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On estimating the scale parameter of the selected uniform population under the entropy loss function

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Abstract. Let $\pi_1, \dots, \pi_k$ be k (≥ 2) independent populations, where π_i denotes the uniform distribution over the interval $(0, \theta_i)$ and $\theta_i > 0$ $(i = 1, \dots, k)$ is an unknown scale parameter. Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be the ordered values of $\theta_1, \dots, \theta_k$. The population $\pi_{(k)}$ $(\pi_{(1)})$ associated with the unknown parameter $\theta_{[k]}$ $(\theta_{[1]})$ is called the best (worst) population. For selecting the best population, we consider a general class of selection rules based on the natural estimators of $\theta_i, i = 1, \dots, k$. Under the entropy loss function, we consider the problem of estimating the scale parameter θ_S of the population selected using a fixed selection rule from this class. We derive the uniformly minimum risk unbiased estimator of θ_S and two natural estimators of θ_S are also considered. We derive a general result for improving a scale invariant estimator of θ_S under the entropy loss function. A simulation study on the performances of various competing estimators of θ_S is also reported. Finally, we provide similar results for the problem of estimating the scale parameter of selected population when the selection goal is that of selecting the worst uniform population.

1 Introduction

Selection and related estimation problems have been extensively studied in the literature. Selection problems primarily deal with the goal of selecting the best (or worst) population among a set of available populations, where the quality of a population is assessed in terms of an unknown parameter associated with it. After the selection has been made using a given selection procedure, one may be interested in estimating the worth of the selected population. In the literature, such problems are referred to as problems of estimation after selection. For detailed discussion on estimation after selection problems, one may refer to Vellaisamy, Kumar and Sharma (1988), Song (1992), Vellaisamy (1992, 1996), Parsian and Farsipour (1999), Misra and van der Meulen (2001), Kumar and Tripathi (2003), Kumar and Gangopadhyay (2005), Nematollahi and Motamed-Shariati (2009, 2012) and Nematollahi and Jozani (2016).

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Most of the work on selection and estimation after selection problems reported in the literature is carried out under the assumption of equal nuisance parameters and/or equal sample sizes and very little research has been carried out under the setup where nuisance parameters and/or sample sizes may be unequal. Such selection problems are exceedingly complex as reported in Hall (1959). For the case of unequal nuisance parameters/sample sizes, some of the contributions are due to Risko (1985), Abughalous and Miescke (1989), Abughalous and Bansal (1994), Misra and Dhariyal (1994), Vellaisamy (1996), Misra and Arshad (2014), Arshad, Misra and Vellaisamy (2015), Arshad and Misra (2015a, 2015b, 2016). Recently, under the entropy loss function and under the natural selection rule which selects the population corresponding to largest (smallest) complete sufficient statistic, Nematollahi and Motamed-Shariati (2012) considered the problem of estimation after selection from uniform populations based on sample of equal sizes. In this paper, we consider unequal sample sizes and a more general class of selection rules thereby extending the results of Nematollahi and Motamed-Shariati (2012).

Let $\pi_1, \dots, \pi_k$ be k (≥ 2) independent populations such that the independent observations $X_{i1}, \dots, X_{in_i}$ from the population π_i have a uniform distribution over the interval $(0, \theta_i), \theta_i > 0, i = 1, \dots, k$. Assume that the parameters $\theta_1, \dots, \theta_k$ are completely unknown. Let $X_i = \max\{X_{i1}, \dots, X_{in_i}\}, i = 1, \dots, k$, so that $\mathbf{X} = (X_1, \dots, X_k)$ is a complete and sufficient statistic for $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in \boldsymbol{\Theta}$; here $\boldsymbol{\Theta}$ $(= \mathbb{R}_+^k)$ denotes the parametric space and $\mathbb{R}_+^k$ denotes the positive part of k-dimensional Euclidean space. The random variables $X_1, \dots, X_k$ are independent and X_i has the probability density function (pdf)

$$f_i(x|\theta_i) = \begin{cases} \dfrac{n_i x^{n_i-1}}{\theta_i^{n_i}}, & \text{if } 0 < x < \theta_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be the ordered values of $\theta_1, \dots, \theta_k$. Let $\pi_{(k)}$ denote the unknown population associated with the largest scale parameter $\theta_{[k]}$, and be called the best population. In case of ties for the best populations, we assume that the population π_j having the largest subscript j among tied populations is tagged as the best population; for example, if $\theta_i = \theta_j = \theta_{[k]}$ and $i < j$ then the population π_j is tagged as the best population. Since X_i is the maximum likelihood estimator of $\theta_i, i = 1, \dots, k$, for the goal of selecting the best population, a natural selection rule $\boldsymbol{\delta}^N$ is to select the population corresponding to $X_{[k]} = \max\{X_1, \dots, X_k\}$. Then the scale parameter of the selected population is $\theta_S = \sum_{i=1}^k \theta_i \{\prod_{j \neq i} I(X_i, X_j)\}$, where $I(a, b) = 1$ if $a \geq b$; $= 0$ otherwise. Under the assumption of equal sample sizes and for the natural selection rule $\boldsymbol{\delta}^N$, Nematollahi and Motamed-Shariati (2012) considered the problem of estimating the scale parameter θ_S of the selected population under the entropy loss function

$$L(\theta_S, \varphi) = \frac{\theta_S}{\varphi} - \ln\left(\frac{\theta_S}{\varphi}\right) - 1, \qquad \varphi \in \mathcal{C}, \tag{1.2}$$

where $\mathcal{C}$ denotes the class of all estimators of θ_S. The authors derived the uniformly minimum risk unbiased (UMRU) estimator of θ_S. For $k = 2$, authors proved the minimaxity of the generalized Bayes estimator and also proved the inadmissibility of the UMRU estimator. In case of unequal sample sizes, it is inappropriate to use the natural selection rule $\boldsymbol{\delta}^N$ for selecting the best population (see Misra and Dhariyal (1994)). For $k = 2$ and for the case of unequal sample sizes, it follows from Arshad and Misra (2015b) (see Concluding Remarks in Arshad and Misra (2015b)) that the selection rule $\boldsymbol{\delta}^{a^*} = (\delta_1^{a^*}, \delta_2^{a^*})$, where

$$\delta_1^{a^*}(\mathbf{X}) = \begin{cases} 1, & \text{if } X_1 > a^* X_2, \\ 0, & \text{if } X_1 \le a^* X_2, \end{cases} \qquad \delta_2^{a^*}(\mathbf{X}) = 1 - \delta_1^{a^*}(\mathbf{X}), \tag{1.3}$$

