_k}\kappa^2\boldsymbol{D}_m(\alpha(\boldsymbol{t}))),$$

where $\boldsymbol{D}_m(\alpha(\boldsymbol{t}))$ is the diagonal matrix of rank m with generic element $\alpha^2(t_i)$, $i = 1, \ldots, m$.

Remark 5.2. For function $\mu \equiv 0$, this corollary is still true for all real $k > 0$.

Now we propose $\widehat{\sigma}_{n,\mu}(t)$ as estimator of $\sigma(t)$, defined as, for $k > 0$,

$$\widehat{\sigma}_{n,\mu}(t) = (\widehat{\alpha}_{n,\mu}(t))^{\frac{1}{k}}. \tag{5.10}$$

Since we generalized Lemma 5.1 previously stated for $\mu \equiv 0$ for a general function μ, Corollary 5.2 give a Central Limit Theorem for $\widehat{\sigma}_{n,\mu}$.

Theorem 5.3. *For all real $k \geq 1$*

1) If $nh^5 \underset{n\to+\infty}{\longrightarrow} 0$,

$$\left(\sqrt{nh}\left(\widehat{\sigma}_{n,\mu}(t_1) - \sigma(t_1)\right), \ldots, \sqrt{nh}\left(\widehat{\sigma}_{n,\mu}(t_m) - \sigma(t_m)\right)\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(\mathbf{0}_m; \widetilde{\sigma}^2_{g_k}\frac{\kappa^2}{k^2}\boldsymbol{D}_m(\sigma(\boldsymbol{t}))\right).$$

2) If $nh^5 \underset{n\to+\infty}{\longrightarrow} \boldsymbol{C} > 0$,

$$\left(\sqrt{nh}\left(\widehat{\sigma}_{n,\mu}(t_1) - \sigma(t_1)\right), \ldots, \sqrt{nh}\left(\widehat{\sigma}_{n,\mu}(t_m) - \sigma(t_m)\right)\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(\boldsymbol{a}_m(\boldsymbol{t}); \widetilde{\sigma}^2_{g_k} \frac{\kappa^2}{k^2} \boldsymbol{D}_m(\sigma(\boldsymbol{t}))\right).$$

where $\boldsymbol{a}_m(\boldsymbol{t})$ *is the column vector of order* m *with generic element*

$$\sqrt{\boldsymbol{C}}\alpha''(t_i)\frac{\chi^2}{k}\sigma^{1-k}(t_i), \quad i = 1, \ldots, m.$$

3) If $nh^5 \underset{n\to+\infty}{\longrightarrow} +\infty$,

$$\frac{1}{h^2}\left(\widehat{\sigma}_{n,\mu}(t) - \sigma(t)\right) \xrightarrow[n\to+\infty]{\mathcal{P}} \alpha''(t)\frac{\chi^2}{k}\sigma^{1-k}(t).$$

Remark 5.3. If $\mu \equiv 0$, this Theorem 5.3 remains true for all real $k > 0$.

Remark 5.4. If we consider the case where $nh^5 \to 0$, the best estimator of $\sigma(t)$ in the sense of minimal variance is obtained for $k = 2$. Furthermore if k is natural and even, the sequence on k formed by the asymptotic variance is an increasing one.

5.2. For the two gfOUp

We assume that in model (2.8) and (2.10) function σ belongs to C^2 on $[\,0;\ 1\,]$ and is strictly positive on this interval. Furthermore for model (2.10), μ is assumed Lipschitz on $\mathbb{R}$.

For $k > 0$ and for fixed $0 < t < 1$, we define

$$\widehat{\alpha}_{n,\lambda}(t) = \frac{2}{n}\sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n})\frac{\left|\Delta_n^{(i)} X_\lambda(t)\right|^k}{\mathrm{E}[|N|^k]}, \tag{5.11}$$

the non-parametric kernel density estimate of the parameter $\alpha(t) = \sigma^k(t)$, where $\Delta_n^{(i)} X_\lambda(t)$ is defined as in (5.2), Section 5.1.1.

An estimator of $\sigma(t)$ is obtained in the same way as in (5.10), that is defined as,

$$\widehat{\sigma}_{n,\lambda}(t) = (\widehat{\alpha}_{n,\lambda}(t))^{1/k}, \tag{5.12}$$

for $k > 0$. We have a Central Limit Theorem for $\widehat{\sigma}_{n,\lambda}(t)$. With the same notations as in Theorem 5.3, one obtains the following theorem.

Theorem 5.4. *For all real* $k \geq 1$

1) If $nh^5 \underset{n\to+\infty}{\longrightarrow} \boldsymbol{C} \geq 0$,

$$\left(\sqrt{nh}\left(\widehat{\sigma}_{n,\lambda}(t_1) - \sigma(t_1)\right), \ldots, \sqrt{nh}\left(\widehat{\sigma}_{n,\lambda}(t_m) - \sigma(t_m)\right)\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(\boldsymbol{a}_m(\boldsymbol{t}); \widetilde{\sigma}^2_{g_k} \frac{\kappa^2}{k^2} \boldsymbol{D}_m(\sigma(\boldsymbol{t}))\right).$$

2) If $nh^5 \underset{n\to+\infty}{\longrightarrow} +\infty$,

$$\frac{1}{h^2}\left(\widehat{\sigma}_{n,\lambda}(t)-\sigma(t)\right)\underset{n\to+\infty}{\overset{\mathcal{P}}{\longrightarrow}}\alpha''(t)\frac{\chi^2}{k}\sigma^{1-k}(t).$$

Let us give a sketch of the proof of Theorem 5.4. In this aim we use an approximation via the process Y or b_H similar to the one given in Lemma 4.5.

Lemma 5.3. *For* $i = 0, 1, \ldots, n-1$,

$$\Delta_n^{(i)} X_\lambda(t) = \sigma(a_n^{(i)}(t))\Delta_n^{(i)} Y^\star(t) + \varepsilon_n^{(i)}(t),$$

with, for any $\delta > 0$,

$$\left|\varepsilon_n^{(i)}(t)\right| \le \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{1-\delta},$$

where $Y^\star$ *stands for* Y *(resp.* b_H*) if* X_λ *is solution of (2.8) (resp. (2.10)), and* $\Delta_n^{(i)} Y^\star(t)$ *is defined by (5.2), Section 5.1.1.*

Lemma 5.3 leads us to Lemma 5.4.

Lemma 5.4. *For all real* $k \ge 1$,

$$\begin{aligned}\widehat{\alpha}_{n,\lambda}(t)-\alpha(t) \equiv \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})g_k\left(\tfrac{\Delta_n^{(i)}Y^\star(t)}{\sigma^\star_{n,1}}\right)\sigma^k(a_n^{(i)}(t))\\ +\left(\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\sigma^k(a_n^{(i)}(t))\right)-\alpha(t), \quad (5.13)\end{aligned}$$

where $\sigma^\star_{n,1}$ *is* $\sigma_{n,1}$ *(resp.* 1*) if* $Y^\star$ *is* Y *(resp.* b_H*), where we recall that* $\sigma_{n,1}$ *is defined by (4.9). The function* g_k *is defined by (4.7).*

Finally, we remark that this last equality is similar to the one given in (5.9), that is $\widehat{\alpha}_{n,\lambda}(t)-\alpha(t) \equiv \widehat{\alpha}_{n,0}(t)-\alpha(t)$, where we recall that $\widehat{\alpha}_{n,0}(t)$ given by (5.7) is the non-parametric kernel density estimate of the parameter $\alpha(t)$ in the fractional pseudo-diffusion models (2.1) with deterministic coefficients and drift $\mu \equiv 0$. In the last equality (5.13), $\Delta_n^{(i)} Y^\star(t)$ plays the role of $\Delta_n^{(i)} X_0(t)$ in (5.9) and $\sigma(a_n^{(i)}(t))$ that of $\sigma_n^{(i)}(t)$.

Corollary 5.2 for $\mu \equiv 0$ and Lemma 5.1 lead to Theorem 5.4.

Remark 5.5. However, this approximation technic does not lead to a result similar to the one of Lemma 5.1 nor to a result similar to that of Corollary 5.2 for $\widehat{\alpha}_{n,\lambda}(t)$.

5.3. For the fDrc

We assume that in model (2.12) function μ is locally Lipschitz on $\mathbb{R}$ and that σ belongs to C^2 on $\mathbb{R}$, is bounded away from zero and that for all $x \in \mathbb{R}$, $|\sigma''(x)| \le P(|x|)$, where P is a polynomial.

For $k > 0$ and for fixed $0 < t < 1$, $\widehat{\alpha}_{n,r}(t)$ is an estimator of $\sigma^k(Y(t))$, say

$$\widehat{\alpha}_{n,r}(t) = \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})\frac{|\Delta_n^{(i)}X_r(t)|^k}{\mathrm{E}[|N|^k]}, \tag{5.14}$$

where $\Delta_n^{(i)}X_r(t)$ is defined as in (5.2), Section 5.1.1.

Mixing techniques of this paper and of Berzin, Latour and León (2014) we prove the following theorem, where the convergence is in the sense of finite dimensional distributions.

Theorem 5.5. *For all real $k \geq 1$*

1. *If $nh^{(2H+1)} \underset{n\to+\infty}{\longrightarrow} 0$, then*

$$\left(\sqrt{nh}(\widehat{\alpha}_{n,r}(t_1) - \sigma^k(Y(t_1))), \ldots, \sqrt{nh}(\widehat{\alpha}_{n,r}(t_m) - \sigma^k(Y(t_m)))\right)$$
$$\xrightarrow[n\to\infty]{\text{Law}} \left(\widetilde{\sigma}_{g_k}\sigma^k(Y(t_1))\,U(t_1), \ldots, \widetilde{\sigma}_{g_k}\sigma^k(Y(t_m))\,U(t_m)\right),$$

where $(U(t_1), \ldots, U(t_m))$ is a centered Gaussian vector, independent of the vector $(Y(t_1), \ldots, Y(t_m))$ such that

$$\mathrm{E}[U(t)U(s)] = \mathbf{1}_{\{t=s\}}\kappa^2.$$

2. *If $nh^{(2H+1)} \underset{n\to+\infty}{\longrightarrow} +\infty$, then*

$$\left(\frac{1}{h^H}(\widehat{\alpha}_{n,r}(t_1) - \sigma^k(Y(t_1))), \ldots, \frac{1}{h^H}(\widehat{\alpha}_{n,r}(t_m) - \sigma^k(Y(t_m)))\right)$$
$$\xrightarrow[n\to\infty]{\text{Law}} \big(\sigma'(Y(t_1))\,k\,\sigma^{k-1}(Y(t_1))\,V(t_1), \ldots,$$
$$\sigma'(Y(t_m))\,k\,\sigma^{k-1}(Y(t_m))\,V(t_m)\big),$$

where $(V(t_1), \ldots, V(t_m))$ is a centered Gaussian vector, independent of the vector $(Y(t_1), \ldots, Y(t_m))$ such that

$$\mathrm{E}\left[V(t)V(s)\right]$$
$$= \mathbf{1}_{\{t=s\}} \lim_{n\to\infty}\left(\frac{n^{2H}}{a^2(n)}\right)\frac{1}{v_{2H}^2(4-4^H)}\,\mathrm{E}\left[\int_{-1}^{+1} K(u)b_H(u)\,\mathrm{d}u\right]^2.$$

3. *If $nh^{(2H+1)} \underset{n\to+\infty}{\longrightarrow} \boldsymbol{C} > 0$, then*

$$\left(\sqrt{nh}(\widehat{\alpha}_{n,r}(t_1) - \sigma^k(Y(t_1))), \ldots, \sqrt{nh}(\widehat{\alpha}_{n,r}(t_m) - \sigma^k(Y(t_m)))\right)$$
$$\xrightarrow[n\to\infty]{\text{Law}} \Big(\widetilde{\sigma}_{g_k}\sigma^k(Y(t_1))\,U(t_1) + \sqrt{\boldsymbol{C}}\sigma'(Y(t_1))\,k\,\sigma^{k-1}(Y(t_1))\,V(t_1), \ldots,$$
$$\widetilde{\sigma}_{g_k}\sigma^k(Y(t_m))\,U(t_m) + \sqrt{\boldsymbol{C}}\sigma'(Y(t_m))\,k\,\sigma^{k-1}(Y(t_m))\,V(t_m)\Big),$$

where $(U(t_1), \ldots, U(t_m))$, $(V(t_1), \ldots, V(t_m))$ and $(Y(t_1), \ldots, Y(t_m))$ are independent.

Let us give a sketch of the proof. First we prove a lemma similar to Lemma 5.3 for which a proof is given in Appendix A.2, that is

Lemma 5.5. *For $i = 0, 1, \dots, n-1$,*

$$\Delta_n^{(i)} X_r(t) = \sigma(Y(a_n^{(i)}(t)))\Delta_n^{(i)} Y(t) + \varepsilon_n^{(i)}(t),$$

with, for any $\delta > 0$,

$$\left|\varepsilon_n^{(i)}(t)\right| \leq \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{H-\delta},$$

where $\Delta_n^{(i)} Y(t)$ is defined by (5.2), Section 5.1.1.

Lemma 5.5 leads us to Lemma 5.6 for which a proof appears in Appendix A.2.

Lemma 5.6. *For all real $k \geq 1$,*

$$\begin{aligned}\widehat{\alpha}_{n,r}(t) - \sigma^k(Y(t)) \equiv{}& \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n}) g_k\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right)\sigma^k(Y(t)) \\ &+ \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})\left(Y(a_n^{(i)}(t)) - Y(t)\right)(\sigma^k)'(Y(t)), \quad (5.15)\end{aligned}$$

where $\sigma_{n,1}$ and g_k are respectively defined by equalities (4.9) and (4.7).

Let g be a general function with $(2+\delta)$-moments with respect to the standard Gaussian measure, even, or odd, with a Hermite rank greater than or equal to one and such that $A_g \neq \emptyset$ (for the definition of A_g, see Section 3). Let f belong to C^2 on $\mathbb{R}$, such that for all $x \in \mathbb{R}$, $|f''(x)| \leq P(|x|)$, where P is a polynomial. With these hypotheses we consider $T_n^{(f,g)}(t)$ the random variable defined by

$$T_n^{(f,g)}(t) = 2\sqrt{\frac{h}{n}}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n}) g(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}) f(Y(t)).$$

Also for p a continuous function on $\mathbb{R}$, let us define the random variable $X_{n,p}(t)$ by

$$X_{n,p}(t) = \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})\left(\frac{Y(a_n^{(i)}(t)) - Y(t)}{h^H}\right) p(Y(t))$$

Theorem 5.5 will ensue from the following theorem, where the convergence is in the sense of the finite dimensional distributions.

Theorem 5.6.

$$\left(T_n^{(f,g)}(t), X_{n,p}(t)\right) \xrightarrow[n\to\infty]{\text{Law}} \left(\widetilde{\sigma}_g\, f(Y(t))\, U(t), p(Y(t))\, V(t)\right),$$

where U, V and Y are as in Theorem 5.5.

6. Simulation study on the estimation of σ

We need to simulate trajectories of processes satisfying

$$X_\mu(t) = X_\mu(0) + Y(t)\sigma(t) - Y(0)\sigma(0) - \int_0^t \sigma'(u)Y(u)\,\mathrm{d}u + \int_0^t \mu(u)\,\mathrm{d}u, \quad (6.1)$$

where $t \in [0;\ 1]$ and $Y(t)$ satisfies (2.2), (2.5) or (2.6). It is a stationary Gaussian process or a process with stationary increments. Simulation is performed using the approach proposed in Berzin, Latour and León (2014). Briefly described, the method uses a partition of the interval $[0;\ 1]$: $0 = t_0 < t_1 < \cdots < t_n = 1$ with $n = 4096$ and $t_k = k/4096$. The values of the process or of the increments are generated at these times t_k, $k = 1, \ldots, 4096$. If we simulate the increments, we just have to sum up these increments to get a trajectory of Y.

Integrals of the following form

$$I(t) = \int_0^t f(u)Y(u)\,\mathrm{d}u \quad (6.2)$$

can be evaluated by the trapezoidal rule. More precisely, we approximate the integral (6.2) by:

$$I\left(\frac{k}{4096}\right) = \frac{1}{4096}\sum_{j=1}^{k}\frac{1}{2}\left[f\left(\frac{k-1}{4096}\right)Y\left(\frac{k-1}{4096}\right) + f\left(\frac{k}{4096}\right)Y\left(\frac{k}{4096}\right)\right],$$

for $k = 1, \ldots, 4096$.

In (6.1), $Y(t)$ can be a fBm, but other processes can also be used. So, we consider the fBm, the sfec process and a third one that is close to the fBm, named *"perturbed* fBm*"*, see (6.4).

Three different function types are considered for $\sigma(t)$: linear, quadratic and sinusoidal. Details for the linear case are explicitly exposed, the other two cases being similar are presented in a shorter way.

For the fBm, we have:

$$\Delta_n^{(1)} b_H(i) = \frac{n^H}{v_{2H}}\left(b_H(\tfrac{i}{n}) - b_H(\tfrac{i-1}{n})\right), \quad i = 1, 2, \ldots, n.$$

This sequence is a centered stationary Gaussian vector. The covariance function is denoted by $\gamma_H(i-j)$, for $i, j = 1, 2, \ldots, n$, and for $x \in \mathbb{R}$, is given by

$$\gamma_H(x) = \frac{1}{2}\left[|x+1|^{2H} - 2\,|x|^{2H} + |x-1|^{2H}\right].$$

The sfec process is interesting too. In the following we will consider this process for which the covariance function is

$$\mathrm{Cov}\,[Y(s), Y(t)] = e^{-|t-s|^{2H}} = r(t-s). \quad (6.3)$$

Consider the centered Gaussian process defined by

$$Y(t) = \int_{-\infty}^{\infty} (e^{it\lambda} - 1)\sqrt{f(\lambda)}\,\mathrm{d}W(\lambda) \quad (6.4)$$

with $f(\lambda) = \frac{1}{\sqrt{2\pi}}\,|\lambda|^{-2H-1} + G(\lambda)$ where G is an even positive integrable function, such that $\int_{-\infty}^{\infty}\lambda^4 G(\lambda)\,\mathrm{d}\lambda < \infty$. The fBm can be written that way with $G \equiv 0$. We also simulate a process of this type with a quite simple function G, a $\mathscr{N}(0;1)$ density. We will refer to this process as a "*perturbed*" fBm, or pfBm.

All the programs were developed in Pascal and run under Mac Os X. Because important computing resources are required in simulation studies, we use the linear congruential generator given in (Langlands, Pouliot and Saint-Aubin, 1994, p. 36) for the uniform deviates in conjunction with a very fast normal deviate generator described as Algorithm M in (Knuth, 1981, p. 122). This method is very efficient. Details about the implementation of these functions are given in Berzin, Latour and León (2014).

Table 1 gives the values of the different parameters used in the simulation experiment. This set up is repeated for the three types of processes we just mentioned.

TABLE 1
Values of the parameters in the simulation experiment

Hurst parameter	$\{0.25; 0.55, 0.75, 0.90\}$
Linear case	$X_\mu(t) = \delta\, t + \frac{1}{2}\,\zeta\, t^2 + (\alpha + \beta t)\, Y(t) - \alpha Y(0) - \beta \int_0^t Y(u)\,\mathrm{d}u + X_\mu(0)$ $(\alpha; \beta; \delta; \zeta) = (1/10, 2, 1, 1/2)$
Quadratic case	$X_\mu(t) = \delta t + \dfrac{1}{2}\zeta t^2 + Y(t)(\alpha + \beta t + \gamma t^2) - \alpha Y(0) - \int_0^t (\beta + 2\gamma u) Y(u)\,\mathrm{d}u + X_\mu(0)$ $(\alpha, \beta, \gamma, \delta, \zeta) = (1/10, 1, 2, 1, 1/2)$
Sinusoidal case	$X_\mu(t) = \delta t + \dfrac{1}{2}\zeta t^2 + Y(t)[\sin(c_1 t - c_2) + c_3] - (c_3 - \sin(c_2))Y(0)$ $-c_1 \int_0^t \cos(c_1 u - c_2) Y(u)\,\mathrm{d}u + X_\mu(0)$ $(c_1, c_2, c_3, \delta, \zeta,) = (1, 1/2, 3\pi, \pi/2, 11/10)$

For 2000 trajectories in each of these situations, we compute the estimator $\sigma(t)$ at points $t_j \in \{k/16 : k = 2, \ldots, 14\}$ and $n = 256$ in (4.1) for the three types of processes referred to by $Y(t)$.

In the three following subsections, the presentation of the model is done using the FBM but the simulation were also performed replacing the FBM by other processes. Graphics are presented with different models from page 952 to page 960 to illustrate that results were similar with different choices of processes.

6.1. The linear case

In (6.1), let $Y(t) = b_H(t)$ and suppose that $\sigma(t) = \alpha + \beta t$, a linear function of time t for which we have $\sigma'(t) = \beta$. Also suppose that $\mu(t) = \delta + \zeta t$. Working with the increments, we can forget the constant $X_\mu(0)$. The model can be written as

$$X_\mu(t) = \delta\, t + \tfrac{1}{2}\,\zeta\, t^2 + (\alpha + \beta t)\, b_H(t) - \beta \int_0^t b_H(u)\,\mathrm{d}u. \tag{6.5}$$

Figures 1 and 2, pages 952 and 953, show a trajectory with the first differences of such a process with numerical values of the parameters being: $(\alpha, \beta, \delta, \zeta) = (1/10, 2, 1, 1/2)$; $H = 0.75$ and 0.25. Changing the values of the parameters can lead to quite different trajectory behaviors.

We would like to illustrate the estimation procedure and assess the quality of the normal approximation given by Theorem 5.3. We use $h = 1/\sqrt{n}$ so,

$$nh^5 \xrightarrow[n\to\infty]{} 0.$$

Note that for $t \in \{t_1, \dots, t_{13}\} = \{k/16 : k = 2, \dots, 14\}$, $\sigma(t) > 0$. We proceed to the non-parametric kernel density estimation of

$$\alpha(t) = \sigma^2(t)$$

say,

$$\widehat{\alpha}_{n,\mu}(t) = \frac{2}{2^{2\ell}} \sum_{i=0}^{2^{2\ell}-1} K\left(-1 + \frac{2i}{2^{2\ell}}\right) (\Delta_n^{(i)} X_\mu(t))^2$$

In our programs we let $\ell = 4$, so $n = 256$.

The kernel K is

$$K(t) = c_1 \times e^{-t^2/2}$$

with $c_1 = 0.584\,368\,567\,256\,817$ to ensure that $\int_{-1}^{1} c_1 e^{-t^2/2}\,\mathrm{d}t = 1$.

Let us return back to Figures 1 and 2. The original path is quite typical of a non-stationary process. The mean is not constant nor the variation is. The second difference of the path produces a process with a constant mean but a linearly increasing standard deviation. These remarks can be done for all the values of H we used.

Some simulation results are presented in the six graphics of Figure 3, page 954. The model used is (6.5) with $H = 0.25$ (and $(\alpha; \beta; \delta; \zeta) = (1/10, 2, 1, 1/2)$). The first two graphics are histograms of $\hat{\sigma}(t)$, for $t = \frac{3}{16}$ and $\frac{14}{16}$. Distributions are a little skewed, but the normal distribution curves are quite close to the histogram.

The third and fourth graphics of Figure 3 are a boxplot diagram and a scattergram of the means of $\hat{\sigma}(t_j)$, $j = 1, \dots, 13$. They clearly show that the standard deviation $\hat{\sigma}(t)$ is linear in t. A weighted least square regression has been performed to fit a straight line to the estimated values $\hat{\sigma}(t)$ as a function of t. The regression (solid) line is almost confounded with the theoretical (dotted) one.

We also wanted to consider single trajectories and see how good was the fit. Arbitrarily we took the 1^{st} and the 101^{th} trajectories. In both cases, the scattergrams of $(t_j, \hat{\sigma}(t_j))$, $j = 1, \dots, 13$, suggest a linear relationship. If we proceed to the estimation of the parameters α and β of the equation $\sigma(t) = \alpha + \beta t$, we find values that are quite close to the theoretical ones.

No matter which type of processes we use, the fBm, the SFEC process or the pfBm, we always obtain results as good as these ones.

6.2. The quadratic case

We also simulate trajectories from the following model:

$$X_\mu(t) = X_\mu(0) + b_H(t)\sigma(t) - \int_0^t \sigma'(u) b_H(u)\,\mathrm{d}u + \int_0^t \mu(u)\,\mathrm{d}u \tag{6.6}$$

where $\{b_H(t), t \in \mathbb{R}\}$ is a FBM with Hurst parameter H in $]\,0;\ 1\,[$, μ and σ belong to C^1 on $[\,0;\ 1\,]$ with $\sigma(t)$ is a quadratic function.

More precisely, in (6.6), suppose that $\sigma(t) = \alpha + \beta t + \gamma t^2$, a quadratic function of time t for which $\sigma'(t) = \beta + 2\gamma t$. For this case too, we suppose that $\mu(t) =$

$\delta + \zeta t$. Working with the increments, we can forget the constant $X_\mu(0)$. The model can be written as

$$X_\mu(t) = \delta t + \frac{1}{2}\zeta t^2 + b_H(t)(\alpha + \beta t + \gamma t^2) - \int_0^t (\beta + 2\gamma u) b_H(u)\,\mathrm{d}u. \tag{6.7}$$

For the simulations, we assign the following numerical values to the parameters:

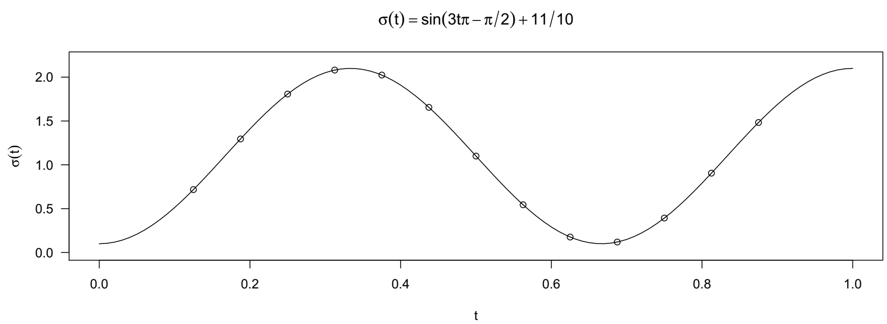
$$(\alpha, \beta, \gamma, \delta, \zeta) = (1/10, 1, 2, 1, 1/2).$$

As we can see on the graphics of Figures 4 to 6, pages 955 to 957, we can state the same comments we gave for the linear case on page 950.

6.3. The sinusoidal case

Let us suppose that in model (6.6) we let $\sigma(t) = \sin(3\pi t - \frac{\pi}{2}) + \frac{11}{10}$, a sinusoidal function of time t for which $\sigma'(t) = 3\pi \sin(3\pi t)$.

The graph of this function is



We also suppose that $\mu(t)$ is linear, that is: $\mu(t) = \delta + \zeta t$. Let $c_1 = 3\pi$, $c_2 = \pi/2$ and $c_3 = 11/10$. We have $\sigma(t) = \sin(c_1 t - c_2) + c_3$ and $\sigma'(t) = c_1 \sin(c_1 t)$.

The model can be written as

$$X_\mu(t) = \delta t + \frac{1}{2}\zeta t^2 + b_H(t)[\sin(c_1 t - c_2) + c_3] - c_1 \int_0^t \sin(c_1 u) b_H(u)\,\mathrm{d}u. \tag{6.8}$$

Now let us give some numerical values to the other parameters

$$(\delta, \zeta) = (1; 1/2).$$

It is quite interesting to see the effect of a very small standard deviation at the very start of the observation time and around 0.7 on the graphics showed from pages 958 to 959. This effect is quite obvious on the original paths as well as on the first and second differences of the path.

Looking at all the graphics on page 960, we can state the same comments we gave for the previous ones.

The reader should notice that the graphics on estimation results were presented using different values of H and different models. Computations have been done replacing the FBM by a SFEC process or a pfBm. The performance of the estimation procedure is the same. The results are very good in all cases.

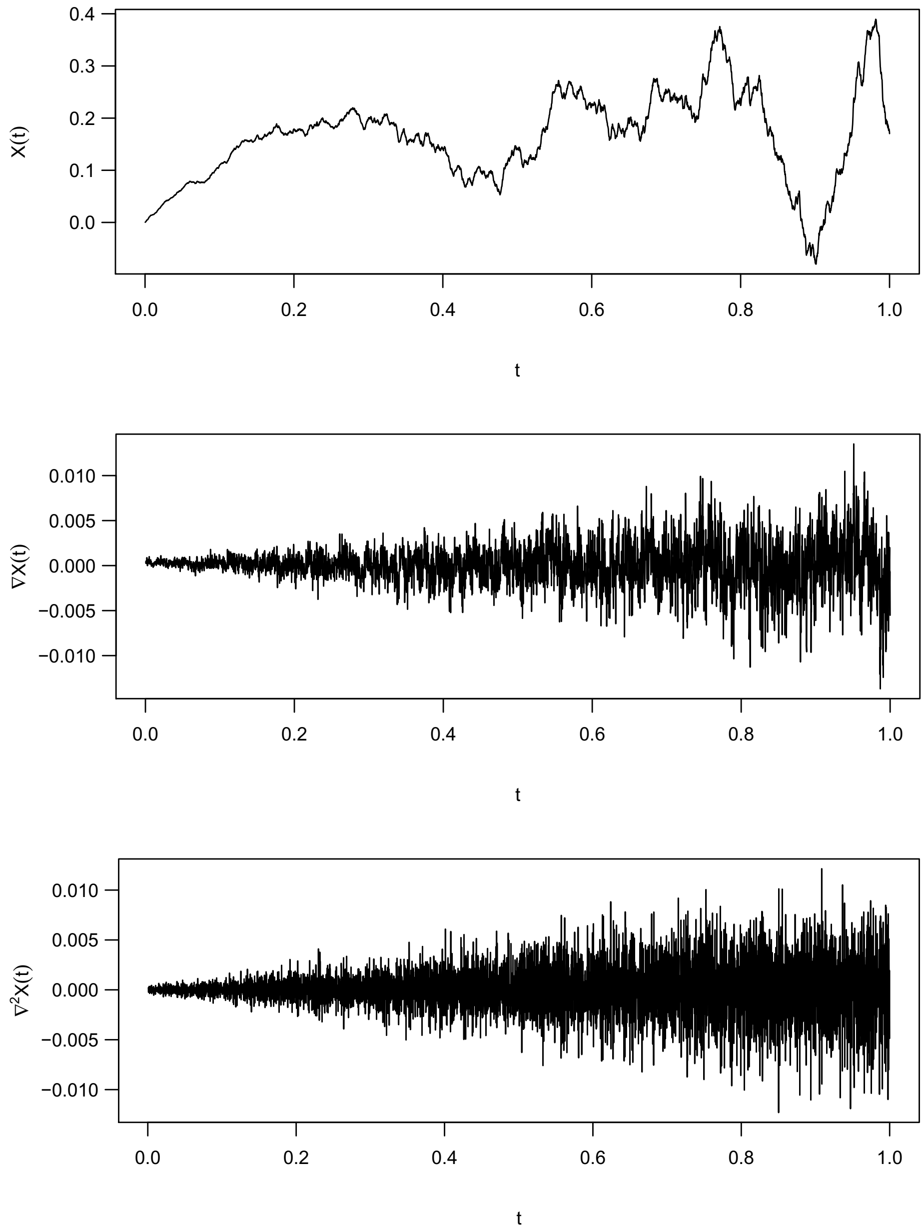


Fig 1. *A trajectory and its first and second differences generated by model (6.5) with* $(\alpha, \beta, \delta, \zeta) = (1/10, 2, 1, 1/2)$ *and* $H = 0.75$.

