Brazilian Journal of Probability and Statistics
2013, Vol. 27, No. 4, 487–501
DOI: 10.1214/11-BJPS178

On some properties of Makeham distribution using generalized record values and its characterization

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Abstract. In this paper, we derive some recurrence relations satisfied by single and product moments of generalized upper record values arising from the Makeham distribution. We obtain similar results for the inverse Makeham distribution using generalized lower record values. Further, we have identified some of these recurrence relations and other properties of generalized record values which characterize Makeham and inverse Makeham distributions.

1 Introduction

The statistical study of record values of a sequence of i.i.d. continuous random variables was first carried out by Chandler (1952). For a survey on important results developed in this area one may refer to Arnold et al. (1998) and Ahsanullah (1988, 1995). Dzuibdziela and Kopocinski (1976) have generalized the concept of record values of Chandler (1952) by random variables of a more generalized nature and called them the kth record values. Since the rth member of the sequence of the classical record values is also known as the rth record value, we may call the record values defined by Dzuibdziela and Kopocinski (1976) also as the generalized record values. For some recent developments on generalized record values and recurrence relations on the moments of generalized record values with special reference to those arising from Pareto, generalized Pareto and Weibull distributions, see Pawlas and Szynal (1999, 2000). In this work we mainly focus on the study of generalized record values arising from the Makeham distribution.

The Makeham distribution is an important life distribution and has been widely used to fit actuarial data [see Marshall and Olkin (2007)]. For a description on the genesis and applications of the Makeham distribution one may refer to Makeham (1860). Aboutahoun and Al-Otaibi (2009) have derived some recurrence relations for the moments of order statistics arising from the doubly truncated Makeham distribution. Recently, Pandit and Math (2009) have dealt with some inference problems of the Makeham distribution.

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with common cumulative distribution function (c.d.f.) $F(x)$ and probability density function (p.d.f.)

Key words and phrases. Characterization, Makeham distribution, generalized record values, recurrence relations, inverse Makeham distribution.

Received September 2010; accepted October 2011.

$f(x)$. For a fixed positive integer k, we define $U_1^{(k)} = 1$ and for $m \geq 1$, $U_{m+1}^{(k)} = \min\{j > U_m^{(k)} : X_{j:j+k-1} > X_{U_m^{(k)}:U_m^{(k)}+k-1}\}$. Then the sequence $\{U_m^{(k)}, m \geq 1\}$ is known as the sequence of the kth upper record times of $\{X_n, n \geq 1\}$. In this case the sequence $\{Y_m^{(k)}, m \geq 1\}$ where $Y_m^{(k)} = X_{U_m^{(k)}:U_m^{(k)}+k-1}$ is called the sequence of kth upper record values or generalized upper record values of $\{X_n, n \geq 1\}$. We may also define $Y_0^{(k)} = 0$. For $k = 1$ we write $U_m^{(1)} = U_m$ and in this case $\{U_m, m \geq 1\}$ is the sequence of the classical upper record times of $\{X_n, n \geq 1\}$ as defined in Chandler (1952). For any positive integer r we write $\mu_{m;k}^{(r)}$ to denote the rth order moment of the mth generalized record value $Y_m^{(k)}$ arising from $\{X_n, n \geq 1\}$. We further write $\mu_{m,t;k}^{(r,s)}$ to represent the product moment of order (r, s) of the mth and tth generalized record values. Thus, we have

$$\mu_{m;k}^{(r)} = E\big[\big(Y_m^{(k)}\big)^r\big],$$

$$\mu_{m,t;k}^{(r,s)} = E\big[\big(Y_m^{(k)}\big)^r\big(Y_t^{(k)}\big)^s\big], \qquad 1 \leq m \leq t-1.$$

For convenience we define

$$\mu_{m,t;k}^{(r,0)} = E\big[\big(Y_m^{(k)}\big)^r\big] = \mu_{m;k}^{(r)}, \qquad 1 \leq m \leq t-1,$$

$$\mu_{m,t;k}^{(0,s)} = E\big[\big(Y_t^{(k)}\big)^s\big] = \mu_{t;k}^{(s)}, \qquad 1 \leq m \leq t-1.$$

Neither recurrence relations on moments of usual record values nor those of generalized record values arising from the Makeham distribution are seen derived in the available literature. Hence, in Section 2, we derive recurrence relations for single and product moments of the kth upper record values (generalized upper record values) arising from the Makeham distribution. We have used a recurrence relation on the single moments of the kth upper record values derived in Section 2 to characterize the Makeham distribution. We establish this characterization property of the Makeham distribution in Section 3. An immediate application of the characterization result is also illustrated in this section. In Section 4 we deal with kth lower record values (generalized lower record values) and obtain recurrence relations for single and product moments of those record values arising from the inverse Makeham distribution. Finally, in Section 5 we generate some charaterization results for the inverse Makeham distribution.

