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On the stick-breaking representation of σ-stable Poisson-Kingman models

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Abstract: In this paper we investigate the stick-breaking representation for the class of σ-stable Poisson-Kingman models, also known as Gibbs-type random probability measures. This class includes as special cases most of the discrete priors commonly used in Bayesian nonparametrics, such as the two parameter Poisson-Dirichlet process and the normalized generalized Gamma process. Under the assumption $\sigma = u/v$, for any coprime integers $1 \leq u < v$ such that $u/v \leq 1/2$, we show that a σ-stable Poisson-Kingman model admits an explicit stick-breaking representation in terms of random variables which are obtained by suitably transforming Gamma random variables and products of independent Beta and Gamma random variables.

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1. Introduction

Random probability measures play a fundamental role in Bayesian nonparametrics as their distributions act as nonparametric priors. In this paper we focus on the stick-breaking representation of discrete random probability measures. On the one hand, stick-breaking representations have proved to be a fruitful tool for defining and investigating statistical models involving complex dependent nonparametric priors. Important contributions in this area are the seminal papers by MacEachern [25] and MacEachern [26] and, among others, De Iorio et al. [3], Duan et al. [4], Dunson and Park [5], Dunson et al. [6], Griffin and Steel [13], Petrone et al. [31] and Teh et al. [42]. On the other hand, stick-breaking representations have displayed great potential in addressing computational issues under the Bayesian nonparametric framework. For instance, recent simulation algorithms developed in the context of hierarchical mixture modeling, such as the blocked Gibbs-sampler by Ishwaran and James [15], the slice sampling by Walker [43] and the retrospective sampling by Papaspiliopoulos and Roberts [29], rely on the stick-breaking representation of the underlying nonparametric prior. Stick-breaking representations have also provided a natural tool for obtaining an approximate evaluation of the distribution of mean functionals of the corresponding prior. See, e.g., Muliere and Tardella [28] and reference therein for details.

The first comprehensive treatment of stick-breaking priors dates back to the paper by Ishwaran and James [15]. There, they introduced a class of stick-breaking priors including as special cases the Dirichlet process by Ferguson [10] and the two parameter Poisson-Dirichlet process by Perman et al. [30]. Specifically, let $\mathcal{P}_0$ be a nonatomic probability measure on a complete and separable metric space $\mathbb{X}$ equipped with the Borel σ-field $\mathscr{X}$. Also, let $(V_i)_{i\geq 1}$ be a sequence of independent random variables such that $\sum_{i\geq 1}\mathbb{E}\left[\log(1-V_i)\right]=-\infty$. Based on such V_i's, define a sequence of random probabilities $(P_i)_{i\geq 1}$ as $P_1=V_1$ and

$$P_i = V_i \prod_{j=1}^{i-1}(1-V_j) \tag{1}$$

for each $i>1$, and let $(Z_i)_{i\geq 1}$ be a sequence of random variables, independent of $(P_i)_{i\geq 1}$, and independent and identically distributed according to $\mathcal{P}_0$. Then,

$$\tilde{P} = \sum_{i\geq 1} P_i \delta_{Z_i},$$

is a stick-breaking prior in the class of Ishwaran and James [15]. For any $\sigma\in[0,1)$ and $\theta>-\sigma$, the stick-breaking representation of the two parameter Poisson-Dirichlet process is recovered by assuming the V_i's to be distributed according to the Beta distribution with parameter $(1-\sigma,\theta+i\sigma)$ for each $i\geq 1$. See Pitman and Yor [33] for a detailed account on other constructive definitions of the two parameter Poisson-Dirichlet process. The stick-breaking representation of the Dirichlet process, which was first derived by Sethuraman [38], is recovered as special case by setting $\sigma = 0$. See also Sethuraman and Tiwari [39] for details.

Apart from the two parameter Poisson-Dirichlet process, most of the discrete random probability measures do not admit a stick-breaking representation in terms of a collection of independent V_i's. As an example, Favaro et al. [8] derived the stick-breaking representation of the normalized inverse Gaussian process introduced in Lijoi et al. [22]. Specifically, let $b > 0$ and let $(V_i)_{i\geq 1}$ be a sequence of dependent random variables such that, for each $i \geq 1$, the conditional distribution of V_i given $(V_1, \ldots, V_{i-1})$ coincides with the distribution of the random variable

$$\frac{X_i}{X_i + Y_i}, \tag{2}$$

where X_i is distributed according to the generalized inverse Gaussian distribution with parameter $(b/\prod_{j=1}^{i-1}(1-V_j), 1, -i/2)$, with $\prod_{j=1}^{0}(1-V_j) \equiv 1$, and Y_i is distributed according to the positive $\frac{1}{2}$-stable distribution. Also, X_i is independent of Y_i. Based on such V_i's define a sequence $(P_i)_{i\geq 1}$ as in (1) and let $(Z_i)_{i\geq 1}$ be a sequence of random variables, independent of $(P_i)_{i\geq 1}$, and independent and identically distributed according to a nonatomic probability measure $\mathcal{P}_0$ on $(\mathbb{X}, \mathscr{X})$. Then

$$\tilde{G}_{\frac{1}{2},b} = \sum_{i\geq 1} P_i \delta_{Z_i},$$

is the normalized inverse Gaussian process. The reader is referred to Favaro et al. [8] and Favaro and Walker [9] for additional details on the stick-breaking representation of the normalized inverse Gaussian process. To the best of our knowledge the normalized inverse Gaussian process provides the first example of a prior admitting a stick-breaking representation in terms of dependent V_i's, and such that for any $i \geq 1$ the conditional distribution of V_i given $(V_1, \ldots, V_{i-1})$ is characterized by means of a straightforward transformation of random variables as in (2).

In this paper we investigate the stick-breaking representation for the class of σ-stable models, or Gibbs-type random probability measures, introduced by Pitman [34]. The former name, which is adopted throughout the paper, originates from their definition, whereas the latter originates from a characterization of the induced predictive distribution. See Gnedin and Pitman [11] for details. The class of σ-stable Poisson-Kingman models forms a large class of discrete random probability measures which are indexed by a parameter $\sigma \in (0, 1)$ and a nonnegative measurable function h. The two parameter Poisson-Dirichlet process and the normalized inverse Gaussian process are σ-stable Poisson-Kingman models under suitable specifications of σ and h. Another noteworthy example is the normalized generalized Gamma process. This is a popular nonparametric prior which coincides with the normalized inverse Gaussian process if $\sigma = 1/2$. See, e.g., James [16], Pitman [34], Lijoi et al. [22], Lijoi et al. [23] and James [19]. The class of σ-stable Poisson-Kingman models is nowadays the subject of a rich and active literature. Indeed σ-stable Poisson-Kingman models share properties that are appealing from both a theoretical and an applied point of view: i) they stand out in terms of mathematical tractability; ii) they are characterized by a flexible parameterization via the function h; iii) they admit an intuitive characterization in terms of their predictive distribution. See De Blasi et al. [2] for an up-to-date review. In particular, in the context of Bayesian nonparametric mixture modeling, σ-stable Poisson-Kingman models have proved to be a valid alternative to the Dirichlet process. See, e.g., Lijoi et al. [22] and Lijoi et al. [23].

A stick-breaking representation for σ-stable Poisson-Kingman models has been recently proposed by Favaro and Walker [9] in order to develop a slice sampling algorithm for hierarchical mixture models. However, such a representation involves a sequence of dependent random variables V_i's whose conditional distributions have not been explicitly characterized in terms of a straightforward transformation of random variables, as in the case of the normalized inverse Gaussian process. In this paper we present a characterization of these conditional distributions under some assumptions on the parameter σ. Specifically, let $T_{\sigma,h}$ be an almost surely finite and positive random variable with density function of the form hf_σ, with f_σ being the positive σ-stable density function. We show that under the assumption $\sigma = u/v$, for any coprime integers $1 \leq u < v$ such that $u/v \leq 1/2$, the conditional distribution of V_i given $T_{\sigma,h}$ and $(V_1, \ldots, V_{i-1})$ coincides with the distribution of a random variable of the form (2) where X_i is a suitable transformation of a product of Beta and Gamma random variables, whereas Y_i is an inverse Gamma random variable. Also, X_i is independent of Y_i for each $i \geq 1$. Our results heavily rely on a well-known representation of f_σ in terms of the G-functions introduced by Meijer [27]. Clearly, by a suitable specification of the function h, an explicit stick-breaking representation for the normalized generalized Gamma process follows as a by-product of our main result. Such a representation thus generalizes the stick-breaking representation of the normalized inverse Gaussian process in Favaro et al. [8], which is recovered by setting $\sigma = 1/2$.

