THE CHAOTIC REPRESENTATION PROPERTY OF COMPENSATED-COVARIATION STABLE FAMILIES OF MARTINGALES[1]

BY PAOLO DI TELLA[2] AND HANS-JÜRGEN ENGELBERT

Technische Universität Dresden and Friedrich-Schiller-Universität Jena

In the present paper, we study the chaotic representation property for certain families $\mathscr{X}$ of square integrable martingales on a finite time interval $[0, T]$. For this purpose, we introduce the notion of compensated-covariation stability of such families. The chaotic representation property will be defined using iterated integrals with respect to a given family $\mathscr{X}$ of square integrable martingales having deterministic mutual predictable covariation $\langle X, Y \rangle$ for all $X, Y \in \mathscr{X}$. The main result of the present paper is stated in Theorem 5.8 below: If $\mathscr{X}$ is a compensated-covariation stable family of square integrable martingales such that $\langle X, Y \rangle$ is deterministic for all $X, Y \in \mathscr{X}$ and, furthermore, the system of monomials generated by $\mathscr{X}$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$, then $\mathscr{X}$ possesses the chaotic representation property with respect to the σ-field $\mathscr{F}_T^{\mathscr{X}}$. We shall apply this result to the case of Lévy processes. Relative to the filtration $\mathbb{F}^L$ generated by a Lévy process L, we construct families of martingales which possess the chaotic representation property. As an illustration of the general results, we will also discuss applications to continuous Gaussian families of martingales and independent families of compensated Poisson processes. We conclude the paper by giving, for the case of Lévy processes, several examples of concrete families $\mathscr{X}$ of martingales including Teugels martingales.

1. Introduction. In his paper [22], Norbert Wiener introduced the notion of *multiple integral* and called it *polynomial chaos*. However, the Wiener polynomials chaos of different order are not orthogonal. In [9], Itô gave another definition of multiple integrals for a general *normal random measure* in such a way that the orthogonality property is achieved. In the same paper, Itô established the relation between orthogonal Hermite polynomials and multiple integrals. This was also done by Cameron and Martin [3] for the special normal random measure induced by a Wiener process. Using this relation, Itô proved that every square integrable functional of a normal random measure can be expanded as an orthogonal sum of

multiple integrals. This property is known in the literature as *chaotic representation property* (CRP). A similar result was shown in Kakutani [13]. In conclusion of [9], Itô pointed out that the multiple integrals of a normal random measure induced by a Wiener process W can be regarded as *iterated stochastic integrals* with respect to W. In the later paper [10], which appeared in 1956, Itô generalized the result of [9] stated for a normal random measure, and in particular for the Wiener process, to the case of an *orthogonal* random measure (cf. Gihman and Skorohod [8], Chapter IV, Section 4) defined as a sum of a normal random measure and a *compensated Poisson random measure*. These random measures are associated with processes with independent increments. For such a random measure, Itô introduced multiple integrals and proved a chaos decomposition for square integrable functionals. We call the multiple integrals introduced in [10] multiple Itô integrals. In [10], no relation between multiple Itô integrals and iterated stochastic integrals is given. However, we note that the relation between multiple Itô integrals and polynomials is established by Segall and Kailath [20].

Multiple Itô integrals have been extensively studied for many decades. A self-contained monograph about Itô-type integrals for *completely random measures*, with particular attention to their combinatorial structure, is Peccati and Taqqu [17]. In [17], the authors also discuss the CRP for centred Gaussian measures and compensated Poisson random measures (which are special cases of completely random measures) and study the relation of multiple integrals with Hermite polynomials for the Gaussian case and Charlier polynomials for the Poisson case.

For a stochastic process X, we denote by $\mathbb{F}^X$ the smallest filtration satisfying the usual conditions such that X is adapted.