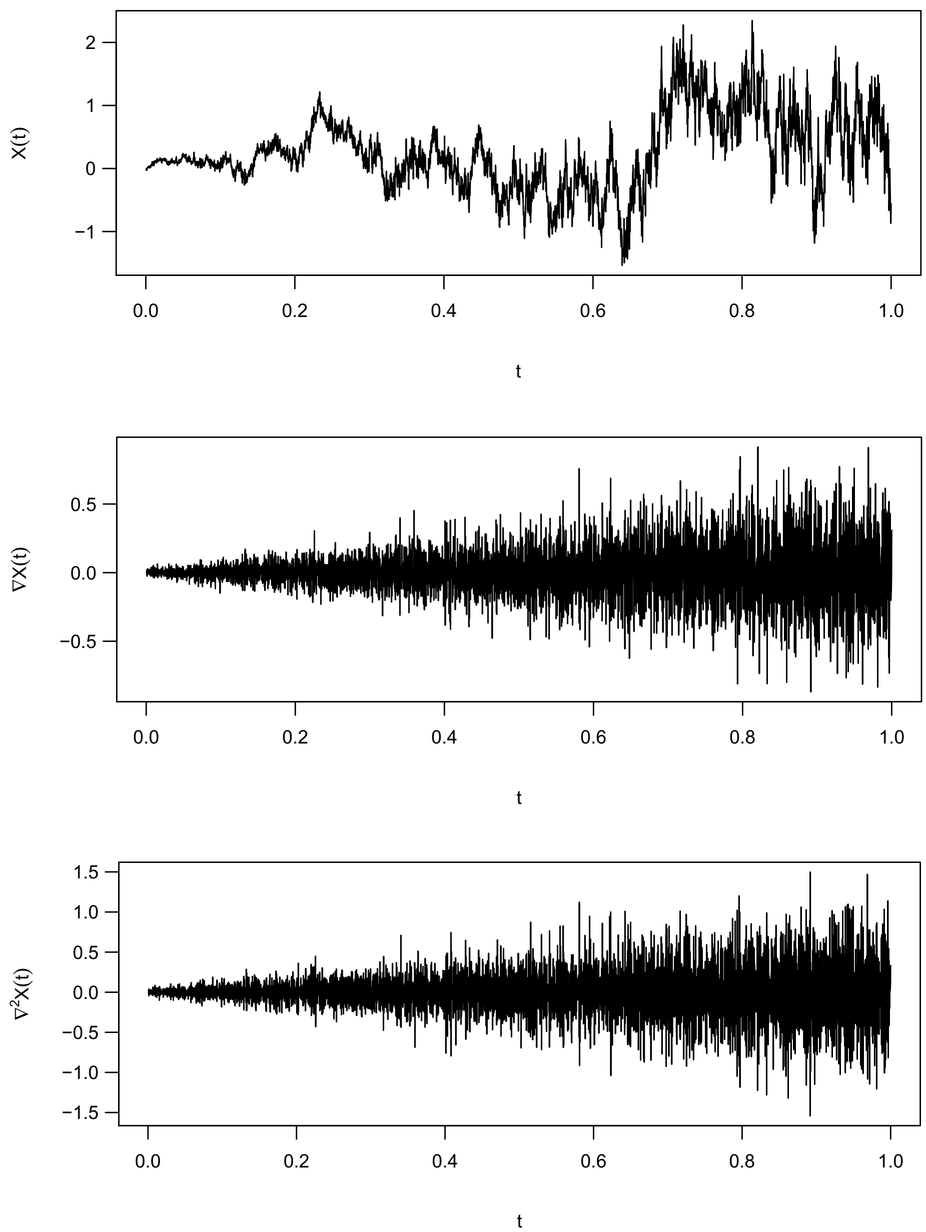


FIG 2. *A trajectory and its first and second differences generated by model (6.5) with* $(\alpha, \beta, \delta, \zeta) = (1/10, 2, 1, 1/2)$ *and* $H = 0.25$.

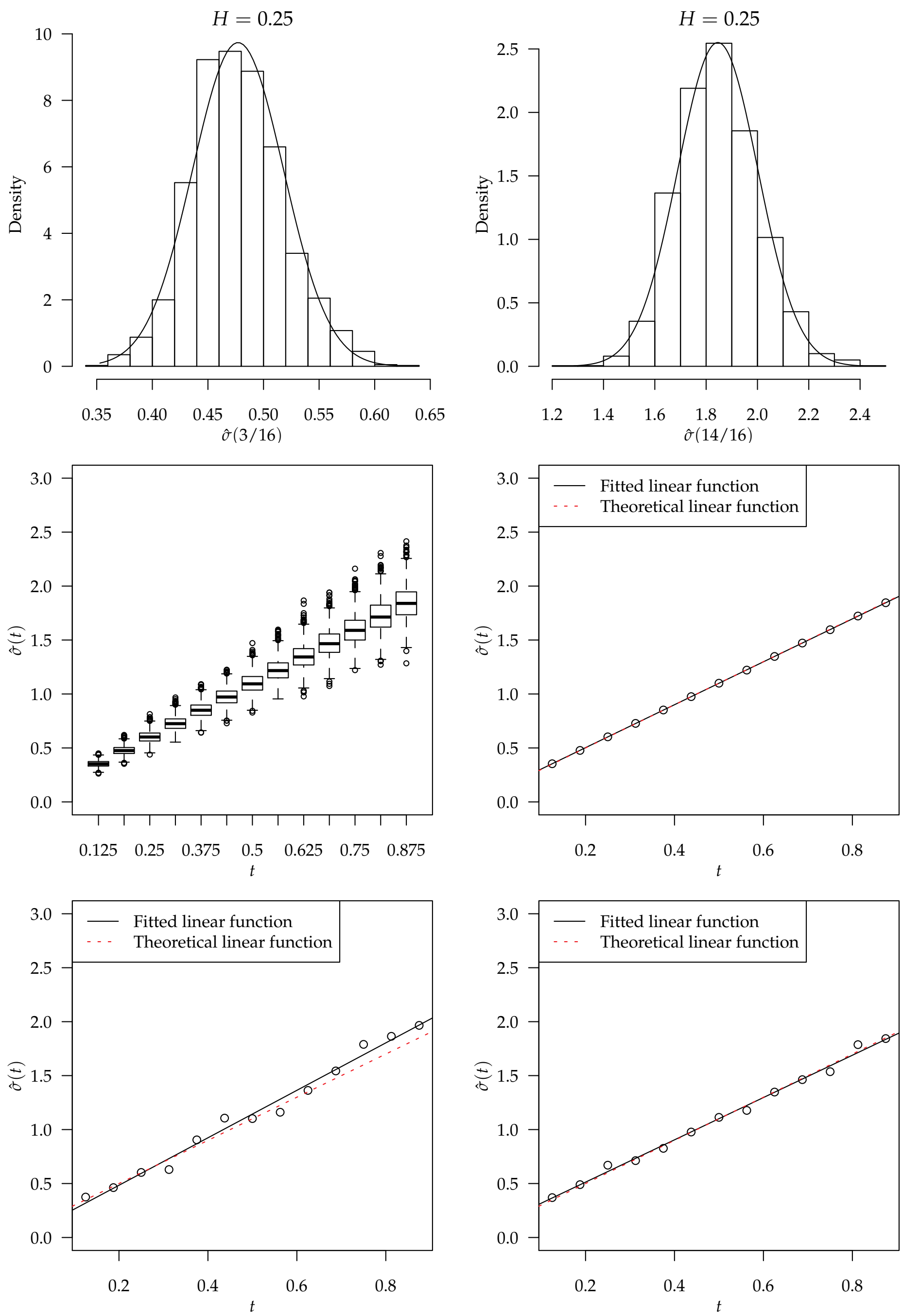


Fig 3. *Linear trend in $\sigma(t)$ with $H = 0.25$. Based on a fBm.*

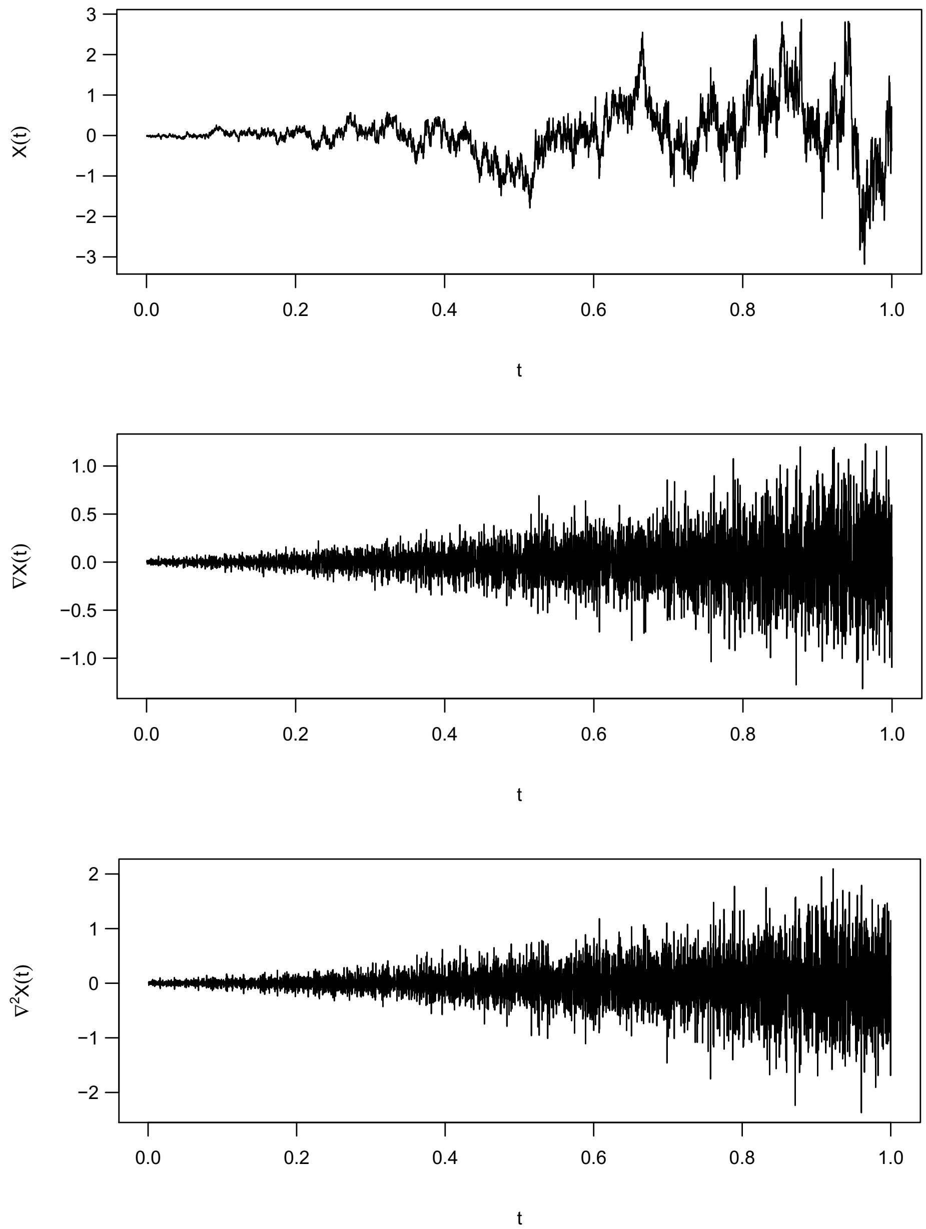


FIG 4. *A trajectory and its first and second differences generated by model (6.6) with* $(\alpha, \beta, \gamma, \delta, \zeta) = (1/10, 1, 2, 1, 1/2)$ *and* $H = 0.25$.

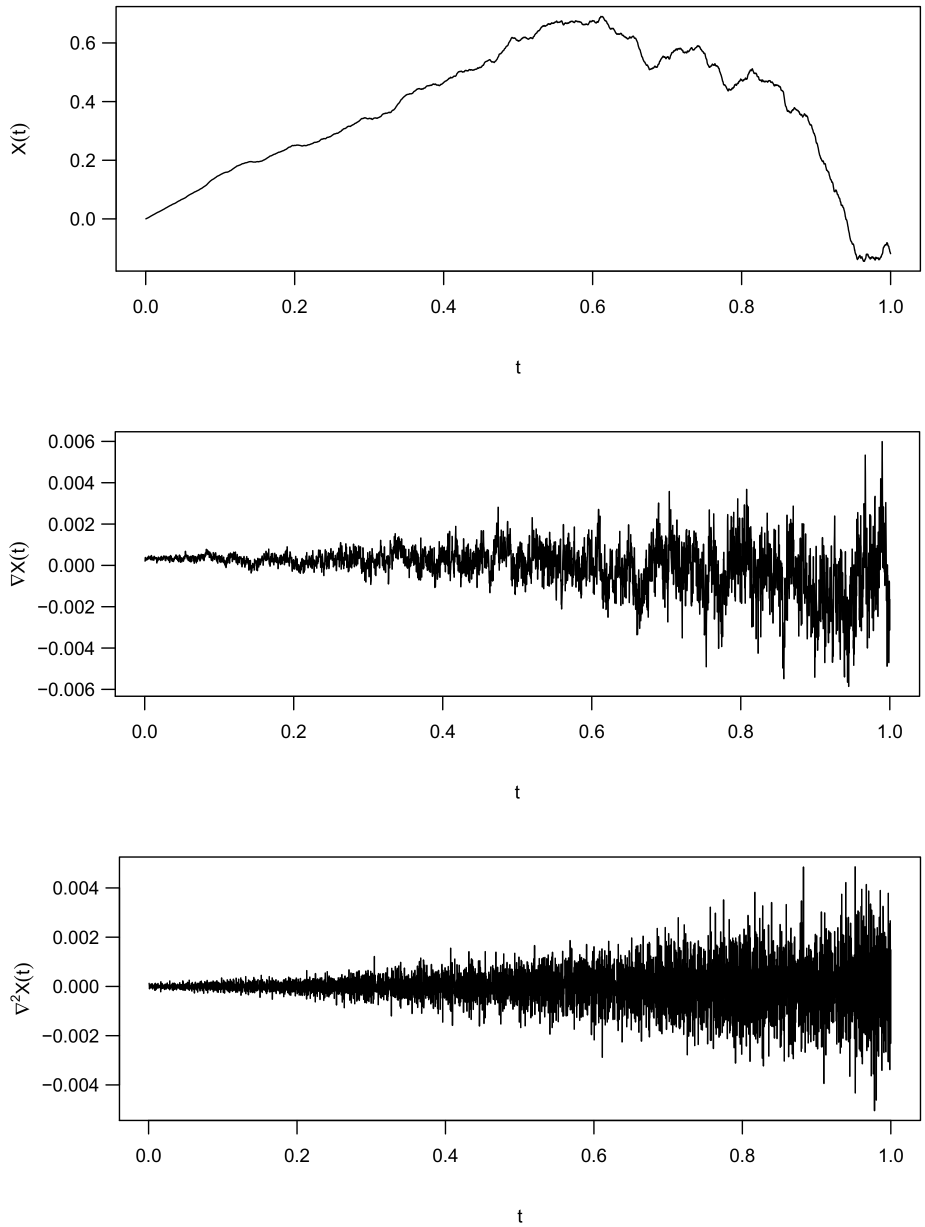


Fig 5. *A trajectory and its first and second differences generated by model (6.6) with* $(\alpha, \beta, \gamma, \delta, \zeta) = (1/10, 1, 2, 1, 1/2)$ *and* $H = 0.90$.

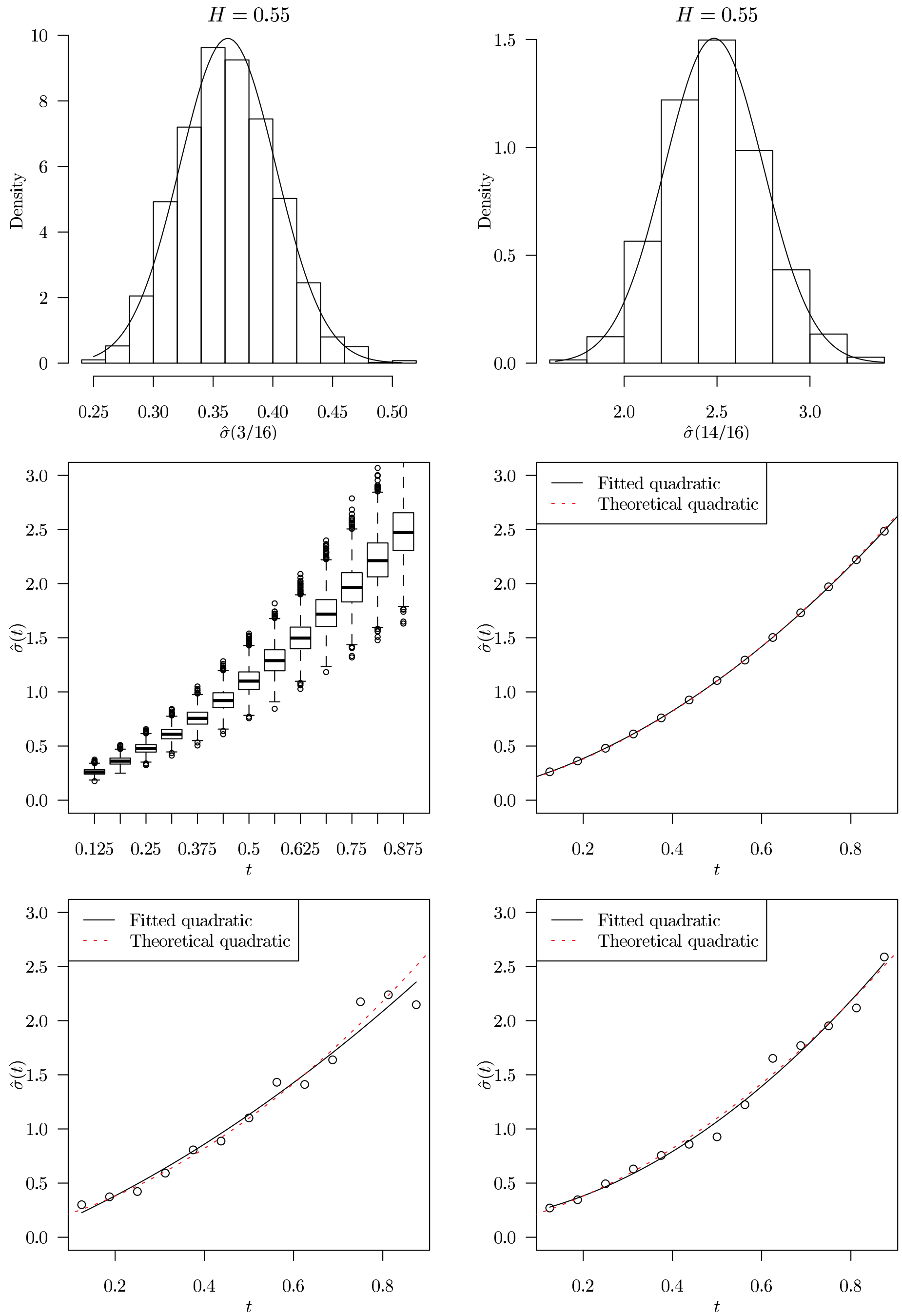


FIG 6. *Quadratic trend in $\sigma(t)$ with $H = 0.55$. Based on a* SFEC *process.*

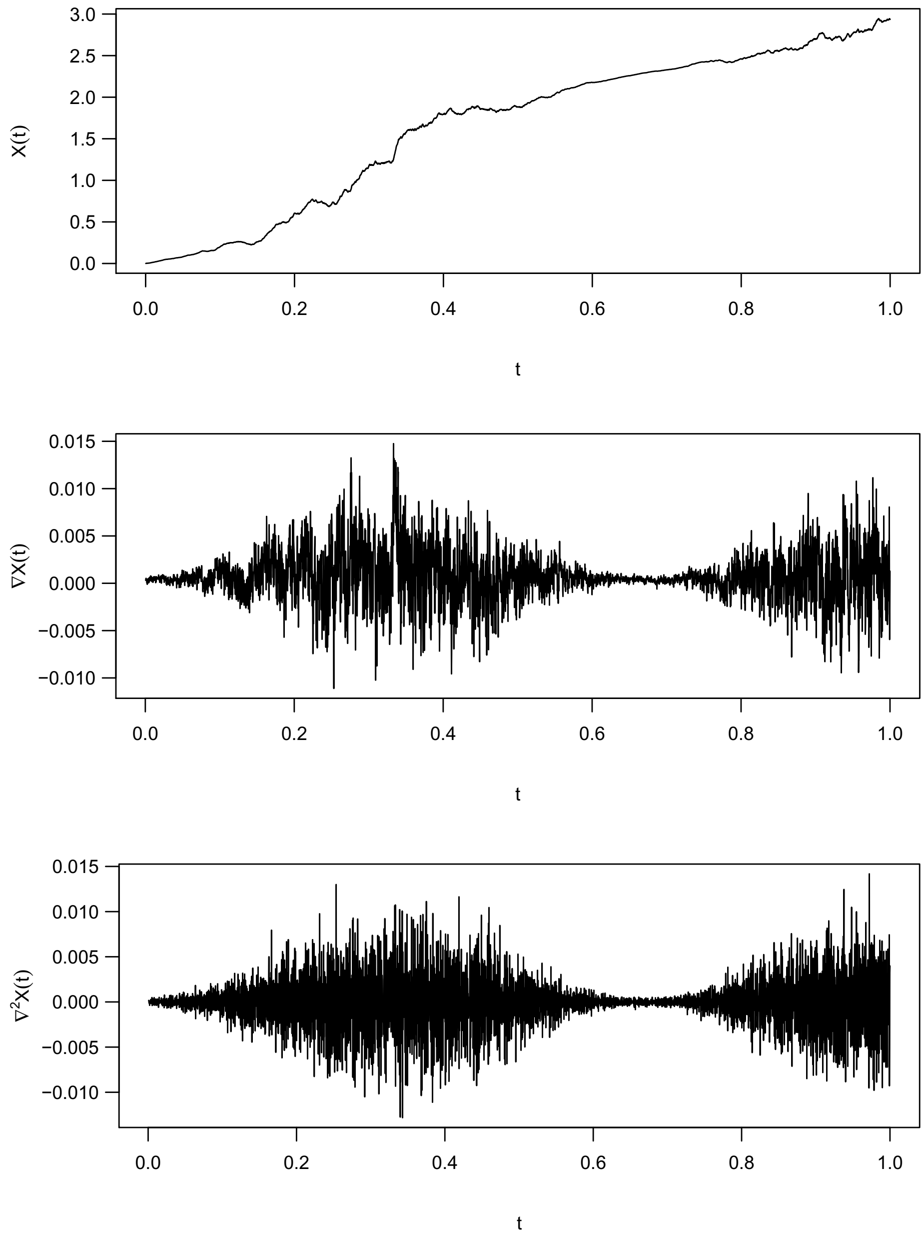


Fig 7. *A trajectory and its first and second differences generated by model (6.8) with* $(\delta, \zeta) = (1; 1/2)$ *and* $H = 0.75$.

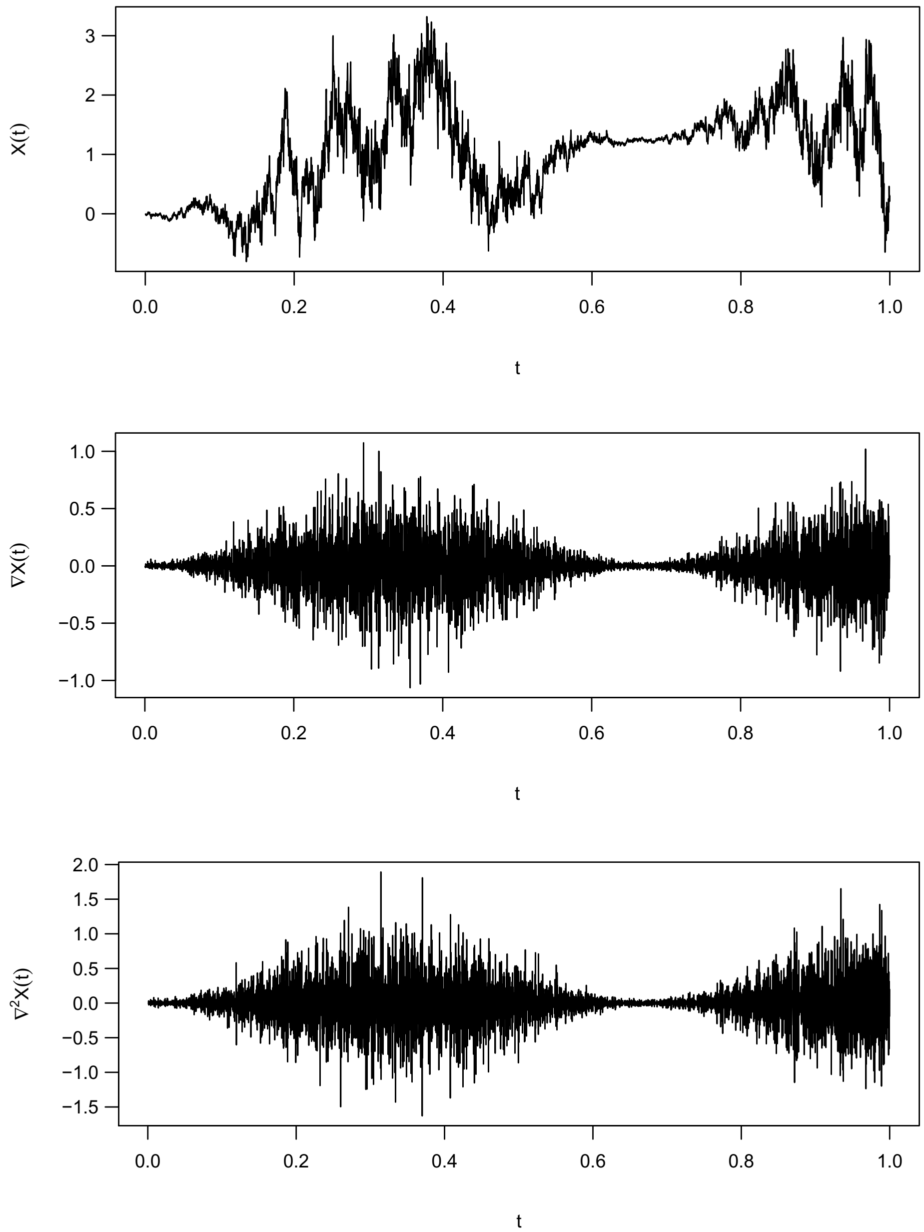


FIG 8. *A trajectory and its first and second differences generated by model (6.8) with* $(\delta, \zeta) = (1; 1/2)$ *and* $H = 0.25$.

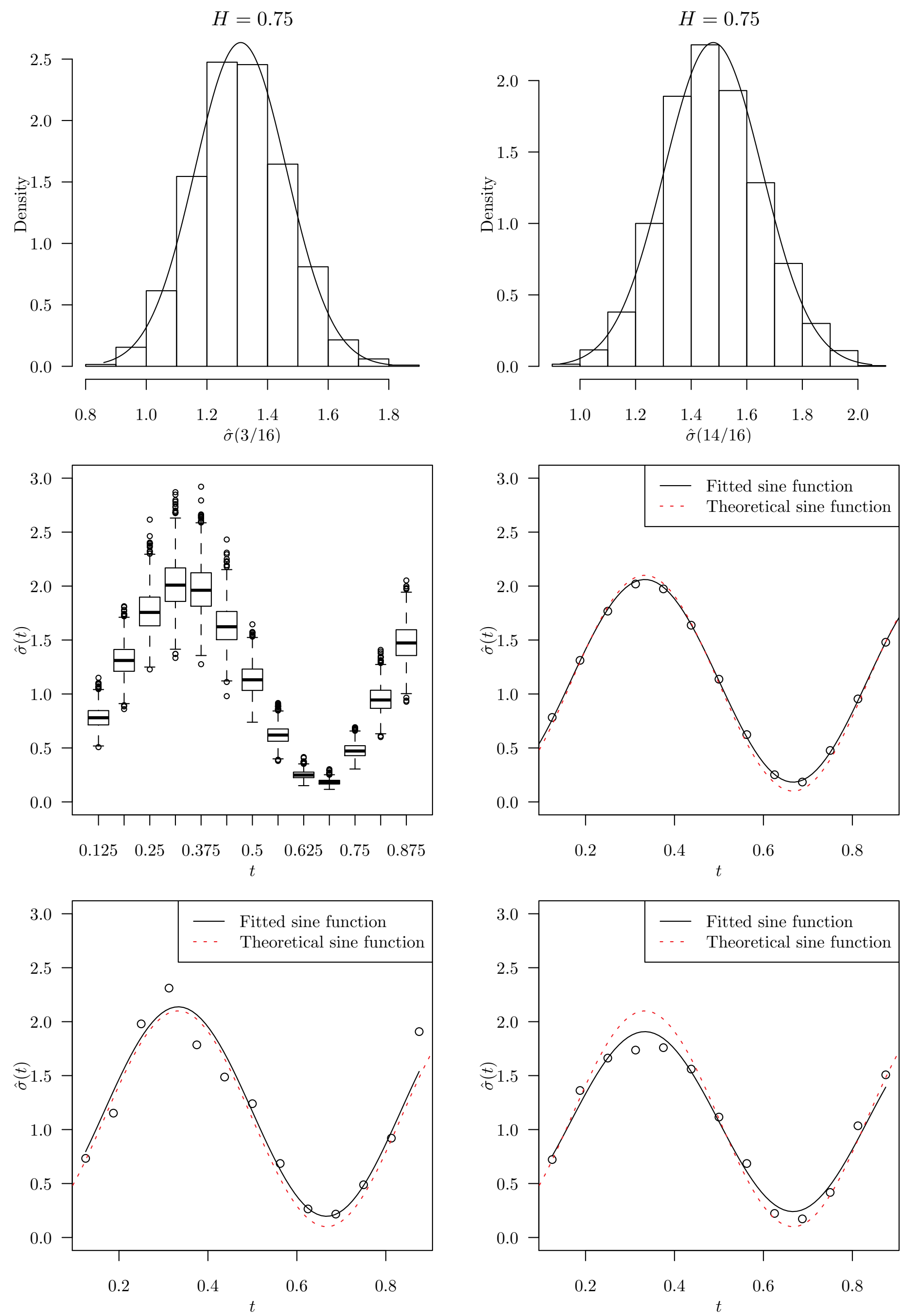


FIG 9. *Sinusoidal trend in $\sigma(t)$ with $H = 0.75$. Based on a pfBm.*

7. Proofs concerning the functional estimation of σ

In the following, we give the proofs of all the results concerning functional estimation of the function σ appearing in models (2.1), (2.8), (2.10) and (2.12). In many cases, mathematical expressions are quite long and complex. We have to break them into sub-expressions delimited by brackets to which we assign labels. We locally refer to these labels later in the text. See (7.3) for example.

