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A pathwise interpretation of the Gorin-Shkolnikov identity

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Abstract

In a recent paper by Gorin and Shkolnikov (2016), they have found, as a corollary to their result relevant to random matrix theory, that the area below a normalized Brownian excursion minus one half of the integral of the square of its total local time, is identical in law with a centered Gaussian random variable with variance $1/12$. In this paper, we give a pathwise interpretation to their identity; Jeulin's identity connecting normalized Brownian excursion and its local time plays an essential role in the exposition.

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1 Introduction

Let $r = \{r_t\}_{0 \le t \le 1}$ be a normalized Brownian excursion, that is, it is identical in law with a standard 3-dimensional Bessel bridge, which has the duration $[0, 1]$, and starts from and ends at the origin; see e.g., [1, Section (2.2)] and references therein for the definition of normalized Brownian excursion and its equivalence in law with standard 3-dimensional Bessel bridge. We denote by $l = \{l_x\}_{x \ge 0}$ the total local time process of r; namely, by the occupation time formula, two processes r and l are related in particular via

$$H(x) := \int_0^1 \mathbf{1}_{\{r_t \le x\}} \, dt = \int_0^x l_y \, dy \quad \text{for all } x \ge 0, \text{ a.s.} \tag{1.1}$$

In a recent paper [3], Gorin and Shkolnikov have found the following remarkable identity in law as a corollary to one of their results:

Theorem 1.1 ([3], Corollary 2.15)**.** *The random variable X defined by*

$$X := \int_0^1 r_t \, dt - \frac{1}{2} \int_0^\infty (l_x)^2 \, dx$$

is a centered Gaussian random variable with variance $1/12$.

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In [3, Proposition 2.14], they show that the expected value of the trace of a random operator indexed by $T > 0$, arising from random matrix theory, admits the representation

$$\sqrt{\frac{2}{\pi T^3}}\mathbb{E}\left[\exp\left(-\frac{T^{3/2}}{2}X\right)\right]$$

for any $T > 0$; in comparison of this expression with the existing literature asserting that the expected value is equal to $\sqrt{2/(\pi T^3)}\exp\left(T^3/96\right)$ for every $T > 0$, they obtain Theorem 1.1 by the analytic continuation and the uniqueness of characteristic functions.

In this paper, we give a proof of Theorem 1.1 without relying on random matrix theory; Jeulin's identity in law ([6, p. 264]):

$$\{r_t\}_{0\le t\le 1} \stackrel{(d)}{=} \left\{\frac{1}{2}l_{H^{-1}(t)}\right\}_{0\le t\le 1} \tag{1.2}$$

with

$$H^{-1}(t) := \inf\{x \ge 0;\, H(x) \ge t\},$$

plays a central role in the proof. For the identity (1.2), we also refer to [1, Proposition 3.6 and Théorème (5.3)].

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 and provide some relevant results.

Proof of Theorem 1.1. Recall from the representation of r by means of a stochastic differential equation (see, e.g., [7, Chapter XI, Exercise (3.11)]) that the process $W = \{W_t\}_{0\le t\le 1}$ defined by

$$W_t := r_t - \int_0^t \left(\frac{1}{r_s} - \frac{r_s}{1-s}\right) ds \tag{2.1}$$

is a standard Brownian motion. We integrate both sides over $[0, 1]$; since

$$\int_0^1 \left|\frac{1}{r_s} - \frac{r_s}{1-s}\right| ds < \infty \quad \text{a.s.} \tag{2.2}$$

(see Remark 2.1 (3) below), we may use Fubini's theorem on the right-hand side to have the identity

$$\int_0^1 W_t\,dt = \int_0^1 r_t\,dt - \int_0^1 ds\left(\frac{1}{r_s} - \frac{r_s}{1-s}\right)\int_s^1 dt,$$

which entails

$$\frac{1}{2}\int_0^1 W_t\,dt = \int_0^1 r_t\,dt - \frac{1}{2}\int_0^1 \frac{1-t}{r_t}\,dt. \tag{2.3}$$

Note that the left-hand side is a centered Gaussian random variable with variance $1/12$. As to the right-hand side, there holds the identity in law

$$\left(\int_0^1 r_t\,dt,\, \frac{1}{2}\int_0^1 \frac{1-t}{r_t}\,dt\right) \stackrel{(d)}{=} \left(\frac{1}{2}\int_0^\infty (l_x)^2\,dx,\, \int_0^1 r_t\,dt\right). \tag{2.4}$$