2 Recurrence relations for single and product moments of kth upper record values

A random variable X is said to have Makeham distribution if its p.d.f. is of the form

$$f(x) = [1 + \theta(1 - e^{-x})]e^{-x-\theta(x+e^{-x}-1)}, \qquad x > 0, \theta > 0. \tag{2.1}$$

The c.d.f. corresponding to (2.1) is given by

$$F(x) = 1 - e^{-x-\theta(x+e^{-x}-1)}. \tag{2.2}$$

It can be seen that

$$f(x) = [1 + (1+\theta)x + \ln \bar{F}(x)]\bar{F}(x), \tag{2.3}$$

where $\bar{F}(x) = 1 - F(x)$.

Let $\{Y_m^{(k)}, m \geq 1\}$, where $Y_m^{(k)} = X_{U_m^{(k)}:U_m^{(k)}+k-1}$ is a sequence of kth upper record values arising from (2.1). Then the p.d.f. of $Y_m^{(k)}, m \geq 1$, is given by [see Dziubdziela and Kopocinski (1976)]

$$f_{Y_m^{(k)}}(x) = \frac{k^m}{(m-1)!}[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^{k-1} f(x) \tag{2.4}$$

and the joint density function of $Y_m^{(k)}$ and $Y_t^{(k)}$, $1 \leq m < t$, $t \geq 2$, is given by

$$\begin{aligned} f_{Y_m^{(k)},Y_t^{(k)}}(x,y) &= \frac{k^t}{(m-1)!(t-m-1)!}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1} \\ &\quad \times [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\bar{F}(y)]^{k-1} f(y), \qquad x < y. \end{aligned} \tag{2.5}$$

Theorem 1. *Fix a positive integer $k \geq 1$. Then for $m \geq 1$ and $r = 1, 2, \ldots,$*

$$\begin{aligned} (r+1)(r+2)\mu_{m;k}^{(r)} &= (k+m)(r+2)\mu_{m;k}^{(r+1)} - k(r+2)\mu_{m-1;k}^{(r+1)} \\ &\quad - m(r+2)\mu_{m+1;k}^{(r+1)} + k(1+\theta)(r+1)\big[\mu_{m;k}^{(r+2)} - \mu_{m-1;k}^{(r+2)}\big]. \end{aligned}$$

Proof. For $m \geq 1$ and $r = 1, 2, \ldots,$ we have from (2.3) and (2.4)

$$\begin{aligned} \mu_{m;k}^{(r)} &= \frac{k^m}{(m-1)!}\int_0^\infty x^r[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^k\,dx \\ &\quad + (1+\theta)\frac{k^m}{(m-1)!}\int_0^\infty x^{r+1}[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^k\,dx \\ &\quad - \frac{k^m}{(m-1)!}\int_0^\infty x^r[-\ln \bar{F}(x)]^m[\bar{F}(x)]^k\,dx. \end{aligned}$$

Now integrating by parts, treating x^r for integration and the rest of the integrand for differentiation and simplifying, we get

$$\begin{aligned} \mu_{m;k}^{(r)} &= \frac{k}{(r+1)}\{\mu_{m;k}^{(r+1)} - \mu_{m-1;k}^{(r+1)}\} \\ &\quad + \frac{k(1+\theta)}{(r+2)}\{\mu_{m;k}^{(r+2)} - \mu_{m-1;k}^{(r+2)}\} \\ &\quad - \frac{m}{(r+1)}\{\mu_{m+1;k}^{(r+1)} - \mu_{m;k}^{(r+1)}\}. \end{aligned}$$

On rearranging the terms of the above expression we get the required result. □

Remark 1. For $k = 1$, we can deduce the relation for single moments of the usual upper record values introduced by Chandler (1952).

Theorem 2. *For* $1 \le m \le t - 2$ *and* $r, s = 1, 2, \ldots,$

$$\begin{aligned}(r+1)(r+2)\mu_{m,t;k}^{(r,s)} &= k(r+2)[\mu_{m,t-1;k}^{(r+1,s)} - \mu_{m-1,t-1;k}^{(r+1,s)}] \\ &\quad + k(1+\theta)(r+1)[\mu_{m,t-1;k}^{(r+2,s)} - \mu_{m-1,t-1;k}^{(r+2,s)}] \\ &\quad + m(r+2)[\mu_{m,t;k}^{(r+1,s)} - \mu_{m+1,t;k}^{(r+1,s)}]\end{aligned} \tag{2.6}$$

and for $m \ge 1, r, s = 1, 2, \ldots,$

$$\begin{aligned}(r+1)(r+2)\mu_{m,m+1;k}^{(r,s)} &= k(r+2)[\mu_{m;k}^{(s+r+1)} - \mu_{m-1,m;k}^{(r+1,s)}] \\ &\quad + k(1+\theta)(r+1)[\mu_{m;k}^{(s+r+2)} - \mu_{m-1,m;k}^{(r+2,s)}] \\ &\quad + m(r+2)[\mu_{m,m+1;k}^{(r+1,s)} - \mu_{m+1;k}^{(s+r+1)}].\end{aligned} \tag{2.7}$$