For reason of simplicity all the results are simultaneously proved for process X_μ (resp. X_λ, resp. X_r) solution of the model (2.1) (resp. (2.8) or (2.10), resp. (2.12)) where the underlying process Y verifies equations (2.2), (2.5) or (2.6).

7.1. For the fDdc

7.1.1. Almost sure convergence of $\Delta_n X_\mu$

Proof of Theorem 4.1. Following the main lines of (Berzin, Latour and León, 2014, Theorem 3.33) gives the proof. It requires Lemma 4.1 whose demonstration follows this one.

In this case, for all interval $[a,b] \subseteq [\,0;\ 1\,]$, almost surely $(\Delta_n X_0)^*$ weakly converges to $\sigma(U) \otimes N$, where U is an uniform random variable on $[\,a;\ b\,]$ independent of N. This fact with Lemma 4.1 imply the following convergence: for any interval $[a,b] \subseteq [\,0;\ 1\,]$, almost surely for all real $k > 0$ we have,

$$\int_a^b \left|(\Delta_n X_0)^*\right|^k(u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} \left(\int_a^b \sigma^k(u)\,\mathrm{d}u\right) \mathrm{E}[|N|^k].$$

This last convergence implies by a density argument, that is, by taking intervals of $[\,0;\ 1\,]$ with rational endpoints and by approximating continuous functions by stepwise functions, that almost surely, for all continuous function h on $[\,0;\ 1\,]$ and for all real $k > 0$,

$$\int_0^1 h(u)\,|(\Delta_n X_0)^*|^k(u)\,\mathrm{d}u \underset{n\to+\infty}{\longrightarrow} \left(\int_0^1 h(u)\,\sigma^k(u)\,\mathrm{d}u\right) \mathrm{E}[|N|^k]. \tag{7.1}$$

Now, let us write the following equality

$$\int_0^1 h(u)\left|(\Delta_n X_0)^*\right|^k(u)\,\mathrm{d}u = \sum_{i=0}^{n-2} |\Delta_n X_0(i)|^k \int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}} h(u)\,\mathrm{d}u. \tag{7.2}$$

Thus we have

$$\begin{aligned}\int_0^1 h(u)\left|(\Delta_n X_0)^*\right|^k(u)\,\mathrm{d}u &= \frac{1}{n-1}\sum_{i=0}^{n-2} h(\tfrac{i}{n})\,|\Delta_n X_0(i)|^k \\ &\quad+ \frac{1}{n-1}\sum_{i=0}^{n-2}\Big(h(\tfrac{i}{n-1}) - h(\tfrac{i}{n})\Big)\,|\Delta_n X_0(i)|^k\end{aligned}$$

$$+\frac{1}{n-1}\sum_{i=0}^{n-2}\left((n-1)\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}(h(u)-h(\tfrac{i}{n-1}))\,\mathrm{d}u\right)|\Delta_n X_0(i)|^k.$$

Using equalities (7.2) and (7.1) for $h \equiv 1$ and since h is uniformly continuous on $[0;\ 1]$, the last two terms almost surely tend to zero. This yields Remark 4.1. Thus Theorem 4.1 follows from Lemma 4.3. □

Proof of Lemma 4.1. Let us suppose that $0 < a < b < 1$. The cases where $0 = a < b < 1$ or where $0 < a < b = 1$ could be treated in a same way.

For n large enough, $a \geq \frac{2}{n}$, $b \leq 1 - \frac{3}{n}$ and $b - a \geq \frac{3}{n}$, so that

$$\begin{aligned}\int_a^b ((\Delta_n X_0)^*)^k(u)\,\mathrm{d}u &= \sum_{i=0}^{n-2}(\Delta_n X_0(i))^k\left(\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}\mathbf{1}_{[a,b]}(u)\,\mathrm{d}u\right)\\ &= \left[\left(\sum_{i=\lfloor na\rfloor-1}^{\lfloor na\rfloor}+\sum_{i=\lfloor nb\rfloor-1}^{\lfloor nb\rfloor}\right)(\Delta_n X_0(i))^k\left(\int_{\frac{i}{n-1}}^{\frac{i+1}{n-1}}\mathbf{1}_{[a,b]}(u)\,\mathrm{d}u\right)\right]_{(1)}\\ &\quad+\left[\left(\frac{1}{n-1}\sum_{i=0}^{\lfloor nb\rfloor-2}(\Delta_n X_0(i))^k-\frac{1}{n-1}\sum_{i=0}^{\lfloor na\rfloor-2}(\Delta_n X_0(i))^k\right)\right]_{(2)}\\ &\quad-\left[\frac{1}{n-1}\sum_{i=\lfloor na\rfloor-1}^{\lfloor na\rfloor}(\Delta_n X_0(i))^k\right]_{(3)}\end{aligned}\tag{7.3}$$

As indicated by the notation, in the following computations, we will use (1), (2) and (3) to refer to the three bracketed terms of (7.3). We need Lemma 7.1 whose proof is given in the Appendix A.1.

Lemma 7.1. *The trajectories of X_0 are $(H-\delta)$-Hölder continuous on $[0;\ 1]$, that is, for any $\delta > 0$, $0 \leq u, v \leq 1$,*

$$|X_0(v)-X_0(u)| \leq \boldsymbol{C}(\omega)\,|v-u|^{H-\delta}.$$

Thus for all $i \in \{0,\ldots,n-2\}$ and for all real $k > 0$,

$$|\Delta_n X_0(i)|^k \leq \boldsymbol{C}(\omega)n^{\delta k},\tag{7.4}$$

for any $\delta > 0$.

Then for n large enough, $\sup\{|(1)|, |(3)|\} \leq \boldsymbol{C}(\omega)n^{\delta k-1}$. If we choose δ small enough, that is $\delta k < 1$, we proved that (1) and (3) converge to zero as $n \to \infty$. Thus, to achieve to proof in this case we have to show that term (2) tends to $(\int_a^b \sigma^k(u)\,\mathrm{d}u)\mathrm{E}[N^k]$.

Now if we look at the case where $a = 0$ and $b = 1$, we want to show that

$$\int_0^1 ((\Delta_n X_0)^*)^k(u)\,\mathrm{d}u = \frac{1}{n-1}\sum_{i=0}^{n-2}(\Delta_n X_0(i))^k \underset{n\to+\infty}{\longrightarrow}\left(\int_0^1\sigma^k(u)\,\mathrm{d}u\right)\mathrm{E}[N^k].$$

Thus to prove Lemma 4.1 it is enough to prove the following convergence: for all $0 < a \leq 1$ and for all $k \in \mathbb{N}^*$,

$$\frac{1}{n-1} \sum_{i=0}^{\lfloor na \rfloor - 2} (\Delta_n X_0(i))^k \xrightarrow[n\to+\infty]{\text{a.s.}} \left(\int_0^a \sigma^k(u) \, \mathrm{d}u \right) \mathrm{E}[N^k]. \tag{7.5}$$

Now, for all $0 < a \leq 1$ and for any function ℓ belonging to C^0 on $[0;\ a]$ (resp. C^1), we have the following equality

$$\int_0^a \ell(u) \, \mathrm{d}u = \frac{1}{n-1} \sum_{i=0}^{\lfloor na \rfloor - 2} \ell(\tfrac{i}{n}) + o(1) \quad (\text{resp.} \ \ o\left(\tfrac{1}{\sqrt{n}}\right)). \tag{7.6}$$

Thus, since σ^k is continuous on $[0, a]$, to prove the convergence in (7.5) it is enough to prove the following one

$$\frac{1}{n} \sum_{i=0}^{\lfloor na \rfloor - 2} g_{(k)}\left(\tfrac{\Delta_n X_0(i)}{\sigma_n(i)}\right) \sigma^k(\tfrac{i}{n}) + \frac{1}{n} \sum_{i=0}^{\lfloor na \rfloor - 2} \left(\sigma_n^k(i) - \sigma^k(\tfrac{i}{n})\right) \frac{(\Delta_n X_0(i))^k}{\sigma_n^k(i)} \xrightarrow[n\to+\infty]{} 0,$$

where the function $g_{(k)}$ is defined by $g_{(k)}(x) = x^k - \mathrm{E}[N^k]$ and $\sigma_n(i)$ by (4.5).

At this step of the proof, we need the following lemma proved in the Appendix A.1.

Lemma 7.2. *For $i = 0, 1, \ldots, n-2$,*

$$\left|\sigma_n^2(i) - \sigma^2(\tfrac{i}{n})\right| \leq \frac{\boldsymbol{C}}{n}.$$

Thus, since σ is strictly positive on $[0;\ 1]$, using Lemma 7.2, we have for real $k > 0$,

$$\begin{aligned} \sigma_n^k(i) - \sigma^k(\tfrac{i}{n}) &= \frac{k}{2}(\sigma_n^2(i) - \sigma^2(\tfrac{i}{n}))\,(\sigma^2(\tfrac{i}{n}) + \theta(\sigma_n^2(i) - \sigma^2(\tfrac{i}{n})))^{\frac{k}{2}-1} \\ &= O(\tfrac{1}{n}), \end{aligned} \tag{7.7}$$

where $0 < \theta < 1$.

Thus, using (7.4) and choosing δ small enough, that is $\delta < \frac{1}{k}$, (7.7) and the strictly positivity of σ on $[0;\ 1]$, we note that Lemma 4.1 proof is achieved if we show that, for all $0 < a \leq 1$ and for all $k \in \mathbb{N}^*$, we have

$$A_n = \frac{1}{n} \sum_{i=0}^{\lfloor na \rfloor - 2} g_{(k)} \left(\frac{\Delta_n X_0(i)}{\sigma_n(i)} \right) \sigma^k(\tfrac{i}{n}) = \frac{1}{\sqrt{n}} S^{(n)}_{g_{(k)}, \sigma^k}(a) \xrightarrow[n\to+\infty]{\text{a.s.}} 0,$$

where we recall that $S^{(n)}_{g_{(k)}, \sigma^k}$ has been defined via (4.6).

The function $g_{(k)}$ has a finite Hermite expansion and the function σ^k is continuous on $[0;\ 1]$. Applying Remark 4.4 on page 937 to the functions $g = g_{(k)}$ and $f = \sigma^k$, we obtain that $\mathrm{E}[A_n^4] = O(\frac{1}{n^2})$.

The Borel-Cantelli Lemma yields Lemma 4.1. ☐

7.1.2. Convergence in law of $S^{(n)}_{g,h}(1)$

Proof of Theorem 4.2 and Remark 4.4. First let us compute the asymptotic variance of $S^{(n)}_{g,h}(1)$.

By using Mehler's formula (3.2), we get

$$\begin{aligned}
\mathrm{E}[S^{(n)}_{g,h}(1)]^2 &= \left[\sum_{\ell=1}^{+\infty} g_\ell^2\, \ell! \frac{1}{n} \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \rho_H^\ell(i-j) h(\tfrac{i}{n})\, h(\tfrac{j}{n})\right]_{(1)} \\
&\quad + \left[\sum_{\ell=1}^{+\infty} g_\ell^2\, \ell! \frac{1}{n} \sum_{\substack{i,j=0\\ i\neq j}}^{n-2} \left[\delta_n^\ell(i,j) - \rho_H^\ell(i-j)\right] h(\tfrac{i}{n})\, h(\tfrac{j}{n})\right]_{(2)} \qquad (7.8)\\
&= (1) + (2),
\end{aligned}$$

where for $i, j = 0, \ldots, n-2$, we have defined $\delta_n(i,j)$ as

$$\begin{aligned}
\delta_n(i,j) &= \mathrm{E}\left[\frac{\Delta_n X_0(i)}{\sigma_n(i)} \frac{\Delta_n X_0(j)}{\sigma_n(j)}\right] \qquad (7.9)\\
&= \rho_H(i-j) + \gamma_n(i,j),
\end{aligned}$$

ρ_H and $\sigma_n(i)$ are respectively defined by equalities (3.1) and (4.5). As suggested by the notation, in the following, we refer to the two terms in brackets in (7.8) by (1) and (2).

Our aim is to prove that

$$\mathrm{E}[S^{(n)}_{g,h}(1)]^2 \underset{n\to+\infty}{\longrightarrow} \sigma_g^2 \left(\int_0^1 h^2(u)\,\mathrm{d}u\right), \qquad (7.10)$$

where we recall that $\sigma_g^2 = \sum_{p=1}^{+\infty} g_p^2 p! \Big(\sum_{r=-\infty}^{+\infty} \rho_H^p(r)\Big)$.

In fact term (1) gives the required limit. To prove this fact, first let us prove that term (2) converges to zero when n tends to infinity.

Since $\delta_n(i,j)$ is a correlation, this term is bounded by 1. Thus, for $\ell \geq 1$ and $i \neq j$, we have the following inequality

$$\begin{aligned}
\left|\delta_n^\ell(i,j) - \rho_H^\ell(i-j)\right| &\leq |\gamma_n(i,j)| \sum_{k=0}^{\ell-1} \left|\rho_H^k(i-j)\right| \left|\delta_n^{(\ell-k-1)}(i,j)\right| \\
&\leq |\gamma_n(i,j)| \sum_{k=0}^{\ell-1} \left|\rho_H^k(i-j)\right| = |\gamma_n(i,j)| \left(\frac{1-\left|\rho_H^\ell(i-j)\right|}{1-|\rho_H(i-j)|}\right) \leq \boldsymbol{C}\, |\gamma_n(i,j)|,
\end{aligned}$$

since for $i \neq j$, we have $1 - |\rho_H(i-j)| \geq \boldsymbol{C} > 0$.

At this step of the proof we need the following lemma proved in the Appendix A.1.

Lemma 7.3. *There exists $M_0 \in \mathbb{N}^*$ such that for all real $M \geq M_0$ and for $n \geq M$, we have for all $i, j \in \{0, 1, \ldots, n-2\}$,*

$$|\gamma_n(i,j)| \leq \frac{\boldsymbol{C}}{n} \left\{ \boldsymbol{C}_M \mathbf{1}_{\{|i-j| \leq M-1\}} + M^{2H-2} \right\}.$$

Thus, since h is bounded on $[\,0;\ 1\,]$, for all real M, $M \geq M_0$ and for $n \geq M$, we have the following inequality

$$|(2)| \leq \boldsymbol{C} \left(\sum_{\ell=1}^{+\infty} g_\ell^2 \ell! \right) \frac{1}{n^2} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \left(\boldsymbol{C}_M \mathbf{1}_{\{|i-j| \leq M-1\}} + M^{2H-2} \right).$$

As $\|g\|_{2,\phi}^2 < +\infty$, for all $M \geq M_0$ we get, $\limsup_{n \to +\infty} |(2)| \leq \boldsymbol{C} M^{2H-2}$, and since $H < 1$, we finally get that (2), the second term of (7.8), converges to 0 as n tends to infinity.

To achieve the computation of the asymptotic variance, we have to compute the limit of term (1) and to show that this limit is (7.10). In this aim, we write

$$(1) = \frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} h(\tfrac{i}{n})\, h(\tfrac{j}{n}) \beta(i-j),$$

where we define for $x \in \mathbb{R}$,

$$\beta(x) = \sum_{\ell=1}^{+\infty} g_\ell^2\, \ell! \rho_H^\ell(x). \tag{7.11}$$

Let $i - j = k$ in the sum of term (1). We get

$$\begin{aligned}
(1) &= \left[\left(\frac{1}{n} \sum_{i=0}^{n-2} h^2(\tfrac{i}{n}) \right) \left(\sum_{k=-\infty}^{+\infty} \beta(k) \right) \right]_{(A)} \\
&\quad - \left[\left(\frac{1}{n} \sum_{i=0}^{n-2} \sum_{k=i+1}^{+\infty} h^2(\tfrac{i}{n}) \beta(k) \right) \right]_{(B)} \\
&\quad - \left[\left(\frac{1}{n} \sum_{i=0}^{n-2} \sum_{k=-\infty}^{i-n+1} h^2(\tfrac{i}{n}) \beta(k) \right) \right]_{(C)} \\
&\quad + \left[\frac{1}{n} \sum_{i=0}^{n-2} \sum_{k=i-n+2}^{i} \left[h(\tfrac{i}{n})\, h(\tfrac{i-k}{n}) - h^2(\tfrac{i}{n}) \right] \beta(k) \right]_{(D)} \\
&= (A) + (B) + (C) + (D).
\end{aligned} \tag{7.12}$$

The four terms in brackets in (7.12) are denoted respectively by (A), (B), (C) and (D) in the following. Let us remark that $\rho_H(x)$ is equivalent to

$$-1/(4 - 2^{2H})\, |x|^{(2H-4)}\, H(2H-1)(2H-2)(2H-3)$$

for large values of $|x|$. So, $\rho_H(x)$ can be bounded in the following way. For $|x|$ large enough, we have

$$|\rho_H(x)| \leq \boldsymbol{C}\,|x|^{2H-4}\,, \tag{7.13}$$

thus, since $\|g\|_{2,\phi}^2 < +\infty$,

$$\sum_{k=-\infty}^{+\infty} |\beta(k)| < +\infty. \tag{7.14}$$

Later, we will use the fact that $\int_{-\infty}^{+\infty} |\beta(x)|\,\mathrm{d}x$ is finite.

Since h is continuous on $[\,0;\ 1\,]$, we can apply (7.6) to function $\ell = h^2$ with $a = 1$ and term (A) gives the required limit (7.10).

To conclude these computations, we show that terms (B), (C) and (D) tend to 0 as n tends to infinity.

Let us look first at term (B). Since h is bounded on $[\,0;\ 1\,]$, for all $M \in \mathbb{N}^*$ and $n \geq M+2$, we have

$$\begin{aligned}|(B)| &\leq \frac{\boldsymbol{C}}{n}\left(\sum_{i=0}^{M-1}\sum_{k=i+1}^{+\infty}|\beta(k)| + \sum_{i=M}^{n-2}\sum_{k=i+1}^{+\infty}|\beta(k)|\right)\\ &\leq \boldsymbol{C}\left(\frac{M}{n}\sum_{k=1}^{+\infty}|\beta(k)| + \sum_{k=M+1}^{+\infty}|\beta(k)|\right).\end{aligned}$$

Using inequality (7.14), we get, for all $M \in \mathbb{N}^*$,

$$\limsup_{n\to+\infty} |(B)| \leq \boldsymbol{C}\sum_{k=M+1}^{+\infty}|\beta(k)|\,,$$

and then, by inequality (7.14), $\lim_{n\to+\infty}(B) = 0$.

A similar proof can be done for term (C). Now, consider the term (D).

On the one hand, by inequality (7.14), for all $\varepsilon > 0$, there exists $M_\varepsilon \in \mathbb{N}$ such that $\sum\limits_{|k|\geq M_\varepsilon} |\beta(k)| \leq \varepsilon$. On the other hand, since h is uniformly continuous on $[\,0;\ 1\,]$, for all $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$, such that for all reals x and y in $[\,0;\ 1\,]$ with $|x-y| \leq \eta_\varepsilon$, we have $|h(x)-h(y)| \leq \varepsilon$. Using that h is bounded on $[\,0;\ 1\,]$ and inequality (7.14), we finally obtain that for all $\varepsilon > 0$, there exists $n_\varepsilon = \left\lfloor \frac{M_\varepsilon}{\eta_\varepsilon} \right\rfloor + 1$, such that for all $n \geq n_\varepsilon$, $|(D)| \leq \boldsymbol{C}\varepsilon$.

Thus $\lim_{n\to+\infty}(D) = 0$, that yields convergence (7.10).

Now we would like to prove that

$$S_{g,h}^{(n)}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(0; \sigma_g^2 \int_0^1 h^2(u)\,\mathrm{d}u\right).$$

For a fixed integer $L \geq 1$, we consider $S_{g_L,h}^{(n)}(1)$ where $g_L(x) = \sum\limits_{\ell=1}^{L} g_\ell H_\ell(x)$.

We shall show that

$$S^{(n)}_{g_L,h}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(0; \sigma^2_{g_L} \int_0^1 h^2(u)\,\mathrm{d}u\right).$$

In this aim, using the chaos representation of Y (see (2.2), (2.5) or (2.6)), for $j = 0, 1, \ldots, n-2$, we can write

$$\frac{\Delta_n X_0(j)}{\sigma_n(j)} = \int_{-\infty}^{+\infty} f^{(n)}(\lambda, j)\,\mathrm{d}W(\lambda),$$

where we define the function $f^{(n)}$ by

$$\sigma_n(j) f^{(n)}(\lambda, j) \\ = a(n) i\lambda\sqrt{f(\lambda)}\left(\left[\int_{\frac{j+1}{n}}^{\frac{j+2}{n}} - \int_{\frac{j}{n}}^{\frac{j+1}{n}}\right]\sigma(u)\exp(i\lambda u)\,\mathrm{d}u\right). \quad (7.15)$$

Now, since $\int_{\mathbb{R}} \left|f^{(n)}(\lambda, j)\right|^2 \mathrm{d}\lambda = 1$, using Itô's formula, see (Major, 1981, p. 30), for fixed $\ell \geq 1$,

$$H_\ell\left(\tfrac{\Delta_n X_0(j)}{\sigma_n(j)}\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^{(n)}(\lambda_1, j) \cdots f^{(n)}(\lambda_\ell, j)\,\mathrm{d}W(\lambda_1) \cdots \mathrm{d}W(\lambda_\ell).$$

To get the asymptotic behavior of $S_{g_L,h}(1)$, we use notations introduced in Slud (1994). For each $\ell \geq 1$, let λ^ℓ denote the Lebesgue measure on $\mathbb{R}^\ell$, $\mathcal{B}(\mathbb{R}^\ell)$, the Borel σ-algebra and consider the complex Hilbert space

$$L^2_{\text{sym}}(\mathbb{R}^\ell, \lambda^\ell) = \{f_\ell \in L^2(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell), \lambda^\ell), f_\ell(x) = \overline{f_\ell(-x)}, \\ f_\ell(x_1, \ldots, x_\ell) = f_\ell(x_{\pi(1)}, \ldots, x_{\pi(\ell)}), \forall \pi \in S_\ell\},$$

where S_ℓ denotes the symmetric group of permutations of $\{1, \ldots, \ell\}$.

For $f_\ell \in L^2_{\text{sym}}(\mathbb{R}^\ell, \lambda^\ell)$, we define

$$I_\ell(f_\ell) = \frac{1}{\ell!}\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_\ell(\lambda_1, \ldots, \lambda_\ell)\,\mathrm{d}W(\lambda_1) \cdots \mathrm{d}W(\lambda_\ell), \quad (7.16)$$

and for $p = 1, \ldots, \ell$, we write $f_\ell \otimes_p f_\ell$ for the p-th contraction of f_ℓ defined as

$$f_\ell \otimes_p f_\ell(\lambda_1, \ldots, \lambda_{2\ell-2p}) = \int_{\mathbb{R}^p} f_\ell(\lambda_1, \ldots, \lambda_{\ell-p}, x_1, \ldots, x_p) \\ f_\ell(\lambda_{\ell-p+1}, \ldots, \lambda_{2\ell-2p}, -x_1, \ldots, -x_p)\,\mathrm{d}\lambda^p(x_1, \ldots, x_p). \quad (7.17)$$

With these notations, we get

$$S^{(n)}_{g_L,h}(1) = \sum_{\ell=1}^{L} I_\ell(h^{(n)}_\ell), \quad (7.18)$$

where $h_\ell^{(n)}$ is

$$h_\ell^{(n)}(\lambda_1,\dots,\lambda_\ell) = g_\ell \ell! \frac{1}{\sqrt{n}} \sum_{j=0}^{n-2} h(\frac{j}{n}) f^{(n)}(\lambda_1, j) \cdots f^{(n)}(\lambda_\ell, j),$$

and $f^{(n)}$ defined by (7.15).

To obtain convergence of $S_{g_L,h}^{(n)}(1)$, we use Theorem 1 of Peccati and Tudor (2005). Convergence in (7.10) gives the required conditions appearing at the beginning of this latter theorem. So we just verify condition (i) in proving the following lemma.

Lemma 7.4. *For fixed ℓ and p, such that $\ell \geq 2$ and $p = 1, \dots, \ell - 1$,*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{2(\ell-p)}} \left| h_\ell^{(n)} \otimes_p h_\ell^{(n)}(\lambda_1,\dots,\lambda_{\ell-p},\mu_1,\dots,\mu_{\ell-p}) \right|^2 \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{\ell-p}\, \mathrm{d}\mu_1 \cdots ,\, \mathrm{d}\mu_{\ell-p} = 0.$$

Proof of Lemma 7.4. Recalling that we defined the correlation $\delta_n(i,j)$ by (7.9), we have

$$\begin{aligned}\int_{\mathbb{R}^{2(\ell-p)}} &\left| h_\ell^{(n)} \otimes_p h_\ell^{(n)}(\lambda_1,\dots,\lambda_{\ell-p},\mu_1,\dots,\mu_{\ell-p}) \right|^2 \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{\ell-p}\, \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_{\ell-p} \\ &= (\ell!)^4 g_\ell^4 \frac{1}{n^2} \sum_{j_1=0}^{n-2} \sum_{j_2=0}^{n-2} \sum_{j_3=0}^{n-2} \sum_{j_4=0}^{n-2} h(\tfrac{j_1}{n}) h(\tfrac{j_2}{n})\, h(\tfrac{j_3}{n})\, h(\tfrac{j_4}{n}) \times \\ &\qquad \delta_n^p(j_1,j_2) \delta_n^p(j_3,j_4)\, \delta_n^{\ell-p}(j_1,j_3) \delta_n^{\ell-p}(j_2,j_4).\end{aligned}$$

Now, since $\delta_n(i,j)$ is a correlation, the function h is bounded on $[0;\ 1]$, $p \geq 1$ and $\ell - p \geq 1$, we just have to prove that $\lim_{n\to+\infty} A_n = 0$, where we define

$$A_n = \frac{1}{n^2} \sum_{j_1=0}^{n-2} \sum_{j_2=0}^{n-2} \sum_{j_3=0}^{n-2} \sum_{j_4=0}^{n-2} |\delta_n(j_1,j_2)|\, |\delta_n(j_3,j_4)|\ |\delta_n(j_1,j_3)|\ |\delta_n(j_2,j_4)|\ .$$

We split the intervals of indices into two parts, B_M and B_M^c, in the following way. For a fixed real $M > 0$, let

$$B_M = \{(j_1,j_2,j_3,j_4) \in \mathbb{N}^4, |j_1 - j_2| > M \quad \text{or} \quad |j_1 - j_3| > M \quad \text{or} \quad |j_2 - j_4| > M\}.$$

Then we can write A_n as the sum of two terms corresponding to B_M and B_M^c respectively.

For the first term, as in the computation of the asymptotic variance of $S_{g,h}^{(n)}(1)$, we can show that $\frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} |\delta_n(i,j)| \leq \boldsymbol{C}$ and that, $|\delta_n(i,j)| \leq 1$.

Furthermore, considering the bound given by (7.13) and Lemma 7.3 page 965, we obtain and use the following bound $|\delta_n(i,j)| \leq \boldsymbol{C} M^{(2H-2)}$, for M large enough and $|i-j| > M$.

For the second term, we bound each of the four functions $|\delta_n(\cdot)|$ by 1, so that for all M large enough we get

$$\overline{\lim_n} A_n \leq \boldsymbol{C} \left(M^{2H-2} + M^3 \overline{\lim_n} (\tfrac{1}{n}) \right) \leq \boldsymbol{C} M^{2H-2},$$

and since $0 < H < 1$ then $\lim_{n \to +\infty} A_n = 0$ and Lemma 7.4 follows. Hence, we proved that

$$S^{(n)}_{g_L,h}(1) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(0; \sigma^2_{g_L}\left(\int_0^1 h^2(u)\,\mathrm{d}u\right)\right).$$

□

Furthermore, $\sum_{p=L+1}^{\infty} g_p^2 p! \xrightarrow[L\to\infty]{} 0$, so we get

$$\lim_{L\to+\infty} \sup_{n\geq 1} \mathrm{E}[S^{(n)}_{g_L,h}(1) - S^{(n)}_{g,h}(1)]^2 = 0.$$

Now, since

$$\mathscr{N}\left(0; \sigma^2_{g_L}\left(\int_0^1 h^2(u)\,\mathrm{d}u\right)\right) \xrightarrow[L\to\infty]{\text{Law}} \mathscr{N}\left(0; \sigma^2_{g}\left(\int_0^1 h^2(u)\,\mathrm{d}u\right)\right),$$

applying Lemma 1.1 of Dynkin (1988), Theorem 4.2 is proved.

Remark 4.4 follows from the following argumentation. First we establish the following inequalities

$$\mathrm{E}\left[\sum_{\ell=1}^{L} I_\ell(h_\ell^{(n)})\right]^4 \leq L^3 \sum_{\ell=1}^{L} \mathrm{E}[I_\ell(h_\ell^{(n)})]^4 \leq \boldsymbol{C}.$$

The last one follows from (V) of Theorem 1 of Peccati and Tudor (2005) and from the fact that for each $\ell \in \{1,\ldots,L\}$, we have $\mathrm{E}[I_\ell(h_\ell^{(n)})]^2 \leq \boldsymbol{C}$, for n large enough. The result is then a consequence of (7.18). □

Proof of Corollary 4.1. We apply Lemma 4.4 and Theorem 4.2. The computation of the coefficients $g_{2p,k}$ of function g_k, given in (4.8), are explicitly made in Berzin and León (2007) and Cœurjolly (2001). Remark 4.6 follows from Remark 4.3 and Theorem 4.2. □

7.2. For the two gfOUp

7.2.1. Almost sure convergence of $\Delta_n X_\lambda$

Proof of Theorem 4.3. As explained in Section 4.2.1, we have just to prove Lemmas 4.5 and 4.6.