Let now X be a square integrable martingale on the finite horizon $[0, T]$, $T > 0$. To approach the problem if the martingale X possesses the CRP with respect to the space $L^2(\Omega, \mathscr{F}_T^X, \mathbb{P})$, it is necessary to define *multiple integrals* with respect to X. In general, this is not an easy task because it is not clear how to associate a suitable random measure with X allowing to introduce multiple integrals. For this reason, it is more convenient to introduce *iterated stochastic integrals* with respect to X. But this, as observed in Meyer [15], pages 321–331, is in general not straightforward. However, if X is a square integrable martingale such that $\langle X, X \rangle_t = t$, $t \geq 0$, and F is a square integrable deterministic function on $[0, T]^n$, then the n-fold iterated integral $J_n(F)_T$ of F up to time T is well defined (cf. Meyer [15], pages 325–327). Note that for $n \neq m$, $J_n(F)_T$ and $J_m(G)_T$ are orthogonal. We recall that the so called Azéma–Yor martingales, which were introduced in Azéma [1] and Azéma and Yor [2], are of this type. Using the *structure equation* as a tool, Émery [7] has shown that some of the Azéma–Yor martingales possess the CRP, meaning that $L^2(\Omega, \mathscr{F}_T^X, \mathbb{P})$ allows a decomposition into the orthogonal sum of the linear subspaces of n-fold iterated integrals associated with X.

In this paper, we deal with the CRP of certain families $\mathscr{X}$ of square integrable martingales instead of only single processes X. We shall restrict ourselves to a

finite time horizon $[0, T]$, $T > 0$, and to the filtration $\mathbb{F}^{\mathscr{X}}$, that is, the smallest filtration satisfying the usual conditions with respect to which $\mathscr{X}$ is a family of adapted processes. The study of the CRP for families $\mathscr{X}$ of square integrable martingales turns out to be of major interest. This is because the CRP for a single square integrable martingale is a strong property and a relatively small class of processes possesses it.

One important example of a family of square integrable martingales possessing the CRP has been considered by Nualart and Schoutens [16]. Let L be a Lévy process with Lévy measure ν. Under the assumption that ν has a finite exponential moment outside the origin, the authors define the family of orthogonalized Teugels martingales, which is a family consisting of countably many orthogonal square integrable $\mathbb{F}^L$-martingales. Then the system of iterated integrals generated by the orthogonalized Teugels martingales is introduced and it is shown that this family of martingales possesses the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ (for the precise definition of the CRP see Definition 3.6 below). Notice that in Nualart and Schoutens [16] the assumption on the Lévy measure is rather strong. Furthermore in Nualart and Schoutens [16] the relationship between the iterated integrals generated by the orthogonalized Teugels martingales and the multiple Itô integrals introduced in [10] is not studied. The problem was also mentioned in Solé, Utzet and Vives [21], where in Proposition 7 the relationship between the iterated integrals generated by the orthogonalized Teugels martingales and the multiple Itô integrals is stated without proof.

The aim of this paper is to study the CRP for certain families $\mathscr{X}$ of square integrable martingales such that the process $\langle X, Y \rangle$ is deterministic whenever X and Y belong to $\mathscr{X}$. Note that the martingales in $\mathscr{X}$ need not have independent increments: If a square integrable martingale X has independent increments, then $\langle X, X \rangle$ is deterministic, the converse is however not true. Most of normal martingales X (i.e., square integrable martingales X such that $\langle X, X \rangle_t = t, t \geq 0$) and, in particular, solutions of the structure equation do not have independent increments (cf. Émery [7] where the very special case of independent increments is discussed on page 74). For the family $\mathscr{X}$ as above the iterated integrals can be defined, and we shall look for sufficient conditions to ensure that $\mathscr{X}$ possesses the CRP as it will be introduced in Definition 3.6 below. We shall require that the family $\mathscr{X}$ is *compensated-covariation stable*, that is, that for every $X, Y \in \mathscr{X}$ the process $[X, Y] - \langle X, Y \rangle$ again belongs to $\mathscr{X}$, $[X, Y]$ denoting the covariation process of X and Y. This property of compensated-covariation stability of families of martingales has been introduced in Di Tella [5] and Di Tella and Engelbert [6], where the *predictable representation property* is studied.

In Section 2, we recall some basic definitions and notation from stochastic analysis needed in the following sections of the paper.

Given a family $\mathscr{X}$ of square integrable martingales with deterministic predictable covariation $\langle X, Y \rangle$ for all X and Y from $\mathscr{X}$, in Section 3 the iterated

integrals are introduced and their properties are studied. Furthermore, the definition of the CRP for such kind of families is given.