and

$$a^* \equiv a^*(n_1, n_2) = \begin{cases} \left(\dfrac{n_1 + n_2}{2n_2}\right)^{\frac{1}{n_1}}, & \text{if } n_1 \le n_2, \\ \left(\dfrac{2n_1}{n_1 + n_2}\right)^{\frac{1}{n_2}}, & \text{if } n_1 > n_2, \end{cases}$$

is minimax rule under the 0–1 loss function and is generalized Bayes rule with respect to non-informative prior. For selecting the best populations, we consider a fixed selection rule $\boldsymbol{\delta}^{\mathbf{a}} \in \mathcal{D}_1 = \{\boldsymbol{\delta}^{\mathbf{a}} : \boldsymbol{\delta}^{\mathbf{a}} = (\delta_1^{\mathbf{a}}, \ldots, \delta_k^{\mathbf{a}}), \mathbf{a} \in \mathbb{R}_+^k\}$, where

$$\delta_i^{\mathbf{a}}(\mathbf{X}) = \begin{cases} 1, & \text{if } a_i X_i > \max\limits_{j \ne i} a_j X_j, \\ 0, & \text{otherwise.} \end{cases}$$

Then the scale parameter of the selected population is

$$\theta_S = \sum_{i=1}^{k} \theta_i \delta_i^{\mathbf{a}}(\mathbf{X}).$$

Let χ $(= \mathbb{R}_+^k)$ denote the sample space and let $A_i = \{\mathbf{x} \in \chi : a_i x_i > a_j x_j, \forall j \ne i, j = 1, \ldots, k\}, i = 1, \ldots, k$. Then the scale parameter θ_S can be written as

$$\theta_S = \sum_{i=1}^{k} \theta_i I_{A_i}(\mathbf{X}), \tag{1.4}$$

where I_A denotes the indicator function of the set A.

Arshad and Misra (2015a) considered estimation of θ_S under the scale invariant squared error loss function. They derived the uniformly minimum variance unbiased estimator of θ_S and proved certain inadmissibility results. For the special choice $a_i = 1, i = 1, \ldots, k$, Nematollahi and Motamed-Shariati (2012) considered estimation of θ_S under the entropy loss function (1.2). They derived the UMRU estimator of θ_S and for the case $k = 2$, they proved that UMRU estimator is inadmissible and the generalized Bayes estimator is minimax. In this paper, for a general $\mathbf{a} \in \mathbb{R}_+^k$, we consider the problem of estimating θ_S under the entropy loss function (1.2). We generalize various results proved in Nematollahi and Motamed-Shariati (2012).

Note that, for $i \in \{1, \dots, k\}$, X_i and $\frac{n_i}{n_i-1}X_i$ are respectively, the maximum likelihood estimator and uniformly minimum risk unbiased estimator (or minimum risk invariant estimator), with respect to the entropy loss function, of θ_i in the component estimation problem. Based on these estimators, two natural estimators of the scale parameter θ_S are given by

$$\varphi_{N,1}(\mathbf{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\mathbf{X}) \quad \text{and} \quad \varphi_{N,2}(\mathbf{X}) = \sum_{i=1}^{k}\left(\frac{n_i}{n_i-1}\right) X_i I_{A_i}(\mathbf{X}).$$

In Section 2, we derive the UMRU estimator of θ_S. In Section 3, we derive a general result for improving a scale invariant estimator of θ_S under the entropy loss function (1.2). Using this result, the estimators better than the UMRU estimator and the natural estimator $\varphi_{N,1}$ are obtained. A subclass of natural type estimators is shown to be inadmissible for estimating θ_S under the entropy loss function. In Section 4, a simulation study on the performances of various competing estimators of θ_S is provided. Section 5 deals with the problem of estimating the scale parameter of the selected uniform population when the selection goal is that of selecting the worst uniform population (population associated with the smallest scale parameter).

2 UMRU estimator

In this section, we will derive the UMRU estimator of θ_S, under the entropy loss function (1.2).

Definition 1 (Nematollahi and Motamed-Shariati (2012)). An estimator $\varphi(\mathbf{X})$ of θ_S is said to be risk unbiased estimator of the random parameter $\eta(\boldsymbol{\theta})$ under the entropy loss function (1.2) if

$$E_{\boldsymbol{\theta}}\left(\frac{1}{\varphi(\mathbf{X})}\right) = E_{\boldsymbol{\theta}}\left(\frac{1}{\eta(\boldsymbol{\theta})}\right), \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}. \tag{2.1}$$

To obtain the UMRU estimator, we need the following lemma given in Nematollahi and Motamed-Shariati (2012).

Lemma 1. *Suppose that $X_1, \dots, X_k$ are independent random variables such that X_i $(i = 1, \dots, k)$ has the pdf (1.1). For $i \in \{1, \dots, k\}$, let $U_i(\cdot)$ be a given real valued function on $\mathbb{R}_+^k$ such that*

(i) $E_{\boldsymbol{\theta}}(U_i(x_1, \dots, x_k)/x_i) < \infty$, $\forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$ *and* $\forall \mathbf{x} \in \mathbb{R}_+^k$,
(ii) $\int_0^{x_i} U_i(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_k) t^{n_i-1}\, dt < \infty$, $\forall \mathbf{x} \in \mathbb{R}_+^k$.

Define the function $V : \mathbb{R}_+^k \to \mathbb{R}_+$, *such that* $\frac{1}{V(\mathbf{X})} = \sum_{i=1}^k \frac{1}{V_i(\mathbf{X})}$, *where*

$$\frac{1}{V_i(\mathbf{x})} = \frac{U_i(\mathbf{x})}{x_i} - \frac{1}{x_i^{n_i+1}} \int_0^{x_i} U_i(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_k) t^{n_i-1}\, dt,$$

$$\mathbf{x} \in \mathbb{R}_+^k, i = 1, 2, \ldots, k.$$

Then, under the entropy loss function (1.2), *the estimator* $V(\mathbf{X})$ *is risk unbiased for* $S_{\mathbf{X}}(\boldsymbol{\theta}) = [\sum_{i=1}^k \frac{1}{\theta_i} U_i(\mathbf{X})]^{-1}$.