Indeed, by Jeulin's identity (1.2), the left-hand side of (2.4) is identical in law with

$$\left(\frac{1}{2}\int_0^1 l_{H^{-1}(t)}\,dt,\, \int_0^1 \frac{1-t}{l_{H^{-1}(t)}}\,dt\right),$$

which is equal, by changing variables with $t = H(x)$, $x \geq 0$, to

$$\begin{aligned}
&\left(\frac{1}{2}\int_0^\infty l_x H'(x)\,dx,\ \int_0^\infty \frac{1-H(x)}{l_x} H'(x)\,dx\right) \qquad (2.5)\\
&= \left(\frac{1}{2}\int_0^\infty (l_x)^2\,dx,\ \int_0^\infty dx \int_0^1 dt\,\mathbf{1}_{\{r_t > x\}}\right)\\
&= \left(\frac{1}{2}\int_0^\infty (l_x)^2\,dx,\ \int_0^1 r_t\,dt\right),
\end{aligned}$$

where the second line follows from the definition (1.1) of H and the third from Fubini's theorem. Therefore combining (2.3) and (2.4) yields

$$\frac{1}{2}\int_0^1 W_t\,dt \overset{(d)}{=} \frac{1}{2}\int_0^\infty (l_x)^2\,dx - \int_0^1 r_t\,dt$$

and concludes the proof. □

We give a remark on the proof. In what follows we denote

$$M(r) = \max_{0\leq t\leq 1} r_t.$$

Remark 2.1. (1) We see from (1.1) that a.s.,

$$\int_0^\infty l_y\,dy = \int_0^{M(r)} l_y\,dy = 1.$$

Therefore, to be more specific, the second integral in (2.5) should be written as

$$\int_0^{M(r)} \frac{1-H(x)}{l_x} H'(x)\,dx.$$

(2) By the time-reversal $\{r_{1-t}\}_{0\leq t\leq 1} \overset{(d)}{=} \{r_t\}_{0\leq t\leq 1}$, two-dimensional random variables in (2.4) are also identical in law with

$$\left(\int_0^1 r_t\,dt,\ \frac{1}{2}\int_0^1 \frac{t}{r_t}\,dt\right).$$

These identities in law indicate in particular that the following four random variables have the same law:

$$\int_0^1 r_t\,dt, \qquad \frac{1}{2}\int_0^\infty (l_x)^2\,dx, \qquad \frac{1}{2}\int_0^1 \frac{1-t}{r_t}\,dt, \qquad \frac{1}{2}\int_0^1 \frac{t}{r_t}\,dt.$$

As to the equivalence in law between the first two random variables, see [2, Theorem 2.1] for its generalization involving an independent uniform random variable on $(0,1)$. The Laplace transform of the law of $\int_0^1 r_t\,dt$ is given in [4, Lemma 4.2] and [1, Proposition (5.5)] in terms of a series expansion.

(3) The a.s. finiteness (2.2) may be deduced from the proof of [5, Théorème (6,40) b)]. For the reader's convenience, we give a proof of (2.2) here, which will be done in a stronger statement that

$$\mathbb{E}\left[\int_0^1 \frac{ds}{r_s}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^1 \frac{r_s}{1-s}\,ds\right] < \infty. \qquad (2.6)$$

To this end, recall the identity in law:

$$\left\{(r_t)^2\right\}_{0\leq t\leq 1} \overset{(d)}{=} \left\{\left(b_t^1\right)^2 + \left(b_t^2\right)^2 + \left(b_t^3\right)^2\right\}_{0\leq t\leq 1}, \qquad (2.7)$$

where $b^i = \{b_t^i\}_{0\le t\le 1}$, $i = 1,2,3$, are independent, standard Brownian bridges. Noting that $a^{-1} = \int_0^\infty (e^{-\lambda a^2}/\sqrt{\pi\lambda})\,d\lambda$ for any $a > 0$, we have by Fubini's theorem and independence of b^i's,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{r_s}\right] &= \int_0^\infty \frac{d\lambda}{\sqrt{\pi\lambda}} \prod_{i=1}^3 \mathbb{E}\left[\exp\left\{-\lambda\left(b_s^i\right)^2\right\}\right]\\ &= \int_0^\infty \frac{d\lambda}{\sqrt{\pi\lambda}} \left(\frac{1}{\sqrt{2\lambda s(1-s)+1}}\right)^3\\ &= \sqrt{\frac{2}{\pi s(1-s)}}\end{aligned}$$

for every $0 < s < 1$, where the last line may be seen by changing variables with $2\lambda s(1-s) = \tan^2\theta$, $0 < \theta < \pi/2$. Therefore by Fubini's theorem,

$$\mathbb{E}\left[\int_0^1 \frac{ds}{r_s}\right] = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \sqrt{2\pi}$$

(cf. [7, Chapter XI, Exercise (3.9)]). In particular, we have the former finiteness in (2.6). As to the latter, we argue along the same lines as in the proof of [5, Proposition (6,37)] to see, by Schwarz's inequality and (2.7), that

$$\begin{aligned}\mathbb{E}\left[r_s\right] &\le \mathbb{E}\left[\left(b_s^1\right)^2 + \left(b_s^2\right)^2 + \left(b_s^3\right)^2\right]^{1/2}\\ &= \sqrt{3s(1-s)}\end{aligned}$$

for any $0 \le s \le 1$, and hence by Fubini's theorem,

$$\mathbb{E}\left[\int_0^1 \frac{r_s}{1-s}\,ds\right] \le \int_0^1 \sqrt{\frac{3s}{1-s}}\,ds,$$

which is finite as claimed.