In Section 4, some more important properties of the iterated stochastic integrals are obtained under the further assumption that $\mathscr{X}$ is a compensated-covariation stable family.

The main result of this paper is proven in Section 5 (see Theorem 5.8 below): If $\mathscr{X}$ is a compensated-covariation stable family of square integrable martingales such that $\langle X, Y \rangle$ is deterministic for all $X, Y \in \mathscr{X}$, and furthermore the system of monomials generated by $\mathscr{X}$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$, then $\mathscr{X}$ possesses the CRP.

Section 6 is devoted to applications of the general results established in the previous sections to Lévy processes. Let L be a Lévy process with Lévy measure ν and Gaussian part W^σ, where $\mathbb{E}[(W_t^\sigma)^2] = \sigma^2 t$, $\sigma^2 \geq 0$, and $\mu := \sigma^2 \delta_0 + \nu$, δ_0 being the Dirac measure in zero. With a deterministic function f in $L^2(\mu)$, we associate a square integrable martingale $X^{(f)}$ by setting $X_t^{(f)} := f(0)W_t^\sigma + \overline{M}(1_{[0,t]}f)$, where $\overline{M}(1_{[0,t]}f)$ denotes the stochastic integral of $1_{[0,t]}f$ with respect to the compensated Poisson random measure $\overline{M}$ associated with the jumps of L. We prove that for a system $\mathscr{T}$ in $L^2(\mu)$ the associated family $\mathscr{X}_{\mathscr{T}} := \{X^{(f)}, f \in \mathscr{T}\}$ possesses the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ *if and only if* $\mathscr{T}$ is total (i.e., its linear hull is dense) in $L^2(\mu)$ (cf. Theorem 6.6 below). A particularly important situation occurs when $\mathscr{T}$ is a complete orthogonal system: In this case $\mathscr{X}_{\mathscr{T}}$ is a family of orthogonal martingales and we shall see that this simplifies the CRP considerably. This is a major generalisation of Nualart and Schoutens [16] because we are able to construct a great variety of families of martingales possessing the CRP for *any* Lévy process, without any assumption on the Lévy measure, therefore also in the case if Teugels martingales cannot be introduced. Then, for a total system $\mathscr{T}$ in $L^2(\mu)$, we investigate the relationship between the iterated integrals generated by $\mathscr{X}$ and the multiple Itô integrals as well as between the CRP and the chaos expansion obtained in Itô [10].

Finally, as an illustration of the general results in Section 7 several applications will be given. We start with Gaussian families of continuous local martingales and pass on to independent families of Poisson processes. Then we proceed with examples for concrete families of martingales constructed from Lévy processes, including the family of Teugels martingales as a particular case.

2. Basic definitions and notation. In this section, we recall some basic definitions and notation from stochastic analysis needed in the following sections of the paper. By $(\Omega, \mathscr{F}, \mathbb{P})$ we denote a *complete* probability space and by $\mathbb{F}$ a filtration satisfying the *usual conditions*. We shall always consider real-valued stochastic processes on a finite time horizon $[0, T]$, $T > 0$.

With a càdlàg process $X = (X_t)_{t \in [0,T]}$, we associate the process $X_- = (X_{t-})_{t \in [0,T]}$ setting $X_{0-} := 0$ and $X_{t-} := \lim_{s \uparrow t} X_s$, $t > 0$. The process $\Delta X =$

$(\Delta X_t)_{t \in [0,T]}$, $\Delta X_t := X_t - X_{t-}$, $t \in [0, T]$, is called *the jump process* of X. Because of the definition of X_{0-}, we always have $\Delta X_0 = X_0$.

In the present paper, $\mathbb{F}$-martingales are always assumed to be càdlàg. For a martingale X, by X^c and X^d we denote the continuous and the purely discontinuous martingale part of X, respectively (cf. Jacod and Shiryaev [12], Theorem I.4.18). We recall that $X_0^c = X_0^d = 0$.