Proof. From (2.5), for $1 \le m \le t - 1$ and $r, s = 1, 2, \ldots,$ we have

$$\mu_{m,t;k}^{(r,s)} = \frac{k^t}{(m-1)!(t-m-1)!} \int_0^\infty y^s [\bar{F}(y)]^{k-1} f(y) I(y)\, dy, \tag{2.8}$$

where

$$I(y) = \int_0^y x^r [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}\, dx. \tag{2.9}$$

Using (2.3) in (2.9), we get

$$\begin{aligned}I(y) &= \int_0^y x^r [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1} [-\ln \bar{F}(x)]^{m-1}\, dx \\ &\quad + (1+\theta) \int_0^y x^{r+1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1} [-\ln \bar{F}(x)]^{m-1}\, dx \\ &\quad - \int_0^y x^r [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1} [-\ln \bar{F}(x)]^{m}\, dx.\end{aligned} \tag{2.10}$$

Integrating (2.10) by parts, treating x^r for integration and the rest of the integrand for differentiation, we get for $t \ge m + 2$,

$$\begin{aligned}I(y) = \int_0^y \Big\{ &\frac{(t-m-1)}{(r+1)} x^{r+1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-2} [-\ln \bar{F}(x)]^{m-1} \\ &- \frac{(m-1)}{(r+1)} x^{r+1} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1} [-\ln \bar{F}(x)]^{m-2}\end{aligned}$$

$$
\begin{aligned}
&+ (1+\theta)\frac{(t-m-1)}{(r+2)}x^{r+2} \\
&\quad \times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-2}[-\ln \bar{F}(x)]^{m-1} \qquad (2.11)\\
&- (1+\theta)\frac{(m-1)}{(r+2)}x^{r+2}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1}[-\ln \bar{F}(x)]^{m-2} \\
&- \frac{(t-m-1)}{(r+1)}x^{r+1}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-2}[-\ln \bar{F}(x)]^{m} \\
&- \frac{m}{(r+1)}x^{r+1}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1}[-\ln \bar{F}(x)]^{m-1}\Bigg\} \frac{f(x)}{\bar{F}(x)}\,dx.
\end{aligned}
$$

Substituting (2.11) in (2.8) and on further simplification, we get

$$
\begin{aligned}
\mu_{m,t;k}^{(r,s)} = &\frac{k}{(r+1)}\{\mu_{m,t-1;k}^{(r+1,s)} - \mu_{m-1,t-1;k}^{(r+1,s)}\} \\
&+ \frac{k(1+\theta)}{(r+2)}\{\mu_{m,t-1;k}^{(r+2,s)} - \mu_{m-1,t-1;k}^{(r+2,s)}\} \\
&+ \frac{m}{(r+1)}\{\mu_{m,t;k}^{(r+1,s)} - \mu_{m+1,t;k}^{(r+1,s)}\}.
\end{aligned}
$$

Now on rearranging the terms in the right side of the above expression we get (2.6).

Further, for $t = m+1$, we have

$$
\begin{aligned}
I(y) = &\frac{y^{r+1}}{(r+1)}[-\ln \bar{F}(y)]^{m-1} + (1+\theta)\frac{y^{r+2}}{(r+2)}[-\ln \bar{F}(y)]^{m-1} \\
&+ \int_0^y \Bigg\{ \frac{m}{(r+1)}x^{r+1}[-\ln \bar{F}(x)]^{m-1} \\
&\qquad - \frac{(m-1)}{(r+1)}x^{r+1}[-\ln \bar{F}(x)]^{m-2} \qquad (2.12)\\
&\qquad - (1+\theta)\frac{(m-1)}{(r+2)}x^{r+2}[-\ln \bar{F}(x)]^{m-2}\Bigg\} \frac{f(x)}{\bar{F}(x)}\,dx \\
&- \frac{y^{r+1}}{(r+1)}[-\ln \bar{F}(y)]^{m}.
\end{aligned}
$$

Substituting (2.12) in (2.8) and on further simplification, we get (2.7). □

Remark 2. On putting $k = 1$, in (2.6) and (2.7) we get the relations for product moments of classical upper record values arising from the Makeham distribution.

3 Characterization results

Now we consider the problem of characterizing the Makeham distribution using the relation in Theorem 1. To establish the result, we require the following result due to Lin (1986).