The paper is structured as follows. In Section 2 we review of the class of σ-stable Poisson-Kingman models and we recall the stick-breaking representation introduced by Favaro and Walker [9]. Section 3 contains the main result of the paper, namely a characterization of the distribution of the V_i's for a $\frac{u}{v}$-stable Poisson-Kingman model, for any coprime integers $1 \leq u < v$ such that $u/v \leq 1/2$. In Section 4 we introduce an exact sampling methods for these V_i's, whereas in Section 5 we present some concluding remarks with a view towards potential applications of our results. The Appendix contains a brief review of G-functions.

2. σ-stable Poisson-Kingman models

We start by recalling the concept of completely random measure (CRM) introduced by Kingman [20]. A CRM $\tilde{\mu}$ is a random element with values on the space of boundedly finite measures on $\mathbb{X}$ and such that for any $A_1, \ldots, A_n$ in $\mathscr{X}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, the random variables $\tilde{\mu}(A_1), \ldots, \tilde{\mu}(A_n)$ are mutually independent. The distribution of a CRM $\tilde{\mu}$ is characterized by means of the Lévy-Khintchine representation of its corresponding Laplace functional transform, i.e.,

$$\mathbb{E}\left[\mathrm{e}^{-\int_{\mathbb{X}} f(x)\tilde{\mu}(dx)}\right] = \exp\left\{-\int_{\mathbb{R}^+\times\mathbb{X}}\left(1-\mathrm{e}^{-sf(y)}\right)\nu(ds,dy)\right\},$$

for any $f : \mathbb{X} \to \mathbb{R}$ such that $\int_{\mathbb{X}} |f(x)|\tilde{\mu}(dx) < +\infty$ almost surely. The measure ν is referred to as the Lévy intensity measure and it uniquely characterizes $\tilde{\mu}$. Kingman [20] showed that a CRM $\tilde{\mu}$ is almost surely discrete and, accordingly, it can be represented in terms of nonnegative random jumps $(\tilde{J}_i)_{i\geq 1}$ at $\mathbb{X}$-valued random locations $(Y_i)_{i\geq 1}$, i.e.,

$$\tilde{\mu} \stackrel{\mathrm{d}}{=} \sum_{i\geq 1} \tilde{J}_i \delta_{Y_i}.$$

For our purposes it is sufficient to consider Lévy intensity measure that can be factorized as $\nu(ds, dy) = \rho(ds)\mathcal{P}_0(dy)$, where ρ is a Lévy measure driving the jump part of $\tilde{\mu}$ and $\mathcal{P}_0$ is a nonatomic probability measure driving the location part of $\tilde{\mu}$. Such a factorization implies the independence between $(\tilde{J}_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ so that, without loss of generality, the locations $(Y_i)_{i\geq 1}$ are assumed to be a sequence of independent $\mathbb{X}$-valued random variables identically distributed according to $\mathcal{P}_0$.

The class of σ-stable Poisson-Kingman models was first introduced in Pitman [34] and further investigated in Gnedin and Pitman [11]. In particular, Gnedin and Pitman [11] introduced the notion of Gibbs-type predictive distribution which characterize the class of σ-stable Poisson-Kingman models. For any $\sigma \in (0, 1)$ let $\tilde{\mu}_\sigma$ be a σ-stable CRM, namely a CRM characterized by the Lévy intensity measure

$$\nu(ds, dy) = \rho_\sigma(ds)\mathcal{P}_0(dy) = \frac{\sigma}{\Gamma(1-\sigma)} s^{-\sigma-1} ds \mathcal{P}_0(dy).$$

Let $T_\sigma = \sum_{i\geq 1} \tilde{J}_i$ be the total mass of $\tilde{\mu}_\sigma$. Since $\int_0^\epsilon \rho_\sigma(s)ds = +\infty$ for any $\epsilon > 0$ one can verify that the random variable T_σ is positive and finite almost surely. In particular T_σ is a positive σ-stable random variable with density function denoted by f_σ. Intuitively, under this setup, one can define an almost surely discrete random probability measure $\tilde{P}_\sigma$ by normalizing $\tilde{\mu}_\sigma$ with respect to T_σ, namely

$$\tilde{P}_\sigma = \frac{\tilde{\mu}_\sigma}{T_\sigma} \stackrel{\mathrm{d}}{=} \sum_{i\geq 1} \tilde{P}_i \delta_{Y_i},$$

with $\tilde{P}_i = \tilde{J}_i/T_\sigma$ and where $(Y_i)_{i\geq 1}$ is a sequence of random variables, independent of $(\tilde{P}_i)_{i\geq 1}$, and independent and identically distributed as $\mathcal{P}_0$. $\tilde{P}_\sigma$ is termed normalized σ-stable process with base distribution $\mathcal{P}_0$ and it was introduced by Kingman [21]. See Regazzini et al. [37] and James et al. [18] for a generalization of P_σ obtained by replacing μ_σ with any CRM μ. This is the class of normalized random measures.

A σ-stable Poisson-Kingman model is defined as a generalization of $\tilde{P}_\sigma$ obtained by suitably deforming (tilting) the distribution of the total mass T_σ. Let

$(P_{(i)})_{i\geq 1}$ be the decreasing rearrangement of $(\tilde{P}_i)_{i\geq 1}$ and let $T_{\sigma,h}$ be a nonnegative random variable with density function $f_{T_{\sigma,h}}(t) = h(t)f_\sigma(t)$ for a nonnegative measurable function h. Denoting by $\mathrm{PK}(\rho_\sigma \,|\, t)$ the conditional distribution of $(P_{(i)})_{i\geq 1}$ given $T_{\sigma,h} = t$, let $\mathrm{PK}(\rho_\sigma, hf_\sigma) = \int_0^{+\infty} \mathrm{PK}(\rho_\sigma \,|\, t) f_{T_{\sigma,h}}(t)dt$ be a mixture distribution which is termed in Pitman [34] as the Poisson-Kingman distribution with Lévy measure ρ_σ and mixing density function hf_σ. For short we refer to the distribution $\mathrm{PK}(\rho_\sigma, hf_\sigma)$ as the σ-stable Poisson-Kingman distribution with parameter h. A σ-stable Poisson-Kingman model with parameter h and base distribution $\mathcal{P}_0$ is the almost surely discrete random probability measure

$$\tilde{P}_{\sigma,h} = \sum_{i\geq 1} P_{(i)}\delta_{Y_i}, \tag{3}$$

where $(P_{(i)})_{i\geq 1}$ is a sequence of random probabilities distributed according to the σ-stable Poisson-Kingman distribution with parameter h, and $(Y_i)_{i\geq 1}$ is a sequence of random variables, independent of $(P_{(i)})_{i\geq 1}$, and independent and identically distributed according to the nonatomic probability measure $\mathcal{P}_0$. According to the definition of $\tilde{P}_{\sigma,h}$, we can write $\tilde{P}_{\sigma,h}(\cdot) = \tilde{\mu}_{\sigma,h}(\cdot)/T_{\sigma,h}$ where $\tilde{\mu}_{\sigma,h}$ is an a.s. discrete random measure with distribution $\mathbb{P}_{\sigma,h}$ absolutely continuous with respect to the distribution $\mathbb{P}_\sigma$ of $\tilde{\mu}_\sigma$, and such that $\mathrm{d}\mathbb{P}_{\sigma,h}(\tilde{\mu})/\mathrm{d}\mathbb{P}_\sigma = h(\tilde{\mu}(\mathbb{X}))$, and $T_{\sigma,h}$ is the total mass of $\tilde{\mu}_{\sigma,h}$ with density function $f_{T_{\sigma,h}}$. Note that the aforementioned normalized σ-stable process is the σ-stable Poisson-Kingman model corresponding to the choice of $h = 1$. We refer to the monograph by Pitman [35] for a comprehensive and stimulating account on σ-stable Poisson-Kingman models.

Example 1. For any $\sigma \in (0,1)$ and $\theta > -\sigma$ the two parameter Poisson-Dirichlet process $\tilde{P}_{\sigma,\theta}$ is a σ-stable Poisson-Kingman model with parameter h of the form

$$h(t) = \frac{\sigma\Gamma(\theta)}{\Gamma(\theta/\sigma)} t^{-\theta}. \tag{4}$$

The normalized σ-stable process coincides with $\tilde{P}_{\sigma,0}$. See Perman et al. [30], Pitman and Yor [33], James [16], Pitman [34] and James [19] for detailed accounts on $\tilde{P}_{\sigma,\theta}$.

Example 2. For any $\sigma \in (0,1)$ and $b > 0$ the normalized generalized Gamma process $\tilde{G}_{\sigma,b}$ is a σ-stable Poisson-Kingman model with parameter h of the form

$$h(t) = \mathrm{e}^{b - b^{1/\sigma} t}. \tag{5}$$

The normalized σ-stable process coincides with $\tilde{G}_{\sigma,0}$, whereas $\tilde{G}_{\frac{1}{2},b}$ is the normalized inverse Gaussian process. See James [16], Pitman [34], Lijoi et al. [22], Lijoi [23], Lijoi et al. [24] and James [19] for detailed accounts on $\tilde{G}_{\sigma,b}$ and applications.