We say that a martingale X is *square integrable* if X_T^2 is integrable. Because of Doob's inequality, this is equivalent to require that $\sup_{t \in [0,T]} X_t^2$ is integrable. By $\mathscr{H}^2 = \mathscr{H}^2(\mathbb{F})$ we denote the set of square integrable martingales and by $\mathscr{H}_0^2$ the subspace of the elements of $\mathscr{H}^2$ starting at 0. For $X \in \mathscr{H}^2$, we put $\|X\|_{\mathscr{H}^2} := \|X_T\|_2$, where $\|\cdot\|_2$ denotes the $L^2(\mathbb{P}) := L^2(\Omega, \mathscr{F}, \mathbb{P})$-norm, and for $X, Y \in \mathscr{H}^2$ we put $(X, Y)_{\mathscr{H}^2} := \mathbb{E}[X_T Y_T]$ which defines a scalar product on $\mathscr{H}^2$. We can identify $(\mathscr{H}^2, \|\cdot\|_{\mathscr{H}^2})$ with the space $(L^2(\mathbb{P}), \|\cdot\|_2)$.

If X and Y belong to $\mathscr{H}^2$, we say that they are *orthogonal* and write $X \perp Y$ if their product XY is a martingale with $X_0 Y_0 = 0$ (cf. [11], Definition 2.10). If $\mathscr{X} \subseteq \mathscr{H}^2$, we say that $Y \in \mathscr{H}^2$ is orthogonal to $\mathscr{X}$ if $Y \perp X$ for every $X \in \mathscr{X}$. We stress that if $X, Y \in \mathscr{H}^2$ are orthogonal, then X_t and Y_t are orthogonal in $L^2(\mathbb{P})$, for every $t \in [0, T]$. However, the converse is, in general, not true (cf. Protter [18], page 181).

By $\mathscr{V}$ we denote the set of adapted càdlàg processes with paths of finite variation on $[0, T]$. If $A \in \mathscr{V}$, then we say that A is a *process of finite variation*. If $A \in \mathscr{V}$, by $\text{Var}(A) = (\text{Var}(A)_t)_{t \in [0,T]}$ we denote the variation process of A. We say that A is of integrable variation if $\text{Var}(A)_T$ is integrable.

For $A \in \mathscr{V}$ we shall make use of the Riemann–Stieltjes integral of a measurable process H with respect to A (cf. Jacod and Shiryaev [12], I, Section 3a): If $\int_0^t |H_s(\omega)| \, d\text{Var}(A)_s(\omega) < +\infty$, we use the notation $H \cdot A_t(\omega) := \int_0^t H_s(\omega) \, dA_s(\omega)$ and otherwise $H \cdot A_t(\omega) := +\infty$. We write $H \cdot A = (H \cdot A_t)_{t \in [0,T]}$ for the integral process. We stress that if $A \in \mathscr{V}$ and H is a measurable process, then $H \cdot A$ belongs to $\mathscr{V}$ if and only if $H \cdot A_t(\omega)$ is finite-valued, that is, $\int_0^t |H_s(\omega)| \, d\text{Var}(A)_s(\omega) < +\infty$, for every $t \in [0, T]$ and $\omega \in \Omega$.

If X and Y belong to $\mathscr{H}^2$ there exists a unique predictable process of integrable variation, denoted by $\langle X, Y \rangle$ and called the *predictable covariation* of X and Y, such that $\langle X, Y \rangle_0 = 0$ and $XY - \langle X, Y \rangle$ is a martingale (cf. Jacod and Shiryaev [12], Theorem I.4.2). Clearly $\mathbb{E}[X_T Y_T - X_0 Y_0] = \mathbb{E}[\langle X, Y \rangle_T]$ and $X, Y \in \mathscr{H}^2$ are orthogonal if and only if $\langle X, Y \rangle = 0$ and $X_0 Y_0 = 0$.

