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# On the distribution of shrinkage parameters of Liu-type estimators

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**Abstract.** In this paper, we derive the density and distribution functions of the estimator of the shrinkage parameters of the Liu and generalized Liu estimators associated with the normal linear regression model. We indicate how these distributions can be used in arriving at a confidence interval for the optimal value of the shrinkage parameter. Since the distributions are difficult to handle, we have carried out some numerical computations to illustrate them.

## 1 Introduction

The multicollinearity problem and its consequences for linear regression in general, and for the least squares estimator in particular, are well known and several alternative estimators have been suggested. One of these is the ordinary ridge regression (ORR) estimator [see Hoerl and Kennard (1970a, 1970b)]. In order to tackle the multicollinearity problem, Liu (1993) combined the Stein (1956) estimator with the ORR estimator and obtained the Liu estimator (Gruber (1998), p. 48; Liu (2003)). Akdeniz and Kaςiranlar (1995) have considered an almost-unbiased generalized Liu estimator, and have suggested a method for the unbiased estimation of bias and mean squared error. In order to determine the value of the biasing parameter $d$ of the Liu estimator, several methods have been suggested by Liu (1993) and Akdeniz and Erol (2003).

The shrinkage parameter determines the bias and mean square error of the shrinkage estimator. As its value is unknown, the usual procedure is to estimate it from the data. It is worth investigating the sampling distribution of this estimator of the shrinkage parameter, which determines the properties of the resulting operational shrinkage estimator.

Even though a number of estimators have been suggested for $d$, very few researchers have looked into their sampling distributions. Akdeniz and Öztürk (2005) have studied the density function of the stochastic shrinkage parameters of the generalized Liu estimator. In this paper we determine the probability density and the probability distribution function of these estimators of $d$. We have discussed a procedure to arrive at the confidence interval for the optimal value of the shrinkage parameter using the sampling distribution. Some numerical computations were also carried out to illustrate the behavior of these distributions.

*Key words and phrases.* Density function, distribution function, least squares estimator, Liu estimator, multicollinearity, shrinkage parameters.

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The outline of the paper is as follows. In Section 2, we obtain the probability density and distribution functions of the estimated shrinkage parameter in the Liu-type estimator in a multiple linear regression model. Section 3 deals with similar results for the generalized Liu estimator. The discussion on the confidence interval for the optimal value of the shrinkage parameter in the Liu-type estimator appears in Section 4. Some numerical computations are obtained in Section 5.

## 2 Distribution of the estimated shrinkage parameter

We consider the standard multiple linear regression model

$$Y = Z\varphi + \varepsilon, \tag{2.1}$$

where $Y$ is an $n \times 1$ vector of observations on the response (or dependent) variable, $Z$ is an $n \times p$ model matrix of observations on $p$ nonstochastic explanatory regressors (or independent) variables, $\varphi$ is a $p \times 1$ vector of unknown parameters associated with the $p$ regressors and $\varepsilon$ is an $n \times 1$ vector of errors with expectation $E(\varepsilon) = 0$ and dispersion matrix $\mathrm{Var}(\varepsilon) = \sigma^2 I_N$. Throughout this paper, we assume that the model matrix $Z$ has full column rank. By means of the spectral theorem we can write

$$\Lambda^{-1/2} Q' Z' Z Q \Lambda^{-1/2} = I,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ is a diagonal matrix with $\lambda_1, \ldots, \lambda_p$ as the eigenvalues of $Z'Z$, and $Q$ is an orthogonal matrix. Thus, we can reparametrize model (2.1) as

$$Y = X\beta + \varepsilon, \tag{2.2}$$

where $X = ZQ\Lambda^{-1/2}$ and $\beta = \Lambda^{1/2} Q'\varphi$; hence, $X'X = I$. The least squares estimator of $\beta$ in (2.2) is

