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Improving Brownian approximations for boundary crossing problems

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Donsker's theorem shows that random walks behave like Brownian motion in an asymptotic sense. This result can be used to approximate expectations associated with the time and location of a random walk when it first crosses a nonlinear boundary. In this paper, correction terms are derived to improve the accuracy of these approximations.

Keywords: asymptotic expansion; Donsker's theorem; excess over the boundary; random walk; stopping times

1. Introduction and main results

Let $X, X_1, X_2, \ldots$ be i.i.d. with mean zero and unit variance; take $S_k = X_1 + \cdots + X_k$, $k \geq 1$, with $S_0 = 0$; and let $W(t)$, $t \geq 0$, be standard Brownian motion. By Donsker's theorem, if W_n is continuous and piecewise linear with

$$W_n(k/n) = S_k/\sqrt{n}, \qquad k = 0, 1, \ldots,$$

then $W_n \Rightarrow W$ in $\mathcal{C}[0, \infty)$ as $n \to \infty$. Let b be a smooth function on $[0, \infty)$ with $b(0) > 0$, such that

$$\tau_0 = \inf\{t \geq 0\colon\ W(t) \geq b(t)\}$$

is finite almost surely, and define

$$\tau_n = \inf\{k/n \geq 0\colon\ W_n(k/n) \geq b(k/n)\}.$$

Defining boundary levels

$$b_k = b_{k,n} = \sqrt{n} b(k/n), \tag{1}$$

this stopping time can be written as

$$\tau_n = \inf\{k \geq 0\colon\ S_k \geq b_k\}/n.$$

As the form suggests, $\tau_n \Rightarrow \tau_0$ as $n \to \infty$. This can be established by introducing $\tilde{\tau}_n = \inf\{t \geq 0\colon\ W(t) \geq b(t)\}$, arguing that $\tau_n - \tilde{\tau}_n \xrightarrow{p} 0$, and using the continuous mapping theorem, Theorem 5.1 of Billingsley [2], to show that $\tilde{\tau}_n \Rightarrow \tau_0$. Note that the Brownian path $W(\cdot)$ will be a continuity point for the transformation $W(\cdot) \rightsquigarrow \tau_0$ whenever $\tau_0 = \inf\{t \geq 0\colon\ W(t) > b(t)\}$, and

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this holds with probability one by the strong Markov property. We also have $W_n(\tau_n)-b(\tau_n)\xrightarrow{p}0$, and so

$$(\tau_n, W_n(\tau_n)) \Rightarrow (\tau_0, W(\tau_0))$$

as $n\to\infty$. Thus if f is a bounded continuous function,

$$Ef(\tau_n, W_n(\tau_n)) \to Ef(\tau_0, W(\tau_0)). \tag{2}$$

For large n, the limit here provides a natural approximation for the expectation on the left-hand side. The main result of this paper provides correction terms of order $1/\sqrt{n}$, improving this approximation. The excess over the boundary,

$$R_n = S_{n\tau_n} - \sqrt{n}b(\tau_n) = \sqrt{n}[W_n(\tau_n) - b(\tau_n)],$$

plays an important role in this analysis. The excess over the boundary also plays a central role in nonlinear renewal theory, where the law of large numbers drives the leading order approximation. See Woodroofe [20] or Siegmund [16] for a discussion and applications to sequential analysis. With the Brownian motion scaling considered in this paper, results on improved approximations and the excess over the boundary are given by Siegmund [15], Siegmund and Yuh [17], Yuh [21] and Hogan [6–9]. Siegmund [16] suggests various applications of this theory to sequential analysis; Broadie *et al.* [3], Broadie *et al.* [4] and Kou [12] use it to study options pricing; and Glasserman and Liu [5] consider its use for inventory control. With the exception of Hogan [6,7], stopping boundaries in these papers are linear.

To appreciate the role of the excess R_n in improving (2), note that if $f(t,x)=h(t)[x-b(t)]$, then $Ef(\tau_n, W_n(\tau_n)) = Eh(\tau_n)R_n/\sqrt{n}$. Hogan [6] derives the limiting joint distribution for R_n and τ_n; they are asymptotically independent, and the limiting distribution for R_n has mean

$$\rho = \frac{ES_{T_0}^2}{2ES_{T_0}}, \tag{3}$$

where T_0 is the ladder time

$$T_0 = \inf\{k>0\colon\ S_k \geq 0\}.$$

Hogan's argument is quite delicate. It is based on conditioning on a stopping time with a boundary just slightly less than the boundary for τ_n. By contrast, the approach pursued here is more global and analytic in character, but relies on smoothness of f and b to a greater extent. Formulas to calculate ρ numerically are given by Siegmund [16] and Keener [11].

An important special case of (2) would be first passage probabilities, $P(\tau_n \leq t)$. The regularity conditions here require differentiable f, so this case is formally excluded (although our result would suggest an approximation). Refined approximations for these probabilities are also suggested by Hogan [6], but his derivation is heuristic and assumes $EX^3=0$.