Proof of Lemma 4.5. For $i = 0, 1, \ldots, n-2$, the random variable $\delta_n X_\lambda(i)$ is defined as in (4.2), Section 4.1.1. Let us suppose that X_λ is solution of (2.8) (resp. (2.10)). Thus using (2.9) (resp. (2.11)), we have

$$\begin{aligned}
\delta_n X_\lambda(i) = \sigma(\tfrac{i}{n})\delta_n Y^\star(i) \\
&+ \left[-2Y^\star(\tfrac{i+1}{n})\left(\sigma(\tfrac{i+1}{n}) - \sigma(\tfrac{i}{n})\right) + Y^\star(\tfrac{i+2}{n})\left(\sigma(\tfrac{i+2}{n}) - \sigma(\tfrac{i}{n})\right)\right]_{(1)} \\
&+ \left[\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\right)\sigma'(u)Y^\star(u)\,\mathrm{d}u\right]_{(2)} \\
&+ \left[\left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} - \int_{\frac{i}{n}}^{\frac{i+1}{n}}\right)\left(\widetilde{\mu}(X_\lambda(u)) - \widetilde{\mu}(X_\lambda(\tfrac{i+1}{n}))\right)\,\mathrm{d}u\right]_{(3)} \qquad (7.19) \\
&= \sigma(\tfrac{i}{n})\delta_n Y^\star(i) + (1) + (2) + (3),
\end{aligned}$$

where $Y^\star$ is Y (resp. b_H), the process $\delta_n Y^\star$ being defined by (4.2), Section 4.1.1. Function $\widetilde{\mu}(x)$ being $-\lambda x$ (resp. $\mu(x)$). The three terms in brackets in (7.19) are denoted respectively by (1), (2) and (3) in the following. First, we consider term (2).

Since function σ belongs to C^2 on $[0;\ 1]$, we write $\sigma'(u) = \sigma'(\frac{i+1}{n}) + (u - \frac{i+1}{n})\sigma''(\frac{i+1}{n} + \theta_u(u - \frac{i+1}{n}))$, with $0 < \theta_u < 1$. We get

$$\begin{aligned}
(2) = \sigma'(\tfrac{i+1}{n})&\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\right)\left(Y^\star(u) - Y^\star(\tfrac{i+1}{n})\right)\,\mathrm{d}u \\
&+ \left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\right)(u - \tfrac{i+1}{n})Y^\star(u)\,\sigma''(\tfrac{i+1}{n} + \theta_u(u - \tfrac{i+1}{n}))\,\mathrm{d}u.
\end{aligned}$$

Using the modulus of continuity of $Y^\star$ (see Lemma 7.1, p. 962 applied to $X_0 = Y^\star$), and the fact that σ' and σ'' are bounded on $[0;\ 1]$, we obtain that

$$|(2)| \leq \boldsymbol{C}(\omega)\left[\left(\tfrac{1}{n}\right)^{H+1-\delta} + \left(\tfrac{1}{n}\right)^2\right] \leq \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{H+1-\delta}.$$

Then we bound term (1).

For $i \in \{0, \ldots, n-2\}$, since σ belongs to C^2 on $[0;\ 1]$, we write $\sigma(\frac{i+1}{n}) - \sigma(\frac{i}{n}) = \frac{1}{n}\sigma'(\frac{i}{n}) + \frac{1}{n^2}\sigma''(\frac{i}{n} + \frac{\theta_1}{n})$ and $\sigma(\frac{i+2}{n}) - \sigma(\frac{i}{n}) = \frac{2}{n}\sigma'(\frac{i}{n}) + \frac{4}{n^2}\sigma''(\frac{i}{n} + \frac{2\theta_2}{n})$, with $0 < \theta_1, \theta_2 < 1$. Thus

$$\begin{aligned}
(1) = \frac{2}{n}\,\sigma'(\tfrac{i}{n})\left(Y^\star(\tfrac{i+2}{n}) - Y^\star(\tfrac{i+1}{n})\right) + \tfrac{4}{n^2}Y^\star(\tfrac{i+2}{n})\,\sigma''(\tfrac{i}{n} + \tfrac{2\theta_2}{n}) \\
- \tfrac{2}{n^2}Y^\star(\tfrac{i+1}{n})\,\sigma''(\tfrac{i}{n} + \tfrac{\theta_1}{n}).
\end{aligned}$$

Using once again Lemma 7.1 page 962 applied to $X_0 = Y^\star$ and the fact that σ' and σ'' are bounded on $[0;\ 1]$, we obtain that for $i \in \{0, \ldots, n-2\}$,

$$|(1)| \leq \boldsymbol{C}(\omega) \left[\left(\tfrac{1}{n}\right)^{H+1-\delta} + \ \left(\tfrac{1}{n}\right)^{2} \right] \leq \boldsymbol{C}(\omega) \left(\tfrac{1}{n}\right)^{H+1-\delta}.$$

For term (3), first, let us remark that the trajectories of X_λ are $(H-\delta)$-Hölder continuous on $[0;\ 1]$. For X_λ solution of (2.8) one can prove this fact using that the function σ belongs to C^1 on $[0;\ 1]$ and that Y almost-surely has $(H-\delta)$-Hölder continuous trajectories on $[0;\ 1]$ (see Lemma 7.1 applied to $X_0 = Y$). If X_λ is solution of (2.10), we refer to Nualart and Răşcanu (2002).

Since $\widetilde{\mu}$ is Lipschitz on $\mathbb{R}$, we get

$$|(3)| \leq \boldsymbol{C}(\omega) \left(\tfrac{1}{n}\right)^{H+1-\delta}.$$

Thus using (A.1), we finally proved that, for any $\delta > 0$,

$$\left|\Delta_n X_\lambda(i) - \sigma(\tfrac{i}{n}) \Delta_n Y^\star(i)\right| \leq \boldsymbol{C}(\omega) \left(\tfrac{1}{n}\right)^{1-\delta}.$$

□

Proof of Lemma 4.6. As in the proof of Lemma 4.3, for $k \geq 1$ we consider inequality (A.2) applied to $x = \Delta_n X_\lambda(i)$ and $y = \sigma(\frac{i}{n}) \Delta_n Y^\star(i)$.

Furthermore, we use that for $i \in \{1, \ldots, n-2\}$, $|\Delta_n Y^\star(i)| \leq \boldsymbol{C}(\omega) n^\delta$ (see inequality (7.4) applied to $X_0 = Y^\star$ with $k = 1$). Finally using that functions h and σ are bounded on $[0;\ 1]$ and Lemma 4.5, since $k \geq 1$, we get

$$|C_n(h)| \leq \boldsymbol{C}_k(\omega) \left(\frac{1}{n}\right)^{1-\delta k}.$$

Choosing δ small enough, that is $\delta < \frac{1}{2k}$, we obtain Lemma 4.6. □

□

7.2.2. Rate of convergence for the increments $\Delta_n X_\lambda$

Proof of Theorem 4.4. As explained in Section 4.2.2, Lemma 4.6 and Remark 4.6 applied in model (2.1) to $X_0 = Y^\star$, replacing function h by $h\sigma^k$ lead to the theorem. □

7.3. For the fDrc

7.3.1. Almost sure convergence of $\Delta_n X_r$

Proof of Theorem 4.5. As mentioned in Section 4.3.1, we just have to prove Lemmas 4.7 and 4.8.

Proof of Lemma 4.7. For $i = 0, 1, \ldots, n-2$, the random variable $\delta_n X_r(i)$ is defined as in (4.2), Section 4.1.1. Thus using (2.13), we have

$$\begin{aligned}
\delta_n X_r(i) = \sigma(Y(\tfrac{i}{n}))\delta_n Y(i) \\
&+ \left[\left(\int_{Y(\frac{i+1}{n})}^{Y(\frac{i+2}{n})} - \int_{Y(\frac{i}{n})}^{Y(\frac{i+1}{n})} \right) \left(\sigma(u) - \sigma(Y(\tfrac{i}{n})) \right) \mathrm{d}u \right]_{(1)} \\
&+ \left[\left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \right) \left(\mu(Y(u)) - \mu(Y(\tfrac{i+1}{n})) \right) \mathrm{d}u \right]_{(2)} \qquad (7.20) \\
&= \sigma(Y(\tfrac{i}{n}))\delta_n Y(i) + (1) + (2),
\end{aligned}$$

where the process $\delta_n Y$ is defined by (4.2), Section 4.1.1. The two terms in brackets in (7.20) are denoted respectively by (1) and (2) in the following. First, we consider term (2). Using that μ is locally Lipschitz on $\mathbb{R}$ and the modulus of continuity of Y (see Lemma 7.1, p. 962 applied to $X_0 = Y$), we get

$$|(2)| \leq \boldsymbol{C}(\omega) \left(\tfrac{1}{n}\right)^{1+H-\delta} \leq \boldsymbol{C}(\omega) \left(\tfrac{1}{n}\right)^{2H-2\delta}.$$

For term (1), using a first order Taylor expansion of the function σ, which belongs to C^1 on $\mathbb{R}$ and using same arguments as before, we get that

$$|(1)| \leq \boldsymbol{C}(\omega) \left(\tfrac{1}{n}\right)^{2H-2\delta}.$$

Using (A.1) finally yields this lemma. □

Proof of Lemma 4.8. As in Lemma 4.3, for $k \geq 1$ we consider inequality (A.2) applied to $x = \Delta_n X_r(i)$ and $y = \sigma(Y(\frac{i}{n}))\Delta_n Y(i)$.

Furthermore we use that for $i \in \{1, \ldots, n-2\}$, $|\Delta_n Y(i)| \leq \boldsymbol{C}(\omega) n^{\delta}$ (see inequality (7.4) applied to function $X_0 = Y$ with $k = 1$). Finally, we use that the functions h and σ are continuous on $\mathbb{R}$ and that Y has continuous trajectories on $[0; 1]$. Applying Lemma 4.7, and since $k \geq 1$ we get

$$|D_n(h)| \leq \boldsymbol{C}_k(\omega) \left(\frac{1}{n}\right)^{H-\delta k}.$$

Choosing δ small enough, that is $0 < \delta < \frac{H-\frac{1}{2}}{k}$, that remains possible since $H > \frac{1}{2}$, we obtain Lemma 4.8. □

The proof of Theorem 4.5 is now completed. □

7.3.2. Rate of convergence for the increments $\Delta_n X_r$

Proof of Theorem 4.6. We decompose $E_n(h)$ as

$$E_n(h) = \frac{n}{n-1} \left(\frac{1}{\sqrt{n}} \sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\sigma^k(Y(\tfrac{i}{n}))g_k\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right)\right)$$

$$
\begin{aligned}
&+ \left[\frac{\sqrt{n}}{\mathrm{E}[|N|^k]} D_n(h) \right]_{(A)} \\
&+ \left[\sqrt{n} \left(\frac{1}{n-1} \sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\sigma^k(Y(\tfrac{i}{n})) \right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. - \int_0^1 h(Y(u))\sigma^k(Y(u))\,\mathrm{d}u \right) \right]_{(B)} \\
&+ \left[\frac{\sqrt{n}}{(n-1)} \sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\sigma^k(Y(\tfrac{i}{n})) \frac{1}{\mathrm{E}[|N|^k]} \left| \frac{\Delta_n Y(i)}{\sigma_{n,1}} \right|^k \left(\sigma_{n,1}^k - 1 \right) \right]_{(C)} \\
&= \frac{n}{n-1} \left(\frac{1}{\sqrt{n}} \sum_{i=0}^{n-2} h(Y(\tfrac{i}{n}))\sigma^k(Y(\tfrac{i}{n})) g_k\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}} \right) \right) + (A) + (B) + (C),
\end{aligned}
\tag{7.21}
$$

where function g_k is defined by (4.7) and $D_n(h)$ is defined by Lemma 4.8 page 939. Obviously, (A), (B) and (C) are the last three terms of (7.21).

Using Lemma 4.8, one gets that $(A) = o_{a.s.}(1)$.

Since h and σ belong to C^1 on $\mathbb{R}$, then for all real $k \geq 1$, $h\sigma^k$ also belongs to C^1 on $\mathbb{R}$. Using that the trajectories of Y are $(H-\delta)$-Hölder continuous on $[0;\ 1]$ (see Lemma 7.1, p. 962, applied to function $X_0 = Y$) and a first order Taylor expansion of $h\sigma^k$, one gets

$$
(B) = O_{a.s.}((\tfrac{1}{n})^{H-\frac{1}{2}-\delta}) = o_{a.s.}(1),
$$

since $H > \frac{1}{2}$.

Now, using inequality (7.4) applied to $X_0 = Y$, (7.7) for $\sigma_{n,1}^k$, we get since h and σ are continuous on $\mathbb{R}$, that for all real $k > 0$,

$$
(C) = O_{a.s.}((\tfrac{1}{n})^{\frac{1}{2}-\delta k}) = o_{a.s.}(1),
$$

by choosing δ small enough, that is $\delta < \frac{1}{2k}$.

So Theorem 4.6 ensues from Theorem 4.7, since the convergence in probability to zero ensures the stable convergence to zero see (Podolskij and Vetter, 2010, Proposition 1). □

Proof of Theorem 4.7. We prove Remark 4.7, page 940, in the case where h belongs to C^4 on $\mathbb{R}$, $|h^{(4)}(x)| \leq P(|x|)$, $H > \frac{1}{4}$ and for any function $g \in L^4(\phi(x)\,\mathrm{d}x)$ with a Hermite rank ≥ 1 and where we suppose that A_g is not an empty set (see Section 3 for definition). Furthermore, we will suppose that g is even, or odd, with a Hermite rank greater than or equal to three. A proof similar to the last one could easily be given to obtain the other cases described in the Remark 4.7. It is sufficient to adapt forthcoming Lemma 7.5 to the new hypotheses that is proved in the Appendix A.1.

In the same manner as in Berzin, Latour and León (2014) the proof will proceed in several steps. Let us define for $0 < t < 1$,

$$S_g^{(n)}(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{\lfloor nt \rfloor - 2} g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right),$$

and

$$T_n(h) = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-2} h(Y(\tfrac{i}{n})) g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right).$$

On the one hand, we prove in forthcoming Lemma 7.6 that $(Y, S_g^{(n)})$ converges to $(Y, \sigma_g W_{\perp Y})$. We will show this lemma after the proof of this theorem.

On the other hand, we will consider a discrete version of $T_n(h)$, defining

$$T_n^{(m)}(h) = \sum_{\ell=0}^{m-1} h(Y(\tfrac{\ell}{m})) \frac{1}{\sqrt{n}} \sum_{i=\lfloor \frac{n\ell}{m} \rfloor - 1}^{\lfloor \frac{n(\ell+1)}{m} \rfloor - 2} g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right).$$

The convergence of $(Y, S_g^{(n)})$ implies that as n goes to infinity

$$T_n^{(m)}(h) \xrightarrow[n\to\infty]{\text{Stable}} T^{(m)}(h) = \sigma_g \sum_{\ell=0}^{m-1} h(Y(\tfrac{\ell}{m})) \left(W_{\perp Y}(\tfrac{\ell+1}{m}) - W_{\perp Y}(\tfrac{\ell}{m})\right),$$

see (Podolskij and Vetter, 2010, Proposition 1).

Furthermore, it is easy to show that $T^{(m)}(h)$ is a Cauchy sequence in $L^2(\Omega)$. It follows, using the asymptotic independence between Y and $W_{\perp Y}$ that

$$T^{(m)}(h) \xrightarrow[m\to\infty]{L^2(\Omega)} \sigma_g \int_0^1 h(Y(u)) \, \mathrm{d}W_{\perp Y}(u).$$

To conclude in proving the convergence of $T_n(h)$, it is sufficient to demonstrate Lemma 7.5. A proof is given in the Appendix A.1.

Lemma 7.5. *Let h belong to C^4 on $\mathbb{R}$, $|h^{(4)}(x)| \le P(|x|)$, $H > \frac{1}{4}$ and function $g \in L^{2+\delta}(\phi(x)\,\mathrm{d}x)$, $\delta > 0$, let $g(x) = \sum_{p=1}^{+\infty} g_p H_p(x)$. Furthermore, we suppose that g is even, or odd, with a Hermite rank greater than or equal to three, then*

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathrm{E}[T_n(h) - T_n^{(m)}(h)]^2 = 0.$$

□

We shall prove the following lemma. As in Berzin, Latour and León (2014) we only make the hypothesis that $g \in L^4(\phi(x)\,\mathrm{d}x)$, with a Hermite rank ≥ 1 and we suppose that A_g is not the empty set (see Section 3 for definition).

Lemma 7.6. *For $0 < H < 1$,*

1)

$$S_g^{(n)} \xrightarrow[n\to\infty]{\text{Law}} \sigma_g W_{\perp Y}.$$

2) Furthermore,

$$(Y, S_g^{(n)}) \xrightarrow[n\to\infty]{\text{Law}} (Y, \sigma_g W_{\perp Y}),$$

where $W_{\perp Y}$ is a standard Brownian motion independent of Y.

Remark 7.1. In fact we will show that in the two assertions the convergence is stable and takes place in the sense of processes convergence.

Proof of Lemma 7.6.

1) For $m \in \mathbb{N}^*$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m \leq 1$, let $\boldsymbol{t} = (t_1, \ldots, t_m)$ and

$$S_g(n\boldsymbol{t}) = \sum_{i=1}^{m} \alpha_i \left(S_g^{(n)}(t_i) - S_g^{(n)}(t_{i-1}) \right),$$

where

$$\alpha_i = \frac{d_i}{\sqrt{\sum\limits_{i=1}^{m} d_i^2 (t_i - t_{i-1})}},$$

while $d_1, \ldots, d_m \in \mathbb{R}$. We want to prove that

$$S_g(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(0; \sigma_g^2).$$

We consider $S_{g_L}(n\boldsymbol{t})$ where $g_L(x) = \sum_{\ell=1}^{L} g_\ell H_\ell(x)$, where $L \geq 1$ is a fixed integer. We will prove that

$$S_{g_M}(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(0; \sigma_{g_L}^2).$$

As in the proof of Theorem 4.2, the chaos representation of the process Y (see (2.2), (2.5) or (2.6)) allows us to write $S_{g_L}(n\boldsymbol{t})$ in the multiple Wiener chaos:

$$S_{g_L}(n\boldsymbol{t}) = \sum_{\ell=1}^{L} I_\ell(h_\ell^{(n,\boldsymbol{t})}),$$

where $h_\ell^{(n,\boldsymbol{t})}$ is

$$h_\ell^{(n,\boldsymbol{t})}(\lambda_1, \ldots, \lambda_\ell) = g_\ell \, \ell! \sum_{i=1}^{m} \alpha_i \frac{1}{\sqrt{n}} \sum_{j=\lfloor nt_{i-1} \rfloor - 1}^{\lfloor nt_i \rfloor - 2} f^{(n)}(\lambda_1, j) \cdots f^{(n)}(\lambda_\ell, j),$$

and where I_ℓ is given by (7.16) and $f^{(n)}$ by (7.15) where we replace the functions σ by 1 and $\sigma_n(j)$ by $\sigma_{n,1}$. First, let us compute the variance of $S_{g_L}(n\boldsymbol{t})$.

$$\mathrm{E}[S_{g_L}(n\boldsymbol{t})]^2 = \sum_{\ell=1}^{L} \frac{1}{\ell!} \int_{\mathbb{R}^\ell} \left| h_\ell^{(n,\boldsymbol{t})}(\lambda_1, \dots, \lambda_\ell) \right|^2 \, \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell$$

$$= \sum_{\ell=1}^{L} \ell! \, g_\ell^2 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \alpha_{i_1} \, \alpha_{i_2} \left(\frac{1}{n} \sum_{j_1=\lfloor nt_{i_1-1} \rfloor -1}^{\lfloor nt_{i_1} \rfloor -2} \sum_{j_2=\lfloor nt_{i_2-1} \rfloor -1}^{\lfloor nt_{i_2} \rfloor -2} \delta_{n,1}^{\ell}(j_1 - j_2) \right),$$

where $\delta_{n,1}(i-j)$ stands for $\delta_n(i,j)$ defined in (7.9) for $X_0 = Y$, that is

$$\begin{aligned} \delta_{n,1}(i-j) &= \mathrm{E}\left[\frac{\Delta_n Y(i)}{\sigma_{n,1}} \frac{\Delta_n Y(j)}{\sigma_{n,1}} \right] \\ &= \rho_H(i-j) + \gamma_{n,1}(i-j), \end{aligned} \tag{7.22}$$

ρ_H and $\sigma_{n,1}$ are respectively defined by (3.1) and (4.9). As in the proof of Theorem 4.2, using Lemma 7.3 for $\gamma_{n,1}(i-j)$, one can prove that

$$\mathrm{E}[S_{g_L}(n\boldsymbol{t})]^2$$

$$\equiv \sum_{\ell=1}^{L} \ell! \, g_\ell^2 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \alpha_{i_1} \, \alpha_{i_2} \left(\frac{1}{n} \sum_{j_1=\lfloor nt_{i_1-1} \rfloor -1}^{\lfloor nt_{i_1} \rfloor -2} \sum_{j_2=\lfloor nt_{i_2-1} \rfloor -1}^{\lfloor nt_{i_2} \rfloor -2} \rho_H^{\ell}(j_1 - j_2) \right),$$

and as in the proof given by Berzin, Latour and León (2014) for the fBm, we use that for $\ell \geq 1$,

$$\begin{aligned} &\frac{1}{n} \sum_{s_1=\lfloor nt_{i-1} \rfloor -1}^{\lfloor nt_i \rfloor -2} \sum_{s_2=\lfloor nt_{j-1} \rfloor -1}^{\lfloor nt_j \rfloor -2} \rho_H^{\ell}(s_1 - s_2) \\ &\quad \underset{n \to +\infty}{\longrightarrow} \begin{cases} (t_i - t_{i-1}) \sum_{r=-\infty}^{+\infty} \rho_H^{\ell}(r), & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{7.23}$$

One finally gets that

$$\lim_{n \to +\infty} \mathrm{E}[S_{g_L}(n\boldsymbol{t})]^2 = \left(\sum_{i=1}^{m} \alpha_i^2 (t_i - t_{i-1}) \right) \left(\sum_{\ell=1}^{L} \ell! \, g_\ell^2 \sum_{r=-\infty}^{+\infty} \rho_H^{\ell}(r) \right) = \sigma_{g_L}^2.$$

To conclude the proof of 1), Theorem 1 of Peccati and Tudor (2005) is used again and as in the proof of Theorem 4.2, it is enough to prove that for fixed ℓ and p, $\ell \geq 2$ and $p = 1, \dots, (\ell - 1)$, $\lim_{n \to +\infty} B_n = 0$, where B_n is

$$B_n = \int_{\mathbb{R}^{2(\ell-p)}} \left| h_\ell^{(n,\boldsymbol{t})} \otimes_p h_\ell^{(n,\boldsymbol{t})}(\lambda_1, \dots, \lambda_{\ell-p}, \mu_1, \dots, \mu_{\ell-p}) \right|^2 \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{\ell-p} \, \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_{\ell-p},$$

remembering that we defined the p-th contractions $\otimes_p$ in (7.17).

Now we compute B_n and we get

$$B_n = (\ell!)^4 g_\ell^4 \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \sum_{i_3=1}^{m} \sum_{i_4=1}^{m} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\alpha_{i_4} \times$$
$$\frac{1}{n^2} \sum_{j_1=\lfloor nt_{i_1-1}\rfloor-1}^{\lfloor nt_{i_1}\rfloor-2} \sum_{j_2=\lfloor nt_{i_2-1}\rfloor-1}^{\lfloor nt_{i_2}\rfloor-2} \sum_{j_3=\lfloor nt_{i_3-1}\rfloor-1}^{\lfloor nt_{i_3}\rfloor-2} \sum_{j_4=\lfloor nt_{i_4-1}\rfloor-1}^{\lfloor nt_{i_4}\rfloor-2}$$
$$\delta_{n,1}^{\ell-p}(j_1-j_2)\,\delta_{n,1}^{\ell-p}(j_3-j_4)\,\delta_{n,1}^{p}(j_1-j_3)\,\delta_{n,1}^{p}(j_2-j_4).$$

Using the same arguments as in the proof of Theorem 4.2, it is easy to see that for M large enough, one has

$$\overline{\lim_{n\to\infty}} B_n \leq \boldsymbol{C}\, M^{2H-2},$$

and then $\lim\limits_{n\to+\infty} B_n = 0$.
Hence, we proved that

$$S_{g_L}(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(0;\sigma_{g_L}^2),$$

where $\boldsymbol{t} = (t_1,\ldots,t_m)$ and $S_{g_L}(n\boldsymbol{t}) = \sum\limits_{i=1}^{m} \alpha_i (S_{g_L}^{(n)}(t_i) - S_{g_L}^{(n)}(t_{i-1}))$.

Furthermore, using that $\lim\limits_{L\to+\infty} \sum\limits_{p=L+1}^{+\infty} g_p^2 p! = 0$, we can prove that

$$\lim_{L\to+\infty} \sup_{n\geq 1} \mathrm{E}[S_g(n\mathbf{t}) - S_{g_L}(n\boldsymbol{t})]^2 = 0,$$

and since

$$\mathcal{N}(0;\sigma_{g_L}^2) \xrightarrow[L\to\infty]{\text{Law}} \mathcal{N}(0;\sigma_g^2),$$

by applying Lemma 1.1 of Dynkin (1988), we proved that

$$S_g(n\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(0;\sigma_g^2).$$

We then obtain assertion 1) about the finite dimensional convergence of $S_g^{(n)}$. Now we will prove the tightness of this process. We need the following lemma proved in Berzin, Latour and León (2014).

Lemma 7.7. *Let G a function in $L^4(\phi(x)\,\mathrm{d}x)$ with a Hermite rank $m\geq 1$ and let $\{X_i\}_{i=1}^{\infty}$ a stationary Gaussian sequence with mean 0, variance 1 and covariance function r. We suppose that, there exists ε, $0<\varepsilon<\frac{1}{3}$ and $j\in\mathbb{N}$, such that for all $i\geq j$, one has $|r(i)|\leq\varepsilon<\frac{1}{3}$. Then for $I\geq 1$,*

$$\mathrm{E}\left[\frac{1}{\sqrt{I}}\sum_{i=1}^{I} G(X_i)\right]^4 \leq \boldsymbol{C}\sum_{i=0}^{I+j} |r^m(i)|\,.$$

Remark 7.2. The proof of this lemma uses (Taqqu, 1977, Proposition 4.2 (i)).

Now for any $t > s$, we have

$$\begin{aligned}
\mathrm{E}\left[S_g^{(n)}(t) - S_g^{(n}(s)\right]^4 &= \frac{1}{n^2}\,\mathrm{E}\left[\sum_{i=\lfloor ns\rfloor-1}^{\lfloor nt\rfloor-2} g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right)\right]^4 \\
&= \frac{1}{n^2}\,\mathrm{E}\left[\sum_{i=1}^{\lfloor nt\rfloor-\lfloor ns\rfloor} g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right)\right]^4,
\end{aligned}$$

since the process Y has stationary increments.
We apply Lemma 7.7 to $g \in L^4(\phi(x)\,\mathrm{d}x)$, $m = 1$, $I = \lfloor nt\rfloor - \lfloor ns\rfloor$ and to the process $\{\frac{\Delta_n Y(i)}{\sigma_{n,1}}\}_{i=1}^{\infty}$ with covariance $r = \delta_{n,1}$ given by (7.22). By using Lemma 7.3 and inequality (7.13), page 966, we remark that this covariance satisfies the hypotheses of the previous lemma and also that $\sum_{i=0}^{I+j} |\delta_{n,1}(i)| \leq \boldsymbol{C}$.
We get

$$\mathrm{E}\left[S_g^{(n)}(t) - S_g^{(n)}(s)\right]^4 \leq \boldsymbol{C}\left(\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}\right)^2.$$

Now, let fixed $t_1 < t < t_2$.
If $t_2 - t_1 \geq \frac{1}{n}$, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
&\mathrm{E}\left[(S_g^{(n)}(t_2) - S_g^{(n)}(t))^2 (S_g^{(n)}(t) - S_g^{(n)}(t_1))^2\right] \\
&\quad\leq \boldsymbol{C}\left(\frac{\lfloor nt_2\rfloor - \lfloor nt\rfloor}{n}\right)\left(\frac{\lfloor nt\rfloor - \lfloor nt_1\rfloor}{n}\right) \\
&\quad\leq \boldsymbol{C}\left(\frac{\lfloor nt_2\rfloor - \lfloor nt_1\rfloor}{n}\right)^2 \leq \boldsymbol{C}\,(t_2 - t_1)^2.
\end{aligned}$$

Now, if $t_2 - t_1 < \frac{1}{n}$, two cases occur. If t_1 and t_2 are in the same interval, that is $t_1, t_2 \in (\frac{k}{n}, \frac{k+1}{n})$, then t_1 and t are in the same interval, and $S_g^{(n)}(t) - S_g^{(n)}(t_1) = 0$. Otherwise, t_1 and t_2 are in contiguous intervals and in this case, then t_1 and t are in the same interval, or t and t_2 are in the same interval. Then in both cases, we have $(S_g^{(n)}(t) - S_g^{(n)}(t_1))(S_g^{(n)}(t_2) - S_g^{(n)}(t)) = 0$.
The tightness of process $S_g^{(n)}$ follows by (Billingsley, 1968, Theorem 15.6). Thus the convergence in assertion 1) takes place in the sense of processes convergence.

2) We can suppose that $g(x) = \sum_{\ell=2}^{+\infty} g_\ell H_\ell(x)$. Indeed, since $\sum_{r=-\infty}^{+\infty} \rho_H(r) = 0$, it follows that $\frac{1}{\sqrt{n}} \sum_{i=0}^{[n\cdot]-2} \frac{\Delta_n Y(i)}{\sigma_{n,1}}$ tends to zero in L^2 as n tends to infinity (see convergence given in part 1) for $m = 1$ and $g = H_1$).

Let $c_0, \dots, c_m$, be real constants. As before, it is enough to establish the limit distribution of

$$\sum_{j=0}^{m} c_j\, Y(t_j) \;+\; S_{g_L}(n\boldsymbol{t}).$$

As in the proof of part 1), Theorem 1 of Peccati and Tudor (2005) allows us to conclude the convergence of finite dimensional distributions of $(Y(t), S_g^{(n)}(t))$. Indeed it is enough to remark that $\sum_{j=0}^{m} c_j\, Y(t_j)$ belongs to the first Wiener chaos and then is a Gaussian random variable with finite variance and that $S_{g_L}(n\boldsymbol{t})$ belongs to the superior order one. Thus assertion 2) of Lemma 7.6 follows.

Furthermore the tightness of the sequence of processes $(Y, S_g^{(n)})$ follows from that of the sequence of process $S_g^{(n)}$ proved in part 1) and implies convergence of $(Y, S_g^{(n)})$ as processes and then ensure the stable convergence of $(Y, S_g^{(n)})$ and of that of $S_g^{(n)}$, see (Podolskij and Vetter, 2010, Proposition 1).