Proposition 1. *Let n_0 be any fixed non-negative integer and let a, b be reals such that $-\infty < a < b < \infty$. Let $g(x) \geq 0$ be an absolutely continuous function with $g'(x) \neq 0$ a.e. on (a, b). Then the sequence of functions $\{[g(x)]^n e^{-g(x)}, n \geq n_0\}$ is complete in $L(a, b)$ if and only if $g(x)$ is strictly monotone on (a, b).*

Theorem 3. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables where each X_n is distributed identically as that of the r.v. X which has an absolutely continuous distribution with support $(0, \infty)$. For a fixed positive integer k, let $\{Y_m^{(k)}\}$ be the sequence of generalized upper record values of $\{X_n\}$. Then, a necessary and sufficient condition for X to be distributed with p.d.f. given by (2.1) is that*

$$(r+1)(r+2)\mu_{m;k}^{(r)} = (k+m)(r+2)\mu_{m;k}^{(r+1)} - k(r+2)\mu_{m-1;k}^{(r+1)} - m(r+2)\mu_{m+1;k}^{(r+1)} + k(1+\theta)(r+1)\big[\mu_{m;k}^{(r+2)} - \mu_{m-1;k}^{(r+2)}\big] \tag{3.1}$$

for $r, m = 1, 2, \ldots$.

Proof. The necessary part follows immediately from Theorem 1.

Conversely, if the recurrence relation (3.1) is satisfied, then on rearranging the terms in (3.1) and using (2.4), we have

$$\begin{aligned}
&(r+1)(r+2)\frac{k^m}{(m-1)!}\int_0^\infty x^r[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^{k-1} f(x)\,dx \\
&\quad = (r+2)\frac{k^{m+1}}{(m-1)!}\int_0^\infty x^{r+1}[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^{k-1} f(x)\,dx \\
&\qquad + m(r+2)\frac{k^m}{(m-1)!}\int_0^\infty x^{r+1}[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^{k-1} f(x)\,dx \\
&\qquad - (r+2)\frac{k^m}{(m-2)!}\int_0^\infty x^{r+1}[-\ln \bar{F}(x)]^{m-2}[\bar{F}(x)]^{k-1} f(x)\,dx \\
&\qquad - m(r+2)\frac{k^{m+1}}{m!}\int_0^\infty x^{r+1}[-\ln \bar{F}(x)]^{m}[\bar{F}(x)]^{k-1} f(x)\,dx \\
&\qquad + (1+\theta)(r+1)\frac{k^{m+1}}{(m-1)!}\int_0^\infty x^{r+2}[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^{k-1} f(x)\,dx \\
&\qquad - (1+\theta)(r+1)\frac{k^m}{(m-2)!}\int_0^\infty x^{r+2}[-\ln \bar{F}(x)]^{m-2}[\bar{F}(x)]^{k-1} f(x)\,dx.
\end{aligned} \tag{3.2}$$

On integrating by parts the second, third and the last integrals on the RHS of (3.2), we get

$$\frac{k^m}{(m-1)!}\int_0^\infty x^r[-\ln \bar{F}(x)]^{m-1}[\bar{F}(x)]^{k-1} \times \{f(x)-[1-(-\ln \bar{F}(x))+(1+\theta)x]\bar{F}(x)\}\,dx = 0.$$

Now from Proposition 1 with $g(x) = -\ln[\bar{F}(x)]$, it follows that

$$f(x) = \{1+\ln \bar{F}(x)+(1+\theta)x\}\bar{F}(x).$$

Consequently, from (2.3) it follows that $f(x)$ has the form (2.1). □

An immediate application of the above characterization theorem is the following. Clearly, one can use (3.1) to express θ in terms of some of the moments of kth record values. Thus, we consider the case of repeated samples (say, N in number) drawn from the given population. Then from each sample observe $Y_{m-1}^{(k)}, Y_m^{(k)}, Y_{m+1}^{(k)}$. Let $R_{m-1,i}, R_{m,i}, R_{m+1,i}$ be the realizations recorded from the ith sample on the generalized upper record values $Y_{m-1}^{(k)}, Y_m^{(k)}, Y_{m+1}^{(k)}$, respectively. Define $S_1 = \sum_{i=1}^N \frac{R_{m,i}^r}{N}$, $S_2 = \sum_{i=1}^N \frac{R_{m,i}^{r+1}}{N}$, $S_3 = \sum_{i=1}^N \frac{R_{m,i}^{r+2}}{N}$, $S_4 = \sum_{i=1}^N \frac{R_{m-1,i}^r}{N}$, $S_5 = \sum_{i=1}^N \frac{R_{m-1,i}^{r+2}}{N}$, $S_6 = \sum_{i=1}^N \frac{R_{m+1,i}^{r+1}}{N}$. Then for sufficiently large N and as a consequence of the above theorem, we can estimate θ by

$$\hat{\theta} = \frac{1}{k(r+1)(S_3-S_5)}[(r+1)(r+2)S_1-(k+m)(r+2)S_2 + k(r+2)S_4+m(r+2)S_6-k(r+1)(S_3-S_5)].$$

If the RHS of the above equation for different $r = 1, 2, \ldots$ is more or less a constant, then in the search of identifying a model to the population from which the data are generated, one may choose the Makeham distribution as an appropriate model. In particular, the above identity further helps to obtain an estimate of θ of the Makeham distribution.