Using the same reasoning as the proof of Theorem 1.1, we may extend Theorem 1.1 to

Proposition 2.2. *For every positive integer n, the random variable*

$$2\int_{[0,1]^n} \min\{r_{t_1},\dots,r_{t_n}\}\,dt_1\cdots dt_n - \frac{n+1}{2}\int_0^\infty (1-H(x))^{n-1}(l_x)^2\,dx$$

has the Gaussian distribution with mean zero and variance $1/(2n+1)$.

Proof. For each fixed n, we multiply both sides of (2.1) by $(1-t)^{n-1}$ and integrate them over $[0,1]$. Then using Fubini's theorem, we obtain

$$\int_0^1 (1-t)^{n-1} W_t\,dt = \frac{n+1}{n}\int_0^1 (1-t)^{n-1} r_t\,dt - \frac{1}{n}\int_0^1 \frac{(1-t)^n}{r_t}\,dt. \tag{2.8}$$

Since the left-hand side may be expressed as $(1/n)\int_0^1 (1-t)^n\,dW_t$, we see that it is a centered Gaussian random variable with variance

$$\frac{1}{n^2}\int_0^1 (1-t)^{2n}\,dt = \frac{1}{n^2(2n+1)}.$$

On the other hand, for the right-hand side of (2.8), we have

$$\begin{aligned}
&\left(\int_0^1 (1-t)^{n-1} r_t \, dt, \int_0^1 \frac{(1-t)^n}{r_t} \, dt \right) \\
&\overset{(d)}{=} \left(\frac{1}{2} \int_0^\infty (1-H(x))^{n-1} (l_x)^2 \, dx, \, 2 \int_{[0,1]^n} \min\{r_{t_1}, \dots, r_{t_n}\} \, dt_1 \cdots dt_n \right).
\end{aligned} \tag{2.9}$$

Indeed, Jeulin's identity (1.2) entails that the left-hand side of (2.9) has the same law as

$$\begin{aligned}
&\left(\frac{1}{2} \int_0^1 (1-t)^{n-1} l_{H^{-1}(t)} \, dt, \, 2 \int_0^1 \frac{(1-t)^n}{l_{H^{-1}(t)}} \, dt \right) \\
&= \left(\frac{1}{2} \int_0^\infty (1-H(x))^{n-1} (l_x)^2 \, dx, \, 2 \int_0^{M(r)} (1-H(x))^n \, dx \right).
\end{aligned}$$

By (1.1), we may rewrite the integral in the second coordinate of the last expression as

$$\begin{aligned}
\int_0^{M(r)} dx \left(\int_0^1 dt \, \mathbf{1}_{\{r_t > x\}} \right)^n &= \int_0^{M(r)} dx \int_{[0,1]^n} dt_1 \cdots dt_n \prod_{i=1}^n \mathbf{1}_{\{r_{t_i} > x\}} \\
&= \int_{[0,1]^n} \min\{r_{t_1}, \dots, r_{t_n}\} \, dt_1 \cdots dt_n,
\end{aligned}$$

where we used Fubini's theorem for the second equality. Therefore we obtain (2.9). Combining (2.8) and (2.9) leads to the conclusion. □

We end this paper with a comment on a relevant fact which is deduced from the proof of Proposition 2.2 and which, as far as we know, has not ever been pointed out.

Remark 2.3. It is known (see, e.g., [1, Equation (5d)]) that

$$M(r) \overset{(d)}{=} \frac{1}{2} \int_0^1 \frac{dt}{r_t};$$

indeed, Jeulin's identity (1.2) entails that

$$\frac{1}{2} \int_0^1 \frac{dt}{r_t} \overset{(d)}{=} \int_0^{M(r)} \frac{1}{l_x} \times l_x \, dx = M(r).$$

Combining this fact with a part of the proof of Proposition 2.2, one sees that the sequence $\{X_n\}_{n=0}^\infty$ of random variables given by

$$X_0 = M(r) \qquad \text{and} \qquad X_n = \int_{[0,1]^n} \min\{r_{t_1}, \dots, r_{t_n}\} \, dt_1 \cdots dt_n \quad \text{for } n \geq 1,$$

is identical in law with

$$\frac{1}{2} \int_0^1 \frac{(1-t)^n}{r_t} \, dt, \quad n = 0, 1, 2, \dots,$$

as well as with

$$\frac{1}{2} \int_0^1 \frac{t^n}{r_t} \, dt, \quad n = 0, 1, 2, \dots$$

by the time-reversal. As an application, one finds that

$$\begin{aligned}
\int_{[0,1]^2} |r_{t_1} - r_{t_2}| \, dt_1 dt_2 &= 2(X_1 - X_2) \\
&\overset{(d)}{=} \int_0^1 \frac{t(1-t)}{r_t} \, dt.
\end{aligned}$$

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