□

8. Proofs concerning the punctual estimation of $\boldsymbol{\sigma}$

8.1. For the fDdc

8.1.1. Bias and variance

Proof of Theorem 5.1. We have to prove Lemma 5.1 and Corollary 5.1 for any function μ and for all real $k \geq 1$. First we prove these results for $\mu \equiv 0$ and for all real $k > 0$. Then an approximation result is given for Lemma 5.1 and Corollary 5.1.

Proof of Lemma 5.1. By (5.9), we have

$$\begin{aligned}
\frac{1}{h^2}\mathrm{E}[\widehat{\alpha}_{n,0}(t) - \alpha(t)] &= \left[\frac{2}{nh^2}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})\,\left((\sigma_n^{(i)}(t))^k - \alpha(t)\right)\right]_{(1)} \\
&\quad + \left[\frac{1}{h^2}\left(\frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n}) - 1\right)\alpha(t)\right]_{(2)} \qquad (8.1)\\
&= (1) + (2),
\end{aligned}$$

where we recall that $\widehat{\alpha}_{n,0}(t)$ (resp. $\alpha(t)$) is defined by (5.7) for $\mu \equiv 0$ (resp. by (5.6)) and $\sigma_n^{(i)}(t)$ is defined by (5.5). The two terms of (8.1) are denoted respectively by (1) and (2) in the following.

We prove that the first term in the above sum, (1), gives the required limit. Thus let us prove that term (2) tends to zero with n. In this aim, we need the following equality. For all function ℓ belonging to C^1 on $[-1;\ 1]$ (resp. C^2 and such that $\ell(1)=\ell(-1)$), we have

$$\int_{-1}^{1} \ell(u)\,\mathrm{d}u = \frac{2}{n}\sum_{i=0}^{n-1} \ell(-1+\tfrac{2i}{n}) + O(\tfrac{1}{n}) \quad \text{(resp. } O\left(\tfrac{1}{n^2}\right)\text{)}. \tag{8.2}$$

We apply this equality to the function $\ell = K$, and since K is a C^2 density function with a compact support in $[-1;\ 1]$ and because σ is bounded on $[0;\ 1]$, we obtain that $(2) = O(\frac{1}{nh})^2 = o(1)$, since $nh \to +\infty$.

Our aim is now to prove that $(1) \to \alpha''(t)\chi^2$, where χ^2 is defined by (3.4). At this step of the proof we need the following lemma proved in Appendix A.2.

Lemma 8.1. *For $i = 0, 1, \ldots, n-1$,*

$$(\sigma_n^{(i)}(t))^2 - \sigma^2(t) = h\,(1-\tfrac{2i}{n})\theta'(t) + \tfrac{1}{2}\,h^2(1-\tfrac{2i}{n})^2\,\theta''(t) + o(h^2),$$

where $\theta(t) = \sigma^2(t)$.

Now we use the following equality similar to (7.7). Since $\sigma(t) > 0$, for real $k > 0$, we have

$$\begin{aligned}(\sigma_n^{(i)}(t))^k - \sigma^k(t) &= \frac{k}{2}\left((\sigma_n^{(i)}(t))^2 - \sigma^2(t)\right)\sigma^{k-2}(t)\\ &\quad + \frac{k}{8}(k-2)\left((\sigma_n^{(i)}(t))^2 - \sigma^2(t)\right)^2\left(\sigma^2(t) + \theta\left((\sigma_n^{(i)}(t))^2 - \sigma^2(t)\right)\right)^{k/2-2},\end{aligned}$$

with $0 < \theta < 1$.

On the one hand, by applying (8.2) to the function $\ell = K$ and since K belongs to C^1 on $[-1;\ 1]$, we have $\frac{2}{n}\sum_{i=0}^{n-1} K(-1+\frac{2i}{n}) < +\infty$. On the other hand, we use Lemma 8.1 and since the function σ belongs to C^2 on $[0;\ 1]$ and $\sigma(t) > 0$, we get

$$\begin{aligned}(1) &= k\sigma^{k-2}(t)\,\theta'(t)\frac{1}{nh}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})(1-\tfrac{2i}{n})\\ &\quad + \frac{k}{2}\theta''(t)\sigma^{k-2}(t)\frac{1}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})(1-\tfrac{2i}{n})^2\\ &\quad + \frac{k}{4}(k-2)(\theta')^2(t)\frac{1}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})(1-\tfrac{2i}{n})^2\\ &\qquad \times \left(\sigma^2(t) + \theta\left((\sigma_n^{(i)}(t))^2 - \sigma^2(t)\right)\right)^{k/2-2} + o(1).\end{aligned}$$

Now we apply (8.2) to $\ell(u) = uK(u)$. Since K belongs to C^1 on $[-1;\ 1]$ and satisfies $\int_{-1}^{+1} uK(u)\,\mathrm{d}u = 0$, we have $\frac{1}{nh}\sum_{i=0}^{n-1} K(-1+\frac{2i}{n})(1-\frac{2i}{n}) = O(\frac{1}{nh}) = o(1)$, since $nh \to +\infty$. Thus, once again using (8.2) applied to the function

$\ell(u) = u^2 K(u)$ belonging to C^1 on $[1;\ 1]$, Lemma 8.1 and the fact that $\sigma(t) > 0$, we finally get

$$(1) \underset{n\to+\infty}{\longrightarrow} \chi^2 \left(\frac{k}{2}\,\theta''(t)\sigma^{k-2}(t) + \frac{k}{4}(k-2)(\theta')^2(t)\sigma^{k-4}(t) \right) = \chi^2 \left(\sigma^k(t)\right)''.$$

That yields Lemma 5.1.

We complete the proof of Lemma 5.1 by considering a general function μ and a real number k, $k \geq 1$.

In this aim, we write,

$$\begin{aligned}&\mathrm{E}[\widehat{\alpha}_{n,\mu}(t) - \alpha(t)] = \mathrm{E}[\widehat{\alpha}_{n,0}(t) - \alpha(t)] \\ &+ \left[\frac{(\sigma_n^{(i)}(t))^k}{\mathrm{E}[|N|^k]} \frac{2}{n} \sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n}) \left(\mathrm{E}\left[\left| \frac{\Delta_n^{(i)} X_\mu(t)}{\sigma_n^{(i)}(t)} \right|^k \right] - \mathrm{E}\left[\left| \frac{\Delta_n^{(i)} X_0(t)}{\sigma_n^{(i)}(t)} \right|^k \right] \right) \right]_{(1)}.\end{aligned}$$

We have to show that $(1) = o(h^2)$.

On the one hand it is easy to show that if U (resp. V) is a Gaussian random variable with mean m (resp. with mean 0) and variance 1, then for all real $k \geq 1$, we have

$$\left| \mathrm{E}[|U|^k] - \mathrm{E}[|V|^k] \right| \leq \boldsymbol{C}_k m^2, \quad \text{when} \quad m \leq m_0.$$

On the other hand, we apply this inequality to $U = \frac{\Delta_n^{(i)} X_\mu(t)}{\sigma_n^{(i)}(t)}$, $V = \frac{\Delta_n^{(i)} X_0(t)}{\sigma_n^{(i)}(t)}$ and to $m = \frac{F_{n,\mu}^{(i)}(t)}{\sigma_n^{(i)}(t)}$, where $F_{n,\mu}^{(i)}(t)$ is defined by (5.4).

By Lemma 8.1 and since $\sigma(t) > 0$, we have, for $i = 0, 1, \ldots, n-1$,

$$\sigma_n^{(i)}(t) \geq \boldsymbol{C} > 0. \tag{8.3}$$

Since σ belongs to C^2 on $[0;\ 1]$, this lemma also implies that

$$\sigma_n^{(i)}(t) \leq \boldsymbol{C}. \tag{8.4}$$

Furthermore, a proof similar to that of Lemma 4.2 stated page 935 gives

$$\left| F_{n,\mu}^{(i)}(t) \right| \leq \boldsymbol{C} \left(\frac{1}{n} \right)^{2-H} \tag{8.5}$$

for $i = 0, 1, \ldots, n-1$. Thus, for all real $k \geq 1$, since K is bounded on $[-1;\ 1]$, we get

$$|(1)| \leq \boldsymbol{C}_k \left(\frac{1}{n} \right)^{2(2-H)}. \tag{8.6}$$

Since $nh \to +\infty$ and $H < 1$, one finally gets that $(1) = o(h^2)$. □

Proof of Corollary 5.1. In this aim, we prove Lemma 5.2.

Proof of Lemma 5.2. We want to compute the asymptotic variance of $S_n^{(g)}(t)$. The proof goes along the lines of the computation of $S_{g,h}^{(n)}(1)$ asymptotic variance made in Theorem 4.2.

By using Mehler's formula (3.2), we have

$$\begin{aligned}
\mathrm{E}[S_n^{(g)}(t)]^2 &= \left[\frac{4h}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} K(-1+\tfrac{2i}{n})\, K(-1+\tfrac{2j}{n})\beta(2h(i-j))\right]_{(1)} \\
&\quad + \Bigg[\sum_{\ell=1}^{+\infty} g_\ell^2\, \ell!\, \frac{4h}{n} \sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \Big(\Big[(\delta_n^{(i,j)}(t,t))^\ell - \rho_H^\ell(2h(i-j))\Big] \\
&\qquad\qquad \times K(-1+\tfrac{2i}{n})\, K(-1+\tfrac{2j}{n})\Big)\Bigg]_{(2)} \\
&= (1)+(2),
\end{aligned}$$

where we define for $t,s \in \,]\,0;\ 1\,[$, $\delta_n^{(i,j)}(t,s)$ as

$$\begin{aligned}
\delta_n^{(i,j)}(t,s) &= \mathrm{E}\left[\frac{\Delta_n^{(i)}X_0(t)}{\sigma_n^{(i)}(t)}\frac{\Delta_n^{(j)}X_0(s)}{\sigma_n^{(j)}(s)}\right] \\
&= \rho_H(n(t-s)+2h(j-i)) + \gamma_n^{(i,j)}(t,s),
\end{aligned} \tag{8.7}$$

where the functions ρ_H and $\sigma_n^{(i)}(t)$ are respectively defined by (3.1) and (5.5), while the function β is defined by (7.11).

As in the proof of Theorem 4.2, term (1) gives the required limit. Thus making the change of variable $i-j=k$ in the sum corresponding to term (1), we obtain

$$\begin{aligned}
(1) &= \left[\left(\tfrac{2}{n}\sum_{i=0}^{n-1} K^2(-1+\tfrac{2i}{n})\right)\left(2h\sum_{k=1-n}^{n-1}\beta(2hk)\right)\right]_{(A)} \\
&\quad + \left[-\tfrac{4h}{n}\sum_{i=0}^{n-2}\sum_{k=i+1}^{n-1} K^2(-1+\tfrac{2i}{n})\beta(2hk)\right]_{(B)} \\
&\quad - \left[\tfrac{4h}{n}\sum_{i=1}^{n-1}\sum_{k=1-n}^{i-n} K^2(-1+\tfrac{2i}{n})\beta(2hk)\right]_{(C)} \\
&\quad + \left[\tfrac{4h}{n}\sum_{i=0}^{n-1}\sum_{k=i-n+1}^{i}\Big(K(-1+\tfrac{2i}{n})\,K(-1+\tfrac{2(i-k)}{n}) - K^2(-1+\tfrac{2i}{n})\Big)\beta(2hk)\right]_{(D)} \\
&= (A)+(B)+(C)+(D)
\end{aligned}$$

At this step of the proof we need the following lemma.

Lemma 8.2. *For all $M \in \mathbb{N}$,*

$$h\sum_{i=\lfloor \frac{M}{h}\rfloor}^{n-1}\beta(ih) \underset{n\to+\infty}{\longrightarrow} \int_M^{+\infty}\beta(x)\,\mathrm{d}x.$$

Remark 8.1. This result remains valid if we replace β by $|\beta|$.

Thus we apply this lemma for $M = 0$ and for the function β. Furthermore, we apply (8.2) to the function $\ell = K^2$ and since the function K belongs to C^1 on $[-1;\ 1]$ and β is even, we obtain that

$$(A) \underset{n\to+\infty}{\longrightarrow} \kappa^2 \widetilde{\sigma}_g^2,$$

where κ^2 and $\widetilde{\sigma}_g^2$ are respectively defined by (3.5) and (3.3).

Now to complete the work required for term (1), we show that terms (B), (C) and (D) tend to zero as n tends to infinity.

First, let us consider term (B). Since the function K is bounded on $[-1;\ 1]$, for all $M \in \mathbb{N}^*$ and n such that $2nh \geq (M+1)$, we have

$$\begin{aligned} |(B)| &\leq \boldsymbol{C}\,\frac{h}{n}\left(\sum_{i=0}^{\lfloor \frac{M}{2h}\rfloor -1}\sum_{k=i+1}^{n-1}|\beta(2hk)| + \sum_{i=\lfloor \frac{M}{2h}\rfloor}^{n-2}\sum_{k=i+1}^{n-1}|\beta(2hk)|\right) \\ &\leq \boldsymbol{C}\left(\frac{1}{n}\left\lfloor\frac{M}{2h}\right\rfloor\left(2h\sum_{k=0}^{n-1}|\beta(2hk)|\right) + \left(2h\sum_{k=\lfloor \frac{M}{2h}\rfloor}^{n-1}|\beta(2hk)|\right)\right). \end{aligned}$$

Applying successively Remark 8.1 for $M = 0$ and then for any $M \in \mathbb{N}^*$ since $nh \to +\infty$, we get that for all $M \in \mathbb{N}^*$,

$$\limsup_{n\to+\infty} |(B)| \leq \boldsymbol{C}\int_M^{+\infty} |\beta(x)|\ \mathrm{d}x.$$

For $|x|$ large enough, we have $|\rho_H(x)| \leq \boldsymbol{C}\,|x|^{(2H-4)}$, thus, since $\|g\|_{2,\phi}^2 < +\infty$,

$$\int_{-\infty}^{+\infty} |\beta(x)|\ \mathrm{d}x < +\infty. \tag{8.8}$$

By inequality (8.8), $\lim\limits_{n\to+\infty} (B) = 0$.

A similar proof could be done for term (C). Now we consider term (D). Using the fact that the function K is bounded on $[-1;\ 1]$ and that the function β is even, we get that, for all $M \in \mathbb{N}^*$ and n such that $2nh \geq (M+1)$,

$$\begin{aligned} |(D)| \leq \boldsymbol{C}\Bigg(&\frac{2h}{n}\sum_{i=0}^{n-1}\sum_{\substack{k=i-n+1\\ |k|\leq\lfloor \frac{M}{2h}\rfloor}}^{i}\left|K(-1+\tfrac{2i}{n}) - K(-1+\tfrac{2(i-k)}{n})\right|\,|\beta(2hk)| \\ &+ 2h\sum_{k=\lfloor \frac{M}{2h}\rfloor}^{n-1}|\beta(2hk)|\Bigg). \end{aligned}$$

Since K is uniformly continuous on $[-1;\ 1]$, for all $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that, for all reals $x, y \in [-1;\ 1]$, such that $|x-y| \leq \eta_\varepsilon$, then $|K(x) - K(y)| \leq \varepsilon$. Thus, let $M \in \mathbb{N}^*$ and $\varepsilon > 0$, since β is an even function there exists

$\alpha_\varepsilon = \sup\{\frac{2M}{\eta_\varepsilon}, (M+1)\}$, such that for $2nh \geq \alpha_\varepsilon$, we have

$$|(D)| \leq \boldsymbol{C}\left[\varepsilon\left(2h\sum_{k=0}^{n-1}|\beta(2hk)|\right) + \left(2h\sum_{k=\lfloor\frac{M}{2h}\rfloor}^{n-1}|\beta(2hk)|\right)\right].$$

By Remark 8.1 applied successively with $M = 0$ and then for any $M \in \mathbb{N}^*$, one finally obtains that for all $M \in \mathbb{N}^*$ and $\varepsilon > 0$,

$$\limsup_{n\to+\infty}|(D)| \leq \boldsymbol{C}\left(\varepsilon + \int_M^{+\infty}|\beta(x)|\,\mathrm{d}x\right),$$

and then by (8.8), $\lim_{n\to+\infty}(D) = 0$.

To complete the proof of Lemma 5.2, let us establish that term (2) tends to zero as n tends to ∞. We fix $M > 0$. Then by using the fact that, if i, j are such $|i-j| \geq \frac{1}{2Mh}$, we get that $1 - |\rho_H(2h(i-j))| \geq \boldsymbol{C}_M > 0$. Thus, as in Theorem 4.2, we obtain the following bound

$$\left|(\delta_n^{(i,j)}(t,t))^\ell - \rho_H^\ell(2h(i-j))\right| \leq \boldsymbol{C}_M\left|\gamma_n^{(i,j)}(t,t)\right|\mathbf{1}_{\{|i-j|\geq\frac{1}{2Mh}\}} + \boldsymbol{C}\,\mathbf{1}_{\{|i-j|<\frac{1}{2Mh}\}}.$$

Thus, since $\|g\|_{2,\phi}^2 < +\infty$ and K is bounded on $[-1;\ 1]$, we have

$$|(2)| \leq \boldsymbol{C}\left(\frac{1}{M} + \boldsymbol{C}_M\frac{4h}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n})\left|\gamma_n^{(i,j)}(t,t)\right|\right).$$

At this step of the proof we need the following lemma proved in Appendix A.2.

Lemma 8.3. *For all $i, j \in \{0, 1, \ldots, n-1\}$, $t, s \in]0, 1[$, we have*

$$\left|\gamma_n^{(i,j)}(t,s)\right| \leq \frac{\boldsymbol{C}}{n}.$$

Now, using the bound given for $\gamma_n^{(i,j)}(t,t)$ in the previous lemma and the fact that K is bounded on $[-1;\ 1]$ we get, for all $M > 0$,

$$\limsup_{n\to+\infty}|(2)| \leq \frac{\boldsymbol{C}}{M}$$

and then, $\lim_{n\to+\infty}(2) = 0$. That yields Lemma 5.2. □

A slight modification of the proof of Lemma 5.2 could be done. Since $\sigma_n^{(i)}(t) \simeq \sigma(t)$ (see Lemma 8.1), we obtain that $\sqrt{nh}(\widehat{\alpha}_{n,0}(t) - \mathrm{E}[\widehat{\alpha}_{n,0}(t)])$ is equivalent in L^2 to $S_n^{(g_k)}(t)\alpha(t)$, where g_k and $\alpha(t)$ are respectively defined by equalities (4.7) and (5.6). That yields Corollary 5.1. □

Now, in the aim of winding up the proof of Theorem 5.1, we have to complete proof of Corollary 5.1 by considering any function μ and real k, $k \geq 1$.

We write the following equality.

$$\sqrt{nh}(\widehat{\alpha}_{n,\mu}(t) - \mathrm{E}[\widehat{\alpha}_{n,\mu}(t)]) = \sqrt{nh}\,(\widehat{\alpha}_{n,0}(t) - \mathrm{E}[\widehat{\alpha}_{n,0}(t)]) + (1) + (2),$$

where

$$\begin{aligned}(1) &= \sqrt{nh}\,\mathrm{E}[\widehat{\alpha}_{n,0}(t) - \widehat{\alpha}_{n,\mu}(t)] \quad \text{and,}\\ (2) &= \sqrt{nh}\,(\widehat{\alpha}_{n,\mu}(t) - \widehat{\alpha}_{n,0}(t)).\end{aligned}$$

We have to show that terms (1) and (2) tend to zero in L^2, when $k \geq 1$.

We already saw in the proof of Lemma 5.1 (see inequality (8.6)), that for all real $k \geq 1$, $|\mathrm{E}[\widehat{\alpha}_{n,0}(t) - \widehat{\alpha}_{n,\mu}(t)]| \leq \boldsymbol{C}_k(\frac{1}{n})^{2(2-H)}$, so that, since $H < 1$,

$$(1) = O\left(\sqrt{h}\left(n^{-(\frac{7}{2}-2H)}\right)\right) = o(1).$$

Now, for term (2), applying Cauchy-Schwarz inequality we get,

$$\mathrm{E}[\widehat{\alpha}_{n,\mu}(t) - \widehat{\alpha}_{n,0}(t)]^2 \leq \frac{1}{\mathrm{E}^2[|N|^k]}\,\frac{4}{n^2}\left(\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})\mathrm{E}^{\frac{1}{2}}\left[\left|\Delta_n^{(i)}X_\mu(t)\right|^k - \left|\Delta_n^{(i)}X_0(t)\right|^k\right]^2\right)^2.$$

Then we apply inequality (A.2) to $x = \Delta_n^{(i)}X_\mu(t)$ and $y = \Delta_n^{(i)}X_0(t)$, so that

$$\mathrm{E}\left[\left|\Delta_n^{(i)}X_\mu(t)\right|^k - \left|\Delta_n^{(i)}X_0(t)\right|^k\right]^2 \leq \boldsymbol{C}_k(F_{n,\mu}^{(i)}(t))^2\left((F_{n,\mu}^{(i)}(t))^{2(k-1)} + \mathrm{E}\left[\left|\Delta_n^{(i)}X_0(t)\right|^{2(k-1)}\right]\right).$$

Finally, the function K being bounded, using inequalities (8.4) and (8.5), we get

$$\begin{aligned}\mathrm{E}[(2)]^2 &\leq \boldsymbol{C}_k nh(\tfrac{1}{n})^{2(2-H)}\left((\tfrac{1}{n})^{2(k-1)(2-H)} + 1\right)\\ &\leq \boldsymbol{C}_k h(\tfrac{1}{n})^{(3-2H)} = o(1),\end{aligned}$$

since $H < 1$. That yields Theorem 5.1. □

8.1.2. Central Limit Theorem

Proofs of Theorem 5.2 and of Corollary 5.2. Let $d_1, \ldots, d_m \in \mathbb{R}$ and

$$S_n^{(g)}(\boldsymbol{t}) = \sum_{i=1}^{m} d_i S_n^{(g)}(t_i).$$

We want to prove that

$$S_n^{(g)}(\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0; \widetilde{\sigma}_g^2\, \kappa^2 \sum_{i=1}^m d_i^2\right),$$

where κ^2 and $\widetilde{\sigma}_g^2$ are respectively defined by (3.5) and (3.3).

As in proof of Theorem 4.2 it is enough to demonstrate that

$$S_n^{(g_L)}(\boldsymbol{t}) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0; \widetilde{\sigma}_{g_L}^2\, \kappa^2 \sum_{i=1}^m d_i^2\right),$$

where $g_L(x) = \sum_{\ell=1}^L g_\ell H_\ell(x)$, where we fixed $L \in \mathbb{N}^*$.

Still as in the proof of Theorem 4.2, the Chaos representation for process Y allows us to write the increments of X_0 as

$$\frac{\Delta_n^{(j)} X_0(t)}{\sigma_n^{(j)}(t)} = \int_{-\infty}^{+\infty} f^{(n)}(\lambda, j, t)\, \mathrm{d}W(\lambda),$$

where we defined function $f^{(n)}$ by

$$\begin{aligned}
&\sigma_n^{(j)}(t) f^{(n)}(\lambda, j, t)\\
&\quad = a(n) i\lambda \sqrt{f(\lambda)} \left(\left[\int_{a_n^{(j)}(t)}^{a_n^{(j)}(t)+\frac{1}{n}} - \int_{a_n^{(j)}(t)-\frac{1}{n}}^{a_n^{(j)}(t)}\right] \sigma(u)\exp(i\lambda u)\, \mathrm{d}u\right).
\end{aligned}$$

In that way, we can write $S_n^{(g_L)}(\boldsymbol{t})$ in the multiple Wiener chaos as

$$S_n^{(g_L)}(\boldsymbol{t}) = \sum_{\ell=1}^L I_\ell(h_\ell^{(n,\boldsymbol{t})}),$$

where the function $h_\ell^{(n,\boldsymbol{t})}$ is

$$\begin{aligned}
&h_\ell^{(n,\boldsymbol{t})}(\lambda_1, \ldots, \lambda_\ell)\\
&\quad = g_\ell \ell! 2\sqrt{\frac{h}{n}} \sum_{i=1}^m d_i \sum_{j=0}^{n-1} K(-1 + \tfrac{2j}{n}) f^{(n)}(\lambda_1, j, t_i) \cdots f^{(n)}(\lambda_\ell, j, t_i),
\end{aligned}$$

and where I_ℓ is given by (7.16).

First, let us compute the variance of $S_n^{(g_L)}(\boldsymbol{t})$.

$$\begin{aligned}
\mathrm{E}[S_n^{(g_L)}(\boldsymbol{t})]^2 &= \sum_{\ell=1}^L \frac{1}{\ell!} \int_{\mathbb{R}^\ell} \left| h_\ell^{(n,\boldsymbol{t})}(\lambda_1, \ldots, \lambda_\ell)\right|^2 \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_\ell\\
&= \sum_{\ell=1}^L \ell! g_\ell^2 \sum_{i_1=1}^m \sum_{i_2=1}^m d_{i_1} d_{i_2} \left(\frac{4h}{n} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} K(-1 + \tfrac{2j_1}{n})\right.\\
&\qquad \left. \times\, K(-1 + \tfrac{2j_2}{n}) (\delta_n^{(j_1,j_2)}(t_{i_1}, t_{i_2}))^\ell\right),
\end{aligned}$$

where $\delta_n^{(i,j)}(t,s)$ is defined by (8.7).

Now for $\ell \geq 1$ and fixed $t, s \in \,]\,0;\ 1\,[$,

$$\frac{4h}{n}\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{n-1} K(-1+\tfrac{2j_1}{n})K(-1+\tfrac{2j_2}{n})(\delta_n^{(j_1,j_2)}(t,s))^{\ell} \tag{8.9}$$

$$\xrightarrow[n\to\infty]{} \begin{cases} \kappa^2 \int\limits_{-\infty}^{+\infty} \rho_H^{\ell}(x)\,\mathrm{d}x, & \text{if } t = s; \\ 0, & \text{otherwise.} \end{cases} \tag{8.10}$$

A proof of this last convergence is obtained, in case where $t = s$, by considering Lemma 5.2. In case where $t \neq s$, the proof is simpler than that of Lemma 5.2. Indeed by using inequality (7.13), we obtain the bound

$$|\rho_H(n(t-s) + 2h(j-i))| \leq \boldsymbol{C}(\tfrac{1}{n})^{4-2H} \leq \frac{\boldsymbol{C}}{n}.$$

Thus, remembering that $\gamma_n^{(i,j)}(t,s)$ is defined by (8.7), using Lemma 8.3 page 984, we get the bound, for $t \neq s$,

$$\left|\delta_n^{(i,j)}(t,s)\right| \leq \frac{\boldsymbol{C}}{n}, \tag{8.11}$$

where $\delta_n^{(i,j)}(t,s)$ is defined by (8.7).

So, $|(8.9)| \leq \boldsymbol{C}h$. Thus we proved

$$\lim_{n\to+\infty} \mathrm{E}[S_n^{(g_L)}(\boldsymbol{t})]^2 = \widetilde{\sigma}_{g_L}^2\, \kappa^2 \sum_{i=1}^{m} d_i^2.$$

To conclude the convergence of $S_n^{(g_L)}(\boldsymbol{t})$ we use again Theorem 1 of Peccati and Tudor (2005) and as in the proof of Theorem 4.2, it is enough to prove that for fixed ℓ and p, $\ell \geq 2$ and $p = 1, \ldots, \ell - 1$, $\lim_{n\to+\infty} B_n = 0$, where B_n is

$$B_n = \int_{\mathbb{R}^{2(\ell-p)}} \left| h_\ell^{(n,\boldsymbol{t})} \otimes_p h_\ell^{(n,\boldsymbol{t})}(\lambda_1, \ldots, \lambda_{\ell-p}, \mu_1, \ldots, \mu_{\ell-p}) \right|^2 \\ \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{\ell-p}\,\mathrm{d}\mu_1 \cdots \mathrm{d}\mu_{\ell-p},$$

where p-th contractions $\otimes_p$ are defined by (7.17).

Now we compute B_n and we get

$$B_n = 16\,(\ell!)^4 g_\ell^4 \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{i_3=1}^{m}\sum_{i_4=1}^{m} d_{i_1} d_{i_2} d_{i_3} d_{i_4} \left(\tfrac{h^2}{n^2}\right) \sum_{j_1=0}^{n-1}\sum_{j_2=0}^{n-1}\sum_{j_3=0}^{n-1}\sum_{j_4=0}^{n-1} \\ K(-1+\tfrac{2j_1}{n})K(-1+\tfrac{2j_2}{n})K(-1+\tfrac{2j_3}{n})K(-1+\tfrac{2j_4}{n}) \left(\delta_n^{(j_1,j_2)}(t_{i_1}, t_{i_2})\right)^p \times \\ \left(\delta_n^{(j_3,j_4)}(t_{i_3}, t_{i_4})\right)^p \left(\delta_n^{(j_1,j_3)}(t_{i_1}, t_{i_3})\right)^{\ell-p} \left(\delta_n^{(j_2,j_4)}(t_{i_2}, t_{i_4})\right)^{\ell-p}.$$

As in the proof of Lemma 7.4, it is enough to prove that $\lim_{n\to+\infty} A_n = 0$, where we define for $i_1, \dots, i_4$ fixed in $\{1, \dots, m\}$,

$$A_n = \frac{h^2}{n^2} \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} \sum_{j_3=0}^{n-1} \sum_{j_4=0}^{n-1} K(-1+\tfrac{2j_1}{n})K(-1+\tfrac{2j_2}{n})K(-1+\tfrac{2j_3}{n})K(-1+\tfrac{2j_4}{n})\times \\ \left|\delta_n^{(j_1,j_2)}(t_{i_1},t_{i_2})\right| \left|\delta_n^{(j_3,j_4)}(t_{i_3},t_{i_4})\right| \left|\delta_n^{(j_1,j_3)}(t_{i_1},t_{i_3})\right| \left|\delta_n^{(j_2,j_4)}(t_{i_2},t_{i_4})\right| .$$

We split the index intervals into two parts, C_M and C_M^c, where for a fixed real number $M > 0$,

$$C_M = \{(j_1,j_2,j_3,j_4) \in \mathbb{N}^4, h\,|j_1 - j_2| > M \quad \text{or} \quad h\,|j_1 - j_3| > M \\ \text{or} \quad h\,|j_2 - j_4| > M\}.$$

In that way, we can write A_n as the sum of two terms corresponding to C_M and C_M^c respectively.

For the first term, we use the fact that, working as for the computation of the asymptotic variance of $S_n^{(g_L)}(\boldsymbol{t})$, see (8.10), we can show that for all $t_1, t_2 \in \,]\,0;\ 1\,[$,

$$\tfrac{h}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n}) \left|\delta_n^{(i,j)}(t_1,t_2)\right| \le \boldsymbol{C}$$

and that, $|\delta_n^{(i,j)}(t_1,t_2)| \le 1$.