One can generate samples from the Makeham distribution using Mathematica software for different values of θ and obtain the kth upper record values and estimate θ as suggested in the application. For $k = 2$ we have simulated 500 such samples, each of size 200 from the Makeham distribution for different values of θ, estimated θ by $\hat{\theta}$ and the results are given in Table 1.

Now, we describe the conditional distribution of a generalized upper record value $Y_t^{(k)}$ given $Y_m^{(k)} = x$ for $m < t$. We make use of this conditional distribution to obtain another characterization property of the Makeham distribution. The joint p.d.f. of $Y_m^{(k)}$ and $Y_t^{(k)}$ is given in (2.5) and the marginal distribution of $Y_m^{(k)}$ is

Table 1 *Estimated value of the parameter θ of the Makeham distribution using simulation*

True value of parameter	m	r	Estimated value
0.5	2	1	0.7837
		2	0.9998
		3	1.3101
	3	1	0.4545
		2	0.4335
		3	0.4352
	4	1	0.7292
		2	0.8900
		3	1.1399
1.0	2	1	1.5389
		2	1.6965
		3	1.7072
	3	1	1.1082
		2	1.3024
		3	1.5490
	4	1	1.3740
		2	1.3271
		3	1.3096
1.5	2	1	1.6012
		2	1.6519
		3	1.8203
	3	1	1.7416
		2	1.8609
		3	1.7010
	4	1	2.1462
		2	2.3940
		3	2.7480

given in (2.4). If we write $h(y|x)$ to denote the conditional p.d.f. of $Y_t^{(k)}$ given $Y_m^{(k)} = x$, $1 \leq m < t$, then we have

$$h(y|x) = \frac{k^{t-m}}{(t-m-1)![\bar{F}(x)]^k}[\ln \bar{F}(x) - \ln \bar{F}(y)]^{t-m-1}[\bar{F}(y)]^{k-1} f(y).$$

Clearly, $h(y|x)$ is distributed as the p.d.f. of the $(t-m)$th generalized upper record value arising from the distribution truncated on the left at x. Thus, we conclude that the conditional p.d.f. of the kth upper record values also shows a property similar to that of the conditional p.d.f. of an order statistic arising from an absolutely continuous distribution given the value of a lower order statistic.

In particular, the conditional p.d.f. of $Y_{m+1}^{(k)}$, given $Y_m^{(k)} = x$, is given by

$$h(y|x) = \frac{k[\bar{F}(y)]^{k-1} f(y)}{[\bar{F}(x)]^k}, \qquad y \geq x. \tag{3.3}$$

Theorem 4. *Let X be an absolutely continuous r.v. with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then X follows the Makeham distribution with $F(x) = 1 - e^{-x-\theta(x+e^{-x}-1)}$, $\theta > 0, 0 < x < \infty$ if and only if*

$$E\big[Y_{m+1}^{(k)} + \theta\big(Y_{m+1}^{(k)} + e^{-Y_{m+1}^{(k)}} - 1\big)\big|Y_m^{(k)} = x\big] = x + \theta[x + e^{-x} - 1] + \frac{1}{k}. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned} &E\big[Y_{m+1}^{(k)} + \theta\big(Y_{m+1}^{(k)} + e^{-Y_{m+1}^{(k)}} - 1\big)\big|Y_m^{(k)} = x\big] \\ &= \frac{k}{[\bar{F}(x)]^k} \int_x^\infty [y + \theta(y + e^{-y} - 1)][\bar{F}(y)]^{k-1} f(y)\, dy. \end{aligned} \tag{3.5}$$

Now, using (2.1) and (2.2) in (3.5) and on further simplification, we get the necessary part.