Let $(X, \mathbb{F})$ be a semimartingale with decomposition $X = X_0 + M + A$, where M (without loss of generality) is *locally* in $\mathscr{H}_0^2$, $A \in \mathscr{V}$ with $A_0 = 0$, and X_0 is $\mathscr{F}_0$-measurable. The continuous martingale part of X, denoted by X^c, is defined by $X^c := M^c$. Note that X^c does not depend on the semimartingale decomposition (cf. Jacod and Shiryaev [12], Proposition I.4.27). With two semimartingales X and Y, we associate the process $[X, Y]$, called covariation of X and Y, defining

$$(1) \qquad [X, Y]_t := \langle X^c, Y^c \rangle_t + \sum_{0 \le s \le t} \Delta X_s \Delta Y_s, \qquad t \in [0, T].$$

It is well known that the process $[X, Y]$ belongs to $\mathscr{V}$ (cf. Jacod [11], Theorem 2.30). We remark that the definition (1) of the covariation $[X, Y]$ implies that $[X, Y]_0 = X_0 Y_0$. If $X, Y \in \mathscr{H}^2$, then $[X, Y]$ is of integrable variation and $\langle X, Y \rangle$ is the compensator of $[X, Y]$, that is, $\langle X, Y \rangle$ is the unique predictable process of integrable variation starting at zero such that $[X, Y] - \langle X, Y \rangle$ is a martingale.

Now we are going to recall the stochastic integral with respect to a martingale $X \in \mathscr{H}^2$. The space of integrands for X is given by $\mathrm{L}^2(X) := \{H \text{ predictable} : \mathbb{E}[H^2 \cdot \langle X, X \rangle_T] < +\infty\}$. For $X \in \mathscr{H}^2$ and $H \in \mathrm{L}^2(X)$, by $H \cdot X$ we denote the stochastic integral of H with respect to X. The stochastic integral of H with respect to X is characterized as it follows: Let $Z \in \mathscr{H}^2$. Then $Z = H \cdot X$ if and only if $Z_0 = H_0 X_0$ and $\langle Z, Y \rangle = H \cdot \langle X, Y \rangle$, for every $Y \in \mathscr{H}^2$. We stress that, if $X, Y \in \mathscr{H}^2$ are orthogonal martingales, then also $H \cdot X$ and $K \cdot Y$ are orthogonal martingales of $\mathscr{H}^2$, for all $H \in \mathrm{L}^2(X)$ and $K \in \mathrm{L}^2(Y)$. The notation $H \cdot X$ is not ambiguous with the one introduced for the stochastic integral with respect to a process of finite variation: If $X \in \mathscr{H}^2 \cap \mathscr{V}$, then $H \cdot X$ coincides with the Riemann–Stieltjes integral (cf. Jacod [11], Remark 2.47).

For a subset $\mathscr{K}$ of a Banach space $(\mathscr{H}, \|\cdot\|)$, by $\mathrm{Span}(\mathscr{K})$ we denote the linear hull of $\mathscr{K}$ and by $\mathrm{cl}(\mathscr{K})_{\mathscr{H}}$ the closure of $\mathscr{K}$ in $\mathscr{H}$.

3. Iterated integrals and chaotic representation property. Let $\mathscr{X} \subseteq \mathscr{H}^2(\mathbb{F})$ be a family of $\mathbb{F}$-martingales. For notational convenience, we represent $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ in parametric form where Λ is an associated index set. In this paper, from now on we shall always assume that $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic, $\alpha, \beta \in \Lambda$. For such a family, we are going to introduce the *iterated integrals*.

Let F_0 be a bounded $\mathcal{F}_0$-measurable function and $F_1, \ldots, F_n$ bounded measurable functions on $([0, T], \mathscr{B}([0, T]))$. We denote by $F := F_0 \otimes \cdots \otimes F_n = \bigotimes_{k=0}^{n} F_k$ the *tensor product* of $F_0, \ldots, F_n$ defined on $\Omega \times [0, T]^n$ and say that F is an *elementary function* of order n.

DEFINITION 3.1. Let $\alpha_1, \ldots, \alpha_n \in \Lambda$ be given and $F = F_0 \otimes \cdots \otimes F_n$ be an elementary function of order n. The *elementary* iterated integral $J_0(F_0)$ of order zero of F_0 and $J_n^{(\alpha_1, \ldots, \alpha_n)}(F)$ of order n of F ($n \geq 1$) with respect to the martingales $(X^{(\alpha_1)}, \ldots, X^{(\alpha_n)})$ is defined inductively as follows: If $n = 0$, then $J_0(F_0) := (J_0(F_0)_t)_{t \in [0, T]}$ is the process defined by $J_0(F_0)_t = F_0$ for $t \in [0, T]$ and, for all $1 \leq m \leq n$,

$$
(2) \quad \begin{aligned} &J_m^{(\alpha_1, \ldots, \alpha_m)}(F_0 \otimes \cdots \otimes F_m)_t \\ &:= \int_0^t J_{m-1}^{(\alpha_1, \ldots, \alpha_{m-1})}(F_0 \otimes \cdots \otimes F_{m-1})_{u-} F_m(u) \, \mathrm{d}X_u^{(\alpha_m)}, \qquad t \in [0, T]. \end{aligned}
$$

In the following lemma, we establish some important properties of the iterated integrals.