The limit in (2) can be found by solving the heat equation. To describe its relevance, let $Y=Y(t,x)$ be a process starting at time t and position x given by

$$Y_s = Y_s(t,x) = x + W(s-t), \qquad s \geq t;$$

let $\tau = \tau(t, x)$ be stopping times given by

$$\tau = \tau(t, x) = \inf\{s \geq t\colon\ Y_s \geq b(s)\};$$

and define

$$u(t, x) = Ef(\tau, Y_\tau), \qquad t \geq 0, x \leq b(t).$$

Noting that $\tau_0 = \tau(0, 0)$ and $W(\tau_0) = Y_{\tau(0,0)}$, the limit $Ef(\tau_0, W(\tau_0))$ in (2) is $u(0, 0)$. By the Feynman–Kac formula (Kac [10]), u satisfies the heat equation

$$u_t + \frac{1}{2}u_{xx} = 0$$

in the region $\{(t, x)\colon\ t \geq 0, x < b(t)\}$, with boundary condition $u(t, b(t)) = f(t, b(t))$. Furthermore, u is the unique solution in a suitable class of functions; see Krylov [13] or Bass [1]. In practice, $u(0, 0)$ can be computed by numerical solution of the heat equation. In the sequel, continuity and differentiability of u will play an important role.

Boundary effects associated with the excess R_n only arise (to order $\mathrm{o}(1/\sqrt{n})$) when f_x and u_x disagree along the boundary. Let $\Delta(t)$ denote the difference

$$\Delta(t) = f_x(t, b(t)) - u_x(t, b(t)-), \qquad t > 0,$$

and decompose f as the sum $f_0 + f_1$ with

$$f_0(t, x) = f(t, x) - \Delta(t)\big(x - b(t)\big)$$

and

$$f_1(t, x) = \Delta(t)\big(x - b(t)\big).$$

Since u and u_x agree with f_0 and $\partial f_0/\partial x$ along the boundary, it seems appropriate to view $u(0, 0)$ as an approximation for $Ef_0(\tau_n, W_n(\tau_n))$. It is then natural and convenient to extend u above the boundary, defining

$$\bar{u}(t, x) = \begin{cases} u(t, x), & x \leq b(t); \\ f_0(t, x), & x > b(t). \end{cases}$$

With this convention, $\bar{u}$ and $\bar{u}_x$ are both continuous at the boundary. Note also that

$$Ef_1(\tau_n, W_n(\tau_n)) = \frac{1}{\sqrt{n}} ER_n\Delta(\tau_n).$$

Theorem 1.1. *Assume*:

1. *The distribution of X is strongly non-lattice* (*or satisfies Cramer's condition C*),

$$\limsup_{|t|\to\infty} |Ee^{\mathrm{i}tX}| < 1,$$

and $EX = 0$, $EX^2 = 1$ *and* $EX^4 < \infty$.

2. *The stopping times τ_n, $n \ge 1$, are uniformly integrable.*
3. *The boundary function b has a bounded first derivative and $b(0) > 0$.*
4. *The function f and its first and second order partial derivatives are bounded and continuous.*
5. *The functions u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_t and u_{tt} are bounded and continuous.*

Then

$$Ef(\tau_n, W_n(\tau_n)) = Ef(\tau_0, W(\tau_0)) + \frac{EX^3}{6\sqrt{n}} E\int_0^{\tau_0} u_{xxx}(t, W(t))\,\mathrm{d}t + \frac{\rho}{\sqrt{n}} E\Delta(\tau_0) + \mathrm{o}\big(1/\sqrt{n}\big)$$

as $n \to \infty$.

The second assumption will hold if $b(t) + \epsilon t \to -\infty$ for some $\epsilon > 0$. If b and f are sufficiently smooth, then the final assumption follows from standard Hölder estimates for solutions of parabolic differential equations; see, for instance, Problem 4.5 of Lieberman [14].

The heat equation for u can be derived, at least informally, by conditioning a short time interval into the future. There is an analogous equation in discrete time. Define

$$\tau_n(t,x) = \inf\big\{t + k/n\colon\ x + S_k/\sqrt{n} \ge b(t+k/n), k = 0, 1, \dots\big\}$$

and

$$u_n(t,x) = Ef_0\big(\tau_n(t,x), x + S_{n\tau_n(t,x)}/\sqrt{n}\big).$$

Conditioning on X_1,

$$u_n(t,x) = \begin{cases} f_0(t,x), & x \ge b(t); \\ Eu_n\big(t + 1/n, x + X/\sqrt{n}\big), & x < b(t). \end{cases} \tag{4}$$

Unfortunately, with integration against the distribution of X, this convolution-type equation is usually less tractable numerically than the heat equation.

Theorem 1.1 evolved from my attempts to improve $\bar{u}$ as an approximation for u_n by imitating the matched asymptotic expansions used to study boundary effects in partial differential equations. The method might also be viewed as martingale approximation, with bounds for potential or renewal measures playing a central role in the proofs.