The stick-breaking representation of $\tilde{P}_{\sigma,h}$ is obtained by the size-biased rearrangement of the decreasing ordered random probabilities $(P_{(i)})_{i\geq 1}$ in (3). We define a new sequence $(P_i)_{i\geq 1}$ of random probabilities such that $P_i = P_{(\pi_i)}$, where, for any positive integer $k \geq 1$ and for all finite sets $j_1, \ldots, j_k$ of distinct positive integers, the conditional probability of the event $\{\pi_i = j_i$ for all $1 \leq$

$i \leq k\}$ given $(P_{(i)})_{i\geq 1}$, is

$$P_{(j_1)} \frac{P_{(j_2)}}{1 - P_{(j_1)}} \cdots \frac{P_{(j_k)}}{1 - P_{(j_1)} \cdots - P_{(j_k)}}.$$

Intuitively, we pick the first atom $Z_1 = Y_{j_1}$ with probability $P_{(j_1)}$, remove it from the atoms under consideration, then we pick the second atom $Z_2 = Y_{j_2}$ with probability proportional to $P_{(j_2)}$, remove it, and so on. The resulting rearrangement of atoms does not affect the distribution of the corresponding σ-stable Poisson-Kingman model, i.e.,

$$\tilde{P}_{\sigma,h} \stackrel{\mathrm{d}}{=} \sum_{i\geq 1} P_i \delta_{Z_i}.$$

The sequence $(P_i)_{i\geq 1}$ is referred to as the size-biased random permutation of $(P_{(i)})_{i\geq 1}$. See, e.g., Pitman [32] for the interplay between size-biased random permutations and almost surely discrete random probability. The next result summarizes Lemma 1 and Lemma 2 in Pitman [34] and it provides the stick-breaking representation of a σ-stable Poisson-Kingman model. See also Perman et al. [30] for details.

Theorem 1. *Let $\tilde{P}_{\sigma,h}$ be a σ-stable Poisson-Kingman model and let $(P_{(i)})_{i\geq 1}$ be the corresponding characterizing sequence of decreasing ordered random probabilities. Then,*

$$\tilde{P}_{\sigma,h} \stackrel{d}{=} \sum_{i\geq 1} P_i \delta_{Z_i},$$

where $(P_i)_{i\geq 1}$ is the size-biased random permutation of $(P_{(i)})_{i\geq 1}$ and $(Z_i)_{i\geq 1}$ is a sequence of independent random variables identically distributed according to the base distribution $\mathcal{P}_0$. Moreover,

$$P_i \stackrel{d}{=} V_i \prod_{j=1}^{i-1} (1 - V_j)$$

where $(V_i)_{i\geq 1}$ is a sequence of random variables such that the conditional distribution of V_i given $T_{\sigma,h}$ and $(V_1, \ldots, V_{i-1})$ has a density function on $(0, 1)$ of the form

$$g_{V_i \mid V_1,\ldots,V_{i-1},T_{\sigma,h}}(v_i \mid v_1, \ldots, v_{i-1}, t) = \frac{\sigma (t w_i v_i)^{-\sigma} f_\sigma(t w_i (1 - v_i))}{\Gamma(1 - \sigma) f_\sigma(t w_i)} \tag{6}$$

with respect to the Lebesgue measure, for each $i \geq 1$ and where $w_i = \prod_{j=1}^{i-1} (1-v_j)$ with the convention that $w_1 = 1$. The sequence $(P_i)_{i\geq 1}$ is independent of the sequence $(Z_i)_{i\geq 1}$.

Theorem 1 provides the distribution of the stick-breaking random variables V_i's given the total mass $T_{\sigma,h}$. This is coherent with the hierarchical structure of the Poisson-Kingman distribution. After specifying the parameter h, the conditional density function of V_i given $(V_1, \ldots, V_{i-1})$ is derived from (6) by integrating out the random variable $T_{\sigma,h}$ with density function $h f_\sigma$. The stick-breaking representations discussed in the Introduction are recovered by a direct application of Theorem 1. In particular, the stick-breaking representation of $\tilde{P}_{\sigma,\theta}$, for any $\sigma \in (0,1)$ and $\theta > -\sigma$ is obtained by combining Theorem 1 with (4). Similarly, the stick-breaking representation of $\tilde{G}_{\sigma,b}$, for any $\sigma \in (0,1)$ and $b > 0$ is obtained by obtained by combining Theorem 1 with (5). As a special case one obtains the stick-breaking representation of the normalized inverse Gaussian process $\tilde{G}_{\frac{1}{2},b}$.

3. A representation of $\frac{u}{v}$-stable Poisson-Kingman models

While providing the stick-breaking representation for the entire class of σ-stable Poisson-Kingman models, Theorem 1 leaves open the problem of finding a straightforward characterization for the conditional density function introduced in (6). So far this problem has been solved under suitable choices of the parameters σ and h, namely $\sigma \in (0,1)$ and h of the form (4), and $\sigma = 1/2$ and h of the form (5). In this section, we present a solution to this problem for the class of $\frac{u}{v}$-stable Poisson-Kingman models, for any coprime integers $1 \leq u < v$ such that $u/v \leq 1/2$.

Our results exploit a well-known representation of the σ-stable density function f_σ in terms of the class of G-functions introduced by Meijer [27]. The reader is referred to the Appendix for a concise account on G-functions. Specifically, according to Zolotarev [44], if $\sigma = u/v$ for any coprime integers $1 \leq u < v$ then one has

$$f_{\frac{u}{v}}(x) = \prod_{i=1}^{u-1} \frac{\Gamma(\frac{i}{u})}{\Gamma(\frac{i}{v})} \prod_{j=u}^{v-1} \frac{1}{\Gamma(\frac{j}{v})} v^{\frac{v}{u}} G_{v-1,u-1}^{0,v-1}\left(\frac{v^v}{u^u} x^u \,\middle|\, \begin{matrix} \left(1 - \frac{1}{u} - \frac{i}{v}\right)_{i=1}^{v-1} \\ \left(1 - \frac{1}{u} - \frac{j}{u}\right)_{j=1}^{u-1} \end{matrix} \right), \tag{7}$$

where $G_{p,q}^{m,n}$ denotes a G-function of order (m,n,p,q). The well-known representation of the density function of the $\frac{1}{2}$-stable random variable is recovered as special case of (7). Indeed if $u = 1$ and $v = 2$, by combining (A.4) with (A.2) one obtains

$$f_{\frac{1}{2}}(x) = \frac{1}{2\Gamma(\frac{1}{2})} x^{-\frac{3}{2}} \mathrm{e}^{-\frac{1}{4x}}. \tag{8}$$

Further special cases of (7) are obtained by suitable specification of u and v and by exploiting properties of G-functions. See the monograph by Chaumont and Yor [1] and references therein for a comprehensive account of $\frac{u}{v}$-stable density functions.

Hereafter we adopt the notation $X \,|\, Y \stackrel{\mathrm{d}}{=} Z$ to indicate that the conditional distribution of X given Y coincides with the distribution of Z. We start by introducing the notion of exponentially tilted random variables. Specifically, let X be a nonnegative random variable distributed according to a distribution F_X and, for any $b > 0$ let Y be a random variable distributed according to a distribution F_Y of the form

$$F_Y(y) = \frac{\mathrm{e}^{-by} F_X(y)}{\int_0^{+\infty} \mathrm{e}^{-bx} F_X(dx)}.$$

We say that F_Y is the exponential tilting with weight b of F_X, and we write $Y \sim \mathcal{E}_{\mathcal{T}}(b, X)$. Accordingly, the random variable Y is termed exponentially tilted random variable. Moreover, for any coprime integers $1 \leq u < v$ such that $u/v < 1/2$, define

$$L_{\frac{u}{v}} = \prod_{i=0}^{u-2} B_{\frac{2i+2}{v}, \frac{i+1}{u} - \frac{2i+2}{v}} \prod_{i=0}^{u-2} B_{\frac{2i+1}{v}, \frac{i+1}{u} - \frac{2i+2}{v}} \prod_{i=1}^{v-2u} G_{\frac{2(u-1)+i}{v}, 1} \tag{9}$$

where $B_{a,b}$ and $G_{a,b}$ denote a Beta random variable with parameter (a, b) and a Gamma random variable with parameter (a, b), respectively. We refer to the monograph by Springer [41] and references therein for a comprehensive and stimulating account on distributional properties of products of independent Beta and Gamma random variables.