Theorem 1. *Under the entropy loss function* (1.2), *the uniformly minimum risk unbiased estimator of the scale parameter* θ_S *of the selected population is given by*

$$\varphi_U(\mathbf{X}) = \sum_{i=1}^k \frac{n_i X_i}{[(n_i - 1) + (\frac{\max_{j \neq i} a_j X_j}{a_i X_i})^{n_i}]} I_{A_i}(\mathbf{X}).$$

Proof. Let $\varphi_U(\mathbf{X})$ be a risk unbiased estimator of $S_{\mathbf{X}}(\boldsymbol{\theta}) = [\sum_{i=1}^k \frac{1}{\theta_i} I_{A_i}(\mathbf{X})]^{-1}$. Using Lemma 1, we have

$$\frac{1}{\varphi_U(\mathbf{X})} = \sum_{i=1}^k \frac{1}{V_i(\mathbf{X})} \qquad \text{(say)},$$

where

$$\begin{aligned}
\frac{1}{V_i(\mathbf{X})} &= \frac{I_{A_i}(\mathbf{X})}{X_i} - \frac{1}{X_i^{n_i+1}} \int_0^{X_i} I_{A_i}(X_1, \ldots, X_{i-1}, t, X_{i+1}, \ldots, X_k) t^{n_i-1}\, dt \\
&= \frac{I_{A_i}(\mathbf{X})}{X_i} - \frac{1}{X_i^{n_i+1}} \int_{\max_{j \neq i} \frac{a_j X_j}{a_i}}^{X_i} t^{n_i-1}\, dt\, I_{A_i}(\mathbf{X}) \\
&= \frac{I_{A_i}(\mathbf{X})}{X_i} - \frac{1}{n_i X_i^{n_i+1}} \left[X_i^{n_i} - \left(\max_{j \neq i} \frac{a_j X_j}{a_i} \right)^{n_i} \right] I_{A_i}(\mathbf{X}) \\
&= \frac{1}{n_i X_i} \left[(n_i - 1) + \left(\frac{\max_{j \neq i} a_j X_j}{a_i X_i} \right)^{n_i} \right] I_{A_i}(\mathbf{X}), \qquad i = 1, \ldots, k.
\end{aligned}$$

Since $\varphi_U(\mathbf{X})$ is a risk unbiased estimator of $S_{\mathbf{X}}(\boldsymbol{\theta})$, it follows that

$$E_{\boldsymbol{\theta}}\left(\frac{1}{\varphi_U(\mathbf{X})} \right) = E_{\boldsymbol{\theta}}\left(\frac{1}{S_{\mathbf{X}}(\boldsymbol{\theta})} \right) = E_{\boldsymbol{\theta}}\left(\sum_{i=1}^k \frac{1}{\theta_i} I_{A_i}(\mathbf{X}) \right) = E_{\boldsymbol{\theta}}\left(\frac{1}{\theta_S} \right).$$

Hence, the estimator $\varphi_U(\mathbf{X})$ is a risk unbiased estimator of θ_S and the result follows on noting that $\mathbf{X} = (X_1, \ldots, X_k)$ is a complete and sufficient statistic. □

Remark 1. Let $X_{[1]} \leq \cdots \leq X_{[k]}$ be the ordered values of $X_1, \ldots, X_k$. For $n_1 = n_2 = \cdots = n_k = n$, and $a_1 = a_2 = \cdots = a_k = 1$, the UMRU estimator of θ_S is given by

$$\varphi_U(\mathbf{X}) = \frac{nX_{[k]}}{[n-1+(\frac{X_{[k-1]}}{X_{[k]}})^n]}.$$

In this case, the UMRU estimator depends only on the last two order statistics. This result was derived by Nematollahi and Motamed-Shariati (2012).

3 Some inadmissibility results

In this section, we will show that the UMRU estimator and the natural estimator $\varphi_{N,1}$ are inadmissible under the entropy loss function (1.2). To obtain a sufficient condition for the inadmissibility of a scale-invariant estimator of θ_S, we need the following lemmas. Lemma 2 is adopted from Arshad and Misra (2015a).

Lemma 2. *Let $X_1, \ldots, X_k$ be independent random variables such that X_i has the probability density function given in (1.1). Let $T_j = \frac{X_j}{X_1}$, $j = 2, \ldots, k$. Then, for a fixed $\mathbf{t} = (t_2, \ldots, t_k) \in \mathbb{R}_+^{k-1}$, the conditional distribution of X_1 given $\mathbf{T} = (t_2, \ldots, t_k)$ is given by*

$$f_{X_1|\mathbf{T}}(x_1|\mathbf{t}) = \begin{cases} \dfrac{(\sum_{j=1}^k n_j) x_1^{\sum_{j=1}^k n_j - 1}}{\theta_{\mathbf{t}}^{\sum_{j=1}^k n_j}}, & \text{if } 0 < x_1 < \theta_{\mathbf{t}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\theta_{\mathbf{t}} = \min\{\theta_1, \min_{j\neq 1} \frac{\theta_j}{t_j}\}$.

Lemma 3. *Let $B_1 = \{(t_2, \ldots, t_k) \in \mathbb{R}_+^{k-1} : t_j < \frac{a_1}{a_j}, j = 2, \ldots, k\}$ and*

$$B_l = \left\{(t_2, \ldots, t_k) \in \mathbb{R}_+^{k-1} : t_l > \max\left(\frac{a_1}{a_l}, \max_{\substack{2\le j\le k \\ j\neq l}} \frac{a_j t_j}{a_l}\right)\right\}, \qquad l = 2, \ldots, k,$$

so that $\{B_1, \ldots, B_k\}$ forms a partition of $\mathbb{R}_+^{k-1}$. Define

$$\phi(\mathbf{t}, \boldsymbol{\theta}) = \sum_{i=1}^k \theta_i E_{\boldsymbol{\theta}}\left(\frac{1}{X_1}\middle|\mathbf{T} = \mathbf{t}\right) I_{B_i}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}_+^{k-1}, \boldsymbol{\theta} \in \boldsymbol{\Theta}.$$

Then

$$\phi(\mathbf{t}, \boldsymbol{\theta}) = \left(\frac{\sum_{j=1}^k n_j}{\sum_{j=1}^k n_j - 1}\right) \sum_{i=1}^k \frac{\theta_i}{\min\{\theta_1, \min_{j\neq 1} \frac{\theta_j}{t_j}\}} I_{B_i}(\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}_+^{k-1}, \boldsymbol{\theta} \in \boldsymbol{\Theta},$$

and

$$\phi_*(\mathbf{t}) = \inf_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \phi(\mathbf{t},\boldsymbol{\theta})$$
$$= \begin{cases} \left(\dfrac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right), & \text{if } \mathbf{t} \in B_1, \\ \left(\dfrac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right) t_l, & \text{if } \mathbf{t} \in B_l, l = 2, \dots, k. \end{cases}$$

Definition 2. An estimator $\varphi(X_1, \dots, X_k)$ of the scale parameter θ_S of selected population is said to be scale-invariant if

$$\varphi(cx_1, \dots, cx_k) = c\varphi(x_1, \dots, x_k), \qquad \forall c > 0 \text{ and } \mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Clearly, a scale-invariant estimator of θ_S will be of the form

$$\varphi(X_1, \dots, X_k) = X_1 \psi(T_2, \dots, T_k),$$

where $T_i = \frac{X_i}{X_1}, i = 2, \dots, k$, and ψ is a non-negative real-valued function on $\mathbb{R}_+^{k-1}$.