To study the error of $u(0,0) = \bar{u}(0,0)$ as an approximation for $Ef_0(\tau_n, W_n(\tau_n))$, define functions

$$e_n(t,x) = \begin{cases} E\bar{u}\big(t + 1/n, x + X/\sqrt{n}\big) - \bar{u}(t,x), & x < b(t); \\ 0, & x \ge b(t). \end{cases} \tag{5}$$

Writing

$$\begin{aligned}
Ef_0(\tau_n, W_n(\tau_n)) &= E\bar{u}((\tau_n, W_n(\tau_n)) \\
&= \bar{u}(0,0) + E\sum_{k=0}^{n\tau_n-1}\bigl[\bar{u}\bigl((k+1)/n, S_{k+1}/\sqrt{n}\bigr) - \bar{u}\bigl(k/n, S_k/\sqrt{n}\bigr)\bigr] \\
&= \bar{u}(0,0) + E\sum_{k=0}^{n\tau_n-1} e_n\bigl(k/n, S_k/\sqrt{n}\bigr),
\end{aligned} \tag{6}$$

a correction term for the approximation $\bar{u}(0,0)$ will be sought by approximating the expected sum in this equation. Details for this calculation are given in Section 2. The approximation for $Ef_1(\tau_n, W_n(\tau_n))$ is derived in Section 3.

2. An approximation for $Ef_0(\tau_n, W_n(\tau_n))$

Lemma 2.1. *Under the assumptions of Theorem 1.1,*

$$\begin{aligned}
E\bar{u}\bigl(t, x + X/\sqrt{n}\bigr) = \bar{u}(t,x) &+ \frac{\bar{u}_{xx}(t,x)}{2n} + \frac{EX^3\bar{u}_{xxx}(t,x)}{6n\sqrt{n}} \\
&+ \mathrm{O}(1/n^2) + \mathrm{O}\biggl(\frac{1/n}{1+n[b(t)-x]^2}\biggr)
\end{aligned} \tag{7}$$

as $n \to \infty$, uniformly for $t \ge 0$, $x < b(t)$. From this,

$$e_n(t,x) = \frac{EX^3\bar{u}_{xxx}(t,x)}{6n\sqrt{n}} + \mathrm{O}(1/n^2) + \mathrm{O}\biggl(\frac{1/n}{1+n[b(t)-x]^2}\biggr)$$

as $n \to \infty$, uniformly for $t \ge 0$, $x < b(t)$.

Proof. By Taylor expansion of u, on $\{x + X/\sqrt{n} \le b(t)\}$ we have

$$\begin{aligned}
\bar{u}\bigl(t, x + X/\sqrt{n}\bigr) = \bar{u}(t,x) &+ \frac{X\bar{u}_x(t,x)}{\sqrt{n}} + \frac{X^2\bar{u}_{xx}(t,x)}{2n} \\
&+ \frac{X^3\bar{u}_{xxx}(t,x)}{6n\sqrt{n}} + \mathrm{O}(X^4/n^2).
\end{aligned} \tag{8}$$

Lagrange's formula for the remainder will involve $\bar{u}_{xxxx}$ at an intermediate value x^* between x and $x + X/\sqrt{n}$, and from this it is clear that this equation holds uniformly for $t \ge 0$, $x < b(t)$. Since $\bar{u}_{xx}(t,x)$ exists unless $x = b(t)$ and is bounded, on $\{x + X/\sqrt{n} > b(t)\}$,

$$\bar{u}\bigl(t, x + X/\sqrt{n}\bigr) = \bar{u}(t,x) + \frac{X\bar{u}_x(t,x)}{\sqrt{n}} + \mathrm{O}(X^2/n), \tag{9}$$

as $n \to \infty$. Again, this will hold uniformly for $t \geq 0$, $x < b(t)$. Noting that

$$\frac{|X|^3}{n\sqrt{n}} \leq \frac{X^2}{n} + \frac{X^4}{n^2},$$

we can combine (8) and (9) to obtain

$$u\big(t, x + X/\sqrt{n}\big) = u(t,x) + \frac{Xu_x(t,x)}{\sqrt{n}} + \frac{X^2 u_{xx}(t,x)}{2n} + \frac{X^3 u_{xxx}(t,x)}{6n\sqrt{n}} \\ + \mathrm{O}(X^4/n^2) + \mathrm{O}(X^2/n) I\big\{X > \sqrt{n}\big(b(t) - x\big)\big\}.$$

The first assertion (7) follows by integrating against the distribution of X, noting that

$$E\big[X^2; X > \sqrt{n}\big(b(t)-x\big)\big] \leq \min\left\{\frac{EX^4}{n[b(t)-x]^2}, EX^2\right\} \\ \leq \frac{1+EX^4}{1+n[b(t)-x]^2}.$$

Here and in the sequel, $E[Y; B] \overset{\text{def}}{=} E(Y1_B)$.