LEMMA 3.2. *Let $n \geq 0$, $\alpha_1, \ldots, \alpha_n \in \Lambda$ and $F = \bigotimes_{k=0}^n F_k$ be an elementary function of order n.*

(i) *The elementary iterated integral $J_n^{(\alpha_1, \ldots, \alpha_n)}(F)$ belongs to $\mathscr{H}^2$ for $n = 0$ and to $\mathscr{H}_0^2$ for $n \geq 1$.*

(ii) *Let moreover be $m \geq 0$, $\beta_1, \ldots, \beta_m \in \Lambda$ and $G = \bigotimes_{k=0}^m G_k$ an elementary function of order m. Then we have $\mathbb{E}[J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t J_m^{(\beta_1, \ldots, \beta_m)}(G)_t | \mathscr{F}_0] = 0$, $t \in [0, T]$, if $n \neq m$, while, if $m = n$,*

$$\mathbb{E}\big[J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t J_n^{(\beta_1, \ldots, \beta_n)}(G)_t | \mathscr{F}_0\big]$$

$$(3) \qquad = F_0 G_0 \int_0^t \int_0^{t_n-} \cdots \int_0^{t_2-} F_1(t_1) G_1(t_1) \cdots F_n(t_n) G_n(t_n) \, d\langle X^{(\alpha_1)}, X^{(\beta_1)}\rangle_{t_1}$$

$$\cdots d\langle X^{(\alpha_n)}, X^{(\beta_n)}\rangle_{t_n}.$$

PROOF. We start proving (i). If $n = 0$, $J_0(F)$ is obviously a (constant) bounded martingale and hence it belongs to $\mathscr{H}^2$. If $n = 1$, the statement follows from the properties of the stochastic integral because $(J_0(F_0)_{t-})_{t \in [0,T]} \in L^2(X^{(\alpha_1)})$ and $J_1^{(\alpha_1)}(F_0 \otimes F_1)_0 = J_0(F_0)_{0-} F_1(0) X_0^{(\alpha_1)} = 0$ in view of the setting $X_{0-} = 0$ for any càdlàg process X. Now we assume that the claim holds for n and we verify it for $n + 1$. From (2) and the definition of the stochastic integral, it is shown as for $n = 1$ that $J_{n+1}^{(\alpha_1, \ldots, \alpha_{n+1})}(F)_0 = 0$. To see that $J_{n+1}^{(\alpha_1, \ldots, \alpha_{n+1})}(F)$ is a square integrable martingale, we only need to verify that the integrand on the right-hand side of (2) for $m = n + 1$ is in $L^2(X^{(\alpha_{n+1})})$:

$$\mathbb{E}\bigg[\int_0^T \big(J_n^{(\alpha_1, \ldots, \alpha_n)}(F_0 \otimes \cdots \otimes F_n)_{u-} F_{n+1}(u)\big)^2 \, d\langle X^{(\alpha_{n+1})}, X^{(\alpha_{n+1})}\rangle_u\bigg]$$

$$\leq \mathbb{E}\big[\big(J_n^{(\alpha_1, \ldots, \alpha_n)}(F_0 \otimes \cdots \otimes F_n)_T\big)^2\big] \int_0^T \big(F_{n+1}(u)\big)^2 \, d\langle X^{(\alpha_{n+1})}, X^{(\alpha_{n+1})}\rangle_u$$

$$< +\infty,$$

where we used that the predictable covariation $\langle X^{(\alpha_{n+1})}, X^{(\alpha_{n+1})}\rangle$ and F_{n+1} are deterministic and the induction hypothesis. Now we show (ii). As a first step, we assume that $m = n$ and we deduce the result by induction. If $n = 0$, there is nothing to prove. Now we assume (3) for n and verify it for $n + 1$. Because from (i) follows that the elementary iterated integrals are in $\mathscr{H}_0^2$, this is an immediate consequence of the relation