Furthermore, considering Lemma 8.3, inequalities (8.11) and (7.13), we obtain and use the following bound:

$$\text{for all } t,s \in \,]\,0;\ 1\,[, \quad \left|\delta_n^{(i,j)}(t,s)\right| \le \boldsymbol{C}(|\rho_H(2h(i-j))| + \tfrac{1}{n}) \le \boldsymbol{C} M^{2H-4}$$

for M large enough, $n \ge n_M$ and $h\,|i-j| > M$.

For the second term, we bound each of the four functions $\delta_n^{(i,j)}$ by 1 so that for M large enough, we get

$$\overline{\lim_{n\to\infty}} A_n \le \boldsymbol{C} \left(M^{(2H-4)} + M^3 \overline{\lim_{n\to\infty}} (\tfrac{1}{nh}) \right) \le \boldsymbol{C} M^{2H-4},$$

since $nh \to +\infty$.

To conclude the proof of Theorem 5.2, we use the fact that $0 < H < 1$ and then $\lim_{n\to+\infty} A_n = 0$.

As mentioned after Corollary 5.1 and Lemma 5.1, Corollary 5.2 follows from the fact that $\sqrt{nh}(\widehat{\alpha}_{n,\mu}(t) - \mathrm{E}[\widehat{\alpha}_{n,\mu}(t)])$ is equivalent in L^2 to $S_n^{(g_k)}(t)\alpha(t)$, where g_k and $\alpha(t)$ are respectively defined by (4.7) and (5.6).

Remark 5.2 follows from the fact that, as mentioned after Lemma 5.1, for all real $k > 0$, $\sqrt{nh}(\widehat{\alpha}_{n,0}(t) - \mathrm{E}[\widehat{\alpha}_{n,0}(t)])$ is equivalent in L^2 to $S_n^{(g_k)}(t)\alpha(t)$. $\square$

Proof of Theorem 5.3. Lemma 5.1 extended to a function μ not necessarily identically null and Corollary 5.2 give the following results. For all real $k \geq 1$,

1) If $nh^5 \underset{n\to+\infty}{\longrightarrow} 0$,

$$\left(\sqrt{nh}(\widehat{\alpha}_{n,\mu}(t_1) - \alpha(t_1)), \ldots, \sqrt{nh}(\widehat{\alpha}_{n,\mu}(t_m) - \alpha(t_m))\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(\mathbf{0}_m; \widetilde{\sigma}^2_{g_k}\kappa^2 \boldsymbol{D}_m(\alpha(\boldsymbol{t}))\right),$$

   where $\boldsymbol{D}_m(\alpha(\boldsymbol{t}))$ is the diagonal matrix of rank m with generic element $\alpha^2(t_i)$, $i = 1, \ldots, m$ and $\mathbf{0}_m$ is the null column vector of order m.

2) If $nh^5 \underset{n\to+\infty}{\longrightarrow} \boldsymbol{C}$,

$$\left(\sqrt{nh}\,(\widehat{\alpha}_{n,\mu}(t_1) - \alpha(t_1)), \ldots, \sqrt{nh}(\widehat{\alpha}_{n,\mu}(t_m) - \alpha(t_m))\right) \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}\left(b_m(\boldsymbol{t}); \widetilde{\sigma}^2_{g_k}\kappa^2 \boldsymbol{D}_m(\alpha(\boldsymbol{t}))\right),$$

   where $b_m(\boldsymbol{t})$ is the column vector of order m with generic element

$$\sqrt{\boldsymbol{C}}\alpha''(t_i)\chi^2, \quad i = 1, \ldots, m.$$

3) If $nh^5 \underset{n\to+\infty}{\longrightarrow} +\infty$,

$$\frac{1}{h^2}\left(\widehat{\alpha}_{n,\mu}(t) - \alpha(t)\right) \xrightarrow[n\to+\infty]{\mathcal{P}} \alpha''(t)\chi^2.$$

   where g_k, $\widetilde{\sigma}^2_{g_k}$, κ^2, $\alpha(t)$ and χ^2 are respectively defined by (4.7), (3.3), (3.5), (5.6) and (3.4).

Using that σ is strictly positive on $[\,0;\ 1\,]$, we get Theorem 5.3.

Remark 5.3 follows from Lemma 5.1 and Remark 5.2. □

Proof of Remark 5.4. Remember that coefficients $g_{2p,k}$ of function g_k are given by (4.8). Thus for real $k > 0$,

$$\begin{aligned}
\frac{\widetilde{\sigma}^2_{g_k}}{k^2} &= \frac{1}{k^2}\sum_{p=1}^{+\infty}\frac{1}{(2p)!}\prod_{i=0}^{p-1}(k-2i)^2\int_{-\infty}^{+\infty}\rho_H^{2p}(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}\rho_H^2(x)\,\mathrm{d}x + \frac{1}{k^2}\sum_{p=2}^{+\infty}\frac{1}{(2p)!}\prod_{i=0}^{p-1}(k-2i)^2\int_{-\infty}^{+\infty}\rho_H^{2p}(x)\,\mathrm{d}x \\
&\geq \frac{1}{2}\int_{-\infty}^{+\infty}\rho_H^2(x)\,\mathrm{d}x = \frac{\widetilde{\sigma}^2_{g_2}}{2^2}.
\end{aligned}$$

Let $k = 2\ell$, $\ell \in \mathbb{N}$, $\ell \geq 1$, then

$$\frac{\widetilde{\sigma}^2_{g_{2\ell}}}{(2\ell)^2} = \sum_{p=1}^{\ell}\tfrac{1}{(2p)!}(2\ell)^{2(p-1)}\prod_{i=0}^{p-1}\left(\tfrac{\ell-i}{\ell}\right)^2\int_{-\infty}^{+\infty}\rho_H^{2p}(x)\,\mathrm{d}x$$

$$\leq \sum_{p=1}^{\ell+1} \tfrac{1}{(2p)!}(2(\ell+1))^{2(p-1)} \prod_{i=0}^{p-1} \left(\tfrac{\ell+1-i}{\ell+1}\right)^2 \int_{-\infty}^{+\infty} \rho_H^{2p}(x)\,\mathrm{d}x$$
$$= \frac{\widetilde{\sigma}^2_{g_{2(\ell+1)}}}{(2(\ell+1))^2}$$

□

8.2. For the two gfOUp

Proof of Theorem 5.4. As we explained in Section 5.2, we have to prove two lemmas, Lemma 5.3 and Lemma 5.4.

Proof of Lemma 5.3. For $i = 0, 1, \ldots, n-1$, we set $a_i = a_n^{(i)}(t)$ (see (5.1) for the definition of $a_n^{(i)}(t)$). We define the random variable $\delta_n^{(i)} X_\lambda(t)$ as in (5.3), Section 5.1.1. Let us suppose that X_λ is a solution of (2.8) (resp. (2.10)). Thus using (2.9) (resp. (2.11)) and similarly as we did in the proof of Lemma 4.5, with this notation we have

$$\begin{aligned}
\delta_n^{(i)} X_\lambda(t) = \sigma(a_i)\delta_n^{(i)} Y^\star(t) &\\
&+ Y^\star(a_i + \tfrac{1}{n})\left(\sigma(a_i + \tfrac{1}{n}) - \sigma(a_i)\right) + Y^\star(a_i - \tfrac{1}{n})\left(\sigma(a_i - \tfrac{1}{n}) - \sigma(a_i)\right)\\
&+ \left(\int_{a_i - \frac{1}{n}}^{a_i} - \int_{a_i}^{a_i + \frac{1}{n}}\right) \sigma'(u) Y^\star(u)\,\mathrm{d}u\\
&+ \left(\int_{a_i}^{a_i + \frac{1}{n}} - \int_{a_i - \frac{1}{n}}^{a_i}\right) (\widetilde{\mu}(X_\lambda(u)) - \widetilde{\mu}(X_\lambda(a_i)))\,\mathrm{d}u
\end{aligned}$$

where $Y^\star$ is Y (resp. b_H) and the process $\delta_n^{(i)} Y^\star$ is defined by (5.3), Section 5.1.1. Here $\widetilde{\mu}(x) = -\lambda x$ (resp. $\mu(x)$).

With arguments very similar to the ones given in Lemma 4.5, using (A.1), we finally proved that, for any $\delta > 0$,

$$\left|\Delta_n^{(i)} X_\lambda(t) - \sigma(a_n^{(i)}(t))\Delta_n^{(i)} Y^\star(t)\right| \leq \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{1-\delta},$$

that yields the lemma. □

Proof of Lemma 5.4. Let us suppose that X_λ is a solution of (2.8) (resp. (2.10)). Remember that $Y^\star$ stands for Y (resp. b_H) and that $\sigma^\star_{n,1}$ is $\sigma_{n,1}$ (resp. 1) where $\sigma_{n,1}$ is defined by (4.9).

For all real $k \geq 1$, we can decompose $\widehat{\alpha}_{n,\lambda}(t) - \alpha(t)$ as

$$\widehat{\alpha}_{n,\lambda}(t) - \alpha(t) = \left[\frac{2}{n}\sum_{i=0}^{n-1} K(-1 + \tfrac{2i}{n}) g_k\Big(\tfrac{\Delta_n^{(i)} Y^\star(t)}{\sigma^\star_{n,1}}\Big)\,\sigma^k(a_n^{(i)}(t))\,(\sigma^\star_{n,1})^k\right]_{(1)}$$

$$+\left[\left(\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\sigma^k(a_n^{(i)}(t))\,(\sigma^\star_{n,1})^k\right)-\alpha(t)\right]_{(2)}$$

$$+\left[\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\frac{1}{\mathrm{E}[|N|^k]}\left(|\Delta_n^{(i)}X_\lambda(t)|^k-\sigma^k(a_n^{(i)}(t))\,|\Delta_n^{(i)}Y^\star(t)|^k\right)\right]_{(3)}, \tag{8.12}$$

where the function g_k is defined by (4.7). The three terms in brackets in (8.12) are denoted respectively by (1), (2) and (3) in the following.

Note that $\mathrm{Var}[\Delta_n^{(i)}Y^\star(t)]=\mathrm{Var}[\Delta_nY^\star(i)]=(\sigma^\star_{n,1})^2$, where $\Delta_nY^\star(i)$ has been defined by (4.1), Section 4.1.1.

As in the proof of Lemma 4.3, for $k\geq 1$ we consider inequality (A.2) applied to $x=\Delta_n^{(i)}X_\lambda(t)$ and $y=\sigma(a_n^{(i)}(t))\Delta_n^{(i)}Y^\star(t)$. Using Lemma 7.1 with $X_0=Y^\star$, we can prove that for $i\in\{1,\dots,n-1\}$,

$$\left|\Delta_n^{(i)}Y^\star(t)\right|\leq \boldsymbol{C}(\omega)n^\delta. \tag{8.13}$$

Furthermore using that the function K is bounded on $[-1;\ 1]$ and Lemma 5.3, since $k\geq 1$, we get

$$|(3)|\leq \boldsymbol{C}_k(\omega)\left(\frac{1}{n}\right)^{1-\delta k}.$$

Choosing δ small enough, that is $\delta<\frac{1}{2k}$, we then obtain that $(3)=o_{a.s.}(\frac{1}{\sqrt{nh}})$. Using (7.7) for $\sigma^\star_{n,1}$ and inequality (8.13), one can prove that

$$(1)+(2)=\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\frac{2i}{n})g_k\Big(\tfrac{\Delta_n^{(i)}Y^\star(t)}{\sigma^\star_{n,1}}\Big)\,\sigma^k(a_n^{(i)}(t))+$$
$$\left(\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\sigma^k(a_n^{(i)}(t))\right)-\alpha(t)+o_{a.s.}\left(\tfrac{1}{\sqrt{nh}}\right)$$

Thus lemma follows. □

Now to finish the proof of the theorem, let us remark that a proof similar to that of Lemma 5.1 and to that of Corollary 5.2 for $\mu\equiv 0$, where we replace $(\sigma_n^{(i)}(t))^k$ by $\sigma^k(a_n^{(i)}(t))$ lead to the following

$$\frac{1}{h^2}\left(\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\sigma^k(a_n^{(i)}(t))-\alpha(t)\right)\underset{n\to+\infty}{\longrightarrow}\alpha''(t)\,\chi^2,$$

where χ^2 is defined by (3.4) and

$$\sqrt{nh}\left(\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})g_k\Big(\tfrac{\Delta_n^{(i)}Y^\star(t)}{\sigma^\star_{n,1}}\Big)\,\sigma^k(a_n^{(i)}(t))\right)\ \equiv\ \sigma^k(t)\,S_{n,\star}^{(g_k)}(t), \tag{8.14}$$

where for any function g described in the notations, we defined

$$S_{n,\star}^{(g)}(t) = 2\sqrt{\frac{h}{n}}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})g\left(\tfrac{\Delta_n^{(i)}Y^\star(t)}{\sigma_{n,1}^\star}\right).$$

Theorem 5.2 with $g = g_k \in L^2(\phi(x)\,\mathrm{d}x)$ and the fact that function σ is strictly positive on interval $[0;1]$ give Theorem 5.4. □

8.3. For the fDrc

Proof of Theorem 5.5. As explained in Section 5.3, we just have to prove Theorem 5.6.

Proof of Theorem 5.6. We defined $T_n^{(f,g)}(t)$ as

$$T_n^{(f,g)}(t) = 2\sqrt{\frac{h}{n}}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})g\left(\tfrac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\right)f(Y(t)),$$

where g is a general function with $(2+\delta)$-moments with respect to the standard Gaussian measure, even, or odd, with a Hermite rank greater than or equal to one and such that $A_g \neq \emptyset$ (for the definition of A_g, see Section 3). Also, f belongs to C^2 on $\mathbb{R}$, such that for all $x \in \mathbb{R}$, $|f''(x)| \le P(|x|)$, where P is a polynomial.

Also we defined the random variable $X_{n,p}(t)$ by

$$X_{n,p}(t) = \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})\left(\frac{Y(a_n^{(i)}(t)) - Y(t)}{h^H}\right)p(Y(t)),$$

for p a continuous function on $\mathbb{R}$.

Let define

$$S_{n,1}^{(g)}(t) = 2\sqrt{\frac{h}{n}}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})g\left(\tfrac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\right),$$

and note that $\mathrm{Var}[\Delta_n^{(i)}Y(t)] = \mathrm{Var}[\Delta_n Y(i)] = (\sigma_{n,1})^2$.

We want to prove that

$$\left(T_n^{(f,g)}(t), X_{n,p}(t)\right) \xrightarrow[n\to\infty]{\text{Law}} \left(\widetilde{\sigma}_g\, f(Y(t))\,U(t), p(Y(t))\,V(t)\right),$$

where U, V and Y are as in Theorem 5.5.

For this, we just have to prove that

$$\left(S_{n,1}^{(g)}(t), X_n(t), Y(t)\right) \xrightarrow[n\to\infty]{\text{Law}} \left(\widetilde{\sigma}_g\, U(t), V(t), Y(t)\right),$$

where we define $X_n(t)$ by $X_{n,p}(t)$ for $p \equiv 1$.

As in the proof of Lemma 7.6, we can suppose that $g(x) = \sum_{\ell=2}^{\infty} g_\ell H_\ell(x)$.

Indeed, since $\int_{-\infty}^{+\infty} \rho_H(x)\,\mathrm{d}x = 0$, it follows by Lemma 5.2 applied to $X_0 = Y$ and $g = H_1$ that $2\sqrt{\frac{h}{n}}\sum_{i=0}^{n-1} K(-1+\frac{2i}{n})\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}$ tends to zero in L^2 as n tends to infinity.

Let $b_1,\ldots,b_m, c_1,\ldots,c_m$ be real constants. First, we will establish the limit distribution of

$$\sum_{j=1}^m b_j\,Y(t_j) \;+\sum_{j=1}^m c_j\,X_n(t_j) \;+\; S_{n,1}^{(g_L)}(\boldsymbol{t}),$$

where we define $S_{n,1}^{(g_L)}(\boldsymbol{t})$ as in the proofs of Theorem 5.2 and of Corollary 5.2 by

$$S_{n,1}^{(g_L)}(\boldsymbol{t}) = \sum_{i=1}^m d_i S_{n,1}^{(g_L)}(t_i),$$

with $d_1,\ldots,d_m \in \mathbb{R}$ and $g_L(x) = \sum_{\ell=2}^L g_\ell H_\ell(x)$ where we fixed $L \in \mathbb{N}^*$.

For this, we remark that the random variable $\sum_{j=1}^m b_j\,Y(t_j) + \sum_{j=1}^m c_j\,X_n(t_j)$ belongs to the first chaos and that the random variable $S_{n,1}^{(g_L)}(\boldsymbol{t})$ belongs to the superior order one and converges by Theorem 5.2 toward the Gaussian centered random variable $\sum_{i=1}^m d_i\,\widetilde{\sigma}_{g_L}U(t_i)$, where $(U(t_1),\ldots,U(t_m))$ is a centered Gaussian vector, such that $\mathrm{E}[U(t)U(s)] = \mathbf{1}_{\{t=s\}}\kappa^2$, and where $\widetilde{\sigma}_{g_L}$ is defined by (3.3). Furthermore, straightforward computations show that for all $0<s,t<1$, $\lim_{n\to+\infty} \mathrm{E}[X_n(t)Y(s)] = 0$ and that

$$\begin{aligned}
&\lim_{n\to+\infty} \mathrm{E}[X_n(t)X_n(s)]\\
&= \mathbf{1}_{\{t=s\}} \lim_{n\to\infty}\left(\frac{n^{2H}}{a^2(n)}\right)\frac{1}{\upsilon_{2H}^2(4-4^H)}\,\mathrm{E}\left[\int_{-1}^{+1} K(u)b_H(u)\,\mathrm{d}u\right]^2\\
&= \alpha(t,s).
\end{aligned}$$

So that the centered Gaussian random variable $\sum_{j=1}^m b_j\,Y(t_j) + \sum_{j=1}^m c_j\,X_n(t_j)$ converges in law to the centered Gaussian random variable $\sum_{j=1}^m b_j\,Y(t_j) + \sum_{j=1}^m c_j\,V(t_j)$, where $(V(t_1),\ldots,V(t_m))$ is a centered Gaussian vector, independent of $(Y(t_1),\,\ldots,\,Y(t_m))$ such that $\mathrm{E}\left[V(t)V(s)\right] = \alpha(t,s)$.

By Theorem 1 of Peccati and Tudor (2005), we finally proved that the random variable

$$\begin{aligned}
&\sum_{j=1}^m b_j\,Y(t_j) \;+\sum_{j=1}^m c_j\,X_n(t_j) \;+\; S_{n,1}^{(g_L)}(\boldsymbol{t})\\
&\qquad \xrightarrow[n\to\infty]{\text{Law}} \sum_{j=1}^m b_j\,Y(t_j) \;+\sum_{j=1}^m c_j\,V(t_j) \;+\sum_{i=1}^m d_i\widetilde{\sigma}_{g_L}U(t_i),
\end{aligned}$$

where $(U(t_1),\ldots,U(t_m))$, $(V(t_1),\ldots,V(t_m))$ and $(Y(t_1),\ldots,Y(t_m))$ are independent.

As in the proof of Theorem 4.2, by using that $\widetilde{\sigma}_{g_L} \to \widetilde{\sigma}_g$ as $L \to +\infty$ and (Dynkin, 1988, Lemma 1.1), we finally proved Theorem 5.6. $\square$

Thus Theorem 5.5 follows. □

Appendix

A.1. Concerning the functional estimation of σ

Proof of Lemma 4.2. For $i = 0, 1, \ldots, n-2$, we have by (4.4),

$$\frac{1}{a(n)} F_{n,\mu}(i) = \left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \right) \left(\mu(u) - \mu(\tfrac{i+1}{n}) \right) \mathrm{d}u = (1)$$

Since μ is Lipschitz on $[\,0;\ 1\,]$, we get

$$(1) = O(n^{-2}).$$

To conclude the proof of this lemma we just have to prove that

$$a(n) = O(n^H). \tag{A.1}$$

For Y, a solution of model (2.2), it is obvious. Now if Y is a solution of models (2.5) or (2.6), using that $r(t) = 1 - L(t)t^{2H}$ for $0 < t \leq 1$ and that $\lim_{t\to 0^+} L(t) = C_0 > 0$, the result comes easily. □

Proof of Lemma 4.3. For $k \geq 1$, we consider the following inequality. For all reals x and y,

$$\left| |x|^k - |y|^k \right| \leq 2^{(k-1)} k\ |x-y| \left(|x-y|^{k-1} + |y|^{k-1} \right). \tag{A.2}$$

We apply this inequality to $x = \Delta_n X_\mu(i)$ and to $y = \Delta_n X_0(i)$. We obtain

$$|A_n(h)| \leq \boldsymbol{C}_k\, \frac{1}{n-1} \sum_{i=0}^{n-2} \left| h(\tfrac{i}{n}) \right| |F_{n,\mu}(i)| \left(|F_{n,\mu}(i)|^{k-1} + |\Delta_n X_0(i)|^{k-1} \right).$$

We use the modulus of continuity of X_0 (see Lemma 7.1, p. 962), and Lemma 4.2. Thus, since $|h|$ is bounded in $[\,0;\ 1\,]$, we get

$$|A_n(h)| \leq \boldsymbol{C}_k(\omega) \left| \left((\tfrac{1}{n})^{k(2-H)} + (\tfrac{1}{n})^{2-H-\delta(k-1)} \right) \right| \leq \boldsymbol{C}_k(\omega) (\tfrac{1}{n})^{2-H-\delta(k-1)}.$$

Obviously, if $k = 1$ then $2 - H - \delta(k-1) > \frac{1}{2}$. Otherwise, choosing δ small enough, that is $\delta < \frac{1}{2(k-1)}$, we then have $2 - H - \delta(k-1) > \frac{1}{2}$. In both cases we get Lemma 4.3. □

Proof of Lemma 4.4. We decompose $B_n(h)$ as

$$B_n(h) \quad = \quad \frac{n}{n-1}\, S^{(n)}_{g_k, h\sigma^k}(1) + \left[\frac{\sqrt{n}}{\mathrm{E}[|N|^k]} A_n(h) \right]_{(A)}$$

$$
\begin{aligned}
&+ \left[\sqrt{n} \left(\frac{1}{n-1} \sum_{i=0}^{n-2} h(\tfrac{i}{n}) \sigma^k(\tfrac{i}{n}) - \int_0^1 h(u)\sigma^k(u)\,\mathrm{d}u \right) \right]_{(B)} \qquad \text{(A.3)}\\
&+ \left[\frac{\sqrt{n}}{(n-1)} \sum_{i=0}^{n-2} h(\tfrac{i}{n}) \frac{1}{\mathrm{E}[|N|^k]} \left| \frac{\Delta_n X_0(i)}{\sigma_n(i)} \right|^k \left(\sigma_n^k(i) - \sigma^k(\tfrac{i}{n}) \right) \right]_{(C)}\\
= \quad & \frac{n}{n-1}\, S^{(n)}_{g_k, h\sigma^k}(1) + (A) + (B) + (C),
\end{aligned}
$$

where the function g_k is defined by (4.7) and $A_n(h)$ is defined by Lemma 4.3 page 936. Obviously, (A), (B) and (C) are the last three terms of (A.3).

Since h and σ belong to C^1 on $[0;\ 1]$ and σ is strictly positive on $[0;\ 1]$, then for all real $k > 0$, $h\sigma^k$ belongs to C^1 on $[0;\ 1]$. Thus we can apply (7.6) to this last function with $a = 1$ so that $(B) = o(1)$.

Now, using inequality (7.4) by choosing δ small enough, that is $\delta < \frac{1}{2k}$, equation (7.7) and the strictly positivity of σ on $[0;\ 1]$, we get, since h is bounded on $[0;\ 1]$, that for all real $k > 0$, $(C) = o_{a.s.}(1)$.

Finally using Lemma 4.3, $(A) = o_{a.s.}(1)$.

Remark follows from the fact that (A) vanishes when $\mu \equiv 0$. $\square$

Proof of Lemma 7.1. For $u, v \geq 0$, by equation (2.7)

$$
X_0(v) - X_0(u) = \sigma(v)(Y(v) - Y(u)) + Y(u)(\sigma(v) - \sigma(u)) - \int_u^v \sigma'(t) Y(t)\,\mathrm{d}t.
$$

We prove that the trajectories of Y are $(H-\delta)$-Hölder continuous on $[0;\ 1]$, in other words, for any $\delta > 0$, $0 \leq u, v \leq 1$,

$$
|Y(v) - Y(u)| \leq \boldsymbol{C}(\omega) \left| v - u \right|^{H-\delta}. \qquad \text{(A.4)}
$$

Suppose that for the moment this inequality is proved. In this case, using that σ belongs to C^1 on $[0;\ 1]$, we obtain, for any $\delta > 0$, $0 \leq u, v \leq 1$,

$$
|X_0(v) - X_0(u)| \leq \boldsymbol{C}(\omega) \left\{ |v-u|^{H-\delta} + |v-u| \right\} \leq \boldsymbol{C}(\omega) \left| v - u \right|^{H-\delta}.
$$

Now, let us prove (A.4).

If Y is solution of equation (2.2) (resp. (2.5) or (2.6)), then for s and $t \in [0;\ 1]$

$$
\begin{aligned}
\mathrm{E}[Y(t) - Y(s)]^2 &= \mathrm{E}[Y^{\star\star}(t) - Y^{\star\star}(s)]^2 \\
&\qquad + \left[4 \int_{-\infty}^{+\infty} \sin^2(\tfrac{\lambda(t-s)}{2}) G(\lambda)\,\mathrm{d}\lambda \right]_{(1)} \\
&= \mathrm{E}[Y^{\star\star}(t) - Y^{\star\star}(s)]^2 + (1),
\end{aligned}
$$

where $Y^{\star\star}$ is a solution of equation (2.2) (resp. (2.5) or (2.6)), with $G \equiv 0$.

On the one hand, if Y is a solution of (2.2), note that $Y^{\star\star} = b_H$ and the fBm with parameter $0 < H < 1$ has $(H-\delta)$-Hölder continuous trajectories on $[0;\ 1]$.

On the other hand, if Y is a solution of (2.5) or (2.6), then $\mathrm{E}[Y^{\star\star}(t) - Y^{\star\star}(s)]^2 = 2(1-r(t-s))$. As continuous function on $]\,0;\ 1\,]$, since $\lim_{t\to 0^+} L(t) = C_0$, L is bounded on $]\,0;\ 1\,]$. Thus $1 - r(x) \leq \boldsymbol{C}\, x^{2H}$, for $x \in [\,0;\ 1\,]$.

Finally, using that $\sin^2(x) \leq \boldsymbol{C} x^2$, for all $x \in \mathbb{R}$, that $\int_{-\infty}^{+\infty} \lambda^2 G(\lambda)\,\mathrm{d}\lambda < +\infty$ and $0 < H < 1$, we get that $(1) \leq \boldsymbol{C}\,|t-s|^{2H}$, in such a way that $\mathrm{E}[Y(t) - Y(s)]^2 \leq \boldsymbol{C}\,|t-s|^{2H}$.

Applying the Kolmogorov result (see Billingsley (1999)), we yield inequality (A.4), which completes proof of lemma. $\square$

Proof of Lemma 7.2. For reason of simplicity let us suppose that $H > \frac{1}{2}$. Similar calculations could be done in case where $H \leq \frac{1}{2}$. For $x \in \mathbb{R}$, let $\widetilde{K}(x)$ be

$$\widetilde{K}(x) = \int_{-\infty}^{+\infty} \lambda^2 G(\lambda) \exp(i\lambda x)\,\mathrm{d}\lambda. \tag{A.5}$$

For Y, a solution of models (2.2), (2.5) or (2.6) and $i = 0, 1, \ldots, n-2$, we have

$$\begin{aligned}\frac{1}{a^2(n)}\left(\sigma_n^2(i) - \sigma^2(\tfrac{i}{n})\right) = \Bigg[\Bigg(&\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} + \int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{i}{n}}^{\frac{i+1}{n}} - 2\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\Bigg) \\ &\left(\sigma(u)\sigma(v) - \sigma^2(\tfrac{i}{n})\right)\left(R(|u-v|) + \widetilde{K}(u-v)\right)\mathrm{d}v\,\mathrm{d}u\Bigg]_{(1)} \\ &+ \left[2\sigma^2(\tfrac{i}{n})\left(\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} - \int_{-\frac{1}{n}}^{0}\int_0^{\frac{1}{n}}\right)\widetilde{K}(u-v)\,\mathrm{d}v\,\mathrm{d}u\right]_{(2)},\end{aligned}$$

where R is defined for $t \in]0,1]$, by

$$R(t) = \begin{cases} H(2H-1)v_{2H}^2 t^{2H-2}, & \text{if } Y \text{ is a solution of (2.2)} \\ -r''(t), & \text{if } Y \text{ is a solution of (2.5) or (2.6)} \end{cases} \tag{A.6}$$

where we recall that v_{2H}^2 is defined by (2.4).

Since σ belongs to C^1 on $[\,0;\ 1\,]$, we write $\sigma(u) = \sigma(\frac{i}{n}) + (u - \frac{i}{n})\sigma'(\frac{i}{n} + \theta_u(u - \frac{i}{n}))$, with $0 < \theta_u < 1$. We do the same for v.