If $x < b(t)$ and $x < b(t+1/n)$, then, by (5) and (7),

$$e_n(t,x) = \bar{u}(t+1/n, x) - u(t,x) + \frac{\bar{u}_{xx}(t+1/n,x)}{2n} + \frac{EX^3 \bar{u}_{xxx}(t+1/n,x)}{6n\sqrt{n}} \\ + \mathrm{O}(1/n^2) + \mathrm{O}\left(\frac{1/n}{1+n[b(t+1/n)-x]^2}\right).$$

In this case, the second assertion follows by the Taylor expansion

$$\bar{u}(t+1/n, x) = \bar{u}(t,x) + \frac{1}{n}\bar{u}_t(t,x) + \mathrm{O}(1/n^2) \\ = \bar{u}(t,x) - \frac{1}{2n}\bar{u}_{xx}(t,x) + \mathrm{O}(1/n^2),$$

and because

$$\frac{1+n[b(t)-x]^2}{1+n[b(t+1/n)-x]^2}$$

is uniformly bounded as b' is bounded. If, instead, $x \geq b(t+1/n)$, but $x < b(t)$, then $n[b(t) - x]^2 \to 0$ and $n[b(t+1/n) - x]^2 \to 0$, and the asymptotic bound holds because $u(t+1/n, x) - u(t,x) = \mathrm{O}(1/n)$. □

Define

$$N_d = N_d(n) = \#\{k < n\tau_n\colon\ S_k > b_k - d\},$$

the number of times the walk is within distance d of the boundary before stopping. The following result is essentially due to Hogan [6]. It slightly improves a bound given in the proof for Lemma 1.1 in his paper.

Lemma 2.2. *With the assumptions of Theorem 1.1, there exists a finite constant $K \geq 0$ such that*

$$EN_d = K(1+d^2),$$

for all $n \geq 1$ and $d > 0$. Also, if

$$M_B(\alpha) = \#\{k \leq \alpha n\tau_n\colon\ b_k - S_k \in B\},$$

then there exists a finite constant $K > 0$ such that

$$EM_B(\alpha) \leq K P\big(M_B(\alpha) \geq 1\big)\big(1 + (\sup B)^2\big),$$

for all $n \geq 1$, all $\alpha > 0$ and all $B \subset \mathbb{R}$.

Proof. Without loss of generality, let d be a positive integer. By the central limit theorem,

$$P\{S_{n^2} > (1 + \|b'\|_\infty)n\} \geq \gamma > 0, \tag{10}$$

for all n sufficiently large, say $n \geq n_0$. Since the τ_n are uniformly integrable (by the second assumption of Theorem 1.1) and $N_d \leq n\tau_n$, we can assume that $n_0 d^2 \leq n$. Define

$$N_{m,d} = \#\{k \leq m\colon\ k < n\tau_n, S_k > b_k - d\},$$

and let

$$\nu_{j,d} = \inf\{m\colon\ N_{m,d} = j\},$$

so the jth time the walk is within d of the boundary happens on step $\nu_{j,d}$. Note that $N_d \geq j + n_0 d^2$ implies the walk is below the boundary at time $\nu_{j,d} + n_0 d^2$, that is,

$$S_{\nu_{j,d}+n_0d^2} < b_{\nu_{j,d}+n_0d^2},$$

which, in turn, implies

$$S_{\nu_{j,d}+n_0d^2} - S_{\nu_{j,d}} < b_{\nu_{j,d}+n_0d^2} - b_{\nu_j} + d \leq d + n_0 d^2 \frac{\|b'\|_\infty}{\sqrt{n}} \leq \sqrt{n_0}\, d(1 + \|b'\|_\infty).$$

But $S_{\nu_{j,d}+n_0d^2} - S_{\nu_{j,d}}$ is independent of $\{N_d \geq k\}$. So using this bound and (10),

$$P(N_d \geq j + n_0 d^2) \leq P(N_d \geq j)(1-\gamma).$$

Iterating this,

$$
\begin{aligned}
&P(N_d \geq 1 + jn_0d^2) \\
&\quad = P\big(N_d \geq 1 + (j-1)n_0d^2 + n_0d^2\big) \\
&\quad \leq P\big(N_d \geq 1 + (j-1)n_0d^2\big)(1-\gamma) \leq \cdots \leq P(N_d \geq 1)(1-\gamma)^j, \qquad j = 0, 1, \ldots.
\end{aligned}
$$

Hence

$$
\begin{aligned}
EN_d &= \int_0^\infty P(N_d \geq x)\,\mathrm{d}x \\
&\leq P(N_d \geq 1)\left[1 + \int_1^\infty (1-\gamma)^{\lfloor (x-1)/(n_0d^2)\rfloor}\,\mathrm{d}x\right] \\
&= P(N_d \geq 1)\left[1 + \frac{n_0d^2}{\gamma}\right].
\end{aligned}
$$

The proof of the bound for $EM_B(\alpha)$ is the same. □

Corollary 2.3. *Let $c_k = c_{k,n}$, $k \geq 0$, $n \geq 1$ be constants. Define*

$$
\Lambda = \sup_{k,n}(b_k - c_k),
$$

and let g be a non-negative function on $(-\infty, \Lambda]$. If $\Lambda < \infty$, $\|g\|_\infty < \infty$, and $g(x) \to 0$ as $x \to -\infty$,

$$
\frac{1}{n} E \sum_{k=0}^{n\tau_n - 1} f(S_k - c_k) \to 0,
$$

as $n \to \infty$. If, in addition, g is non-decreasing,

$$
E \sum_{k=0}^{n\tau_n - 1} g(S_k - c_k) \leq K\left[g(\Lambda) + 2\int_{-\infty}^0 |x| g(x+\Lambda)\,\mathrm{d}x\right],
$$

where K is the constant in Lemma 2.2.