Conversely, assume that (3.4) holds. Then

$$\begin{aligned} &\int_x^\infty [y + \theta(y + e^{-y} - 1)][\bar{F}(y)]^{k-1} f(y)\, dy \\ &= \left\{x + \theta[x + e^{-x} - 1] + \frac{1}{k}\right\} \frac{[\bar{F}(x)]^k}{k}. \end{aligned} \tag{3.6}$$

Differentiating both sides of (3.6) and simplifying, we get

$$-\frac{d}{dx} \ln \bar{F}(x) = [1 + \theta(1 - e^{-x})],$$

which on further simplification leads to $\bar{F}(x) = e^{-x-\theta(x+e^{-x}-1)}$ and this proves the theorem. □

4 Recurrence relations for single and product moments of *k*th lower record values from the inverse Makeham distribution

A r.v. X is said to have an inverse Makeham distribution if its p.d.f. is of the form

$$f(x) = \frac{[1 + \theta(1 - e^{-1/x})]}{x^2} e^{-1/x-\theta[1/x+e^{-1/x}-1]}, \qquad x > 0, \theta > 0. \tag{4.1}$$

Clearly, the c.d.f. corresponding to the p.d.f. (4.1) is given by

$$F(x) = e^{-1/x-\theta[1/x+e^{-1/x}-1]}.$$

It should be noted that if Y follows a Makeham distribution with p.d.f. (2.1), then $X = 1/Y$ follows the inverse Makeham distribution defined by the p.d.f. (4.1). For an inverse Makeham distribution $f(x)$ and $F(x)$ are connected by the relation

$$x^2 f(x) = \left[1 + \frac{(1+\theta)}{x} + \ln F(x)\right] F(x). \tag{4.2}$$

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with p.d.f. $f(x)$ and c.d.f. $F(x)$. For a fixed $k \geq 1$, we define the sequence $\{L_m^{(k)}, m \geq 1\}$ of kth lower record times of $\{X_n, n \geq 1\}$ [as introduced by Pawlas and Szynal (1998)] as $L_1^{(k)} = 1$, $L_{m+1}^{(k)} = \min\{j > L_m^{(k)} : X_{k:L_m^{(k)}+k-1} > X_{k:j+k-1}\}$.

Then the sequence $\{Z_m^{(k)}, m \geq 1\}$ where $Z_m^{(k)} = X_{k:L_m^{(k)}+k-1}$ is called the sequence of generalized lower record values or kth lower record values of $\{X_n, n \geq 1\}$. For convenience we shall also take $Z_0^{(k)} = 0$. For $k = 1$, if we write $L_m^{(1)} = L_m$, then $L_m, m = 1, 2, \ldots,$ defines the usual lower record times of $\{X_n, n \geq 1\}$. In this case $Z_m^{(1)} = X_{L_m}, m \geq 1$, is the usual sequence of lower record values of $\{X_n, n \geq 1\}$. We shall define for positive integers r and s,

$$\begin{aligned}
\nu_{m;k}^{(r)} &= E\big[\big(Z_m^{(k)}\big)^r\big], \\
\nu_{m,t;k}^{(r,s)} &= E\big[\big(Z_m^{(k)}\big)^r \big(Z_t^{(k)}\big)^s\big], \qquad 1 \leq m \leq t-1, \\
\nu_{m,t;k}^{(r,0)} &= E\big[\big(Z_m^{(k)}\big)^r\big] = \nu_{m;k}^{(r)}, \qquad 1 \leq m \leq t-1, \\
\nu_{m,t;k}^{(0,s)} &= E\big[\big(Z_t^{(k)}\big)^s\big] = \nu_{t;k}^{(s)}, \qquad 1 \leq m \leq t-1.
\end{aligned}$$

The p.d.f. of $Z_m^{(k)}, m \geq 1$, is given by

$$f_{Z_m^{(k)}}(x) = \frac{k^m}{(m-1)!}[-\ln F(x)]^{m-1}[F(x)]^{k-1} f(x) \tag{4.3}$$

and the joint density function of $Z_m^{(k)}$ and $Z_t^{(k)}$, $1 \leq m < t$, $t \geq 2$, is given by

$$\begin{aligned}
f_{Z_m^{(k)}, Z_t^{(k)}}(x, y) &= \frac{k^t}{(m-1)!(t-m-1)!}[\ln F(x) - \ln F(y)]^{t-m-1} \\
&\quad \times [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [F(y)]^{k-1} f(y), \qquad x > y.
\end{aligned} \tag{4.4}$$

Theorem 5. *Fix a positive integer $k \geq 1$. Then for $m \geq 1$ and $r = 1, 2, \ldots,$*

$$\begin{aligned}
r(r+1)\nu_{m;k}^{(r+2)} &= kr\nu_{m-1;k}^{(r+1)} - (k+m)r\nu_{m;k}^{(r+1)} + mr\nu_{m+1;k}^{(r+1)} \\
&\quad + k(1+\theta)(r+1)\big[\nu_{m-1;k}^{(r)} - \nu_{m;k}^{(r)}\big].
\end{aligned} \tag{4.5}$$