As already seen in the proof of Lemma 7.1, L is bounded on $]0,1]$. Furthermore, L has two continuous derivatives except at the origin where, $tL'(t) = O(1)$ and $t^2 L''(t) = O(1)$, as $t \to 0^+$. So that for $t \in]\,0;\ 1\,]$, we have

$$|R(t)| \leq \boldsymbol{C} t^{2H-2}. \tag{A.7}$$

Using that σ and σ' are bounded on $[\,0;\ 1\,]$, that $\widetilde{K}$ is bounded on $\mathbb{R}$ since $\int_{-\infty}^{+\infty} \lambda^2 G(\lambda)\,\mathrm{d}\lambda < +\infty$ and the fact that $0 < H < 1$, we get

$$
\begin{aligned}
(1) = \Bigg(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} + \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{i}{n}}^{\frac{i+1}{n}} -2 \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\Bigg) \sigma'(\tfrac{i}{n} + \theta_u(u - \tfrac{i}{n}))\times \\
(R(|u-v|) + \widetilde{K}(u-v)) \left\{(u - \tfrac{i}{n})\sigma'(\tfrac{i}{n} + \theta_u(u-\tfrac{i}{n}))\right. \\
\left[\sigma(\tfrac{i}{n}) + \sigma'(\tfrac{i}{n} + \theta_v(v - \tfrac{i}{n}))(v - \tfrac{i}{n})\right] + \\
\left.\sigma'(\tfrac{i}{n} + \theta_v(v - \tfrac{i}{n}))(v - \tfrac{i}{n})\sigma(\tfrac{i}{n})\right\} \,\mathrm{d}v\,\mathrm{d}u = O\left(\tfrac{1}{n^{2H+1}}\right).
\end{aligned}
$$

Since $\int_{-\infty}^{+\infty} |\lambda|^3 G(\lambda)\,\mathrm{d}\lambda < +\infty$, the function $\widetilde{K}$, defined by equation (A.5), belongs to C^1 on $\mathbb{R}$. So, to tackle term (2), we use the first order Taylor approximation of $\widetilde{K}$, that is, $\widetilde{K}(u-v) = \widetilde{K}(0) + (u-v)\widetilde{K}'(\theta_{u,v}(u-v))$, for $u, v \in \mathbb{R}$, with $0 < \theta_{u,v} < 1$. In this way

$$
(2) = 2\sigma^2(\tfrac{i}{n}) \left(\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} - \int_{-\frac{1}{n}}^{0} \int_0^{\frac{1}{n}}\right) (u-v)\widetilde{K}'(\theta_{u,v}(u-v))\,\mathrm{d}v\,\mathrm{d}u.
$$

Since $\widetilde{K}'$ is bounded on $\mathbb{R}$ and σ in bounded on $[\,0;\ 1\,]$, we get that $(2) = O(n^{-3}) = O(n^{-(2H+1)})$, since $0 < H < 1$.

Equality (A.1) gives lemma. □

Proof of Lemma 7.3. For reason of simplicity, let us suppose that $H > \frac{1}{2}$. A similar proof could be done in case where $H \le \frac{1}{2}$.

Recall that functions R, $\widetilde{K}$ and ρ_H are respectively defined by equalities (A.6), (A.5) and (3.1). We remember that if Y is a solution of equation (2.2) (resp. (2.5), resp. (2.6)), then the notation $Y^{\star\star}$ stands for the solution process of equation (2.2) (resp. (2.5), resp. (2.6)), with $G \equiv 0$. With these notations, we have

$$
\begin{aligned}
\gamma_n(i,j) = \Bigg[a^2(n)\Bigg(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{j+1}{n}}^{\frac{j+2}{n}} - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{j+1}{n}}^{\frac{j+2}{n}} + \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}}\Bigg) \\
\left(\sigma(u)\sigma(v) - \sigma(\tfrac{i}{n})\sigma(\tfrac{j}{n})\right)\left(R(|u-v|) + \widetilde{K}(u-v)\right)\frac{\mathrm{d}v}{\sigma_n(i)}\frac{\mathrm{d}u}{\sigma_n(j)}\Bigg]_{(1)} \\
+ \Bigg[a^2(n)\frac{\sigma(\frac{i}{n})\sigma(\frac{j}{n})}{\sigma_n(i)\sigma_n(j)}\Bigg(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{j+1}{n}}^{\frac{j+2}{n}} - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{j+1}{n}}^{\frac{j+2}{n}} + \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}}\Bigg) \\
\widetilde{K}(u-v)\,\mathrm{d}v\,\mathrm{d}u\Bigg]_{(2)} \\
+ \Bigg[\Bigg(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{j+1}{n}}^{\frac{j+2}{n}} - \int_{\frac{i+1}{n}}^{\frac{i+2}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} - \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{j+1}{n}}^{\frac{j+2}{n}} + \int_{\frac{i}{n}}^{\frac{i+1}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}}\Bigg) \\
a^2(n)\left(\frac{\sigma(\frac{i}{n})\sigma(\frac{j}{n})}{\sigma_n(i)\sigma_n(j)} - 1\right) R(|u-v|)\,\mathrm{d}v\,\mathrm{d}u\Bigg]_{(3)}
\end{aligned}
$$

$$\Bigg[\left(\frac{a^2(n)}{n^{2H}}\times\lim_{n\to\infty}\left(\frac{a^2(n)}{n^{2H}}\right)^{-1}-1\right)\times\lim_{n\to\infty}\left(\frac{a^2(n)}{n^{2H}}\right)n^{2H}$$
$$\left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{j+1}{n}}^{\frac{j+2}{n}}-\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}}-\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{j+1}{n}}^{\frac{j+2}{n}}+\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}}\right)R(|u-v|)\,\mathrm{d}v\,\mathrm{d}u\Bigg]_{(4)}$$
$$+\left[n^{2H}\lim_{n}\left(\frac{a^2(n)}{n^{2H}}\right)\mathrm{E}[\delta_nY^{\star\star}(i)\delta_nY^{\star\star}(j)]-\rho_H(i-j)\right]_{(5)}$$

where the process $\delta_n Y^{\star\star}$ is defined by (4.2), Section 4.1.1.

If $i=j$, we get $\gamma_n(i,j)=0$ and the proof of the lemma is trivial. Thus we suppose $i\neq j$. We fix an integer $M\geq 4$. We look at the term (1).

As in the proof of Lemma 7.2, we write $\sigma(u)=\sigma(\frac{i}{n})+(u-\frac{i}{n})\,\sigma'(\frac{i}{n}+\theta_u(u-\frac{i}{n}))$, and we do the same for v.

Then we use the fact that since σ is strictly positive on $[\,0;\ 1\,]$, by (7.7) for $k=1$, we have, for $i=0,\dots,n-2$ and n large enough,

$$\sigma_n(i)\geq \boldsymbol{C}>0. \tag{A.8}$$

The functions σ and σ' are bounded on $[\,0;\ 1\,]$. More, the function $\widetilde{K}$ is bounded on $\mathbb{R}$ since $\int_{-\infty}^{+\infty}\lambda^2G(\lambda)\,\mathrm{d}\lambda<+\infty$. So, using (A.7) and (A.1), we see that there exists $M_0\in\mathbb{N}^*$, such that for all $M\geq M_0$, for all $n\geq M$,

$$\begin{aligned}
|(1)| &\leq \frac{\boldsymbol{C}}{n^{1-2H}}\left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{j+1}{n}}^{\frac{j+2}{n}}+\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}}+\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{j+1}{n}}^{\frac{j+2}{n}}+\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}}\right)\\
&\qquad\qquad\qquad\qquad\left(|v-u|^{2H-2}+1\right)\mathrm{d}v\,\mathrm{d}u\\
&\leq \frac{\boldsymbol{C}}{n}\left\{\boldsymbol{C}_M\mathbf{1}_{\{|i-j|\leq(M-1)\}}+|i-j|^{(2H-2)}\,1_{\{|i-j|\geq M\}}+\frac{1}{n^{2-2H}}\right\}\\
&\leq \frac{\boldsymbol{C}}{n}\left\{\boldsymbol{C}_M\mathbf{1}_{\{|i-j|\leq(M-1)\}}+M^{2H-2}\right\}.
\end{aligned}$$

For term (2), we use the fact that since $\int_{-\infty}^{+\infty}|\lambda|^3\,G(\lambda)\,\mathrm{d}\lambda<+\infty$, the function $\widetilde{K}$ belongs to C^1 on $\mathbb{R}$. As in the proof of Lemma 7.2, we use the first order Taylor expansion of $\widetilde{K}$, that is, for $u,v\in\mathbb{R}$, $\widetilde{K}(u-v)=\widetilde{K}(\frac{i}{n}-\frac{j}{n})+(u-\frac{i}{n}-(v-\frac{j}{n}))\widetilde{K}'((\frac{i}{n}-\frac{j}{n})+\theta_{i,j,u,v}(u-\frac{i}{n}-(v-\frac{j}{n})))$, with $0<\theta_{i,j,u,v}<1$. Thus

$$(2)=a^2(n)\frac{\sigma(\frac{i}{n})\sigma(\frac{j}{n})}{\sigma_n(i)\sigma_n(j)}\left(\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{j+1}{n}}^{\frac{j+2}{n}}-\int_{\frac{i+1}{n}}^{\frac{i+2}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}}-\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{j+1}{n}}^{\frac{j+2}{n}}+\int_{\frac{i}{n}}^{\frac{i+1}{n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}}\right)$$
$$(u-\tfrac{i}{n}-(v-\tfrac{j}{n}))\widetilde{K}'((\tfrac{i}{n}-\tfrac{j}{n})+\theta_{i,j,u,v}(u-\tfrac{i}{n}-(v-\tfrac{j}{n})))\,\mathrm{d}v\,\mathrm{d}u.$$

By (A.1) and (A.8) and the fact that functions $\widetilde{K}'$ and σ are respectively bounded on $\mathbb{R}$ and on $[\,0;\ 1\,]$, we get that for $n\geq M$,

$$|(2)|\leq\boldsymbol{C}\left(\frac{1}{n^{3-2H}}\right)\leq\boldsymbol{C}\,\frac{1}{n}\,M^{2H-2},$$

since $H<1$.

Let us look at the term (3). Using (7.7) for $k=1$ and inequality (A.8), it is easy to see that for n large enough, $|\frac{\sigma(\frac{i}{n})\sigma(\frac{j}{n})}{\sigma_n(i)\sigma_n(j)}-1| \leq \frac{\boldsymbol{C}}{n}$. Thus, using (A.7) and (A.1), as for term (1), we obtain that there exists $M_0 \in \mathbb{N}^*$, such that for all $M \geq M_0$, for all $n \geq M$,

$$|(3)| \leq \frac{\boldsymbol{C}}{n}\left\{\boldsymbol{C}_M \mathbf{1}_{\{|i-j| \leq (M-1)\}} + M^{(2H-2)}\right\}.$$

Finally we look at terms (4) and (5). Note that if Y is a solution of (2.2), these two terms vanish. So, we now look at Y, a solution of (2.5) or (2.6).

For term (4), using that $r(t) = 1 - L(t)t^{2H}$ for $0 < t \leq 1$ and that $\lim_{t\to 0^+} L(t) = C_0$, we obtain

$$\lim_{n\to\infty}\left(\frac{n^{2H}}{a^2(n)}\right) = 2\,(4-4^H)C_0. \tag{A.9}$$

Furthermore, knowing that $L(t) - C_0 = O(t^{2H})$ as $t \to 0^+$, we get

$$\left|\frac{a^2(n)}{n^{2H}} 2\,(4-4^H)\,C_0 - 1\right| \leq \boldsymbol{C}\left(\frac{1}{n}\right)^{2H}. \tag{A.10}$$

Now, as for term (1), we use (A.7) and obtain

$$\begin{aligned} |(4)| &\leq \frac{\boldsymbol{C}}{n^{2H}}\left\{\boldsymbol{C}_M \mathbf{1}_{\{|i-j| \leq (M-1)\}} + M^{(2H-2)}\right\} \\ &\leq \frac{\boldsymbol{C}}{n}\left\{\boldsymbol{C}_M \mathbf{1}_{\{|i-j| \leq (M-1)\}} + M^{(2H-2)}\right\}, \end{aligned}$$

since $H > \frac{1}{2}$.

To complete the proof of this lemma, we look at term (5). If $|i-j| = 1$ or $|i-j| = 2$, using the equality $L(t) - C_0 = O(t^{2H})$ as $t \to 0^+$ and (A.9), it is easy to prove that $|(5)| \leq \frac{\boldsymbol{C}}{n^{2H}} \leq \frac{\boldsymbol{C}}{n}$, since $H > \frac{1}{2}$. So let us suppose that $|i-j| \geq 3$. Since Y is a solution of equations (2.5) or (2.6), using (A.9) we have

$$(5) = \frac{n^{2H}}{2\,(4-4^H)\,C_0}\left(-6F(\tfrac{i-j}{n}) + 4F(\tfrac{i-j-1}{n}) + 4F(\tfrac{i-j+1}{n}) - F(\tfrac{i-j-2}{n}) - F(\tfrac{i-j+2}{n})\right),$$

where the function F is defined by

$$F(x) = (L(x) - C_0)\,|x|^{2H}, \quad \text{for} \quad x \in \mathbb{R}^*. \tag{A.11}$$

On the one hand, let us suppose that $3 \leq |i-j| \leq (M-1)$. As already seen in Lemma 7.1, L is bounded on $]\,0;\ 1\,]$ and $L(t) - C_0 = O(t^{2H})$ as $t \to 0^+$, so for $t \in]0,1]$, we have $|L(t) - C_0| \leq \boldsymbol{C}t^{2H}$. Thus we get

$$|(5)| \leq \boldsymbol{C}_M(\tfrac{1}{n^{2H}}) \leq \boldsymbol{C}_M(\tfrac{1}{n}),$$

since $H > \frac{1}{2}$.

On the other hand, if $|i-j| \geq M$, we use a third order Taylor expansion of F about $x = (i-j)/n$. We get that there exists $0 < \theta_k < 1$, for $k = 1,2,3,4$ such that

$$(5) = \tfrac{n^{2H-3}}{3\,(4-4^H)\,C_0}\left(-F'''(\tfrac{i-j}{n} - \tfrac{\theta_1}{n}) + F'''(\tfrac{i-j}{n} + \tfrac{\theta_2}{n}) + 2F'''(\tfrac{i-j}{n} - 2\tfrac{\theta_3}{n}) - 2F'''(\tfrac{i-j}{n} + 2\tfrac{\theta_4}{n})\right).$$

Since by hypothesis for $t \in]0,1]$, we have $|F'''(t)| \leq \boldsymbol{C} t^{(4H-3)}$, thus there exists $M_0 \in \mathbb{N}^*$, such that for all $M \geq M_0$ and $n \geq M$,

$$|(5)| \leq \boldsymbol{C}\left[\frac{1}{n^{2H}} \cdot \frac{1}{M^{(3-4H)}} \mathbf{1}_{H<\frac{3}{4}} + \frac{1}{n^{3-2H}} \mathbf{1}_{H \geq \frac{3}{4}}\right] \leq \boldsymbol{C}(\tfrac{1}{n})(\tfrac{1}{M})^{(2-2H)},$$

since $H > \frac{1}{2}$.

That yields Lemma 7.3. □

Proof of Lemma 7.5. First we compute $\mathrm{E}[T_n(h)]^2$. In this aim, we decompose this expectation into two terms S_1 and S_2 where

$$S_1 = \frac{1}{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathrm{E}\left[h(Y(\tfrac{i}{n}))\, h(Y(\tfrac{j}{n}))\, g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right)\, g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right]$$

$$S_2 = \frac{1}{n} \sum_{i=0}^{n-2} \mathrm{E}\left[h^2(Y(\tfrac{i}{n}))\, g^2\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right)\right],$$

where we recall that $\sigma_{n,1}$ is defined by (4.9).

Let us consider S_1. We fix $i, j \in \{0, 1, \ldots, n-2\}$, $i \neq j$ and we consider the change of variables

$$\begin{aligned} Y(\tfrac{i}{n}) &= Z_{1,n}(i,j) + A_{1,n}(i,j)\,\frac{\Delta_n Y(i)}{\sigma_{n,1}} + A_{2,n}(i,j)\,\frac{\Delta_n Y(j)}{\sigma_{n,1}}, \\ Y(\tfrac{j}{n}) &= Z_{2,n}(i,j) + B_{1,n}(i,j)\,\frac{\Delta_n Y(i)}{\sigma_{n,1}} + B_{2,n}(i,j)\,\frac{\Delta_n Y(j)}{\sigma_{n,1}}, \end{aligned}$$

with $(Z_{1,n}(i,j), Z_{2,n}(i,j))$ a zero mean Gaussian vector independent of $(\Delta_n Y(i), \Delta_n Y(j))$ and

$$A_{1,n}(i,j) = \frac{\mathrm{E}[Y(\frac{i}{n})\frac{\Delta_n Y(i)}{\sigma_{n,1}}] - \delta_{n,1}(i-j)\,\mathrm{E}[Y(\frac{i}{n})\frac{\Delta_n Y(j)}{\sigma_{n,1}}]}{1 - \delta^2_{n,1}(i-j)},$$

$$A_{2,n}(i,j) = \frac{\mathrm{E}[Y(\frac{i}{n})\frac{\Delta_n Y(j)}{\sigma_{n,1}}] - \delta_{n,1}(i-j)\,\mathrm{E}[Y(\frac{i}{n})\frac{\Delta_n Y(i)}{\sigma_{n,1}}]}{1 - \delta^2_{n,1}(i-j)},$$

where $\delta_{n,1}(i-j)$ is defined in (7.22).

Two similar formulas hold for $B_{1,n}(i,j)$ and $B_{2,n}(i,j)$.

For $|i-j| \geq 1$, one has $1-\rho_H^2(i-j) \geq \boldsymbol{C} > 0$, so that by Lemma 7.3 applied to $X_0 = Y$, one can show that for $|i-j| \geq 1$, $1-\delta_{n,1}^2(i-j) \geq \boldsymbol{C} > 0$. Thus a straightforward computation shows that for $i \neq j$

$$\max_{k=1,2} |A_{k,n}(i,j), B_{k,n}(i,j)| \leq \boldsymbol{C}\, n^{-H}\,. \tag{A.12}$$

Writing the Taylor expansion of h one has,

$$\begin{aligned} h(Y(\tfrac{i}{n})) = \sum_{k=0}^{3} \frac{1}{k!}\, h^{(k)}(Z_{1,n}(i,j)) \left[A_{1,n}(i,j)\, \frac{\Delta_n Y(i)}{\sigma_{n,1}} + A_{2,n}(i,j)\, \frac{\Delta_n Y(j)}{\sigma_{n,1}} \right]^k \\ + \frac{1}{4!}\, h^{(4)}(\theta_{1,n}(i,j)) \left[A_{1,n}(i,j)\, \frac{\Delta_n Y(i)}{\sigma_{n,1}} + A_{2,n}(i,j)\, \frac{\Delta_n Y(j)}{\sigma_{n,1}} \right]^4, \end{aligned}$$

with $\theta_{1,n}(i,j)$ between $Y(\frac{i}{n})$ and $Z_{1,n}(i,j)$.

A similar formula holds for $f(Y(\frac{j}{n}))$.

We can decompose S_1 as the sum of twenty-five terms. We use the notations J_{j_1,j_2} for the corresponding sums, where $j_1, j_2 = 0, \ldots, 4$ are the subscripts involving $h^{(j_1)}$ and $h^{(j_2)}$. We only consider J_{j_1,j_2} with $j_1 \leq j_2$. Then we obtain the following:

(A) One term of the form

$$J_{0,0} = \frac{1}{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathrm{E}\left[h(Z_{1,n}(i,j)) h(Z_{2,n}(i,j))\right] \mathrm{E}\left[g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right].$$

We recall that we defined function β in (7.11) by

$$\beta(k) = \mathrm{E}[g(\Delta_n b_H(0)) g(\Delta_n b_H(k))] = \sum_{p=1}^{+\infty} g_p^2\, p!\, \rho_H^p(k).$$

Using the fact that $|h''(x)| \leq P(|x|)$ and inequality (A.12), we can prove that for $i \neq i$,

$$\left|\mathrm{E}[h(Z_{1,n}(i,j)) h(Z_{2,n}(i,j))] - \mathrm{E}[h^2(Y(\tfrac{i}{n}))]\right| \leq \boldsymbol{C} \left| \frac{i-j}{n} \right|^H.$$

Also by using that $g \in L^2(\phi(x)\,\mathrm{d}x)$ and Mehler's formula, one can prove that

$$\left| \mathrm{E}\left[g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right] \right| \leq \boldsymbol{C}\, |\delta_{n,1}(i-j)|\,.$$

Finally, using Lemma 7.3 applied to $X_0 = Y$, and the fact that since $H < 1$, $\sum_{k=-\infty}^{+\infty} |k|^H\, |\rho_H(k)| < +\infty$ (see inequality (7.13)), one obtains

$$J_{0,0} \equiv \frac{1}{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathrm{E}[h^2(Y(\tfrac{i}{n}))]\, \mathrm{E}\left[g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right].$$

Now, as in the proof of Theorem 4.2, one can prove since $i \neq j$, that

$$\left| \mathrm{E}\left[g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\frac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right] - \beta(i-j) \right| \leq \boldsymbol{C} \left|\gamma_{n,1}(i-j)\right|,$$

where we recall that $\gamma_{n,1}$ is defined by (7.22). By using Lemma 7.3 applied to $X_0 = Y$, we finally get that

$$J_{0,0} \equiv \frac{1}{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathrm{E}[h^2(Y(\tfrac{i}{n}))]\, \beta(i-j).$$

Then we follow the proof given in Berzin, Latour and León (2014) to obtain that

$$\lim_{n \to +\infty} J_{0,0} = \left(\sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \beta(k) \right) \left(\int_0^1 \mathrm{E}[h^2(Y(u))]\, \mathrm{d}u \right). \tag{A.13}$$

(B) One term of the form $J_{0,1} \equiv 0$ by a symmetry argument: if $\mathcal{L}(U,V) = N(0,\Sigma)$ then $\mathrm{E}[U\, g(U)\, g(V)] = 0$ for g even or odd.

(C) Two terms of the form

$$J_{0,2} = \frac{1}{2\,n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathrm{E}[h(Z_{1,n}(i,j)) h''(Z_{2,n}(i,j))] \times$$
$$\mathrm{E}\left[g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\frac{\Delta_n Y(j)}{\sigma_{n,1}}\right) \left(B_{1,n}(i,j) \frac{\Delta_n Y(i)}{\sigma_{n,1}} + B_{2,n}(i,j) \frac{\Delta_n Y(j)}{\sigma_{n,1}} \right)^2 \right].$$

Since $|\delta_{n,1}(i-j)| \leq 1$, g is even, or odd with Hermite rank greater than or equal to three, then

$$\left| \mathrm{E}\left[g\left(\frac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\frac{\Delta_n Y(j)}{\sigma_{n,1}}\right) \left(B_{1,n}(i,j) \frac{\Delta_n Y(i)}{\sigma_{n,1}} + B_{2,n}(i,j) \frac{\Delta_n Y(j)}{\sigma_{n,1}} \right)^2 \right] \right|$$
$$\leq \boldsymbol{C} \left(\max_{k=1,2} B_{k,n}^2(i,j) \right) |\delta_{n,1}(i-j)|.$$

Using (A.12), Lemma 7.3 applied to $X_0 = Y$ and since $\frac{1}{n} \sum_{i=0}^{n} \sum_{j=0}^{n} |\rho_H(i-j)| \leq 2 \sum_{i=0}^{+\infty} |\rho_H(i)| < +\infty$, we get $J_{0,2} = O(n^{-2H}) = o(1)$.

(D) Two terms of the form $J_{0,3} \equiv 0$ by a symmetry argument: if $\mathcal{L}(U,V) = N(0,\Sigma)$ then $\mathrm{E}[(aU + bV)^3\, g(U)\, g(V)] = 0$ for any two constants a and b and for function g even or odd.

(E) Three terms of the form

$$J_{0,4} = \frac{1}{4!\,n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \mathrm{E}\left[h(Z_{1,n}(i,j)) h^{(4)}(\theta_{2,n}(i,j)) g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right) \right.$$
$$\left. \times \left\{ B_{1,n}(i,j) \frac{\Delta_n Y(i)}{\sigma_{n,1}} + B_{2,n}(i,j) \frac{\Delta_n Y(j)}{\sigma_{n,1}} \right\}^4 \right].$$

Therefore

$$|J_{0,4}| \leq \boldsymbol{C} \frac{1}{n} \sum_{\substack{i,j=0 \\ i \neq j}}^{n-2} \left(\max_{k=1,2} B_{k,n}^4(i,j) \right).$$

Finally using (A.12) once again, one obtains

$$J_{0,4} = O(n^{-(4H-1)}) = o(1),$$

since $H > \frac{1}{4}$.

Using the same type of arguments as for (C), (D), (E) we can prove that the other terms are all $o(1)$. Thus using (A.13) we proved that

$$\lim_{n \to +\infty} S_1 = \left(\sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \beta(k) \right) \left(\int_0^1 \mathrm{E}[h^2(Y(u))]\,\mathrm{d}u \right).$$

Let us now consider S_2. Similar computations, holding i fixed and doing a regression of $Y(\frac{i}{n})$ on $\frac{\Delta_n Y(i)}{\sigma_{n,1}}$, give that

$$\lim_{n \to +\infty} S_2 = \beta(0) \left(\int_0^1 \mathrm{E}[h^2(Y(u))]\,\mathrm{d}u \right).$$

Thus we proved that

$$\lim_{n \to +\infty} \mathrm{E}\left[T_n(h)\right]^2 = \sigma_g^2 \left(\int_0^1 \mathrm{E}[h^2(Y(u))]\,\mathrm{d}u \right). \tag{A.14}$$

Now let us compute $\mathrm{E}[T_n^{(m)}(h)]^2$. We decompose the last expression into two terms: $S_1 + S_2$, where

$$S_1 = \sum_{\substack{\ell_1,\ell_2=0 \\ \ell_1 \neq \ell_2}}^{m-1} \frac{1}{n} \sum_{i=\lfloor \frac{n\ell_1}{m} \rfloor - 1}^{\lfloor \frac{n(\ell_1+1)}{m} \rfloor - 2} \sum_{j=\lfloor \frac{n\ell_2}{m} \rfloor - 1}^{\lfloor \frac{n(\ell_2+1)}{m} \rfloor - 2} \mathrm{E}\left[h(Y(\tfrac{\ell_1}{m}))\, h(Y(\tfrac{\ell_2}{m})) \right.$$
$$\left. \times\, g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right) \right],$$

and

$$S_2 = \sum_{\ell=0}^{m-1} \frac{1}{n} \sum_{i=\lfloor \frac{n\ell}{m} \rfloor -1}^{\lfloor \frac{n(\ell+1)}{m} \rfloor -2} \sum_{j=\lfloor \frac{n}{\ell} m \rfloor -1}^{\lfloor \frac{n(\ell+1)}{m} \rfloor -2} \mathrm{E}\left[h^2(Y(\tfrac{\ell}{m})) g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right].$$

First we look at the first term. For fixed $\ell_1 \neq \ell_2$ and i, j (in this case i is necessarily different from j), we use the regression of $(Y(\frac{\ell_1}{m}), Y(\frac{\ell_2}{m}))$ on $(\frac{\Delta_n Y(i)}{\sigma_{n,1}}, \frac{\Delta_n Y(j)}{\sigma_{n,1}})$. In the same way, we can prove that

$$\lim_{n\to+\infty} S_1 = \sum_{\substack{\ell_1,\ell_2=0\\ \ell_1\neq\ell_2}}^{m-1} \mathrm{E}\left[h(Y(\tfrac{\ell_1}{m}))h(Y(\tfrac{\ell_2}{m}))\right] \times \lim_{n\to+\infty} \frac{1}{n} \sum_{i=\lfloor \frac{n\ell_1}{m} \rfloor -1}^{\lfloor \frac{n(\ell_1+1)}{m} \rfloor -2} \sum_{j=\lfloor \frac{n\ell_2}{m} \rfloor -1}^{\lfloor \frac{n(\ell_2+1)}{m} \rfloor -2} \beta(i-j) = 0$$

where last equality follows from convergence seen in (7.23).

Then for the second term S_2, for fixed ℓ, i, j, using a regression of $Y(\frac{\ell}{m})$ on $(\frac{\Delta_n Y(i)}{\sigma_{n,1}}, \frac{\Delta_n Y(i)}{\sigma_{n,1}})$ if $i \neq j$ and on $\frac{\Delta_n Y(i)}{\sigma_{n,1}}$ otherwise, as before similar straightforward calculations show that

$$\lim_{n\to+\infty} \mathrm{E}[T_n^{(m)}(h)]^2 = \lim_{n\to+\infty} S_2 = \sigma_g^2 \left(\frac{1}{m} \sum_{\ell=0}^{m-1} \mathrm{E}\left[h^2(Y(\tfrac{\ell}{m}))\right]\right),$$

and then

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathrm{E}[T_n^{(m)}(h)]^2 = \sigma_g^2 \left(\int_0^1 \mathrm{E}[h^2(Y(u))]\,\mathrm{d}u \right). \tag{A.15}$$

To conclude the proof of the lemma, we have to compute $\mathrm{E}[T_n(h)T_n^{(m)}(h)]$.

$$\mathrm{E}[T_n(h)T_n^{(m)}(h)] = \sum_{\ell=0}^{m-1} \frac{1}{n} \sum_{i=0}^{n-2} \sum_{j=\lfloor \frac{n\ell}{m} \rfloor -1}^{\lfloor \frac{n(\ell+1)}{m} \rfloor -2} \mathrm{E}\left[h(Y(\tfrac{\ell}{m}))\, h(Y(\tfrac{i}{n}))\, g\left(\tfrac{\Delta_n Y(i)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n Y(j)}{\sigma_{n,1}}\right)\right].$$

For fixed ℓ, i, j, using a regression of $(Y(\frac{i}{n}), Y(\frac{\ell}{m}))$ on $(\frac{\Delta_n Y(i)}{\sigma_{n,1}}, \frac{\Delta_n Y(j)}{\sigma_{n,1}})$ if $i \neq j$ and on $\frac{\Delta_n Y(i)}{\sigma_{n,1}}$ otherwise, as before similar straightforward calculations show that

$$\lim_{n\to+\infty} \mathrm{E}[T_n(h)T_n^{(m)}(h)] = \sigma_g^2 \left(\sum_{\ell=0}^{m-1} \int_{\frac{\ell}{m}}^{\frac{\ell+1}{m}} \mathrm{E}[h(Y(u))\, h(Y(\tfrac{\ell}{m}))]\,\mathrm{d}u \right),$$

so that

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\mathrm{E}[T_n(h)T_n^{(m)}(h)] = \sigma_g^2\left(\int_0^1 \mathrm{E}[h^2(Y(u))]\,\mathrm{d}u\right). \tag{A.16}$$

(A.14), (A.15) and (A.16) yield the lemma. □

A.2. Concerning the punctual estimation of σ

Proof of Lemma 8.1. For simplicity reasons, let us suppose that $H > \frac{1}{2}$. Similar calculations could be done for $H \leq \frac{1}{2}$.

We fix $i \in \{0, 1, \dots, n-1\}$ and we set $a_i = a_n^{(i)}(t)$ (see (5.1) for the definition of $a_n^{(i)}(t)$). With this notation, we have

$$\begin{aligned}
&\frac{1}{a^2(n)}\left((\sigma_n^{(i)}(t))^2 - \sigma^2(t)\right)\\
&= \Bigg[\left(\int_{a_i}^{a_i+\frac{1}{n}}\int_{a_i}^{a_i+\frac{1}{n}} + \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_i-\frac{1}{n}}^{a_i} -2\int_{a_i-\frac{1}{n}}^{a_i}\int_{a_i}^{a_i+\frac{1}{n}}\right)\\
&\qquad \left(\sigma(u)\sigma(v) - \sigma^2(t)\right)\left(R(|u-v|) + \widetilde{K}(u-v)\right)\mathrm{d}v\,\mathrm{d}u\Bigg]_{(1)}\\
&\qquad + \left[2\sigma^2(t)\left(\int_{a_i}^{a_i+\frac{1}{n}}\int_{a_i}^{a_i+\frac{1}{n}} - \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_i}^{a_i+\frac{1}{n}}\right)\widetilde{K}(u-v)\,\mathrm{d}v\,\mathrm{d}u\right]_{(2)}\\
&= (1) + (2),
\end{aligned}$$

where the functions R and $\widetilde{K}$ are respectively defined by (A.6) and (A.5).