When this corollary is used later, c_k will be either b_k or b_{k+1}. When $c_k = b_k$, Λ is zero, and when $c_k = b_{k+1}$, $\Lambda \leq \|b'\|_\infty$.

Proof of Corollary 2.3. For the first assertion, for any $d > 0$,

$$
g(S_k - c_k) \leq \|g\|_\infty I\{S_k > b_k - d\} + \sup_{x \in (-\infty, \Lambda - d]} g(x).
$$

Summing over k and bounding the expectation using Lemma 2.2,

$$\frac{1}{n}E\sum_{k=0}^{n\tau_n-1} g(S_k-c_k) \le \frac{1}{n}\|g\|_\infty K(1+d^2) + \sup_{x\in(-\infty,\Lambda-d]} g(x)E\tau_n,$$

and the result follows because d can be arbitrarily large.

In the second assertion, we can assume without loss of generality that g is right continuous and write

$$g(y) = \int I\{x \le y\}\,\mathrm{d}g(x).$$

By Fubini's theorem and Lemma 2.2,

$$\begin{aligned}
E\sum_{k=0}^{n\tau_n-1} g(S_k-c_k) &\le E\sum_{k=0}^{n\tau_n-1} g(S_k-b_k+\Lambda)\\
&\le \int E\sum_{k\ge 0} I\{x<S_k-b_k+\Lambda, k<n\tau_n\}\,\mathrm{d}g(x)\\
&= \int EN_{\Lambda-x}\,\mathrm{d}g(x)\\
&\le K\int_{-\infty}^{\Lambda}[1+(\Lambda-x)^2]\,\mathrm{d}g(x)\\
&= K\left[g(\Lambda)+2\int_{-\infty}^{0}|x|g(x+\Lambda)\,\mathrm{d}x\right].
\end{aligned}$$

□

The second assertion in Corollary 2.3 is useless when the integral in the bound diverges, but, in certain cases, it gives sharper results than the first assertion. The next corollary considers a specific function of interest later.

Corollary 2.4. *With the assumptions of Theorem 1.1,*

$$E\sum_{k=0}^{n\tau_n-1}\frac{1}{1+[b_k-S_k]^2} = \mathrm{O}(\log n)$$

as $n\to\infty$.

Proof. If $0\le b_k-S_k\le\sqrt{n}$,

$$\frac{1}{1+[b_k-S_k]^2} = \frac{1}{1+n} + \int\frac{I\{b_k-S_k<x<\sqrt{n}\}2x}{(1+x^2)^2}\,\mathrm{d}x,$$

and so

$$\sum_{k=0}^{n\tau_n-1} \frac{1}{1+[b_k-S_k]^2} \le \tau_n + \int_0^{\sqrt{n}} \frac{2xN_x}{(1+x^2)^2}\,\mathrm{d}x.$$

Using Lemma 2.2,

$$E\sum_{k=0}^{n\tau_n-1} \frac{1}{1+[b_k-S_k]^2} = E\tau_n + \mathrm{O}(1)\int_0^{\sqrt{n}} \frac{2x}{1+x^2}\,\mathrm{d}x = \mathrm{O}(\log n). \qquad \square$$

The final corollary gives uniform integrability for moments of R_n.

Corollary 2.5. *With the assumptions of Theorem 1.1, if $E|X|^{p+2} < \infty$, R_n^p, $n \ge 1$, are uniformly integrable.*

Proof. Conditioning on $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$, if $c > 0$,

$$\begin{aligned} E[R_n^p; R_n \ge c] &= \sum_{k\ge 0} E[(S_k + X_{k+1} - b_{k+1})^p; k < n\tau_n, S_k + X_{k+1} - b_{k+1} \ge c] \\ &= E\sum_{k<n\tau_n} g(S_k - b_{k+1}), \end{aligned}$$

where

$$g(x) = E[(x+X)^p; x + X \ge c].$$

This function is increasing and right continuous. Taking $\Lambda = \sup_{k,n}(b_k - b_{k+1}) \le \|b'\|_\infty$, by Fubini's theorem,

$$\begin{aligned} \int_{-\infty}^0 |x| g(x+\Lambda)\,\mathrm{d}x &= -E\int x(x+\Lambda+X)^p I\{c - X - \Lambda \le x < 0\}\,\mathrm{d}x \\ &= E\left[\frac{(X+\Lambda)^{p+2} - c^{p+2}}{(p+1)(p+2)} + \frac{c^{p+1}(X+\Lambda-c)}{p+1}; X + \Lambda \ge c\right]. \end{aligned}$$

This expectation tends to zero as $c \to \infty$ by dominated convergence, as does $g(\Lambda)$, and uniform integrability follows from the bound in Corollary 2.3. $\square$

Theorem 2.6. *Under the assumptions of Theorem 1.1,*

$$Ef_0(\tau_n, W_n(\tau_n)) = \bar{u}(0,0) + \frac{EX^3}{6\sqrt{n}} E\int_0^{\tau_0} \bar{u}_{xxx}(t, W(t))\,\mathrm{d}t + \mathrm{o}\big(1/\sqrt{n}\big).$$