$$\hat{\beta} = X'Y. \tag{2.3}$$

The Liu estimator of $\beta$ (Liu (1993)) is

$$\hat{\beta}_d = (X'X + I)^{-1}(X'Y + d\hat{\beta}).$$

Since $X'X = I$, this estimator reduces to

$$\hat{\beta}_d = \frac{1}{2}(1 + d)X'Y,$$

where $d \in (-\infty, \infty)$. The mean squared error (MSE) matrix of an estimator $b^*$ for $\beta$ is defined as

$$\mathrm{MSE}(b^*) = \mathrm{Cov}(b^*) + (\mathrm{Bias}(b^*))(\mathrm{Bias}(b^*))',$$

where $\text{Bias}(b^*) = E(b^*) - \beta$. Also, we can define the scalar mean squared error as

$$\text{mse}(b^*) = \text{tr}\,\text{MSE}(b^*) = \text{tr}\,\text{Cov}(b^*) + \|E(b^*) - \beta\|^2,$$

where tr denotes the trace. Now, in the model (2.2)

$$\text{mse}(\hat{\beta}_d) = \frac{1}{4}(1+d)^2 \sum_{i=1}^{p} (\sigma^2 + \beta_i^2) - d \sum_{i=1}^{p} \beta_i^2. \tag{2.4}$$

By minimizing (2.4) with respect to $d$, we have

$$d_{\text{opt}} = 1 - \frac{2p\sigma^2}{p\sigma^2 + \beta'\beta}. \tag{2.5}$$

Replacing $\beta_i^2$ and $\sigma^2$ by their unbiased estimators, we get the estimator of $d_{\text{opt}}$ as

$$\hat{d} = 1 - 2p \frac{\hat{\sigma}^2}{p\hat{\sigma}^2 + \hat{\beta}'\hat{\beta}}, \tag{2.6}$$

with $\hat{\sigma}^2 = \frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{(n-p)}$ and $\hat{\beta}$ defined as in (2.3). Thus we get an operational version of the Liu estimator. Now, we derive the probability density and the distribution function of the estimate of the shrinkage parameter given in (2.6).

**Theorem 1.** *The density function of* $\hat{d}$ *is*

$$f(\hat{d}) = \frac{e^{-\delta/2} p^{p/2} (n-p)^{(n-p)/2}}{B(1/2, (n-p)/2)} \frac{2^{p/2}/(1-\hat{d})^{p/2+1}}{(2p/(1-\hat{d}) + n - p)^{n/2}}$$
$$\times \sum_{j=1}^{\infty} \left[ \frac{\delta p/(1-\hat{d})}{(2p/(1-\hat{d}) + n - p)} \right]^j \frac{\Gamma(n/2 + j)\Gamma(1/2)}{\Gamma(j+1)\Gamma((n-p+1)/2)\Gamma(p/2+j)},$$

*where* $\delta = \beta'\beta/\sigma^2$ *and* $0 < \hat{d} < 1$.

**Proof.** Write $\hat{d} = 1 - 2/G$, where

$$G = \frac{(n-p)Y'NY}{pY'MY} = \frac{(n-p)Z'NZ}{pZ'MZ},$$

with $Z = Y/\sigma$ and $N = XX'$, $M = I - N$. The ranks of $M$ and $N$ are $n-p$ and $p$, respectively. Since $M$ and $N$ are idempotent and $MN = 0$, $Z'MZ$ and $Z'NZ$ are independently distributed as central and noncentral $\chi^2$, that is, $Z'MZ \sim \chi^2_{(n-p)}$, $Z'NZ \sim \chi^2_p(\delta)$, where $\delta = \beta'X'X\beta/\sigma^2$ is the noncentrality parameter (Akdeniz, Stan and Werner (2006)). Then, $G = (\hat{\beta}'\hat{\beta}/p\hat{\sigma}^2) \sim F_{p,(n-p)}(\delta)$ where

$\delta = \beta'\beta/\sigma^2$ (Rubio and Firinguetti (2002)). Thus,

$$f(G) = \frac{e^{-\delta/2} p^{p/2} (n-p)^{(n-p)/2}}{B(1/2, (n-p)/2)} \frac{G^{(p/2)-1}}{(pG+n-p)^{n/2}} \times \sum_{j=1}^{\infty} \left[ \frac{\delta p G/2}{(pG+n-p)} \right]^j \frac{\Gamma(n/2+j)\Gamma(1/2)}{\Gamma(j+1)\Gamma((n-p+1)/2)\Gamma(p/2+j)},$$

where

$$\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t}\, dt, \qquad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The proof is completed by a change of variables, using $\hat{d} = 1 - 2/G$. □