For any coprime integers $1 \leq u < v$ such that $u/v < 1/2$, the density function of the stick-breaking random variables of a $\frac{u}{v}$-stable Poisson-Kingman model is given by (6) with $\sigma = u/v$. We present a novel characterization of such a density function in terms of a suitable transformation of two independent random variables. Let $(V_{\frac{u}{v},i})_{i\geq 1}$ be a sequence of dependent random variables such that

$$(V_{\frac{u}{v},i} \,|\, V_{\frac{u}{v},1}, \ldots, V_{\frac{u}{v},i-1}, T_{\frac{u}{v},h}) \stackrel{\mathrm{d}}{=} \frac{X_i}{X_i + Y_i} \tag{10}$$

where, for each $i \geq 1$, the random variables X_i and Y_i are assumed to be independent and

$$\frac{1}{X_i} \sim \mathcal{E}_{\mathcal{T}} \left(\frac{\frac{u^2}{v^{\frac{v}{u}} T_{\frac{u}{v},h}}}{\prod_{j=1}^{i-1}(1 - V_{\frac{u}{v},j})}, \frac{1}{L_{\frac{u}{v}}^{1/u}} \right) \tag{11}$$

and

$$Y_i \sim \mathcal{IG} \left(1 - \frac{u}{v}, \frac{\frac{u^2}{v^{\frac{v}{u}} T_{\frac{u}{v},h}}}{\prod_{j=1}^{i-1}(1 - V_{\frac{u}{v},j})} \right), \tag{12}$$

with the convention that an empty product is defined to be one, namely $\prod_{j=1}^{0}(1 - V_{\frac{u}{v},j}) = 1$. The problem of exact sampling from the exponentially tilted random variable (11), and hence from the random variable (10), is discussed in the next section. The next theorem introduces an explicit stick-breaking representation for $\frac{u}{v}$-stable Poisson-Kingman models, for any coprime integers $1 \leq u < v$ such that $u/v < 1/2$.

Theorem 2. *For any coprime integers $1 \leq u < v$ such that $u/v < 1/2$, let $(V_{\frac{u}{v},i})_{i\geq 1}$ be the sequence of random variables defined according to (10). Then,*

$$\tilde{P}_{\frac{u}{v},h} \stackrel{d}{=} \sum_{i\geq 1} V_{\frac{u}{v},i} \prod_{j=1}^{i-1} (1 - V_{\frac{u}{v},j}) \delta_{Z_i}, \tag{13}$$

where $(Z_i)_{i\geq 1}$ is a sequence of independent random variables identically distributed according to the base distribution $\mathcal{P}_0$ and independent of the sequence $(V_{\frac{u}{v},i})_{i\geq 1}$.

Proof. Without loss of generality we focus on the distribution of the random variable $V_{\frac{u}{v},1}$ conditionally on $T_{\frac{u}{v},h}=t$. In particular, by combining (6) and (7) one obtains

$$g_{V_{\frac{u}{v},1}|T_{\frac{u}{v},h}}(v_1\,|\,t)=\frac{\frac{u}{v}(tv_1)^{-\frac{u}{v}}G^{0,v-1}_{v-1,u-1}\left(\frac{v^v}{u^u}t^u(1-v_1)^u\,\middle|\,\begin{array}{c}\left(1-\frac{1}{u}-\frac{i}{v}\right)_{i=1}^{v-1}\\ \left(1-\frac{1}{u}-\frac{i}{u}\right)_{i=1}^{u-1}\end{array}\right)}{\Gamma(1-\frac{u}{v})G^{0,v-1}_{v-1,u-1}\left(\frac{v^v}{u^u}t^u\,\middle|\,\begin{array}{c}\left(1-\frac{1}{u}-\frac{i}{v}\right)_{i=1}^{v-1}\\ \left(1-\frac{1}{u}-\frac{i}{u}\right)_{i=1}^{u-1}\end{array}\right)}. \tag{14}$$

The first part of the proof consists in determining the density function of the random variable X_1 in (11). Specifically, by means of Theorem 9 in Springer and Thompson [40] the density function of $\tilde{X}_1=1/X_1$ is represented as follows

$$\begin{aligned}
&g_{\tilde{X}_1}(x_1)\\
&=\frac{\frac{\mathrm{e}^{-\frac{\frac{u^2}{v}}{v\,u\,t}x_1}}{x_1^{u+1}}G^{v-2,0}_{2(u-1),v-2}\left(\frac{1}{x_1^u}\,\middle|\,\begin{array}{c}\left(\frac{i+1}{u}-1\right)_{i=0}^{u-2},\left(-\frac{1}{v}+1-\frac{u-i-1}{u}-1\right)_{i=0}^{u-2}\\ \left(\frac{2i+1}{v}-1\right)_{i=0}^{u-2},\left(\frac{2i+2}{v}-1\right)_{i=0}^{u-2},\left(\frac{2(u-1)+i}{v}-1\right)_{i=1}^{v-2u}\end{array}\right)}{\int_0^{+\infty}\frac{\mathrm{e}^{-\frac{\frac{u^2}{v}}{v\,u\,t}x_1}}{x_1^{u+1}}G^{v-2,0}_{2(u-1),v-2}\left(\frac{1}{x_1^u}\,\middle|\,\begin{array}{c}\left(\frac{i+1}{u}-1\right)_{i=0}^{u-2},\left(-\frac{1}{v}+1-\frac{u-i-1}{u}-1\right)_{i=0}^{u-2}\\ \left(\frac{2i+1}{v}-1\right)_{i=0}^{u-2},\left(\frac{2i+2}{v}-1\right)_{i=0}^{u-2},\left(\frac{2(u-1)+i}{v}-1\right)_{i=1}^{v-2u}\end{array}\right)dx_1}\\
&=\frac{\mathrm{e}^{-\frac{\frac{u^2}{v}}{v\,u\,t}x_1}x_1^{u-1}G^{v-2,0}_{2(u-1),v-2}\left(\frac{1}{x_1^u}\,\middle|\,\begin{array}{c}\left(\frac{i+1}{u}+1\right)_{i=0}^{u-2},\left(-\frac{1}{v}+1-\frac{u-i-1}{u}+1\right)_{i=0}^{u-2}\\ \left(\frac{2i+1}{v}+1\right)_{i=0}^{u-2},\left(\frac{2i+2}{v}+1\right)_{i=0}^{u-2},\left(\frac{2(u-1)+i}{v}+1\right)_{i=1}^{v-2u}\end{array}\right)}{\frac{u^{u-\frac{1}{2}}}{(2\pi)^{\frac{u-1}{2}}}\left(\frac{\frac{u^2}{v}}{v\,u\,t}\right)^{-u}G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{\frac{u^2}{v}}{v\,u\,t}\right)^u}\,\middle|\,\begin{array}{c}\left(\frac{i-u}{u}\right)_{i=1}^{u},\left(-\frac{i}{v}\right)_{i=1}^{v-2}\\ \left(-\frac{i}{u}\right)_{i=1}^{u-1},\left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1}\end{array}\right)}
\end{aligned}$$

where the last identity is obtained by applying (A.5) and (A.4) in order to solve the integral with respect to x_1. Then, the density function of X_1 follows by making the transformation $X_1=1/\tilde{X}_1$. The proof is completed by proving that (14) coincides with the density function of the random variable $W_1=X_1/(X_1+Y_1)$ where Y_1 is the random variable in (12). Recall that Y_1 is independent of X_1. Accordingly, by making the transformation, the density function of W_1 coincides with

$$\begin{aligned}
&g_{W_1}(w_1)\\
&=\frac{\frac{(2\pi)^{\frac{u-1}{2}}}{u^{u-\frac{1}{2}}}\left(\frac{u^2}{v\frac{v}{u}t}\right)^u w_1^{-u-1}(1-w_1)^{-1+\frac{u}{v}-1}\left(\frac{u^2}{v\frac{v}{u}t}\right)^{1-\frac{u}{v}}}{\Gamma\left(1-\frac{u}{v}\right)G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v\frac{v}{u}t}\right)^u}\,\middle|\,\begin{array}{c}\left(\frac{i-u}{u}\right)_{i=1}^{u},\left(-\frac{i}{v}\right)_{i=1}^{v-2}\\ \left(-\frac{i}{u}\right)_{i=1}^{u-1},\left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1}\end{array}\right)}\\
&\quad\times\int_0^{+\infty}z^{-u-1+\frac{u}{v}-1}\mathrm{e}^{-\left(\frac{\frac{u^2}{v\frac{v}{u}t}}{zw_1}+\frac{\frac{u^2}{v\frac{v}{u}t}}{z(1-w_1)}\right)}
\end{aligned}$$

$$\times\, G^{v-2,0}_{2(u-1),v-2}\left((zw_1)^u \,\middle|\, \begin{matrix} \left(1+\frac{i}{u}\right)_{i=1}^{u-1}, \left(1-\frac{1}{v}-\frac{i-u}{u}\right)_{i=1}^{u-1} \\ \left(1+\frac{i}{v}\right)_{i=1}^{v-2} \end{matrix}\right) dz$$