Proof. Because

$$\frac{1}{n}\sum_{k=0}^{n(T\wedge\tau_n)-1}\bar{u}_{xxx}\big(k/n,\,S_k/\sqrt{n}\big)=\int_0^{T\wedge\tau_n}\bar{u}_{xxx}\big(\lfloor nt\rfloor/n,\,W_n(\lfloor nt\rfloor/n)\big)\,\mathrm{d}t,$$

$$\frac{1}{n}E\sum_{k=0}^{n(T\wedge\tau_n)-1}\bar{u}_{xxx}\big(k/n,\,S_k/\sqrt{n}\big)-E\int_0^{T\wedge\tau_n}\bar{u}_{xxx}(t,\,W_n(t))\,\mathrm{d}t\to 0$$

by dominated convergence, since $\max_{k<nT}|X_k|/\sqrt{n}\to 0$ almost surely. So, by Donsker's theorem,

$$E\int_0^{T\wedge\tau_n}\bar{u}_{xxx}(t,\,W_n(t))\,\mathrm{d}t\to E\int_0^{T\wedge\tau}\bar{u}_{xxx}(t,\,W(t))\,\mathrm{d}t.$$

Then, since u_{xxx} is uniformly bounded and $\tau_n,\,n\ge 1$ are uniformly integrable,

$$\frac{1}{n}E\sum_{k=0}^{n\tau_n-1}\bar{u}_{xxx}\big(k/n,\,S_k/\sqrt{n}\big)\to E\int_0^{\tau_0}\bar{u}_{xxx}(t,\,W(t))\,\mathrm{d}t.$$

The theorem now follows from (6) using the formula for e_n in Lemma 2.1 and the asymptotic bound in Corollary 2.4. □

3. An approximation for $ER_n\Delta_n(\tau_n)$

Theorem 3.1. *Under the assumptions of Theorem 1.1,*

$$ER_n\Delta(\tau_n)\to\rho E\Delta(\tau_0),$$

where ρ is the limiting mean excess defined in (3).

The proof of this result, like that for Theorem 2.6, is based on a telescoping sum argument, but now the summands involve functions related to fluctuation theory for random walks. For $x\le 0$, define stopping times

$$T_x=\inf\{k\ge 1:\;x+S_k\ge 0\},$$

and define

$$H(x)=\begin{cases}x-\rho, & x\ge 0;\\ E[S_{T_x}+x]-\rho; & x<0.\end{cases}$$

Conditioning on X_1, for $x<0$

$$H(x)=EH(x+X).\tag{11}$$

In particular, on $\{S_k<b_k\}$,

$$E[H(S_{k+1}-b_k)\big|\mathcal{F}_k]=H(S_k-b_k).\tag{12}$$

Now

$$R_n\Delta(\tau_n) = \rho\Delta(\tau_n) + \Delta(\tau_n)H\big[S_{n\tau_n} - \sqrt{n}b(\tau_n)\big],$$

and by a telescoping sum argument,

$$\begin{aligned}
&E\Delta(\tau_n)H\big[S_{n\tau_n} - \sqrt{n}b(\tau_n)\big] - \Delta(0)H\big[-\sqrt{n}b(0)\big] \\
&\quad = E\sum_{k=0}^{n\tau_n-1}\left[\Delta\left(\frac{k+1}{n}\right)H(S_{k+1}-b_{k+1}) - \Delta\left(\frac{k}{n}\right)H(S_k-b_k)\right] \\
&\quad = E\sum_{k=0}^{n\tau_n-1}\left[\Delta\left(\frac{k+1}{n}\right)H(S_{k+1}-b_{k+1}) - \Delta\left(\frac{k}{n}\right)H(S_{k+1}-b_k)\right],
\end{aligned}$$

with the last equality from (12), since $\{k < n\tau_n\} \in \mathcal{F}_k$. The magnitude of the final expectation here is bounded by the sum of

$$E\sum_{k=0}^{n\tau_n-1}\left|\Delta\left(\frac{k+1}{n}\right) - \Delta\left(\frac{k}{n}\right)\right| |H(S_{k+1}-b_{k+1})| \tag{13}$$

and

$$E\sum_{k=0}^{n\tau_n-1}\left|\Delta\left(\frac{k}{n}\right)\right| |H(S_{k+1}-b_{k+1}) - H(S_{k+1}-b_k)|. \tag{14}$$

Using Corollaries 2.3 and 2.5, it is easy to show that (13) tends to zero as $n \to \infty$. To show that (14) also tends to zero, we need a few results from renewal theory and the fluctuation theory for random walk.

Let $Y, Y_1, Y_2, \ldots$ be i.i.d. with $Y \sim S_{T_0}$, the first ascending ladder height for S_k, $k \geq 1$, and let $V_k = Y_1 + \cdots + Y_k$ with $V_0 \stackrel{\text{def}}{=} 0$. Then $EY^3 < \infty$ and the characteristic function for Y satisfies Cramer's condition. Define

$$\mu(B) = \sum_{k=0}^{\infty} P(V_k \in B),$$

so μ is the renewal measure for the random walk V_k, $k \geq 0$. By Wald's identity, for $x < 0$,

$$\mu((-\infty, -x)) = ES_{T_x}/EY.$$

So, for $\epsilon > 0$ and $x < -\epsilon/2$,

$$H(x+\epsilon/2) - H(x-\epsilon/2) = \epsilon - EY\mu([-x-\epsilon/2, -x+\epsilon/2)). \tag{15}$$

The following lemma follows immediately from these equations and Theorem 3 of Stone [19].