**Theorem 2.** *The distribution function of $\hat{d} = 1 - 2/H$, $H = \hat{\beta}'\hat{\beta}/p\hat{\sigma}^2$ is*

$$\sum_{j=1}^{\infty} \frac{(\delta/2)^j e^{-\delta/2}}{j! B(p/2+j, (n-p)/2)} B_r\big(p/2+j, (n-p)/2\big),$$

*where*

$$B_r(a,b) = \int_0^r t^{a-1} (1-t)^{b-1}\, dt, \qquad r = \frac{2p}{n(1-x)+p(1+x)}.$$

**Proof.** We have

$$F(x) = P(\hat{d} \le x) = P\left(H \le \frac{2}{1-x}\right).$$

But,

$$H \sim F_{p,(n-p)}(\delta) = \left[\frac{(n-p)}{p}\right] G^*_{p,(n-p)}(\delta),$$

where $G^*_{p,(n-p)}(\delta) = \frac{\chi_2^p(\delta)}{\chi^2_{(n-p)}}$ (Johnson, Kotz and Balakrishnan (1995)). So,

$$\begin{aligned} P\left(H \le \frac{2}{1-x}\right) &= P\left(((n-p)/p)G^* \le \frac{2}{1-x}\right) \\ &= \int_0^{2p/((n-p)(1-x))} \sum_{j=1}^{\infty} \frac{(\delta/2)^j e^{-\delta/2}}{j!} \\ &\quad \times \frac{g^{p/2+j-1}}{B(p/2+j, (n-p)/2)(1+g)^{n/2+j}}\, dg \end{aligned} \tag{2.7}$$

$$= \sum_{j=1}^{\infty} \frac{(\delta/2)^j e^{-\delta/2}}{j!B(p/2+j,(n-p)/2)}$$

$$\times \int_0^{2p/((n-p)(1-x))} \frac{g^{p/2+j-1}}{(1+g)^{n/2+j}}\, dg.$$

Let $m = \frac{g}{1+g}$, so $g = \frac{m}{1-m}$, $dg = \frac{dm}{(1-m)^2}$. We may write the integral as

$$\begin{aligned}\int_0^{2p/((n-p)(1-x))} \frac{g^{p/2+j-1}}{(1+g)^{n/2+j}}\, dg &= \int_0^r \frac{(m/(1-m))^{p/2+j-1}}{(1+m/(1-m))^{n/2+j}} \frac{dm}{(1-m)^2} \\ &= \int_0^r m^{p/2+j-1}(1-m)^{(n-p)/2-1} dm \quad (2.8) \\ &= B_r\big(p/2+j,(n-p)/2\big).\end{aligned}$$

The proof is completed by substituting (2.8) into (2.7). □

## 3 Generalized Liu estimator

The orthogonal version of the multiple linear regression model (2.1) is

$$Y = ZQQ'\varphi + \varepsilon = X\alpha + \varepsilon,$$

where $\varepsilon \sim N(0,\sigma^2 I)$, $X = ZQ$, $X'X = \Lambda = \text{diag}(\lambda_i,\ldots,\lambda_p)$ and $\alpha = Q'\varphi$.