The following theorem provide a sufficient condition for the inadmissibility of a scale-invariant estimator of θ_S under the entropy loss function (1.2). The proof of the theorem is an application of the technique of Brewster and Zidek (1974).

Theorem 2. *Suppose $\varphi(\mathbf{X}) = X_1\psi(\mathbf{T})$ is a given scale-invariant estimator of θ_S, where $\mathbf{T} = (T_2, \dots, T_k) = (\frac{X_2}{X_1}, \dots, \frac{X_k}{X_1})$, and $\psi(\cdot)$ is a real-valued function defined on $\mathbb{R}_+^{k-1}$. Let $P_{\boldsymbol{\theta}}(\psi(\mathbf{T}) < \phi_*(\mathbf{T})) \geq 0, \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, with strictly inequality for some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, where $\phi_*(\cdot)$ is as defined in Lemma 3. Then, under the entropy loss function (1.2), the estimator φ is inadmissible and is dominated by the estimator $\varphi_1(\mathbf{X}) = X_1\psi_1(\mathbf{T})$, where $\psi_1(\mathbf{t}) = \max\{\psi(\mathbf{t}), \phi_*(\mathbf{t})\}$.*

Proof. Consider the risk difference

$$R(\boldsymbol{\theta}, \varphi) - R(\boldsymbol{\theta}, \varphi_1) = E_{\boldsymbol{\theta}}\big(D_{\boldsymbol{\theta}}(\mathbf{t})\big),$$

where, for $\mathbf{t} \in \mathbb{R}_+^{k-1}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,

$$D_{\boldsymbol{\theta}}(\mathbf{t}) = E_{\boldsymbol{\theta}}\big(L\big(\boldsymbol{\theta}, X_1\psi(\mathbf{T})\big) - L\big(\boldsymbol{\theta}, X_1\psi_1(\mathbf{T})\big)|\mathbf{T} = \mathbf{t}\big)$$
$$= \left(\frac{1}{\psi(\mathbf{t})} - \frac{1}{\psi_1(\mathbf{t})}\right) E_{\boldsymbol{\theta}}\left(\frac{\theta_S}{X_1}\Big|\mathbf{T} = \mathbf{t}\right) - \ln\left(\frac{\psi_1(\mathbf{t})}{\psi(\mathbf{t})}\right)$$
$$= \left(\frac{1}{\psi(\mathbf{t})} - \frac{1}{\psi_1(\mathbf{t})}\right) \phi(\mathbf{t}, \boldsymbol{\theta}) - \ln\left(\frac{\psi_1(\mathbf{t})}{\psi(\mathbf{t})}\right).$$

Here $\phi(\mathbf{t}, \boldsymbol{\theta})$ is as defined in Lemma 3. Clearly, for a fixed $\mathbf{t} \in \mathbb{R}_+^{k-1}$, if $\psi(\mathbf{t}) \geq \phi_*(\mathbf{t})$, then $D_{\boldsymbol{\theta}}(\mathbf{t}) = 0, \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$. Also, if $\psi(\mathbf{t}) < \phi_*(\mathbf{t})$, then

$$\begin{aligned} D_{\boldsymbol{\theta}}(\mathbf{t}) &= \left(\frac{1}{\psi(\mathbf{t})} - \frac{1}{\phi_*(\mathbf{t})}\right)\phi(\mathbf{t}, \boldsymbol{\theta}) - \ln\left(\frac{\phi_*(\mathbf{t})}{\psi(\mathbf{t})}\right) \\ &\geq \left(\frac{1}{\psi(\mathbf{t})} - \frac{1}{\phi_*(\mathbf{t})}\right)\phi_*(\mathbf{t}) - \ln\left(\frac{\phi_*(\mathbf{t})}{\psi(\mathbf{t})}\right) \qquad \text{(using Lemma 3)} \\ &\geq 0. \end{aligned}$$

Since $P_{\boldsymbol{\theta}}(\psi(\mathbf{T}) < \phi_*(\mathbf{T})) \geq 0, \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, with strictly inequality for some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, we conclude that

$$R(\boldsymbol{\theta}, \varphi) \geq R(\boldsymbol{\theta}, \varphi_1), \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta},$$

and there is a strict inequality for some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Hence the result follows. □

It is easy to verify that the UMRU estimator φ_U and the natural estimator $\varphi_{N,1}$, respectively, can be written as $\varphi_U(\mathbf{X}) = X_1 \psi_U(\mathbf{T})$ and $\varphi_{N,1}(\mathbf{X}) = X_1 \psi_{N,1}(\mathbf{T})$, where

$$\psi_U(\mathbf{t}) = \begin{cases} \dfrac{n_1}{n_1 - 1 + (\frac{\max_{j \neq 1} a_j t_j}{a_1})^{n_1}}, & \text{if } \mathbf{t} \in B_1, \\ \dfrac{n_l t_l}{n_l - 1 + (\frac{\max\{a_1, \max_{j \neq l, j \neq 1} a_j t_j\}}{a_l t_l})^{n_l}}, & \text{if } \mathbf{t} \in B_l, l = 2, \ldots, k, \end{cases}$$

and

$$\psi_{N,1}(\mathbf{t}) = \begin{cases} 1, & \text{if } \mathbf{t} \in B_1, \\ t_l, & \text{if } \mathbf{t} \in B_l, l = 2, \ldots, k. \end{cases}$$

Now, using Theorem 2, we have the following results.