For term (1), we make the change of variables $u = x + a_i$ and $v = y + a_i$ in the corresponding integral. We get

$$\begin{aligned}
(1) = \left(\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} + \int_{-\frac{1}{n}}^0\int_{-\frac{1}{n}}^0 -2\int_{-\frac{1}{n}}^0\int_0^{\frac{1}{n}}\right)\left(\sigma(x+a_i)\sigma(y+a_i) - \sigma^2(t)\right)\\
\times\left(R(|x-y|) + \widetilde{K}(x-y)\right)\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Since σ belongs to C^2 on $[0;\ 1]$, σ'' is uniformly continuous on this interval. Thus, since $nh \to +\infty$, we can write

$$\sigma(x+a_i) = \sigma(t) + (h(1-\tfrac{2i}{n}) + x)\sigma'(t) + \tfrac{1}{2}h^2(1-\tfrac{2i}{n})^2\sigma''(t) + o(h^2).$$

We do the same for y.

Furthermore, since $\int_{-\infty}^{+\infty}\lambda^4 G(\lambda)\,\mathrm{d}\lambda < +\infty$, the function $\widetilde{K}$ belongs to C^2 on $\mathbb{R}$. Also, and since the function G is even, we have $\widetilde{K}'(0) = 0$. Thus, for $x, y \in [0;\ 1]$, we work with the second order Taylor expansion of $\widetilde{K}$ about 0, that is, $\widetilde{K}(x-y) = \widetilde{K}(0) + \frac{1}{2}(x-y)^2\widetilde{K}''(\theta_{x,y}(x-y))$, where $0 < \theta_{x,y} < 1$.

Thus, since $nh \to +\infty$, $0 < H < 1$, σ belongs to C^2 on $[0;\ 1]$ and $\widetilde{K}''$ is bounded on $\mathbb{R}$, by (A.1) we have,

$$a^2(n) \times (1) = 2\sigma(t)\sigma'(t)h(1 - \tfrac{2i}{n}) + \frac{1}{2}h^2(1 - \tfrac{2i}{n})^2 \left(2\sigma(t)\sigma''(t) + 2(\sigma')^2(t)\right)$$
$$+ \ a^2(n)\sigma(t)\sigma'(t)\left(\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} + \int_{-\frac{1}{n}}^{0}\int_{-\frac{1}{n}}^{0} - 2\int_{-\frac{1}{n}}^{0}\int_0^{\frac{1}{n}}\right) R(|x-y|)\,(x+y)\,\mathrm{d}y\,\mathrm{d}x$$
$$+ \ o(h^2).$$

Now, recall that $\theta(t) = \sigma^2(t)$. Since last integral is equal to zero because the function R is even, by (A.1) we finally get the result if we prove that term (2) is $o(h^2/n^{2H})$.

As before, for term (2), with a second order Taylor expansion of $\widetilde{K}$ about 0, and since $\widetilde{K}''$ is bounded on $\mathbb{R}$, we finally get that $(2) = O(1/n^4) = o(h^2/n^{2H})$, since $0 < H < 1$ and $nh \to +\infty$.

Discussion about Remark 5.1 page 941. Note that we choose to work with $\Delta_n^{(i)} X_\mu(t)$ (see 5.2), and the last integral in this case is equal to zero. If we work with (5.8) as in Section 3.1, we obtain, in case where $H > \frac{1}{2}$, the following integral (the case where $H \leq \frac{1}{2}$ is similar),

$$a^2(n)\sigma(t)\sigma'(t)\left(\int_{\frac{1}{n}}^{\frac{2}{n}}\int_{\frac{1}{n}}^{\frac{2}{n}} + \int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} - 2\int_0^{\frac{1}{n}}\int_{\frac{1}{n}}^{\frac{2}{n}}\right) R(|x-y|)\,(x+y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \frac{2}{n}\sigma(t)\,\sigma'(t),$$

and this term does not vanish.

A way to explain this fact consists in noting that when we work with $\Delta_n^{(i)} X_0(t)$, the second order increments of X_0 can be written as the second derivative of a convolution of X_0 with a particular density, say

$$X_0(s + \tfrac{1}{n}) - 2X_0(s) + X_0(s - \tfrac{1}{n}) = n^{-2} X''_{0,n}(s),$$

where $X_{0,n}(s) = n\varphi(\cdot n) * X_0(s)$ and $\varphi(x) = \mathbf{1}_{[-1,0]} * \mathbf{1}_{[0;\ 1]}(x)$, that is

$$\varphi(x) = \begin{cases} 1 + x, & \text{if } x \in [-1, 0], \\ 1 - x, & \text{if } x \in [0;\ 1], \\ 0 & \text{otherwise.} \end{cases}$$

We remark that φ is even so, the term that we are looking at,

$$\frac{-a^2(n)}{n^3}\sigma(t)\sigma'(t)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi'(x)\varphi'(y)R(\frac{|x-y|}{n})\,(x+y)\,\mathrm{d}y\,\mathrm{d}x,$$

vanishes.

Now if we work with (5.8), the density φ is

$$\varphi(x) = \mathbf{1}_{[-1,0]} * \mathbf{1}_{[-1,0]}(x) = \begin{cases} 2+x, & \text{if } x \in [-2,-1], \\ -x, & \text{if } x \in [-1,0], \\ 0 & \text{otherwise,} \end{cases}$$

and φ is not even. $\square$

Proof of Lemma 8.3. The proof is made according to the same approach used in Lemma 7.3, and we suppose that $H > \frac{1}{2}$.

For $i, j \in \{0, 1, \ldots, n-1\}$ we set, $a_i = a_n^{(i)}(t)$ and $a_j = a_n^{(j)}(s)$, and we suppose that $a_i \neq a_j$. If not, $\gamma_n^{(i,j)}(t,s) = 0$ and lemma is obvious.

Recall that functions R, $\widetilde{K}$ and ρ_H are respectively defined by equalities (A.6), (A.5) and (3.1) and $Y^{\star\star}$ is the process solution of equations (2.2), (2.5) or (2.6) with $G \equiv 0$.

With this notation, we have

$$\begin{aligned}
&\gamma_n^{(i,j)}(t,s) \\
&= \Bigg[a^2(n)\left(\int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j}^{a_j+\frac{1}{n}} - \int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j-\frac{1}{n}}^{a_j} - \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j}^{a_j+\frac{1}{n}} + \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j-\frac{1}{n}}^{a_j}\right) \\
&\qquad (\sigma(u)\sigma(v) - \sigma(a_i)\sigma(a_j))\left(R(|u-v|) + \widetilde{K}(u-v)\right)\frac{\mathrm{d}v}{\sigma_n^{(i)}(t)}\frac{\mathrm{d}u}{\sigma_n^{(j)}(s)}\Bigg]_{(1)} \\
&\quad + \Bigg[a^2(n)\frac{\sigma(a_i)\sigma(a_j)}{\sigma_n^{(i)}(t)\sigma_n^{(j)}(s)}\left(\int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j}^{a_j+\frac{1}{n}} - \int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j-\frac{1}{n}}^{a_j}\right. \\
&\qquad\qquad \left. - \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j}^{a_j+\frac{1}{n}} + \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j-\frac{1}{n}}^{a_j}\right)\widetilde{K}(u-v)\,\mathrm{d}v\,\mathrm{d}u\Bigg]_{(2)} \\
&\quad + \Bigg[\left(\int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j}^{a_j+\frac{1}{n}} - \int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j-\frac{1}{n}}^{a_j} - \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j}^{a_j+\frac{1}{n}} + \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j-\frac{1}{n}}^{a_j}\right) \\
&\qquad\qquad a^2(n)\left(\frac{\sigma(a_i)\sigma(a_j)}{\sigma_n^{(i)}(t)\sigma_n^{(j)}(s)} - 1\right)R(|u-v|)\,\mathrm{d}v\,\mathrm{d}u\Bigg]_{(3)} \\
&\quad + \Bigg[\left(\frac{a^2(n)}{n^{2H}} \times \lim_{n\to\infty}\left(\frac{a^2(n)}{n^{2H}}\right)^{-1} - 1\right) \times \lim_{n\to\infty}\left(\frac{a^2(n)}{n^{2H}}\right) n^{2H} \\
&\qquad \left(\int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j}^{a_j+\frac{1}{n}} - \int_{a_i}^{a_i+\frac{1}{n}}\int_{a_j-\frac{1}{n}}^{a_j} - \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j}^{a_j+\frac{1}{n}} + \int_{a_i-\frac{1}{n}}^{a_i}\int_{a_j-\frac{1}{n}}^{a_j}\right) \\
&\qquad\qquad R(|u-v|)\,\mathrm{d}v\,\mathrm{d}u\Bigg]_{(4)} \\
&\quad + \left[n^{2H}\lim_{n}\left(\frac{a^2(n)}{n^{2H}}\right)\mathrm{E}[\delta_n^{(i)}Y^{\star\star}(t)\delta_n^{(j)}Y^{\star\star}(s)] - \rho_H(n(t-s) + 2h(j-i))\right]_{(5)}
\end{aligned}$$

$= (1) + (2) + (3) + (4) + (5),$

where the process $\delta_n^{(i)} Y^{\star\star}$ is defined by (5.3), Section 5.1.1.

Let us look at term (1). The first order Taylor expansion of σ about a_i is $\sigma(u) = \sigma(a_i) + (u - a_i)\sigma'(a_i + \theta_u(u - a_i))$, $0 < \theta_u < 1$.

Using (8.3), the fact that σ belongs to C^1 on $[0;\ 1]$ and that $\widetilde{K}$ is bounded, (A.1) and (A.7), it is easy to see that

$$|(1)| \leq \left(\int_{a_i}^{a_i+\frac{1}{n}} \int_{a_j}^{a_j+\frac{1}{n}} + \int_{a_i}^{a_i+\frac{1}{n}} \int_{a_j-\frac{1}{n}}^{a_j} + \int_{a_i-\frac{1}{n}}^{a_i} \int_{a_j}^{a_j+\frac{1}{n}} + \int_{a_i-\frac{1}{n}}^{a_i} \int_{a_j-\frac{1}{n}}^{a_j} \right)$$
$$\boldsymbol{C}\left(\frac{1}{n^{(1-2H)}}\right) \times \left(|u-v|^{(2H-2)} + 1\right) \mathrm{d}v\, \mathrm{d}u \leq \boldsymbol{C}\left(\tfrac{1}{n}\right),$$

since $0 < H < 1$.

For term (2), the first order Taylor expansion of the function $\widetilde{K}$ about $(a_i - a_j)$ gives for $u, v \in \mathbb{R}$,

$$\widetilde{K}(u-v) = \widetilde{K}(a_i - a_j) + (u - a_i - (v - a_j))\widetilde{K}'(a_i - a_j + \theta_{i,j,u,v}(u - a_i - (v - a_j))),$$

with $0 < \theta_{i,j,u,v} < 1$.

Since $\widetilde{K}'$ is bounded on $\mathbb{R}$, the function σ belongs to C^1 on $[0;\ 1]$, by inequality (8.3) and (A.1), we get $(2) = O(\frac{1}{n^{(3-2H)}}) = O(\frac{1}{n})$, since $0 < H < 1$.

Let us look at term (3). Proceeding as in Lemma 7.2 and Lemma 8.1, since σ belongs to C^1 on $[0;\ 1]$, we can show on the one hand that for $i = 0, 1, \ldots, n-1$, we have $|(\sigma_n^{(i)}(t))^2 - \sigma^2(a_i)| \leq \boldsymbol{C}(\frac{1}{n})$.

On the other hand, by using inequalities (8.3) and (8.4), this implies that

$$\left| \frac{\sigma(a_i)\sigma(a_j)}{\sigma_n^{(i)}(t)\sigma_n^{(j)}(s)} - 1 \right| \leq \boldsymbol{C}\left(\tfrac{1}{n}\right).$$

Thus by using (A.1) and inequality (A.7), we get that $(3) = O(n^{-1})$.

To end the proof of this lemma we look at term (4) and (5) that vanish if Y is a solution of equation (2.2). So, let us suppose that Y is a solution of equations (2.5) or (2.6).

By using inequalities (A.9), (A.10) and (A.7), we get that $(4) = O(n^{-2H}) = O(n^{-1})$, since $H > \frac{1}{2}$.

Finally, we look at term (5). As in Lemma 7.3, if $|a_i - a_j| = \frac{1}{n}$ or $|a_i - a_j| = \frac{2}{n}$, we can prove that $(5) = O(n^{-2H}) = O(n^{-1})$, since $H > \frac{1}{2}$.

So let us suppose that $|a_i - a_j| \neq \frac{1}{n}$ and that $|a_i - a_j| \neq \frac{2}{n}$, then

$$(5) = \frac{n^{2H}}{2\,(4 - 4^H)\,C_0} \left(-6F(a_i - a_j) + 4F(a_i - a_j - \tfrac{1}{n}) + 4F(a_i - a_j + \tfrac{1}{n}) \right.$$
$$\left. - F(a_i - a_j - \tfrac{2}{n}) - F(a_i - a_j + \tfrac{2}{n}) \right),$$

where function F has been defined by (A.11).

On the one hand, if $|a_i - a_j| \le \frac{3}{n}$, we get that $|(5)| \le \boldsymbol{C}(\frac{1}{n})^{2H} \le \boldsymbol{C}(\frac{1}{n})$, since $H > \frac{1}{2}$.

On the other hand, if $|a_i - a_j| > \frac{3}{n}$, then using a third order Taylor expansion of F about $(a_i - a_j)$ we obtain

$$|(5)| \le \boldsymbol{C}\left[\tfrac{1}{n^{2H}}\mathbf{1}_{H<\frac{3}{4}} + \tfrac{1}{n^{3-2H}}\mathbf{1}_{H\ge\frac{3}{4}}\right] \le \boldsymbol{C}(\tfrac{1}{n}),$$

since $H > \frac{1}{2}$. That yields Lemma 8.3. □

Proof of Lemma 5.5. We fix $i \in \{0, 1, \ldots, n-1\}$ and we set $a_i = a_n^{(i)}(t)$ (see (5.1) for the definition of $a_n^{(i)}(t)$). The random variable $\delta_n^{(i)} X_r(t)$ is defined as in (5.3), Section 5.1.1. With these notations, we have as in Lemma 4.7,

$$\begin{aligned}\delta_n^{(i)} X_r(t) = \sigma(Y(a_i))\, \delta_n^{(i)} Y(t) \\ &+ \left[\left(\int_{Y(a_i)}^{Y(a_i+\frac{1}{n})} - \int_{Y(a_i-\frac{1}{n})}^{Y(a_i)}\right)(\sigma(u) - \sigma(Y(a_i)))\,\mathrm{d}u\right]_{(1)} \\ &+ \left[\left(\int_{a_i}^{a_i+\frac{1}{n}} - \int_{a_i-\frac{1}{n}}^{a_i}\right)(\mu(Y(u)) - \mu(Y(a_i)))\,\mathrm{d}u\right]_{(2)} \qquad \text{(A.17)} \\ = \sigma(Y(a_i))\delta_n^{(i)} Y(t) + (1) + (2),\end{aligned}$$

where the process $\delta_n^{(i)} Y$ is defined by (5.3), Section 5.1.1. The two terms in brackets in (A.17) are denoted respectively by (1) and (2) in the following. As in the proof of Lemma 4.7, using that σ belongs to C^1 on $\mathbb{R}$, that μ is locally Lipschitz on $\mathbb{R}$ and the modulus of continuity of Y (see Lemma 7.1 p. 962, applied to $X_0 = Y$), we get

$$|(1)| \le \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{2H-2\delta},$$

and

$$|(2)| \le \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{1+H-\delta} \le \boldsymbol{C}(\omega)\left(\tfrac{1}{n}\right)^{2H-2\delta}.$$

Using (A.1) we finally yield this lemma. □

Proof of Lemma 5.6. For $i \in \{0, 1, \ldots, n-1\}$ we set $a_i = a_n^{(i)}(t)$ (see (5.1) for the definition of $a_n^{(i)}(t)$). With this notation and as for the proof of Lemma 5.4 for all real $k \ge 1$, we can decompose $\widehat{\alpha}_{n,r}(t) - \sigma^k(Y(t))$ as

$$\begin{aligned}\widehat{\alpha}_{n,r}(t) - \sigma^k(Y(t)) &= \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n}) g_k\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right)\sigma^k(Y(t)) \\ &+ \frac{2}{n}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n})(\sigma^k)'(Y(t))(Y(a_n^{(i)}(t)) - Y(t))\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\frac{1}{\mathrm{E}[|N|^k]}\left(\left|\Delta_n^{(i)}X_r(t)\right|^k-\sigma^k(Y(a_n^{(i)}(t)))\left|\Delta_n^{(i)}Y(t)\right|^k\right)\right]_{(1)}\\
&\quad+\left[\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\frac{1}{\mathrm{E}[|N|^k]}\left|\tfrac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\right|^k\sigma^k(Y(a_n^{(i)}(t)))\ (\sigma_{n,1}^k-1)\right]_{(2)}\\
&\quad+\left[\left(\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})-1\right)\sigma^k(Y(t))\right]_{(3)}\\
&\quad+\Bigg[\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\Big(\sigma^k(Y(a_n^{(i)}(t)))-\sigma^k(Y(t))\\
&\qquad\qquad-(\sigma^k)'(Y(t))(Y(a_n^{(i)}(t))-Y(t))\Big)\Bigg]_{(4)}\\
&+\left[\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})g_k\left(\tfrac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\right)\left(\sigma^k(Y(a_n^{(i)}(t)))-\sigma^k(Y(t))\right)\right]_{(5)}, \qquad (A.18)
\end{aligned}
$$

where $\sigma_{n,1}$ and g_k are respectively defined by equalities (4.9) and (4.7).

The five terms in brackets in (A.18) are denoted respectively by (1) to (5) in the following. As in Lemma 4.3, for $k\geq 1$ we consider inequality (A.2) applied to $x=\Delta_n^{(i)}X_r(t)$ and to $y=\sigma(Y(a_n^{(i)}(t)))\Delta_n^{(i)}Y(t)$. Using inequality (8.13) for $Y^\star=Y$, that functions K is bounded on $[-1;\ 1]$ and Lemma 5.5, since $k\geq 1$, we get

$$|(1)|\leq \boldsymbol{C}_k(\omega)\left(\frac{1}{n}\right)^{H-\delta k}.$$

In the same manner and using (7.7) for $\sigma_{n,1}^k$, we get

$$|(2)|\leq \boldsymbol{C}_k(\omega)\left(\frac{1}{n}\right)^{1-\delta k}\leq \boldsymbol{C}_k(\omega)\left(\frac{1}{n}\right)^{H-\delta k}.$$

For term (3) we use (8.2) for $\ell=K\in C^1$ on $[-1;\ 1]$ and then we obtain

$$|(3)|\leq \boldsymbol{C}_k(\omega)\left(\frac{1}{n}\right)\leq \boldsymbol{C}_k(\omega)\left(\frac{1}{n}\right)^{H-\delta k}.$$

Choosing δ small enough, that is $0<\delta<(H-\frac{1}{2})/k$, that remains possible since $H>\frac{1}{2}$, we then obtain that $(1)+(2)+(3)=o_{a.s.}(1/\sqrt{nh})$.

For term (4), a second order Taylor expansion of σ^k about the point $Y(t)$ gives

$$(4)=\frac{2}{n}\sum_{i=0}^{n-1}K(-1+\tfrac{2i}{n})\tfrac{1}{2}\left(Y(a_n^{(i)}(t))-Y(t)\right)^2(\sigma^k)''(Y(t)+\theta(Y(a_n^{(i)}(t))-Y(t))),$$

where $0<\theta<1$.

Using the modulus of continuity of Y (see Lemma 7.1 p. 962, applied to $X_0 = Y$), we get

$$|(4)| \leq \boldsymbol{C}_k(\omega) h^{2H-2\delta},$$

and then $(4) = o_{a.s.}(h^H)$, for $\delta > 0$ small enough.

For the fifth term, we will prove that $(5) = o_P(1/\sqrt{nh})$ through the following lemma.

Lemma A.1.

$$\mathrm{E}\left[2\sqrt{\frac{h}{n}}\sum_{i=0}^{n-1} K(-1+\tfrac{2i}{n}) g\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right)\left(f(Y(a_n^{(i)}(t))) - f(Y(t))\right)\right]^2 = o(1),$$

for f belonging to C^2 on $\mathbb{R}$, such that for all $x \in \mathbb{R}$, $|f''(x)| \leq P(|x|)$, where P is a polynomial and for a general function g with $(2+\delta)$-moments with respect to the standard Gaussian measure, even, or odd, with a Hermite rank greater than or equal to one and such that $A_g \neq \emptyset$ (for the definition of A_g, see Section 3).

Proof of Lemma A.1. This expectation contains four terms. We will compute only one term, the others could be treated in a similar way. This term is

$$(1) = \frac{4h}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n})\times$$
$$\mathrm{E}\left[g\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n^{(j)} Y(t)}{\sigma_{n,1}}\right) f(Y(a_n^{(i)}(t))) f(Y(a_n^{(j)}(t)))\right]$$

As in the proof of Lemma 7.5, we decompose this term into two terms S_1 and S_2, where

$$S_1 = \frac{4h}{n}\sum_{\substack{i,j=0\\ 2h|i-j|\geq\delta_0}}^{n-1} K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n})\times$$
$$\mathrm{E}\left[g\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n^{(j)} Y(t)}{\sigma_{n,1}}\right) f(Y(a_n^{(i)}(t))) f(Y(a_n^{(j)}(t)))\right]$$

while

$$S_2 = \frac{4h}{n}\sum_{\substack{i,j=0\\ 2h|i-j|<\delta_0}}^{n-2} K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n})\times$$
$$\mathrm{E}\left[g\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n^{(j)} Y(t)}{\sigma_{n,1}}\right) f(Y(a_n^{(i)}(t))) f(Y(a_n^{(j)}(t)))\right],$$

where $\delta_0 > 0$ is a fixed number.

Using that g has $(2+\delta)$-moments with respect to the standard Gaussian measure and that $|f''(x)| \leq P(|x|)$, we get that

$$|S_2| \leq \boldsymbol{C}\delta_0.$$

Let us consider S_1.

As in the proof of Lemma 7.5, we make the following regression. We fix $i,j\in\{0,1,\ldots,n-1\}$ such that $2h\,|i-j|\geq\delta_0$ and we consider the change of variables

$$\begin{aligned} Y(a_n^{(i)}(t)) &= Z_{1,n}(i,j)+A_{1,n}(i,j)\,\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}+A_{2,n}(i,j)\,\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}},\\ Y(a_n^{(j)}(t)) &= Z_{2,n}(i,j)+B_{1,n}(i,j)\,\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}+B_{2,n}(i,j)\,\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}, \end{aligned}$$

with $(Z_{1,n}(i,j),Z_{2,n}(i,j))$ a zero mean Gaussian vector independent of $(\Delta_n^{(i)}Y(t),\Delta_n^{(j)}Y(t))$ and

$$\begin{aligned} A_{1,n}(i,j) &= \frac{\mathrm{E}[Y(a_n^{(i)}(t))\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}]-\delta_{n,1}^{(i-j)}\,\mathrm{E}[Y(a_n^{(i)}(t))\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}]}{1-(\delta_{n,1}^{(i-j)})^2},\\ A_{2,n}(i,j) &= \frac{\mathrm{E}[Y(a_n^{(i)}(t))\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}]-\delta_{n,1}^{(i-j)}\,\mathrm{E}[Y(a_n^{(i)}(t))\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}]}{1-(\delta_{n,1}^{(i-j)})^2}, \end{aligned}$$

where $\delta_{n,1}^{(i-j)}$ stands for $\delta_n^{(i,j)}(t,t)$ defined in (8.7) for $X_0=Y$, that is

$$\begin{aligned} \delta_{n,1}^{(i-j)} &= \mathrm{E}\left[\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\,\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}\right]\\ &= \rho_H(2h(j-i))+\gamma_{n,1}^{(i-j)}, \end{aligned}$$

where the function ρ_H is defined by (3.1).

Two similar formulas hold for $B_{1,n}(i,j)$ and $B_{2,n}(i,j)$.

For $2h\,|i-j|\geq\delta_0$, using Lemma 8.3 applied to $X_0=Y$, one can show that $(1-(\delta_{n,1}^{(i-j)})^2)\geq \boldsymbol{C}_{\delta_0}>0$. Thus a straightforward computation shows that for $2h\,|i-j|\geq\delta_0$

$$\max_{k=1,2}|A_{k,n}(i,j),B_{k,n}(i,j)|\leq \boldsymbol{C}_{\delta_0}\,n^{-H}\,. \tag{A.19}$$

Writing the Taylor expansion of f one has,

$$\begin{aligned} f(Y(a_n^{(i)}(t))) &= f(Z_{1,n}(i,j))\\ &\quad+\left[A_{1,n}(i,j)\,\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}+A_{2,n}(i,j)\,\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}\right]f'(Z_{1,n}(i,j))\\ &\quad+\frac{1}{2!}\,f''(\theta_{1,n}(i,j))\left[A_{1,n}(i,j)\,\frac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}+A_{2,n}(i,j)\,\frac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}\right]^2, \end{aligned}$$

with $\theta_{1,n}(i,j)$ between $Y(a_n^{(i)}(t))$ and $Z_{1,n}(i,j)$.

A similar formula holds for $f(Y(a_n^{(j)}(t)))$.

We can decompose S_1 as the sum of nine terms. We use the notations J_{j_1,j_2} for the corresponding sums, where $j_1, j_2 = 0,\ 1,\ 2$ are the subscripts involving $h^{(j_1)}$ and $h^{(j_2)}$. We only consider J_{j_1,j_2} with $j_1 \leq j_2$. Then we obtain the following:

(A) One term of the form

$$J_{0,0} = \frac{4h}{n} \sum_{\substack{i,j=0 \\ 2h|i-j|\geq\delta_0}}^{n-1} K(-1+\tfrac{2i}{n})\, K(-1+\tfrac{2j}{n})\times \\ \mathrm{E}\left[f(Z_{1,n}(i,j))f(Z_{2,n}(i,j))\right] \mathrm{E}\left[g\left(\tfrac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}\right)\right].$$

Recall that we defined function β in (7.11) by

$$\beta(k) = \mathrm{E}[g(\Delta_n b_H(0))g(\Delta_n b_H(k))] = \sum_{p=1}^{+\infty} g_p^2\, p!\, \rho_H^p(k).$$

With these notations, using the fact that $|f''(x)| \leq P(|x|)$ and inequality (A.19), we can prove that for $2h\,|i-j| \geq \delta_0$,

$$\left|\mathrm{E}[f(Z_{1,n}(i,j))f(Z_{2,n}(i,j))] - \mathrm{E}[f^2(Y(t)]\right| \leq \boldsymbol{C}_{\delta_0}\, h^H.$$

Furthermore as in the proof of Lemma 5.2, one can prove that

$$\left|\mathrm{E}\left[g\left(\tfrac{\Delta_n^{(i)}Y(t)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n^{(j)}Y(t)}{\sigma_{n,1}}\right)\right] - \beta(2h(i-j))\right| \leq \boldsymbol{C}_{\delta_0} \left|\gamma_{n,1}^{(i-j)}\right| \leq \boldsymbol{C}_{\delta_0}(\tfrac{1}{n}),$$

last inequality follows from Lemma 8.3 applied to $X_0 = Y$.

Finally using a proof similar to the one of Lemma 5.2 we can prove that

$$\lim_{n\to+\infty} \frac{4h}{n} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n})\,|\beta(2h(i-j))| \\ = \kappa^2 \int_{-\infty}^{\infty} |\beta(x)|\, \mathrm{d}x < +\infty,$$

by inequality (7.13) and where κ^2 is defined by (3.5). Thus

$$J_{0,0} = \left(\frac{4h}{n} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} K(-1+\tfrac{2i}{n})K(-1+\tfrac{2j}{n})\beta(2h(i-j))\right) \mathrm{E}[f^2(Y(t)] + J_{0,0}^{\star},$$

where $\overline{\lim}_{n\to+\infty} \left|J_{0,0}^{\star}\right| \leq \boldsymbol{C}\delta_0$.

(B) One term of the form $J_{0,1} \equiv 0$ by a symmetry argument: if $\mathcal{L}(U, V) = N(0,\Sigma)$ then $\mathrm{E}[U\,g(U)\,g(V)] = 0$ for g even or odd.

(C) Two terms of the form

$$J_{0,2} = \frac{4h}{n} \sum_{\substack{i,j=0 \\ 2h|i-j| \geq \delta_0}}^{n-1} K(-1 + \tfrac{2i}{n})\, K(-1 + \tfrac{2j}{n}) \mathrm{E}\left[f(Z_{1,n}(i,j)) f''(\theta_{2,n}(i,j)) \times \right.$$

$$\left. g\left(\tfrac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}}\right) g\left(\tfrac{\Delta_n^{(j)} Y(t)}{\sigma_{n,1}}\right) \left(B_{1,n}(i,j) \frac{\Delta_n^{(i)} Y(t)}{\sigma_{n,1}} + B_{2,n}(i,j) \frac{\Delta_n^{(j)} Y(t)}{\sigma_{n,1}} \right)^2 \right].$$

By inequality (A.19), one has

$$|J_{0,2}| \leq \boldsymbol{C}_{\delta_0} \frac{nh}{n^{2H}} \leq \boldsymbol{C}_{\delta_0} \frac{h}{n^{(2H-1)}},$$

and since $H > \frac{1}{2}$, $\lim_{n \to +\infty} J_{0,2} = 0$.

Similarly one can prove that the other terms tend to zero.

Finally we proved that

$$(1) = \left(\frac{4h}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} K(-1 + \tfrac{2i}{n}) K(-1 + \tfrac{2j}{n}) \beta(2h(i-j)) \right) \mathrm{E}[f^2(Y(t)] + J_{0,0}^{\star\star},$$

where $\overline{\lim}_{n \to +\infty} \left| J_{0,0}^{\star\star} \right| \leq \boldsymbol{C} \delta_0$.

Since $\delta_0 > 0$ is fixed as small as we want, as in the proof of Lemma 5.2 we get

$$\lim_{n \to +\infty} (1) = \kappa^2 \left(\int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}x \right) \mathrm{E}[f^2(Y(t)].$$

In a similar way we could prove that the three other terms appearing in the expectation expressed in Lemma A.1 have the same last limit.

Thus lemma A.1 follows. □

This leads to Lemma 5.6. □

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