$$= \frac{\frac{(2\pi)^{\frac{u-1}{2}}}{u^{u-\frac{1}{2}}}\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^u w_1^{-u-1}(1-w_1)^{-1+\frac{u}{v}-1}\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^{1-\frac{u}{v}}}{\Gamma\left(1-\frac{u}{v}\right) G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^u} \,\middle|\, \begin{matrix} \left(\frac{i-u}{u}\right)_{i=1}^{u}, \left(-\frac{i}{v}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1} \end{matrix}\right)}$$

$$\times \int_0^{+\infty} y^{u+1-\frac{u}{v}-1} \mathrm{e}^{-y\left(\frac{\frac{u^2}{v^{\frac{v}{u}}t}}{w_1}+\frac{\frac{u^2}{v^{\frac{v}{u}}t}}{1-w_1}\right)}$$

$$\times\, G^{0,v-2}_{v-2,2(u-1)}\left(\frac{y^u}{w_1^u} \,\middle|\, \begin{matrix} \left(-\frac{i}{v}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1} \end{matrix}\right) dy$$

where the last identity is obtained by applying (A.4) and by the change of variable $y = z^{-1}$. Then, (A.5) allows to solve the integral with respect to y. In particular one obtains

$$g_{W_1}(w_1)$$

$$= \frac{\frac{(2\pi)^{\frac{u-1}{2}}}{u^{u-\frac{1}{2}}}\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^u w_1^{-u-1}(1-w_1)^{-1+\frac{u}{v}-1}\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^{1-\frac{u}{v}}}{\Gamma\left(1-\frac{u}{v}\right) G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^u} \,\middle|\, \begin{matrix} \left(\frac{i-u}{u}\right)_{i=1}^{u}, \left(-\frac{i}{v}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1} \end{matrix}\right)}$$

$$\times \frac{u^{u+1-\frac{u}{v}-\frac{1}{2}}}{(2\pi)^{\frac{u-1}{2}}}\left(\frac{\frac{u^2}{v^{\frac{v}{u}}t}}{w_1}+\frac{\frac{u^2}{v^{\frac{v}{u}}t}}{1-w_1}\right)^{-u-1+\frac{u}{v}}$$

$$\times\, G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{\frac{u^u}{w_1^u}}{\left(\frac{\frac{u^2}{v^{\frac{v}{u}}t}}{w_1}+\frac{\frac{u^2}{v^{\frac{v}{u}}t}}{1-w_1}\right)^u} \,\middle|\, \begin{matrix} \left(\frac{i-u-1+\frac{u}{v}}{u}\right)_{i=1}^{u}, \left(-\frac{i}{v}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1} \end{matrix}\right)$$

$$= \frac{\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^{u-1} w_1^{-\frac{u}{v}} u^{2-u-\frac{u}{v}}}{\Gamma\left(1-\frac{u}{v}\right) G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^u} \,\middle|\, \begin{matrix} \left(\frac{i-u}{u}\right)_{i=1}^{u}, \left(-\frac{i}{v}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}\right)_{i=1}^{u-1} \end{matrix}\right)}$$

$$\times\, G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v^{\frac{v}{u}}t}\right)^u}(1-w_1)^u \,\middle|\, \begin{matrix} \left(\frac{i-u-1+\frac{u}{v}}{u}-\frac{1-u}{u}\right)_{i=1}^{u}, \left(-\frac{i}{v}-\frac{1-u}{u}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}-\frac{1-u}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}-\frac{1-u}{u}\right)_{i=1}^{u-1} \end{matrix}\right).$$

Finally, an application of (A.4) to the G-function in the denominator of the last expression leads to

$$
\begin{aligned}
&g_{W_1}(w_1) \\
&= \frac{\left(\frac{u^2}{v^{\frac{v}{u}} t}\right)^{\frac{u}{v}} w_1^{-\frac{u}{v}} u^{1-\frac{u}{v}-\frac{u}{v}}}{\Gamma\left(1-\frac{u}{v}\right) G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v^{\frac{v}{u}} t}\right)^u} \left| \begin{array}{l} \left(\frac{i-1}{u}-\frac{1}{v}\right)_{i=1}^{u}, \left(-\frac{i}{v}-\frac{1-u}{u}-\frac{1}{v}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}-\frac{1}{v}-\frac{1-u}{u}\right)_{i=1}^{u-1}, \left(\frac{i-u}{u}-\frac{1-u}{u}\right)_{i=1}^{u-1} \end{array}\right.\right)} \\
&\quad \times G^{0,v-2+u}_{v-2+u,2(u-1)}\left(\frac{u^u}{\left(\frac{u^2}{v^{\frac{v}{u}} t}\right)^u}(1-w_1)^u \left| \begin{array}{l} \left(\frac{i-u-1+\frac{u}{v}}{u}-\frac{1-u}{u}\right)_{i=1}^{u}, \left(-\frac{i}{v}-\frac{1-u}{u}\right)_{i=1}^{v-2} \\ \left(-\frac{i}{u}-\frac{1-u}{u}\right)_{i=1}^{u-1}, \left(\frac{1}{v}+\frac{i-u}{u}-\frac{1-u}{u}\right)_{i=1}^{u-1} \end{array}\right.\right) \\
&= \frac{\frac{u}{v} t^{-\frac{u}{v}} w_1^{-\frac{u}{v}}}{\Gamma\left(1-\frac{u}{v}\right) G^{0,v-1}_{v-1,u-1}\left(\frac{v^v}{u^u} t^u \left| \begin{array}{l} \left(1-\frac{1}{u}-\frac{i}{v}\right)_{i=1}^{v-1} \\ \left(1-\frac{1}{u}-\frac{i}{u}\right)_{i=0}^{u-1} \end{array}\right.\right)} \\
&\quad \times G^{0,v-1}_{v-1,u-1}\left(\frac{v^v}{u^u} t^u (1-w_1)^u \left| \begin{array}{l} \left(1-\frac{1}{u}-\frac{i}{v}\right)_{i=1}^{v-1} \\ \left(1-\frac{1}{u}-\frac{i}{u}\right)_{i=1}^{u-1} \end{array}\right.\right)
\end{aligned}
$$

where the last identity is obtained by applying (A.3) to the G-functions. The resulting g_{W_1} coincides with the conditional density function of the random variable $V_{\frac{u}{v},1}$ given $T_{\frac{u}{v},h} = t$ in (14). Note that, according to (6), for each $i > 1$ the density function of

$$V_{\frac{u}{v},i} \,|\, V_{\frac{u}{v},1} = v_1, \ldots, V_{\frac{u}{v},i-1} = v_{i-1}, T_{\frac{u}{v},h} = t$$

coincides with the density function (14) in which t is replaced by $t\prod_{j=1}^{i-1}(1-v_j)$. Therefore, such a density function coincides with the density function of $W_i = X_i/(X_i+Y_i)$ for each $i > 1$, where X_i and Y_i are the random variables in (11) and (12), respectively. $\square$

We conclude this section by providing an explicit stick-breaking representation for $\frac{1}{2}$-stable Poisson-Kingman models. Unfortunately, technicalities exploited in the proof on Theorem 2 do not work for the case $\sigma = 1/2$. Indeed, note that the random variable (9) is not defined for $u = 1$ and $v = 2$. Hence, different arguments need to be considered. Let $(V_{\frac{1}{2},i})_{i\geq 1}$ be a sequence of random variables defined as

$$(V_{\frac{1}{2},i} \,|\, V_{\frac{1}{2},1}, \ldots, V_{\frac{1}{2},i-1}, T_{\frac{1}{2},h}) \stackrel{\mathrm{d}}{=} \frac{X_i}{X_i+Y_i} \tag{15}$$

where, for each $i \geq 1$, the random variables X_i and Y_i are assumed to be independent and such that

$$X_i^2 \sim \mathcal{G}\left(\frac{3}{4}, 1\right) \tag{16}$$

and

$$Y_i^2 \sim \mathcal{IG}\left(\frac{1}{4}, \frac{\overline{\frac{1}{4^3 T^2_{\frac{1}{2},h}}}}{\prod_{j=1}^{i-1}(1-V_{\frac{1}{2},j})^2}\right), \tag{17}$$

with the convention that the empty product is defined to be unity and with $\mathcal{G}$ and $\mathcal{IG}$ denoting the Gamma distribution and the inverse Gamma distribution, respectively. The next theorem introduces an explicit stick-breaking representation for the class of $\frac{1}{2}$-stable Poisson-Kingman models. It provides a generalization of Proposition 1 in Favaro et al. [8] to the entire class of $\frac{1}{2}$-stable Poisson-Kingman models.

Theorem 3. *Let $(V_{\frac{1}{2},i})_{i\geq 1}$ be the sequence of random variables defined according to (15). Then,*

$$\tilde{P}_{\frac{1}{2},h} \stackrel{d}{=} \sum_{i\geq 1} V_{\frac{1}{2},i}\prod_{j=1}^{i-1}(1-V_{\frac{1}{2},j})\delta_{Z_i}$$

where $(Z_i)_{i\geq 1}$ is a sequence of independent random variables identically distributed according to the base distribution $\mathcal{P}_0$ and independent of the sequence $(V_{\frac{1}{2},i})_{i\geq 1}$.