Corollary 1. *Under the loss function* (1.2), *the UMRU estimator* $\varphi_U(\mathbf{X})$ *is inadmissible for estimating* θ_S *and is dominated by the estimator* $\varphi_U^*(\mathbf{X}) = X_1 \psi_U^*(\mathbf{t})$, *where* $\psi_U^*(\mathbf{t}) = \max\{\psi_U(\mathbf{t}), \phi_*(\mathbf{t})\}, \mathbf{t} \in \mathbb{R}_+^{k-1}$, *and* $\phi_*(\cdot)$ *is as defined in Lemma* 3.

Corollary 2. *Under the loss function* (1.2), *the natural estimator* $\varphi_{N,1}(\mathbf{X})$ *is inadmissible for estimating* θ_S *and is dominated by the estimator*

$$\varphi_{N,1}^*(\mathbf{X}) = \left(\frac{\sum_{j=1}^k n_j}{\sum_{j=1}^k n_j - 1}\right)\varphi_{N,1}(\mathbf{X}).$$

Remark 2. For $k = 2, n_1 = n_2 = n$ and $a_1 = a_2 = 1$, it follows from Remark 1 that the UMRU estimator of θ_S is

$$\varphi_U(\mathbf{X}) = \frac{nX_{[2]}}{[n - 1 + (\frac{X_{[1]}}{X_{[2]}})^n]}. \tag{3.2}$$

Under the loss function (1.2), Nematollahi and Motamed-Shariati (2012) proved that the UMRU estimator, given in (3.2), is inadmissible. Thus, Corollary 1 generalizes their result.

Now we will prove that a subclass of natural estimators is inadmissible for estimating θ_S, under the entropy loss function (1.2). The following lemma will be useful in deriving the next result (see Lemma 3 in Arshad and Misra (2015a) and Theorem 3.1 in Rajesh, Misra and Singh (1998)).

Lemma 4. *Let $\{G_\alpha : \alpha > 0\}$ be a family of distribution functions defined by*

$$G_\alpha(z) = \begin{cases} 0, & \text{if } z < 0, \\ z^\alpha, & \text{if } 0 \le z < 1, \alpha > 0, \\ 1, & \text{if } z \ge 1. \end{cases}$$

Then, for any non-decreasing function $\phi(z)$ and $0 < \alpha_1 < \alpha_2 < \infty$,

$$\int_0^1 \phi(z)\,\mathrm{d}G_{\alpha_1}(z) \le \int_0^1 \phi(z)\,\mathrm{d}G_{\alpha_2}(z).$$

Theorem 3. *Let $\min\{n_1, \dots, n_k\} > 1$. For a fixed $i \in \{1, \dots, k\}$, let $c_i \equiv c_i(n_1, \dots, n_k)$ be a positive constant and let $\mathbf{c} = (c_1, \dots, c_k) \in \mathbb{R}_+^{k-1}$. Assume that $c_i \in (0, \frac{\sum_{j=1}^k n_j}{\sum_{j=1}^k n_j - 1}) \cup (\frac{n_i}{n_i - 1}, \infty)$, for some $i \in \{1, \dots, k\}$. Then, under the entropy loss function (1.2), the natural type estimator*

$$\varphi_{\mathbf{c}}(\mathbf{X}) = \sum_{j=1}^k c_j X_j I_{A_j}(\mathbf{X}),$$

is inadmissible for estimating θ_S.

Proof. Suppose that $c_i \in (0, \frac{\sum_{j=1}^k n_j}{\sum_{j=1}^k n_j - 1})$, for some fixed $i \in \{1, \dots, k\}$. Clearly the natural type estimators $\varphi_{\mathbf{c}}$ satisfied the sufficient condition for inadmissibility given in Theorem 2. Thus, it follows from Theorem 2 that the natural type estimators $\varphi_{\mathbf{c}}$ are inadmissible and are dominated by the estimator

$$\varphi_{\mathbf{c}}^*(\mathbf{X}) = \frac{\sum_{j=1}^k n_j}{\sum_{j=1}^k n_j - 1} \sum_{j=1}^k X_j I_{A_j}(\mathbf{X}).$$

Now suppose that $c_i \in (\frac{n_i}{n_i-1}, \infty)$, for some fixed $i \in \{1, \dots, k\}$. The risk function of the natural type estimator $\varphi_{\mathbf{c}}$ is given by

$$\begin{aligned} R(\boldsymbol{\theta}, \varphi_{\mathbf{c}}) &= E_{\boldsymbol{\theta}}\left(\frac{\theta_S}{\varphi_{\mathbf{c}}(\mathbf{X})} - \ln\left(\frac{\theta_S}{\varphi_{\mathbf{c}}(\mathbf{X})}\right) - 1\right) \\ &= \sum_{j=1}^{k} E_{\boldsymbol{\theta}}\left(\left\{\frac{\theta_i}{c_j X_j} - \ln\left(\frac{\theta_i}{c_j X_j}\right) - 1\right\} I_{A_j}(\mathbf{X})\right) \qquad (3.3) \\ &= \sum_{j=1}^{k} M_j(\boldsymbol{\theta}, c_j), \qquad \text{(say)}. \end{aligned}$$

For $j \in \{1, \dots, k\}$, let $Z_j = \frac{X_j}{\theta_j}$. Then $Z_1, \dots, Z_k$ are independent random variables and Z_j has the distribution function $G_{n_j}(\cdot)$, defined in Lemma 4. For a fixed value of $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $M_i(\boldsymbol{\theta}, c)$ is minimum at $c = c_i^*(\boldsymbol{\theta})$, where

$$c_i^*(\boldsymbol{\theta}) = \theta_i \frac{E_{\boldsymbol{\theta}}(X_i^{-1} I_{A_i}(\mathbf{X}))}{E_{\boldsymbol{\theta}}(I_{A_i}(\mathbf{X}))} = \frac{E_{\boldsymbol{\theta}}(Z_i^{-1} I_{D_i}(\mathbf{Z}))}{E_{\boldsymbol{\theta}}(I_{D_i}(\mathbf{Z}))}, \qquad i = 1, \dots, k.$$

Here, for $i \in \{1, \dots, k\}$, $D_i = \{\mathbf{z} \in (0, 1)^k : a_i\theta_i z_i > a_j\theta_j z_j, \forall j \neq i, j = 1, \dots, k\}$. Therefore,

$$E_{\boldsymbol{\theta}}\big(Z_i^{-1} I_{D_i}(\mathbf{Z})\big) = \frac{n_i}{n_i - 1}\int_0^1 \prod_{j\neq i}^{k} G_{n_j}\left(\frac{a_i\theta_i z}{a_j\theta_j}\right) dG_{n_i-1}(z)$$

and

$$E_{\boldsymbol{\theta}}\big(I_{D_i}(\mathbf{Z})\big) = \int_0^1 \prod_{j\neq i}^{k} G_{n_j}\left(\frac{a_i\theta_i z}{a_j\theta_j}\right) dG_{n_i}(z).$$