$$\mathbb{E}[J_{n+1}^{(\alpha_1, \ldots, \alpha_{n+1})}(F)_t J_{n+1}^{(\beta_1, \ldots, \beta_{n+1})}(G)_t | \mathscr{F}_0]$$

$$= \mathbb{E}[\langle J_{n+1}^{(\alpha_1, \ldots, \alpha_{n+1})}(F), J_{n+1}^{(\beta_1, \ldots, \beta_{n+1})}(G)\rangle_t | \mathscr{F}_0]$$

$$= \int_0^t \mathbb{E}\bigg[J_n^{(\alpha_1, \ldots, \alpha_n)}\bigg(\bigotimes_{k=0}^n F_k\bigg)_{u-} J_n^{(\beta_1, \ldots, \beta_n)}\bigg(\bigotimes_{k=0}^n G_k\bigg)_{u-} \bigg| \mathscr{F}_0\bigg] F_{n+1}(u) G_{n+1}(u)$$

$$\times d\langle X^{(\alpha_{n+1})}, X^{(\beta_{n+1})}\rangle_u$$

(following from the properties of the predictable covariation of stochastic inte-
grals and the definition of the iterated integrals) and the induction hypothesis.
This completes the proof of (ii) for the case $n = m$. Now we consider the case
$n \neq m$, say $n = m + p$, $p > 0$. We proceed by induction on m. If $m = 0$, then
(ii) follows from (i). It remains to prove the statement for $m + 1$ under the as-
sumption that it is fulfilled for m. To this end, we notice that if $X, Y \in \mathscr{H}^2$ are
such that $\mathbb{E}[X_t Y_t | \mathcal{F}_0] = 0$ for every $t \in [0, T]$, then $\mathbb{E}[X_{t-} Y_{t-} | \mathcal{F}_0] = 0$ for ev-
ery $t \in [0, T]$. Indeed, as a consequence of Doob's inequality, $\sup_{t \in [0, T]} |X_t Y_t| \leq$
$1/2 \sup_{t \in [0, T]} X_t^2 + 1/2 \sup_{t \in [0, T]} Y_t^2$ is integrable and the conclusion follows from
Lebesgue's theorem on dominated convergence. To complete the proof of the in-
duction step, now we have only to recall (i) and that $\mathbb{E}[X_t Y_t | \mathcal{F}_0] = \mathbb{E}[\langle X, Y \rangle_t | \mathcal{F}_0]$
for every $X, Y \in \mathscr{H}_0^2$ and $t \in [0, T]$ and to apply the induction hypothesis. $\square$

For $t \in [0, T]$ and $n \geq 1$, we introduce the sets

$$(4) \quad M_t^{(n)} := \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq t\}, \qquad \overline{M}_t^{(n)} := \Omega \times M_t^{(n)}.$$

For every $\alpha \in \Lambda$, we also introduce the finite measure $m^{(\alpha)}$ on $([0, T], \mathscr{B}([0, T]))$
generated by the right-continuous increasing function $\langle X^{(\alpha)}, X^{(\alpha)} \rangle$. For any
$\alpha_1, \ldots, \alpha_n \in \Lambda$, we define the product measure $m_{\mathbb{P}}^{(\alpha_1, \ldots, \alpha_n)} := \mathbb{P} \otimes \bigotimes_{k=1}^n m^{(\alpha_k)}$
on $(\Omega \times [0, T]^n, \mathcal{F}_0 \times \mathscr{B}([0, T]^n))$.