Proof. Without loss of generality we focus on the distribution of the random variable $V_{\frac{1}{2},1}$ conditionally on $T_{\frac{1}{2},h} = t$. In particular, by combining (6) and (8) with $u = 1$ and $v = 2$, one has

$$\begin{aligned} g_{V_{\frac{1}{2},1}|T_{\frac{1}{2},h}}(v_1 \,|\, t) &= \frac{\frac{1}{2}(tv_1)^{-\frac{1}{2}} G^{0,1}_{1,0}\left(2^2 t(1-v_1)\middle|\begin{matrix} -\frac{1}{2} \\ - \end{matrix}\right)}{\Gamma\left(\frac{1}{2}\right) G^{0,1}_{1,0}\left(2^2 t\middle|\begin{matrix} -\frac{1}{2} \\ - \end{matrix}\right)} \\ &= \frac{\frac{1}{2}t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} v_1^{-\frac{1}{2}}(1-v_1)^{-\frac{1}{2}-1}\mathrm{e}^{-\frac{v_1}{2^2 t(1-v_1)}} \end{aligned} \tag{18}$$

where the last identity is obtained by combining (A.4) with (A.2). Note that (18) can be written in terms of modified Bessel function $K_{\frac{1}{2}}$, i.e.,

$$g_{V_{\frac{1}{2},1}|T_{\frac{1}{2},h}}(v_1 \,|\, t) = \frac{2^{-\frac{3}{2}}t^{-1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}(1-v_1)^{-2} K_{\frac{1}{2}}\left(\frac{v_1}{2^2 t(1-v_1)}\right). \tag{19}$$

The proof is completed by proving that (19) coincides with the density function of the random variable $W_1 = X_1/(X_1+Y_1)$ where X_1^2 and Y_1^2 are the random variables in (16) and (17), respectively. Recall that Y_1 is independent X_1. Accordingly, by making the transformation, the density function of W_1 coincides with

$$g_{W_1}(w_1) = \frac{w_1^{\frac{3}{2}-1}(1-w_1)^{-\frac{1}{2}-1} 2\left(\frac{1}{4^3 t^2}\right)^{\frac{1}{4}}}{\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}$$

$$\times \int_0^{+\infty} z^{\frac{3}{2}-\frac{1}{2}-1} \mathrm{e}^{-\left((zw_1)^2+\frac{\frac{1}{4^3t^2}}{(z(1-w_1))^2}\right)} dz$$

$$= \frac{w_1^{\frac{3}{2}-1}(1-w_1)^{-\frac{1}{2}-1} 2\left(\frac{1}{4^3t^2}\right)^{\frac{1}{4}}}{\frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}$$

$$\times \left(\frac{\frac{1}{4^3t^2}}{(1-w_1)^2w_1^2}\right)^{\frac{1}{4}} K_{\frac{1}{2}}\left(2\left(\frac{\frac{w_1^2}{4^3t^2}}{(1-w_1)^2}\right)^{\frac{1}{2}}\right)$$

where the last identity is obtained by Equation 3.478.1 in Gradshteyn and Ryzhik [12]. Hence,

$$g_{W_1}(w_1) = \frac{t^{-1}}{\frac{4}{2^{\frac{1}{2}}}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}(1-w_1)^{-2} K_{\frac{1}{2}}\left(\frac{w_1}{4t(1-w_1)}\right). \tag{20}$$

The resulting g_{W_1} coincides with the conditional density function of the random variable $V_{\frac{1}{2},1}$ given $T_{\frac{1}{2},h} = t$ in (19). Note that, according to (6), for each $i > 1$ the density function of

$$V_{\frac{1}{2},i} \,|\, V_{\frac{1}{2},1} = v_1, \ldots, V_{\frac{1}{2},i-1} = v_{i-1}, T_{\frac{1}{2},h} = t$$

coincides with the density function (20) in which t is replaced by $t\prod_{j=1}^{i-1}(1-v_j)$. Therefore, such a density function coincides with the density function of $W_i = X_i/(X_i + Y_i)$ for each $i > 1$, where X_i^2 and Y_i^2 are the random variables in (16) and (17), respectively. □

Extending the stick-breaking representation in Theorem 2 to the class of $\frac{u}{v}$-stable Poisson-Kingman models, for any coprime integers $1 \le u < v$ such that $u/v > 1/2$, still remains an open problem. In this respect, the proof of Theorem 2 suggests that the stick-breaking random variables $(V_{\frac{u}{v},i})_{i\ge1}$, for any coprime integers $1 \le u < v$ such that $u/v > 1/2$, can not be directly represented in terms of a transformation of the form (10) with X_i being the product of independent Beta and Gamma random variables and Y_i, independent of X_i, being an inverse Gamma random variable. Different transformations or, alternatively, different distributions need to be considered and investigated. Work in these directions is ongoing.

4. Exact sampling of $(V_{\frac{u}{v},i} \,|\, V_{\frac{u}{v},1}, \ldots, V_{\frac{u}{v},i-1}, T_{\frac{u}{v},h})_{i\ge1}$

Theorems 2 and Theorem 3 provide a simple strategy for sampling the random probabilities $P_{\frac{u}{v},i} = V_{\frac{u}{v},i}\prod_{j=1}^{i-1}(1-V_{\frac{u}{v},j})$ in (13), for any coprime integers $1 \le u < v$ such that $u/v \le 1/2$. While sampling the $V_{\frac{1}{2},i}$'s is straightforward, particular attention must be given to the $V_{\frac{u}{v},i}$'s for any $u/v < 1/2$. Without loss of generality here we focus on the distribution of the random variable $V_{\frac{u}{v},1}$ conditionally on the total mass $T_{\frac{u}{v},h} = t$. More specifically we consider the problem of generating, exactly, random variates for the distribution of $1/X_1$ in (11). If we set $\lambda = u^2/\left(v^{\frac{v}{u}}t\right)$, then we aim at sampling from a random variable with density function

$$f(x) \propto e^{-\lambda x} m(x)$$

where $m(x)$ is the density function of $L_{u/v}^{-1/u}$ and $L_{u/v}$ is the random variable in (9) whose density function is denoted by $m_{L_{u/v}}(x)$. By a simple rejection sampling with proposal $m(x)$ one would get an acceptance rate equal $C = \int_0^\infty e^{-\lambda x} m(x)dx$ that, depending on λ, can be extremely small. In our context this is a major issue due to the randomness of $T_{\frac{u}{v},h}$ and, in turn, of λ. Recall that $T_{\frac{u}{v},h}$ is a random variable with density function $h(t)f_{\frac{u}{v}}(t)$ with $f_{\frac{u}{v}}$ being the $\frac{u}{v}$-stable density function. To overtake this issue we devise an improved rejection sampling which is based on the choice of a suitable proposal that takes into account λ.

Hereafter we present our improved rejection sampling for $V_{\frac{u}{v},1}$, for any $1 \le u < v$ such that $u/v < 1/2$, conditionally on the total mass $T_{\frac{u}{v},h} = t$. First we observe that $\mathrm{e}^{-\lambda x} \le \mathrm{e}^{-\lambda x_0(1-\log(x_0))} x^{-\lambda x_0}$ for every $x_0 \ge 0$. Then we consider the family of proposals

$$g_{x_0}(x) \propto \mathrm{e}^{-\lambda x_0(1-\log(x_0))} x^{-\lambda x_0} m(x).$$

The acceptance rate corresponding to the proposal g_{x_0} coincides with C/C_{x_0}, where

$$\begin{aligned} C_{x_0} &= \int_0^\infty g_{x_0}(x)dx \\ &= \mathrm{e}^{-\lambda x_0(1-\log(x_0))} \int_0^\infty x^{-\lambda x_0} m(x)dx \\ &= \mathrm{e}^{-\lambda x_0(1-\log(x_0))} \int_0^\infty \ell^{\lambda x_0/u} m_{L_{u/v}}(\ell)d\ell, \end{aligned}$$

where the last identity is obtained by means of the change of variable $\ell = x^{-u}$. The last integral coincides with the moment of order $\lambda x_0/u$ of $L_{\frac{u}{v}}$ in (9) and, therefore,

$$\begin{aligned} C_{x_0} = \mathrm{e}^{-\lambda x_0(1-\log(x_0))} &\prod_{i=1}^{v-2u} \frac{\Gamma\left(\frac{2u-2+i}{v} + \frac{\lambda x_0}{u}\right)}{\Gamma\left(\frac{2u-2+i}{v}\right)} \\ &\times \prod_{i=0}^{u-2} \frac{\Gamma\left(\frac{i+1}{u}\right)\Gamma\left(\frac{2i+2}{v} + \frac{\lambda x_0}{u}\right)\Gamma\left(\frac{i+1}{u} - \frac{1}{v}\right)\Gamma\left(\frac{2i+1}{v} + \frac{\lambda x_0}{u}\right)}{\Gamma\left(\frac{i+1}{u} + \frac{\lambda x_0}{u}\right)\Gamma\left(\frac{2i+2}{v}\right)\Gamma\left(\frac{i+1}{u} - \frac{1}{v} + \frac{\lambda x_0}{u}\right)\Gamma\left(\frac{2i+1}{v}\right)}. \end{aligned}$$

In order to maximize the acceptance rate C/C_{x_0} we choose x_0 so to minimize C_{x_0}. To this end we observe that, for our purpose, it is enough to evaluate C_{x_0} on a sufficiently large and fine grid of values x_0 and pick the value, denoted by x^*, that gives rise to the smallest C_{x_0}. This, in turn, leads us to consider g_{x^*} as proposal.