Lemma 3.2. *As $x \to -\infty$,*

$$H(x) = -\frac{E(Y+x)_+^2}{2(EY)^2} + \mathrm{o}\biggl(\frac{\log|x|}{x^2}\biggr).$$

Also,

$$H(x+\epsilon/2) - H(x-\epsilon/2) = -\frac{\epsilon E(Y+x)_+}{EY} + \mathrm{o}\biggl(\frac{\log|x|}{|x|^3}\biggr),$$

as $x \to -\infty$, uniformly for $\epsilon > 0$ in any bounded set.

To show that (14) is small we will need the following lemma, similar to Lemma 2.2, but bounding the expected number of visits to smaller sets.

Lemma 3.3. *Let $I_k = I_k(n) = (c_k, d_k)$ be intervals with*

$$\sup_{n,k} d_k \le K,$$

and

$$\sup_{n,k}(d_k - c_k) \le \frac{K}{\sqrt{n}},$$

for some $K \in (0, \infty)$. If

$$W = W_n = \#\{k \le n\tau_n \colon b_k - S_k \in I_k\},$$

then $EW \to 0$ as $n \to \infty$.

Proof. For $\alpha > 0$, let

$$W_0 = \#\bigl\{k \le \min\{3\alpha n, n\tau_n\} \colon b_k - S_k \in I_k\bigr\},$$

the contribution to the count in W from indices $k \le 3\alpha n$. Then $W_0 \le M_{(-\infty,K]}(3\alpha)$. By Donsker's theorem,

$$\begin{aligned}\limsup_{n\to\infty} P\bigl(M_{(-\infty,K]}(3\alpha) \ge 1\bigr) &\le \limsup_{n\to\infty} P\bigl(W_n(t) \ge b(t) - K/\sqrt{n},\ \text{for some } t \le 3\alpha\bigr)\\ &= P\bigl(W(t) \ge b(t),\ \text{for some } t \le 3\alpha\bigr),\end{aligned}$$

which tends to 0 as $\alpha \downarrow 0$. So by Lemma 2.2, $\limsup EW_0$ will be arbitrarily small if α is chosen suitably small. Thus this lemma will hold if

$$EW - EW_0 = \sum_{k > 3\alpha n} P(b_k - S_k \in I_k, k \le n\tau_n) \to 0,$$

as $n \to \infty$ for any fixed $\alpha > 0$.

Let $n^* = \lfloor \alpha n \rfloor$. By the local limit theorem of Stone [18] (or Edgeworth expansion), for some K_0,

$$P(x - S_{n^*} \in I_k) \leq K_0/n, \tag{16}$$

for all $x \in \mathbb{R}$, $n \geq 1$ and $k \geq 1$. If $j < k$, $k - j < \sqrt{n}$, $k \leq n\tau_n$ and $b_k - S_k \in I_k$, then

$$b_k - S_k \leq K \quad \text{and} \quad S_j < b_j.$$

Together, these imply

$$S_k - S_j \geq b_k - b_j - K \geq -\frac{k-j}{\sqrt{n}} \|b'\|_\infty - K \geq -K - \|b'\|_\infty.$$

Thus, if n is large enough that $\alpha n > \sqrt{n}$ and if $k > 3\alpha n$,

$$\begin{aligned} &P(b_k - S_k \in I_k, k \leq n\tau_n) \\ &\quad \leq P\big(b_k - S_k \in I_k, n\tau_n \geq k/3, S_k - S_j > -K - \|b'\|_\infty, 0 < k - j < \sqrt{n}\big). \end{aligned} \tag{17}$$

To use this bound, let

$$\mathcal{G} = \sigma\big(X_1, \ldots, X_{\lfloor k/3 \rfloor}, X_{\lfloor k/3 \rfloor + n^* + 1}, \ldots, X_k\big).$$

Writing

$$b_k - S_k = b_k - S_{\lfloor k/3 \rfloor} - \big(S_k - S_{\lfloor k/3 \rfloor + n^*}\big) - \big(S_{\lfloor k/3 \rfloor + n^*} - S_{\lfloor k/3 \rfloor}\big),$$

since $b_k - S_{\lfloor k/3 \rfloor} - (S_k - S_{\lfloor k/3 \rfloor + n^*})$ is $\mathcal{G}$ measurable and

$$S_{\lfloor k/3 \rfloor + n^*} - S_{\lfloor k/3 \rfloor} \,\big|\, \mathcal{G} \sim S_{n^*},$$

by (16),

$$P(b_k - S_k \in I_k | \mathcal{G}) \leq K_0/n.$$

Since events $\{n\tau_n \geq k/3\}$ and $\{S_k - S_j > -K - \|b'\|_\infty, 0 < k - j < \sqrt{n}\}$ are independent and both lie in $\mathcal{G}$, using (17) and conditioning on $\mathcal{G}$,

$$\begin{aligned} P(b_k - S_k \in I_k, k \leq n\tau_n) &\leq \frac{K_0}{n} P(n\tau_n \geq k/3) P\big(S_k - S_j > -K - \|b'\|_\infty, 0 < k - j < \sqrt{n}\big) \\ &= \frac{K_0}{n} P(n\tau_n \geq k/3) P\big(S_j > -K - \|b'\|_\infty, 0 < j < \sqrt{n}\big). \end{aligned}$$