Clearly, $\prod_{j\neq i}^{k} G_{n_j}(\frac{a_i\theta_i z}{a_j\theta_j})$ is a non-decreasing function of $z \in \mathbb{R}_+$. Using Lemma 4, we get

$$\begin{aligned} &\int_0^1 \prod_{j\neq i}^{k} G_{n_j}\left(\frac{a_i\theta_i z}{a_j\theta_j}\right) dG_{n_i-1}(z) \le \int_0^1 \prod_{j\neq i}^{k} G_{n_j}\left(\frac{a_i\theta_i z}{a_j\theta_j}\right) dG_{n_i}(z) \\ &\quad\Rightarrow \quad \frac{n_i - 1}{n_i} E_{\boldsymbol{\theta}}\big(Z_i^{-1} I_{D_i}(\mathbf{Z})\big) \le E_{\boldsymbol{\theta}}\big(I_{D_i}(\mathbf{Z})\big) \\ &\quad\Rightarrow \quad c_i^*(\boldsymbol{\theta}) = \frac{E_{\boldsymbol{\theta}}(Z_i^{-1} I_{D_i}(\mathbf{Z}))}{E_{\boldsymbol{\theta}}(I_{D_i}(\mathbf{Z}))} \le \frac{n_i}{n_i - 1}, \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}, i = 1, \dots, k. \end{aligned}$$

Note that, for a fixed $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and a fixed i, $M_i(\boldsymbol{\theta}, c)$ is a decreasing function of $c \in (0, c_i^*)$ and is an increasing function of $c \in [c_i^*, \infty)$ with $c_i^* \le \frac{n_i}{n_i-1}$. It follows that, for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $c \geq \frac{n_i}{n_i-1}$, $M_i(\boldsymbol{\theta}, c)$ is an increasing function of c. Consequently,

$$M_i(\boldsymbol{\theta}, c_i) > M_i\left(\boldsymbol{\theta}, \frac{n_i}{n_i-1}\right), \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}.$$

$$\begin{aligned}\Rightarrow \quad R(\boldsymbol{\theta}, \varphi_{\mathbf{c}}) &= \sum_{j=1}^{k} M_j(\boldsymbol{\theta}, c_j) \\ &> \sum_{\substack{j=1 \\ j\neq i}}^{k} M_j(\boldsymbol{\theta}, c_j) + M_i\left(\boldsymbol{\theta}, \frac{n_i}{n_i-1}\right) \\ &= R(\boldsymbol{\theta}, \varphi_{\mathbf{d}}), \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta},\end{aligned}$$

where $\varphi_{\mathbf{d}}(\mathbf{X}) = \sum_{\substack{j=1 \\ j\neq i}}^{k} c_j X_j I_{A_j}(\mathbf{X}) + \frac{n_i}{n_i-1} X_i I_{A_i}(\mathbf{X})$. □

The computation of various competing estimators of θ_S is illustrated through the following example.

Example. The data in Table 1 is reported in Lawless (1982; page 138). The data represent failure times (in minutes) for two types of electrical insulation in an experiment in which the insulation was subjected to a continuously voltages stress. Arshad and Misra (2015a) considered data in Table 1 and fitted the uniform distributions. They shifted the location of the data by its minimum value. The shifted data from the populations π_1 and π_2 are fitted to uniform $U(0, 200.8)$ and uniform $U(0, 139.6)$ distributions, respectively. Suppose the quality of the electrical insulation is measured in terms of average failure time, that is, the population $\pi_1 \equiv U(0, \theta_1)$ is better than the population $\pi_2 \equiv U(0, \theta_2)$ if $\theta_1 > \theta_2$, and the population π_2 is better than the population π_1 if $\theta_1 \leq \theta_2$. For the goal of selecting the better electrical insulation, we use the minimax selection rule $\boldsymbol{\delta}^{a^*}$ given in (1.3). Since the minimax selection rule $\boldsymbol{\delta}^{a^*}$ depends on the sample sizes n_1 and n_2, we consider the following two cases:

Case I: Taking $n_1 = 4$ and $n_2 = 11$. We use the first 4 observations from π_1 and 11 observations from π_2 (excluding 0). From the above data, we have $a^* = 0.9087$ and $\mathbf{x} = (x_1, x_2) = (200.8, 139.6)$. Clearly, $x_1 = 200.8 > a^* x_2 = 126.85$. Thus, the various estimates of θ_S are given by $\varphi_U(\mathbf{x}) = \varphi_U^*(\mathbf{x}) = 254.23$, $\varphi_{N,1}(\mathbf{x}) = 200.8$, $\varphi_{N,1}^*(\mathbf{x}) = 215.14$ and $\varphi_{N,2}(\mathbf{x}) = 267.73$.

Table 1 *Failure times (in minutes) for two types of electrical insulation*

Population	Observations
Type A (π_1)	219.3, 79.4, 86.0, 150.2, 21.7, 18.5, 121.9, 40.5, 147.1, 35.1, 42.3, 48.7
Type B (π_2)	21.8, 70.7, 24.4, 138.6, 151.9, 75.3, 12.3, 95.5, 98.1, 43.2, 28.6, 46.9

Case II: Taking $n_1 = 11$ and $n_2 = 4$. We use 11 observations from π_1 and the first 4 observations from π_2, we have $a^* = 1.1005$ and $\mathbf{x} = (x_1, x_2) = (200.8, 126.3)$. The various estimates of θ_S are $\varphi_U(\mathbf{x}) = \varphi_U^*(\mathbf{x}) = 220.49$, $\varphi_{N,1}(\mathbf{x}) = 200.8$, $\varphi_{N,1}^*(\mathbf{x}) = 215.14$ and $\varphi_{N,2}(\mathbf{x}) = 220.88$.