Table 1
Empirical acceptance rate (e.a.r.) *and log-improvement* (l.i.) $\log(C_{x^*}/C)$

		$u/v = 1/3$	$u/v = 1/15$	$u/v = 7/15$
$\lambda = 10^{-6}$	e.a.r.	0.98	0.31	1.00
	l.i.	2.29×10^{-5}	4.64	2.14×10^{-6}
$\lambda = 10^{-3}$	e.a.r.	0.87	0.30	0.99
	l.i.	2.08×10^{-2}	8.07	2.14×10^{-3}
$\lambda = 1$	e.a.r.	0.71	0.29	0.94
	l.i.	2.10	13.66	1.50
$\lambda = 10^{3}$	e.a.r.	0.70	0.28	0.93
	l.i.	63.9	22.7	6.15×10^{3}
$\lambda = 10^{6}$	e.a.r.	0.70	0.27	0.92
	l.i.	2.00×10^{4}	37.6	2.89×10^{5}

The original problem of sampling from an exponentially tilted distribution f has boiled down to the problem of sampling from the polynomially tilted distribution g_{x^*}. Moreover, g_{x^*} coincides with the density function of a random variable $L_{x^*}^{-1/u}$, where

$$L_{x^*} = \prod_{i=0}^{u-2} B_{\frac{2i+2}{v}+\frac{\lambda x^*}{u},\frac{i+1}{u}-\frac{2i+2}{v}} \prod_{i=0}^{u-2} B_{\frac{2i+1}{v}+\frac{\lambda x^*}{u},\frac{i+1}{u}-\frac{2i+2}{v}} \prod_{i=1}^{v-2u} G_{\frac{2(u-1)+i}{v}+\frac{\lambda x^*}{u},1}.$$

By summarizing, in order to exactly generate a random variate for the distribution of the random variable $1/X_1$ in (11), we propose the following improved rejection sampling:

1. find x^* that minimizes C_{x_0};
2. for each $i = 0, \dots, u-2$, sample independently β_i from $B_{\frac{2i+2}{v}+\frac{\lambda x^*}{u},\frac{i+1}{u}-\frac{2i+2}{v}}$ and β_i' from $B_{\frac{2i+1}{v}+\frac{\lambda x^*}{u},\frac{i+1}{u}-\frac{2i+2}{v}}$;
3. for each $i = 1, \dots, v-2u$, sample independently γ_i from $G_{\frac{2(u-1)+i}{v}+\frac{\lambda x^*}{u},1}$;
4. compute $w = (\prod_{i=0}^{u-2} \beta_i\beta_i' \prod_{i=1}^{v-2u} \gamma_i)^{-1/u}$;
5. accept w as a realization of the random variable $1/X_1$ with probability $\mathrm{e}^{-\lambda(w-x^*)}(w/x^*)^{\lambda x^*}$.

Once X_1 is generated, then we can easily generate a random variate for the distribution of the random variable Y_1 in (12). Finally, we combine X_1 and Y_1 according to (10) and we obtain a random variate for the distribution of $V_{\frac{u}{v},1} \,|\, T_{\frac{u}{v},h} = t$.

Simulation results show that the proposed method is very effective and, importantly, outperforms the simple rejection sampling with proposal m when the parameter λ is large. We have analyzed and compared the performance of the two algorithms for several combinations of values of u/v and λ. In Table 1 we report, for $u/v = 1/3, 1/15, 7/15$ and $\lambda = 10^i$ with $i \in \{-6, -3, 0, 3, 6\}$, the empirical acceptance rate of the improved rejection sampling and, as a comparison between improved and simple rejection sampling, $\log(C_{x^*}/C)$. The empirical acceptance rate is estimated by means of 10000 independent samples from (11). In all the cases we analyzed, the improved rejection sampling turns out to be more efficient. Moreover, whereas the two algorithms have similar performances for small λ, the improved rejection sampling has an acceptance rate significantly larger than the simple rejection sampling when λ is large. For example, when $u/v = 1/3$ and $\lambda = 10^6$, the simple algorithm proves essentially useless while the estimated acceptance rate for the improved algorithm is 0.70. Similar observations hold for the other two values of u/v we considered. It is interesting to notice that the larger u/v is, e.g. $u/v = 7/15$, the more evident the difference between the performance of the two algorithms can be. On the other hand, when u/v is small, e.g. $u/v = 1/15$, the role of λ in determining the efficiency of the algorithms is less crucial.

The procedure for sampling $V_{\frac{u}{v},1} \,|\, T_{\frac{u}{v},h} = t$ can be applied iteratively in order to generate random variates for the distribution of the random variable $V_{\frac{u}{v},i} \,|\, V_{\frac{u}{v},1} = v_1, \ldots, V_{\frac{u}{v},i-1} = v_{i-1}, T_{\frac{u}{v},h} = t$. We only need to updated the parameters of the distributions of X_i and Y_i according to (11) and (12), respectively. Moreover, once h is specified, this sampling procedure leads to random variates for the distribution of $P_{\frac{u}{v},i} = V_{\frac{u}{v},i} \prod_{j=1}^{i-1}(1 - V_{\frac{u}{v},j})$. To this end, an additional sampling step is required in order to generate random variates for the distribution of $T_{\frac{u}{v},h}$.

5. Discussion

In Section 3 we introduced a completely explicit stick-breaking representation for the class of $\frac{u}{v}$-stable Poisson-Kingman models, for any coprime integers $1 \le u < v$ such that $u/v \le 1/2$. In particular Theorem 2 and Theorem 3 provide an alternative definition, in terms of the intuitive stick-breaking metaphor, for $\frac{u}{v}$-stable Poisson-Kingman models. This result is interesting in that: i) it introduces a novel class of stick-breaking random probability measures which does not belong to the class of stick-breaking priors originally proposed by Ishwaran and James [15]; it provides new insights on the large class of $\frac{u}{v}$-stable Poisson-Kingman models: most of their properties are known by now but an explicit stick-breaking representation is missing; iii) it shows that stick-breaking representations with dependent weights can be determined; iv) it provides an alternative stick-breaking characterization, in terms of the latent random variable $T_{\frac{u}{v},h}$, for the two parameter Poisson-Dirichlet process and the normalized inverse Gaussian process.

A stick-breaking representation of the normalized generalized Gamma process $\tilde{G}_{\frac{u}{v},b}$, for any coprime integers $1 \le u < v$ such that $u/v \le 1/2$, arises as special case of Theorem 2 and Theorem 3 by setting h of the form (5). Such a result thus extends remarkably the stick-breaking representation for the normalized inverse Gaussian process in Favaro et al. [8]. Let us denote by g the parameter characterizing $\tilde{G}_{\frac{u}{v},b}$, namely $g(t) = \exp\{b - b^{v/u}t\}$. Then, the random variable $T_{\frac{u}{v},g}$ is distributed according to an exponentially tilted positive

$\frac{u}{v}$-stable distribution, i.e.,

$$\mathbb{P}[T_{\frac{u}{v},g} \in dt] = \mathrm{e}^{b-b^{v/u}t} f_{\frac{u}{v}}(t)dt. \tag{21}$$

The distribution (21) can be easily sampled by resorting to the fast rejection algorithm proposed by Hofert [14]. Therefore, by means of the sampling methods described in Section (4), Theorem 2 and Theorem 3 provide a natural tool for sampling the stick-breaking random variables $V_{\frac{u}{v},i}$'s defining $\tilde{P}_{\frac{u}{v},g}$. This, in turn, can be exploited to obtain an approximate evaluation of the distribution of the random variable

$$F_{\frac{u}{v},g} = \int_{\mathbb{X}} f(x)\tilde{P}_{\frac{u}{v},g}(dx) = \sum_{i\geq 1} V_{\frac{u}{v},i} \prod_{j=1}^{i-1}(1-V_{\frac{u}{v},j}) f(Z_i), \tag{22}$$

for any measurable linear function $f : \mathbb{X} \to \mathbb{R}$. The approximation being determined by the choice of a suitable truncation level for the series (22). The random variable (22) is typically referred to as the mean functional of $\tilde{P}_{\frac{u}{v},g}$. In such a context Muliere and Tardella [28] first investigated the use of the stick-breaking representation of the Dirichlet process process in order to obtain an approximate evaluation of the distribution of its mean functionals. See James et al. [17] for a generalization to the mean functionals of the two parameter Poisson-Dirichlet process.