The generalized Liu estimator (Liu (1993)) is defined by

$$\hat{\alpha}_D = (\Lambda + I)^{-1}(X'Y + D\hat{\alpha}),$$

where$D = \text{diag}(d_1,\ldots,d_p)$ and $\hat{\alpha} = \Lambda^{-1}X'Y$. If $d_i$ is fixed, then $\text{MSE}(\hat{\alpha}_{d_i})$ is minimized at (Akdeniz and Kaςiranlar (1995))

$$d_{i\,(\text{opt})} = \frac{\lambda_i(\alpha_i^2 - \sigma^2)}{\lambda_i\alpha_i^2 + \sigma^2}, \qquad i = 1,2,\ldots,p.$$

Replacing $\alpha_i$ and $\sigma^2$ by their unbiased estimators $\hat{\alpha_i} = \lambda_i^{-1}X'Y$ and $\hat{\sigma}^2 = \frac{(Y-X\hat{\beta})'(Y-X\hat{\beta})}{(n-p)}$, we obtain the estimate of $d_i$,

$$\hat{d_i} = \frac{\lambda_i(\hat{\alpha}_i^2 - \hat{\sigma}^2)}{\lambda_i\hat{\alpha}_i^2 + \hat{\sigma}^2}.$$

**Theorem 3.** *The density function of $\hat{d_i}$ is*

$$f(\hat{d_i}) = \frac{e^{-v\delta_i/2}(1+\lambda_i)^{v-1}((1+\lambda_i)+(1-\hat{d_i}))^{-3v/2}/(1-\hat{d_i})^{1-v/2}}{\Gamma(v/2)}$$

$$\times \sum_{j=1}^{\infty}\left[\frac{v\delta_i(1+\lambda_i)}{4[(1+\lambda_i)+(1-\hat{d_i})]}\right]^j \frac{\Gamma(3v/2+j)}{\Gamma(j+1)\Gamma(v+j)},$$

*where* $\delta_i = (\lambda_i^{-1/2}\alpha_i/\sigma)^2$, $0 < \hat{d}_i < 1$ *and* $v = n - p$.

**Proof.** We can write

$$\hat{d}_i = \frac{\lambda_i(\hat{\alpha}_i^2 - \hat{\sigma}^2)}{\lambda_i\hat{\alpha}_i^2 + \hat{\sigma}^2} = 1 - (1+\lambda_i)\frac{V}{R_i + V},$$

where $R_i = vz_i^2 \sim \chi_v^2(v\delta_i)$, $z_i = \frac{\hat{\alpha}_i}{\sigma/\lambda_i^{1/2}} \sim N(\delta_i^{1/2}, 1)$ and $V = \frac{v\hat{\sigma}^2}{\sigma^2} \sim \chi_i^2$, $v = n - p$. Let $J = R_i + V \sim \chi_{2v}^2(v\delta_i)$. Therefore, $Y = \frac{J/2v}{V/v} \sim F_{2v,v}(v\delta_i, 0)$ (Rohatgi and Saleh (2001)). Thus,

$$f(y) = \frac{e^{-v\delta_i/2}v^{3v/2}2^v y^{v-1}}{\Gamma(v/2)} \times \sum_{j=1}^{\infty} \frac{(2v^2\delta_i y/2)^j\Gamma(3v/2+j)\Gamma(1/2)}{\Gamma(j+1)\Gamma(v+j)(2vy+v)^{3v/2+j}},$$

$$0 < y < \infty.$$

Since $\hat{d}_i = 1 - \frac{(1+\lambda_i)V}{J} = 1 - \frac{(1+\lambda_i)}{2y}$, by changing the variables

$$f(\hat{d}_i) = \frac{e^{-v\delta_i/2}(1+\lambda_i)^{v-1}((1+\lambda_i)+(1-\hat{d}_i))^{-3v/2}/(1-\hat{d}_i)^{1-v/2}}{\Gamma(v/2)} \times \sum_{j=1}^{\infty}\left[\frac{v\delta_i(1+\lambda_i)}{4[(1+\lambda_i)+(1-\hat{d}_i)]}\right]^j \frac{\Gamma(3v/2+j)}{\Gamma(j+1)\Gamma(v+j)}, \qquad 0 < \hat{d}_i < 1,$$

the proof is completed. □

Akdeniz and Öztürk (2005) derived the density function of the stochastic shrinkage parameters of the generalized Liu estimator using a different approach.