Proof. For $m \geq 1$ and $r = 1, 2, \ldots,$ we have from (4.2) and (4.3)

$$\nu_{m;k}^{(r+2)} = \frac{k^m}{(m-1)!}\int_0^\infty x^r[-\ln F(x)]^{m-1}[F(x)]^k\,dx$$
$$+ (1+\theta)\frac{k^m}{(m-1)!}\int_0^\infty x^{r-1}[-\ln F(x)]^{m-1}[F(x)]^k\,dx$$
$$- \frac{k^m}{(m-1)!}\int_0^\infty x^r[-\ln F(x)]^m[F(x)]^k\,dx.$$

On integrating by parts treating x^r for integration and the rest of the integrand for differentiation and simplifying, we get relation (4.5). □

Theorem 6. *For $1 \leq m \leq t-2$ and $r, s = 1, 2, \ldots,$*

$$r(r+1)\nu_{m,t;k}^{(r+2,s)} = kr\big[\nu_{m-1,t-1;k}^{(r+1,s)} - \nu_{m,t-1;k}^{(r+1,s)}\big] + mr\big[\nu_{m+1,t;k}^{(r+1,s)} - \nu_{m,t;k}^{(r+1,s)}\big]$$
$$+ k(1+\theta)(r+1)\big[\nu_{m-1,t-1;k}^{(r,s)} - \nu_{m,t-1;k}^{(r,s)}\big]$$

and for $m \geq 1, r, s = 1, 2, \ldots,$

$$r(r+1)\nu_{m,m+1;k}^{(r+2,s)} = kr\big[\nu_{m-1,m;k}^{(r+1,s)} - \nu_{m;k}^{(s+r+1)}\big] + mr\big[\nu_{m+1;k}^{(s+r+1)} - \nu_{m,m+1;k}^{(r+1,s)}\big]$$
$$+ k(1+\theta)(r+1)\big[\nu_{m-1,m;k}^{(r,s)} - \nu_{m;k}^{(s+r)}\big].$$

Proof. The proof follows exactly in the same manner as that of Theorem 2 and hence is omitted. □

5 Characterization results for the inverse Makeham distribution

Theorem 7. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables which are distributed identically with that of the r.v. X which has an absolutely continuous distribution with support $(0, \infty)$. For a fixed positive integer k, let $\{Z_m^{(k)}\}$ be the sequence of generalized lower record values of $\{X_n\}$. Then, a necessary and sufficient condition for X to be distributed with p.d.f. given by (4.1) is that*

$$r(r+1)\nu_{m;k}^{(r+2)} = kr\nu_{m-1;k}^{(r+1)} - (k+m)r\nu_{m;k}^{(r+1)} + mr\nu_{m+1;k}^{(r+1)}$$
$$+ k(1+\theta)(r+1)\big[\nu_{m-1;k}^{(r)} - \nu_{m;k}^{(r)}\big] \qquad (5.1)$$

for $r, m = 1, 2, \ldots.$

Proof. The necessary part follows immediately from Theorem 5.

Conversely, if relation (5.1) is satisfied, then proceeding in a manner similar to the proof given for the sufficient part of Theorem 3, we get

$$\frac{k^m}{(m-1)!}\int_0^\infty x^{r-1}[-\ln F(x)]^{m-1}[F(x)]^{k-1}$$
$$\times \{x^3 f(x) - [x - x(-\ln F(x)) + (1+\theta)]F(x)\}\,dx = 0.$$

Using Proposition 1, with $g(x) = -\ln F(x)$, it follows that

$$x^2 f(x) = \left[1 + \frac{(1+\theta)}{x} + \ln F(x)\right]F(x),$$

which proves by (4.2) that $f(x)$ has the form (4.1) and this proves the theorem. □

An immediate application of the above characterization theorem is described below. Clearly, one can use (5.1) to express θ in terms of some of the moments of the kth record values. Thus, we consider the case of repeated samples (say, N in number) drawn from the given population. Then from each sample observe $Z_{m-1}^{(k)}, Z_m^{(k)}, Z_{m+1}^{(k)}$. Let $V_{m-1,i}, V_{m,i}, V_{m+1,i}$ be the realizations recorded from the ith sample on the generalized lower record values $Z_{m-1}^{(k)}, Z_m^{(k)}, Z_{m+1}^{(k)}$, respectively. Define $S_1^* = \sum_{i=1}^N \frac{V_{m,i}^r}{N}$, $S_2^* = \sum_{i=1}^N \frac{V_{m,i}^{r+1}}{N}$, $S_3^* = \sum_{i=1}^N \frac{V_{m,i}^{r+2}}{N}$, $S_4^* = \sum_{i=1}^N \frac{V_{m-1,i}^r}{N}$, $S_5^* = \sum_{i=1}^N \frac{V_{m-1,i}^{r+1}}{N}$, $S_6^* = \sum_{i=1}^N \frac{V_{m+1,i}^{r+1}}{N}$. Then as a consequence of the above theorem and for sufficiently large N, we can estimate θ by

$$\hat{\theta} = \frac{1}{k(r+1)(S_4^* - S_1^*)}[r(r+1)S_3^* - krS_5^* + (k+m)rS_2^*$$
$$- mrS_6^* - k(r+1)(S_4^* - S_1^*)].$$

If the RHS of the above equation for different $r = 1, 2, \ldots$ is more or less a constant, then in the search of identifying a model to the population from which the data are generated, one may choose the inverse Makeham distribution as an appropriate model. In particular, the above identity further helps to obtain an estimate of θ of the inverse Makeham distribution.