The stick-breaking representation of $\tilde{G}_{\frac{u}{v},b}$ also appears in the posterior characterization of $\tilde{G}_{\frac{u}{v},b}$ given the observed data. Indeed, let $(X_1,\ldots,X_n)$ be an observed sample featuring j distinct values $X_1^*,\ldots,X_j^*$ with corresponding frequencies $n_1,\ldots,n_j$. Also, let Y_n be a random variable whose density function, conditionally on $(X_1,\ldots,X_n)$ is $f_{Y_n}(y) \propto y^{n-1}(1+y)^{ju/v-n}\mathrm{e}^{-(1+y)^{u/v}}$. Then, by exploiting Theorem 1 in James et al. [18], it can be shown that conditionally on $(X_1,\ldots,X_n)$ and on Y_n, the normalized generalized Gamma process coincides in distribution with

$$\frac{T_{\frac{u}{v},g^*}}{T_{\frac{u}{v},g^*}+\sum_{i=1}^{j} J_i}\tilde{G}_{\frac{u}{v},Y_n+1} + \sum_{i=1}^{j}\frac{J_i}{T_{\frac{u}{v},g^*}+\sum_{i=1}^{j} J_i}\delta_{X_i^*} \tag{23}$$

where $\tilde{G}_{\frac{u}{v},Y_n+1}(\cdot) = \tilde{\mu}_{\sigma,g^*}(\cdot)/T_{\frac{u}{v},g^*}$ with $g^*(t) = \exp\{(Y_n+1)-(Y_n+1)^{v/u}\}$, and where the random variables $J_1,\ldots,J_j,T_{\frac{u}{v},g^*}$ are independent and such that J_i is distributed according to a Gamma distribution with parameter $(Y_n+1, n_i-u/v)$, for any $i=1,\ldots,j$. We refer to James et al. [18] for additional details on the posterior distribution of the normalized generalized Gamma process. Therefore, according to the characterization (23), our stick-breaking representation can be still exploited in the posterior inference under the normalized generalized Gamma process prior.

We conclude by pointing out a few problems that naturally arise from Theorem 2 and Theorem 3. First of all, as we stated in Section 3, an open problem consists in extending Theorem 2 to $\frac{u}{v}$-stable Poisson-Kingman models, for any coprime integers $1 \leq u < v$ such that $u/v > 1/2$. More generally, we aim at extending Theorem 2 to σ-stable Poisson-Kingman models, for any $\sigma \in (0,1)$. In this respect, an explicit stick-breaking representation for the normalized generalized Gamma process $\tilde{G}_{\sigma,b}$, for any $\sigma \in (0,1)$, has been recently proposed by James [19]. Another open problem consists in investigating the applicability of our results. On the one hand, we believe that, from a modeling point of view, our explicit stick-breaking representations paves the way for the definition of complex models based on $\frac{u}{v}$-stable Poisson-Kingman models by simply replacing the stick-breaking constructed Dirichlet process, most notably within the context of dependent models for nonparametric regression. On the other hand, from a computational point of view, our explicit stick-breaking representations can be used to extend various recently proposed simulation algorithms, based on stick-breaking constructions, in order to cover also the case of $\frac{u}{v}$-stable Poisson-Kingman models.

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Appendix

G-functions, also known as Meijer G-function, are flexible special functions introduced by Meijer [27] that incorporate as special cases most of the mathematical functions such as the elementary functions, trigonometric functions, Bessel functions and generalized hypergeometric functions. A G-function of order (m,n,p,q), where $0 \le m \le q$, $0 \le n \le p$ and $p \le q-1$ is defined by the contour integral

$$G_{p,q}^{m,n}\left(x \middle| \begin{array}{c} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{array}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} \frac{\prod_{j=1}^{m}\Gamma(s+b_j)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(s+a_j)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)} ds \tag{A.1}$$

where c denotes a real constant defining the so-called Bromwich path separating the poles of $\Gamma(s+b_j)$ from those of $\Gamma(1-a_k-s)$ and where the empty product is defined to be unity. The reader is referred to the monograph by Erdély [7] for a more thorough definition of the class of G-functions, including conditions for the convergence of the integral (A.1). In what follows we set $\mathbf{a}_p = (a_1,\dots,a_p)$ and $\mathbf{b}_q = (b_1,\dots,b_q)$.

A comprehensive collection of properties of G-functions is provided in Prudnikov et al. [36]. In order to make our paper self-contained, we recall those properties of G-functions that are used in our proofs. Equation 8.4.3.1 in Prudnikov et al. [36] shows that an exponential function can be written in terms of G-function as

$$\exp\{-x\} = G_{0,1}^{1,0}\left(x \middle| \begin{array}{c} - \\ 0 \end{array}\right). \tag{A.2}$$

Equation 8.2.2.8 in Prudnikov et al. [36] is an example of the formulae for lowering the order, i.e.,

$$G_{p,q}^{m,n}\left(x \middle| \begin{array}{c} \mathbf{a}_p \\ \mathbf{b}_{q-1}, a_1 \end{array}\right) = G_{p-1,q-1}^{m,n-1}\left(x \middle| \begin{array}{c} a_2, \dots, a_p \\ \mathbf{b}_{q-1} \end{array}\right), \tag{A.3}$$

or

$$G_{p+1,q+1}^{m+1,n}\left(x \middle| \begin{array}{c} \mathbf{a}_p, 1-r \\ 0, \mathbf{b}_{q-1} \end{array}\right) = (-1)^r G_{p+1,q+1}^{m,n+1}\left(x \middle| \begin{array}{c} 1-r, \mathbf{a}_p \\ \mathbf{b}_q, 1 \end{array}\right),$$

for any $r \geq 0$. Equation 8.2.2.8 in Prudnikov et al. [36] is an example of the translation formulae, i.e.,

$$x^\alpha G_{p,q}^{m,n}\left(x \middle| \begin{array}{c} \mathbf{a}_p \\ \mathbf{b}_q \end{array}\right) = G_{p,q}^{m,n}\left(x \middle| \begin{array}{c} \mathbf{a}_p + \alpha \\ \mathbf{b}_q + \alpha \end{array}\right), \tag{A.4}$$

or

$$G_{p,q}^{m,n}\left(\frac{1}{x} \middle| \begin{array}{c} \mathbf{a}_p \\ \mathbf{b}_q \end{array}\right) = G_{q,p}^{n,m}\left(x \middle| \begin{array}{c} 1-\mathbf{b}_q \\ 1-\mathbf{a}_p \end{array}\right).$$

Formulae for lowering the order of G-functions and translation formulae represent two of the most important classes of properties for manipulating G-functions. Other classes of properties are the so-called properties of symmetry, degeneracy and contiguity. See, e.g., Meijer [27], Erdély [7] and Prudnikov et al. [36] for details.

Equation 2.24.1.1 in Prudnikov et al. [36] provides an integral of general form involving a power function and the product of two G-functions. Such an integral includes as special cases many integrals involving combinations of elementary and special functions. An important special case is given by Equation 2.24.1.3 in Prudnikov et al. [36], i.e.,

$$\int_0^{+\infty} x^{\alpha-1} \mathrm{e}^{-\sigma x} G_{p,q}^{m,n}\left(w x^{l/k} \middle| \begin{array}{c} \mathbf{a}_p \\ \mathbf{b}_q \end{array}\right) dx \tag{A.5}$$
$$= \frac{k^\mu l^{\alpha-1/2} \sigma^{-\alpha}}{(2\pi)^{(l-1)/2+c^*(k-1)}} G_{kp+l,kq}^{km,kn+l}\left(\frac{w^k l^l}{\sigma^l k^{k(q-p)}} \middle| \begin{array}{c} \Delta(l, 1-\alpha), \Delta(k, (\mathbf{a}_p)) \\ \Delta(k, \mathbf{b}_q) \end{array}\right)$$

where

$$c^* = m + n - \frac{p+q}{2}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2} + 1$$

$$\Delta(k,a) = \left(\frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}\right)$$

$$\Delta(k,\mathbf{a}_p) = \left(\frac{\mathbf{a}_p}{k}, \frac{\mathbf{a}_p+1}{k}, \ldots, \frac{\mathbf{a}_p+k-1}{k}\right).$$

Equation (A.5) holds under suitable conditions on the parameters $\mathbf{a}_p$ $\mathbf{b}_q$. We refer to Section 2.24.1 in Prudnikov et al. [36] for a comprehensive account on these conditions.

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