**Theorem 4.** *The distribution function of* $\hat{d}_i$ *is*

$$F(x_i) = \sum_{j=1}^{\infty} \frac{(v\delta_i/2)^j e^{-v\delta_i/2}}{j!B(v+j, v/2)} B_{r_i}(v+j, v/2),$$

*where* $r_i = \frac{1+\lambda_i}{2-x_i+\lambda_i}$.

**Proof.** The proof is similar to the proof of Theorem 2. □

## 4 Confidence interval for shrinkage parameter

A confidence interval for the optimal value of shrinkage parameter $d_{\text{opt}}$ in (2.5) can be obtained using the destribution theory of $\hat{d}$ presented in Section 2.

By using the method discussed by Venables (1975), it is possible to find a $(1-\alpha)100\%$ confidence interval for the noncentrality parameter $\delta$ of a noncentral $F$ distribution. Let $(\delta_1(x), \delta_2(x))$ be such a confidence interval for $\delta$.

Now, exploring the relationship between $d_{\text{opt}}$ and $\delta$, a $(1-\alpha)100\%$ confidence interval for $d_{\text{opt}}$ will be

$$(1-2p/(\delta_1(\hat{d})+p), 1-2p/(\delta_2(\hat{d})+p)).$$

## 5 Numerical study

In this section we present some numerical calculations to illustrate the behavior of the distribution of the stochastic shrinkage parameters, under different models. To achieve different degrees and patterns of collinearity (Firinguetti and Rubio (2000)) the explanatory variables were generated by

$$x_{ij} = (1-a_j^2)^{1/2} z_{ij} + a_j z_{ip}, \qquad j = 1, 2, \ldots, p-1, \; i = 1, 2, \ldots, n,$$

where $z_{ij} \sim U(0, 1), \; j = 1, 2, \ldots, p, i = 1, 2, \ldots, n$.

Two $X$ matrices were specified, each with $p = 4$ explanatory variables and $n = 50$ observations. We then specify the following sets of $a_j$ values:

$$A1 = (0.2, 0.3, 0.4, 0.5),$$

$$A2 = (0.99, 0.95, 0.65, 0.60).$$

Four values of $\sigma^2$ are considered: (2.5, 5, 10, 20).

The coefficients $\beta_i'$s are selected as $(1, 1, 1, 1)'$. Numerical calculations were carried out under different degrees of collinearity and different levels of signal-to-noise ratio $\delta = \frac{\beta' X' X \beta}{\sigma^2}$.

Plots of the densities of the estimate of the shrinkage parameter for the Liu estimator are given in Figures 1–3. From Figure 1, it may be noted that the density function is sensitive to the value of $\sigma^2$ and to the degree of the multicollinearity between the variables but the value of $\sigma^2$ has more effect on the density function. Also, when $\hat{d}$ increases, the density function decreases in all cases. We have no guarantee that the estimated values of $d$ will be positive always or between 0 and 1 (Akdeniz and Erol (2003)). However, we restrict our attention to the range (0, 1) only. It may be noted that the distribution has the shape of a folded normal.

Figures 2 and 3, show that the density function of $d_i$ depends on the eigenvalue (noncentral $\chi^2$ parameter) and is also affected by the degree of correlation; this is clearly visible from the shapes of the density function. The value of $\sigma^2$ does not affect the behavior of the density function for all $d_i$. The degree of correlation has more effect on the behavior of the shrinkage parameter, as is clear from all the figures.