4 Numerical comparison

In this section, we compare the risk of the various competing estimators of θ_S under the entropy loss function (1.2). For $k = 2$, it is easy to verify that the risks of the competing estimators, that is, the UMRU estimator φ_U, the estimator φ_U^* (that improves upon the UMRU estimator), the natural estimator $\varphi_{N,1}$ and $\varphi_{N,2}$, and the estimator $\varphi_{N,1}^*$ (that improves upon the natural estimator $\varphi_{N,1}$), of θ_S are the functions of $\theta = \frac{\theta_2}{\theta_1}$. For selecting the best uniform population, we consider the minimax selection rule $\boldsymbol{\delta}^{a^*}$ (see (1.3)), which depends on the sample sizes n_1 and n_2. Clearly, the minimax selection rule $\boldsymbol{\delta}^{a^*}$ is not the same for different configurations of the sample sizes. We have compared the risk functions of the five competing estimators of θ_S for various values of θ and for various configurations of sample sizes. For notational convenience, let $R_1(\theta) = R(\boldsymbol{\theta}, \varphi_U)$, $R_2(\theta) = R(\boldsymbol{\theta}, \varphi_U^*)$, $R_3(\theta) = R(\boldsymbol{\theta}, \varphi_{N,1})$, $R_4(\theta) = R(\boldsymbol{\theta}, \varphi_{N,1}^*)$, $R_5(\theta) = R(\boldsymbol{\theta}, \varphi_{N,2})$ denote the risk functions of the various estimators. The risks of these estimators are plotted for $(n_1, n_2) \in \{(2, 3), (3, 2), (4, 5), (5, 4)\}$. The following observations are made from Figures 1–4.

(i) The UMRU estimator φ_U dominates the natural estimator $\varphi_{N,1}$.
(ii) The estimator φ_U^* provides only marginal improvement over the UMRU estimator φ_U.
(iii) The estimator $\varphi_{N,1}^*$ provides significant improvement over the natural estimator $\varphi_{N,1}$.
(iv) The natural estimator $\varphi_{N,2}$ is not comparable with other competing estimators.

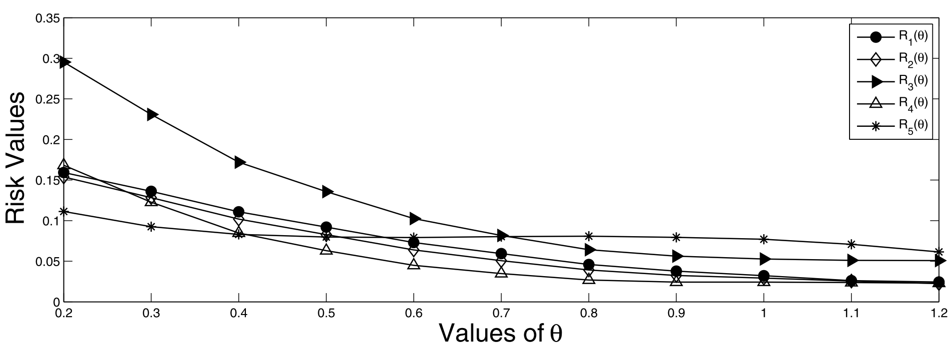

Figure 1 *Risk values of various estimators for* $(n_1, n_2) = (2, 3)$.

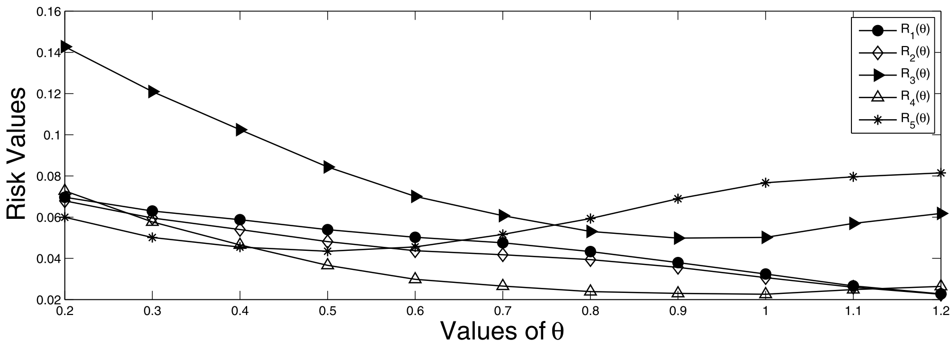

Figure 2 *Risk values of various estimators for* $(n_1, n_2) = (3, 2)$.

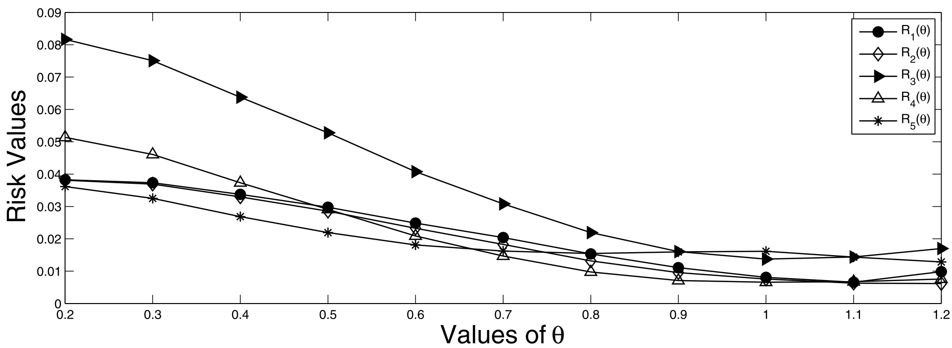

Figure 3 *Risk values of various estimators for* $(n_1, n_2) = (4, 5)$.

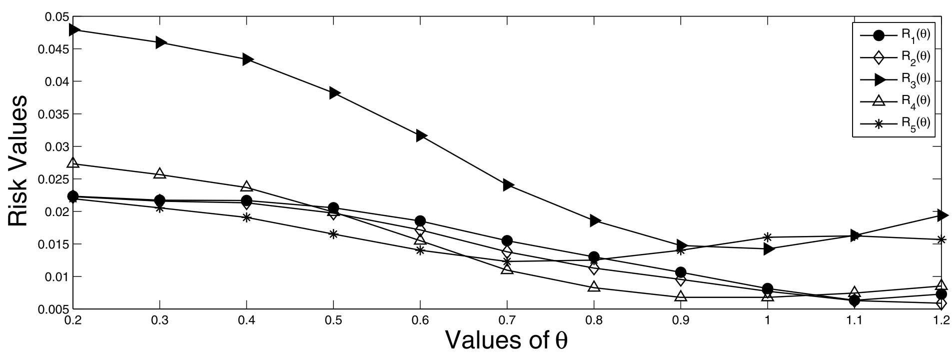

Figure 4 *Risk values of various estimators for* $(n_1, n_2) = (5, 4)$.