One can generate samples from the inverse Makeham distribution using Mathematica software for different values of θ and obtain the kth lower record values and estimate θ as suggested in the application. For $k = 2$ we have simulated 500 such samples, each of size 200 from the inverse Makeham distribution for different values of θ, estimated θ by $\hat{\theta}$, and the results are given in Table 2.

Now, we consider the conditional distribution of a generalized lower record value $Z_t^{(k)}$ given $Z_m^{(k)} = x$ for $m < t$. The joint p.d.f. of $Z_m^{(k)}$ and $Z_t^{(k)}$ is given in (4.4) and the marginal distribution of $Z_m^{(k)}$ is given in (4.3). If we write $g(y|x)$ to denote this conditional density, then the conditional p.d.f. of $Z_t^{(k)}$ given $Z_m^{(k)} =$

Table 2 *Estimated value of the parameter θ of the inverse Makeham distribution using simulation*

True value of parameter	m	r	Estimated value
0.5	2	1	0.7782
		2	0.9846
		3	1.2202
	3	1	0.4742
		2	0.4545
		3	0.4576
	4	1	0.7386
		2	0.8676
		3	0.9821
1.0	2	1	1.5124
		2	1.6825
		3	1.7126
	3	1	1.1146
		2	1.3224
		3	1.5643
	4	1	1.3634
		2	1.3158
		3	1.3024
1.5	2	1	1.6143
		2	1.6596
		3	1.8313
	3	1	1.7314
		2	1.8219
		3	1.7123
	4	1	2.1386
		2	2.3822
		3	2.7246

x, $1 \le m < t$, then we have

$$g(y|x) = \frac{k^{t-m}}{(t-m-1)![F(x)]^k}[\ln F(x) - \ln F(y)]^{t-m-1}[F(y)]^{k-1} f(y).$$

Clearly, $g(y|x)$ is distributed as the p.d.f. of the $(t-m)$th generalized lower record value arising from the distribution truncated on the right at x. Thus, we conclude that the conditional p.d.f. of the kth lower record values also shows a property similar to that of the conditional p.d.f. of an order statistic arising from an absolutely continuous distribution given the value of a higher order statistic.

In particular, the conditional p.d.f. of $Z_{m+1}^{(k)}$, given $Z_m^{(k)} = x$, is given by

$$g(y|x) = \frac{k[F(y)]^{k-1} f(y)}{[F(x)]^k}. \tag{5.2}$$

Theorem 8. *Let X be an absolutely continuous r.v. with p.d.f. $f(x)$ and c.d.f. $F(x)$. Then X follows an inverse Makeham distribution with $F(x) = e^{-1/x-\theta[1/x+e^{-1/x}-1]}$ if and only if*

$$\begin{aligned} E\bigg[\frac{1}{Z_{m+1}^{(k)}} + \theta\bigg(\frac{1}{Z_{m+1}^{(k)}} + e^{-1/Z_{m+1}^{(k)}} - 1\bigg)\bigg|Z_m^{(k)} = x\bigg] \\ = \frac{1}{x} + \theta\bigg(\frac{1}{x} + e^{-1/x} - 1\bigg) + \frac{1}{k}. \end{aligned} \tag{5.3}$$

Proof. Using (5.2), the necessary part follows by direct computation.

Conversely, assume that (5.3) holds. Then,

$$\begin{aligned} \int_0^x \bigg[\frac{1}{y} + \theta\bigg(\frac{1}{y} + e^{-1/y} - 1\bigg)\bigg][F(y)]^{k-1} f(y)\,dy \\ = \bigg\{\frac{1}{x} + \theta\bigg(\frac{1}{x} + e^{-1/x} - 1\bigg) + \frac{1}{k}\bigg\}\frac{[F(x)]^k}{k}. \end{aligned} \tag{5.4}$$

Differentiating both sides of (5.4) and simplifying, we get

$$\frac{d}{dx}\ln F(x) = \frac{1 + \theta(1 - e^{-1/x})}{x^2},$$

which on further simplification leads to $F(x) = e^{-1/x-\theta[1/x+e^{-1/x}-1]}$. □

Acknowledgment

The authors acknowledge their gratefulness to the learned referee for many of his constructive comments on an earlier version of this paper.

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