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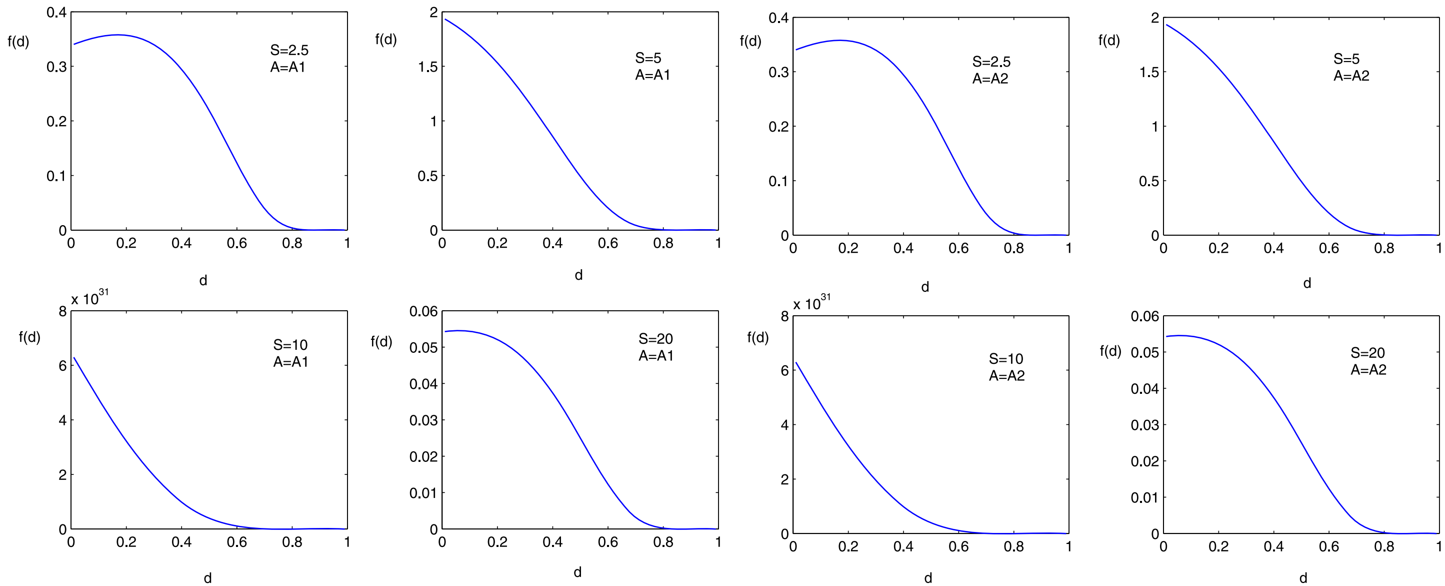


**Figure 1** *The density function of the estimate of shrinkage parameter in Liu estimator when* $A = A1, A2$ *and for* $S = \sigma^2 = 2.5, 5, 10, 20.$