(v) Although the natural estimator $\varphi_{N,2}$ and the UMRU estimator φ_U are not dominated by the estimator $\varphi^*_{N,1}$, but overall performance of the estimator

$\varphi^*_{N,1}$ is satisfactory. Thus, the estimator $\varphi^*_{N,1}$ may be recommended for use in practical applications.

5 Estimation after selection of worst uniform population

Let $\pi_{(1)}$ be the unknown uniform population associated with the smallest scale parameter $\theta_{[1]} = \min\{\theta_1, \dots, \theta_k\}$ and is called the worst population. For selecting the worst population, Arshad and Misra (2015a) considered the class $\mathcal{D}_2 = \{\boldsymbol{\delta}^{\mathbf{b}} : \boldsymbol{\delta}^{\mathbf{b}} = (\delta_1^{\mathbf{b}}, \dots, \delta_k^{\mathbf{b}}), \mathbf{b} \in \mathbb{R}_+^k\}$ of natural selection rules, where

$$\delta_i^{\mathbf{b}}(\mathbf{X}) = \begin{cases} 1, & \text{if } b_i X_i < \min_{j \neq i} b_j X_j, \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathbf{b} = (b_1, \dots, b_k)$. Then the scale parameter of the selected population is

$$\theta_J = \sum_{i=1}^{k} \theta_i I_{F_i}(\mathbf{X}),$$

where $F_i = \{\mathbf{x} \in \mathbb{R}_+^k : b_i x_i < b_j x_j, \forall j \neq i, j = 1, \dots, k\}, i = 1, \dots, k$.

In this section, we consider the problem of estimation after selection of the worst uniform population under the entropy loss function (1.2). For estimation of θ_J, we consider the following two natural estimators based on the maximum likelihood estimator and uniformly minimum risk unbiased estimator (or minimum risk invariant estimator) of θ_i in the component estimation problem:

$$\varphi_{N,1}^W(\mathbf{X}) = \sum_{i=1}^{k} X_i I_{F_i}(\mathbf{X}), \quad \text{and} \quad \varphi_{N,2}^W(\mathbf{X}) = \sum_{i=1}^{k} \left(\frac{n_i}{n_i - 1} \right) X_i I_{F_i}(\mathbf{X}).$$

Now we will provide some results (without proofs) similar to the results derived in Sections 2 and 3. The following theorem is an analog of Theorem 1.

Theorem 4. *The UMRU estimator of the scale parameter θ_J of the selected population is given by*

$$\varphi_U^W(\mathbf{X}) = \sum_{i=1}^{k} \left[\frac{n_i X_i}{n_i - \sum_{l=1}^{k} \frac{n_i b_l}{n_l b_i} (\frac{b_i X_i}{b_l X_l})^{n_l + 1}} \right] I_{F_i}(\mathbf{X}).$$

Remark 3. Let $X_{[1]} \le \cdots \le X_{[k]}$ be the ordered values of $X_1, \dots, X_k$. For $n_1 = \cdots = n_k = n$ and $b_1 = \cdots = b_k = 1$, it follows from Theorem 4 that the UMRU estimator of θ_J is

$$\varphi_U^W(\mathbf{X}) = \frac{n X_{[1]}}{n - \sum_{l=1}^{k} (\frac{X_{[1]}}{X_{[l]}})^{n+1}}.$$

This result was derived by Nematollahi and Motamed-Shariati (2012). Thus, Theorem 4 generalizes their result.

The following lemma (analog of Lemma 3) will be use in Theorem 5.

Lemma 5. *Let* $G_1 = \{(t_2, \ldots, t_k) \in \mathbb{R}_+^{k-1} : t_j > \frac{a_1}{a_j}, j = 2, \ldots, k\}$ *and*

$$G_l = \left\{(t_2, \ldots, t_k) \in \mathbb{R}_+^{k-1} : t_l < \min\left(\frac{a_1}{a_l}, \min_{\substack{2 \le j \le k \\ j \ne l}} \frac{a_j t_j}{a_l}\right)\right\}, \qquad l = 2, \ldots, k,$$

so that $\{G_1, \ldots, G_k\}$ *forms a partition of* $\mathbb{R}_+^{k-1}$. *Define*

$$M(\mathbf{t}, \boldsymbol{\theta}) = \sum_{i=1}^{k} \theta_i E_{\boldsymbol{\theta}}\left(\frac{1}{X_1}\Big|\mathbf{T} = \mathbf{t}\right), \qquad \mathbf{t} \in \mathbb{R}_+^{k-1}, \boldsymbol{\theta} \in \boldsymbol{\Theta}.$$

Then

$$M_*(\mathbf{t}) = \inf_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} M(\mathbf{t}, \boldsymbol{\theta}) = \begin{cases} \left(\frac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right), & \text{if } \mathbf{t} \in G_1, \\ \left(\frac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right) t_l, & \text{if } \mathbf{t} \in G_l, l = 2, \ldots, k. \end{cases}$$

The following result is an analog of Theorem 2 and provides a sufficient condition for the inadmissibility of a scale-invariant estimator of θ_J under the entropy loss function (1.2).

Theorem 5. *Let* $\varphi(\mathbf{X}) = X_1 \psi(\mathbf{T})$ *be a scale-invariant estimator of* θ_J, *where* $\mathbf{T} = (T_2, \ldots, T_k) = (\frac{X_2}{X_1}, \ldots, \frac{X_k}{X_1})$, *and* $\psi(\cdot)$ *is a real-valued function defined on* $\mathbb{R}_+^{k-1}$. *Let* $\varphi_J^*(\mathbf{X}) = X_1 \max\{\psi(\mathbf{T}), M_*(\mathbf{T})\}$, *where* M_* *is as defined in Lemma 5. Then, under the entropy loss function (1.2), the estimator* $\varphi(\mathbf{X})$ *is inadmissible for estimating* θ_J *and is dominated by* φ_J^*, *provided that* $P_{\boldsymbol{\theta}}(\mathbf{T} : \psi(\mathbf{T}) < M_*(\mathbf{T})) \ge 0, \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$, *with strictly inequality for some* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

The following corollary is a consequence of Theorem 5.

Corollary 3. *Under the entropy loss function (1.2), the natural estimator* $\varphi_{N,1}^W(\mathbf{X})$ *is inadmissible for estimating* θ_J *and is dominated by the estimator*

$$\varphi_J^*(\mathbf{X}) = \left(\frac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right) \varphi_{N,1}^W(\mathbf{X}).$$

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