The second probability in this bound tends to zero, and so

$$EW - EW_0 = \mathrm{o}(1) \frac{1}{n} \sum_{k > 3\alpha n} P(n\tau_n > k/3) = \mathrm{o}(1) E\tau_n \to 0,$$

proving the lemma. □

Proof of Theorem 3.1. From the discussion and bounds above, the desired result will hold if (14) tends to zero, or if

$$E\sum_{k=0}^{n\tau_n-1}|H(S_{k+1}-b_{k+1})-H(S_{k+1}-b_k)|\to 0.$$

The expectation of the final term in the sum tends to zero, for if $S_{n\tau_n}-b_{n\tau_n-1}>0$, the summand is $|b_{n\tau_n}-b_{n\tau_n-1}|\le\|b'\|_\infty/\sqrt{n}$; and if $S_{n\tau_n}-b_{n\tau_n-1}\le 0$, $S_{n\tau_n}-b_{n\tau_n}$ is within some multiple of $1/\sqrt{n}$ of zero and the expectation will tend to zero by Lemma 3.3. So if $c_k=(b_k+b_{k-1})/2$ and $\epsilon_k=|b_k-b_{k-1}|\le\|b'\|_\infty/\sqrt{n}$, we need to show that

$$E\sum_{k=1}^{n\tau_n-1}|H(S_k-c_k+\epsilon_k/2)-H(S_k-c_k-\epsilon_k/2)|\to 0.$$

Using Lemma 3.2, for some constant K_0,

$$|H(x+\epsilon/2)-H(x-\epsilon/2)|\le\epsilon g(x)+\frac{K_0}{|x|^{5/2}},$$

for all $x<0$ and all $\epsilon\in[0,\|b'\|_\infty]$, where $g(x)=E(Y+x)_+/EY$. Using this,

$$\begin{aligned}
&E\sum_{k=1}^{n\tau_n-1}|H(S_k-c_k+\epsilon_k/2)-H(S_k-c_k-\epsilon_k/2)|I\{b_k-S_k\ge K_1\}\\
&\quad\le\frac{\|b'\|_\infty}{\sqrt{n}}E\sum_{k=1}^{n\tau_n-1}g(S_k-c_k)+K_0E\sum_{k=1}^{n\tau_n-1}\min\{K_1^{-5/2},(b_k-S_k)^{-5/2}\}.
\end{aligned}$$

By Corollary 2.3,

$$\begin{aligned}
E\sum_{k=1}^{n\tau_n-1}g(S_k-c_k)&\le K\left[\frac{E(Y+\Lambda)_+}{EY}+\frac{2}{EY}E\int|x|(Y+x+\Lambda)_+I\{x<0\}\,\mathrm{d}x\right]\\
&=K\left[\frac{E(Y+\Lambda)_+}{EY}+\frac{E(Y+\Lambda)^3}{3EY}\right],
\end{aligned}$$

which is finite since $EY^3<\infty$. And by the same corollary,

$$\begin{aligned}
E\sum_{k=1}^{n\tau_n-1}\min\{K_1^{-5/2},(b_k-S_k)^{-5/2}\}&\le K\left[K_1^{-5/2}+2\int^0|x|\min\{K_1^{-5/2},|x|^{-5/2}\}\,\mathrm{d}x\right]\\
&=K\left[K_1^{-1/2}+\frac{4}{3}K_1^{-3/2}+K_1^{-5/2}\right].
\end{aligned}$$

Since this bound tends to zero as $K_1 \to \infty$, the theorem will hold if

$$E \sum_{k=1}^{n\tau_n - 1} |H(S_k - c_k + \epsilon_k/2) - H(S_k - c_k - \epsilon_k/2)| I\{b_k - S_k < K_1\}$$

converges to zero for any fixed K_1. Also, using Lemma 3.3, we can include the restriction $b_k - S_k > \|b'\|_\infty/\sqrt{n}$ in the indicator. Using (15), it will then be sufficient to show

$$E \sum_{k=1}^{n\tau_n - 1} \epsilon_k I\{b_k - S_k < K_1\} \to 0$$

and

$$E \sum_{k=1}^{n\tau_n - 1} \mu([c_k - S_k - \epsilon_k/2, c_k - S_k + \epsilon_k/2]) I\left\{\frac{\|b'\|_\infty}{\sqrt{n}} < b_k - S_k < K_1\right\} \to 0.$$

The first of these follows immediately from Lemma 2.2. Using Fubini's theorem, the second expression equals

$$\int E \sum_{k=1}^{n\tau_n - 1} I\left\{\frac{\|b'\|_\infty}{\sqrt{n}} < b_k - S_k < K_1, |S_k - c_k + x| \le \epsilon_k/2\right\} \mathrm{d}\mu(x).$$

By Lemma 3.3, the integrand here tends to zero, uniformly in x. Since the range of integration remains bounded, the integral must tend to zero, proving the theorem. □

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