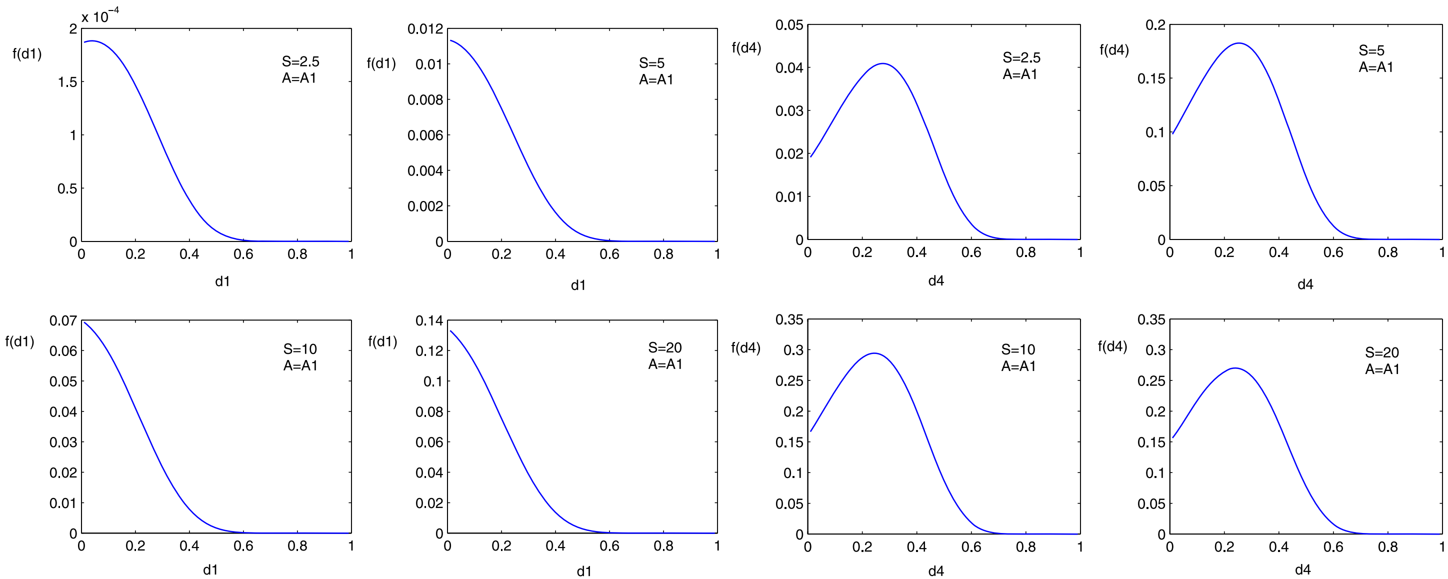


**Figure 2** *The density function for $d_1$ and $d_4$ of the generalized shrinkage parameter of Liu estimator when $A = A1$ and for $S = \sigma^2 = 2.5, 5, 10, 20$.*

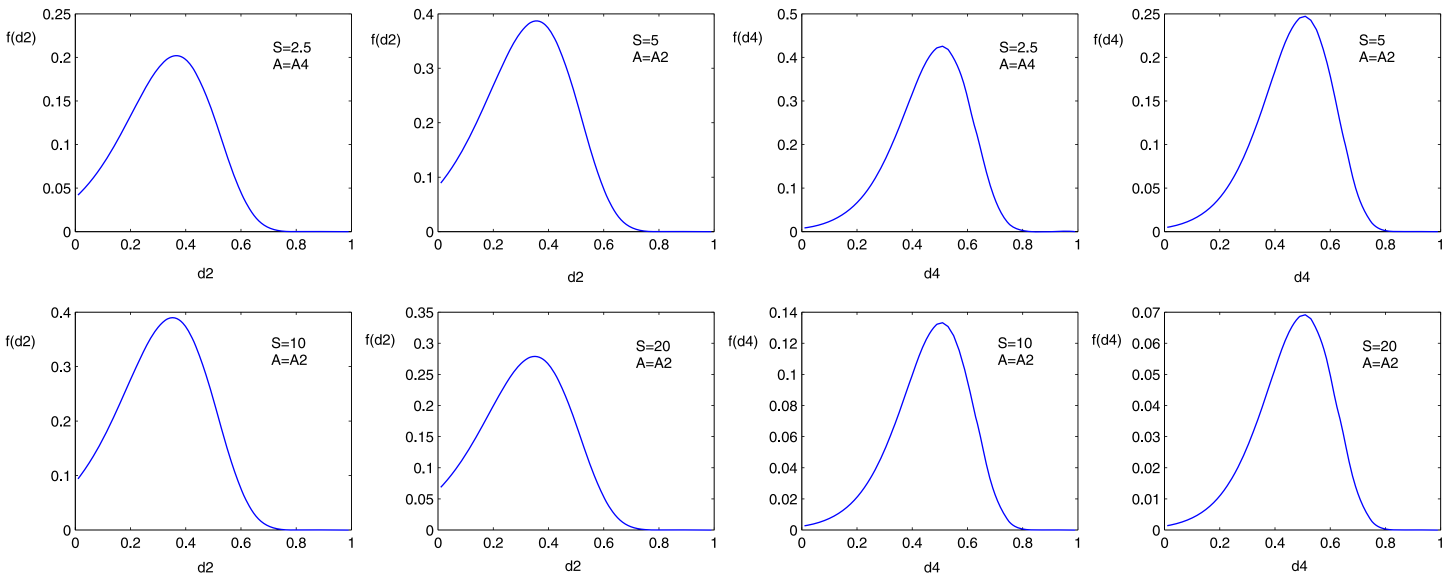


**Figure 3** *The density function for $d_2$ and $d_4$ of the generalized shrinkage parameter of Liu estimator when $A = A2$ and for $S = \sigma^2 = 2.5, 5, 10, 20$.*

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