For $n \geq 1$, we denote by $\mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$ the linear subspace of $L^2(\overline{M}_t^{(n)}, m_{\mathbb{P}}^{(\alpha_1, \ldots, \alpha_n)})$
generated by the elementary functions F of order n restricted to $\overline{M}_t^{(n)}$. Apply-
ing the expectation to (3), from the resulting isometry relation it easily follows
that $J_n^{(\alpha_1, \ldots, \alpha_n)}(\cdot)_t$ can be uniquely extended *linearly* to $\mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$. Clearly, rela-
tion (3) extends to all $F \in \mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$ and $G \in \mathscr{E}_{n,t}^{(\beta_1, \ldots, \beta_n)}$. In particular, the map-
ping $J_n^{(\alpha_1, \ldots, \alpha_n)}(\cdot)_t$ linearly extended to $\mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$ is a linear and isometric map-
ping from $L^2(\overline{M}_t^{(n)}, m_{\mathbb{P}}^{(\alpha_1, \ldots, \alpha_n)})$ into $L^2(\mathbb{P})$. Setting $\alpha_k = \beta_k$, $k = 1, \ldots, n$, and
$F = G \in \mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$ in the extended isometry relation (3) and then taking the ex-
pectation, yields

$$(5) \qquad \qquad \| J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t \|_{L^2(\mathbb{P})}^2 = \| F \|_{L^2(\overline{M}_t^{(n)}, m_{\mathbb{P}}^{(\alpha_1, \ldots, \alpha_n)})}^2.$$

On the other side, the linear space $\mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$ is dense in $L^2(\overline{M}_t^{(n)}, m_{\mathbb{P}}^{(\alpha_1, \ldots, \alpha_n)})$
and therefore, the linear mapping $F \mapsto J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t$, $F \in \mathscr{E}_{n,t}^{(\alpha_1, \ldots, \alpha_n)}$, can
uniquely be extended to an isometry on the space $L^2(\overline{M}_t^{(n)}, m_{\mathbb{P}}^{(\alpha_1, \ldots, \alpha_n)})$ with val-
ues in $L^2(\mathbb{P})$, for every $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in \Lambda$. We denote this extension by
$J_n^{(\alpha_1, \ldots, \alpha_n)}(\cdot)_t$, $t \in [0, T]$. If $n = 0$, then the iterated integrals $J_0(F_0)_t$ of order
zero evaluated at time $t \in [0, T]$ are defined just as the identity $J_0(F_0)_t = F_0$ for
$F_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$, the closure of the space of elementary functions of order zero.

DEFINITION 3.3. Let $n \geq 1$, $\alpha_1, \ldots, \alpha_n \in \Lambda$ and $F \in L^2(\overline{M}_t^{(n)}, m_\mathbb{P}^{(\alpha_1, \ldots, \alpha_n)})$. We call the stochastic process $J_n^{(\alpha_1, \ldots, \alpha_n)}(F) := (J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t)_{t \in [0, T]}$ the n-fold iterated stochastic integral of F with respect to $(X^{(\alpha_1)}, X^{(\alpha_2)}, \ldots, X^{(\alpha_n)})$. If $n = 0$, we say that the constant square integrable martingale $J_0(F_0) = (J_0(F_0)_t)_{t \in [0, T]}$ is the 0-fold iterated integral of $F_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$.

Using the linearity and isometry of the iterated integral, we obtain the following straightforward extension of Lemma 3.2 to arbitrary $F \in L^2(\overline{M}_t^{(n)}, m_\mathbb{P}^{(\alpha_1, \ldots, \alpha_n)})$ and $G \in L^2(\overline{M}_t^{(m)}, m_\mathbb{P}^{(\beta_1, \ldots, \beta_m)})$.

PROPOSITION 3.4. Let $n \geq 1$, $\alpha_1, \ldots, \alpha_n \in \Lambda$ and $F \in L^2(\overline{M}_t^{(n)}, m_\mathbb{P}^{(\alpha_1, \ldots, \alpha_n)})$.

(i) The iterated integral $J_n^{(\alpha_1, \ldots, \alpha_n)}(F)$ belongs to $\mathcal{H}_0^2$.

(ii) Let moreover $m \geq 1$, $\beta_1, \ldots, \beta_m \in \Lambda$ and $G \in L^2(\overline{M}_t^{(m)}, m_\mathbb{P}^{(\beta_1, \ldots, \beta_m)})$. Then, for every $t \in [0, T]$, we have: If $n \neq m$, then
$$\mathbb{E}[J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t J_m^{(\beta_1, \ldots, \beta_m)}(G)_t | \mathcal{F}_0] = 0,$$
while, if $m = n$,
$$\mathbb{E}[J_n^{(\alpha_1, \ldots, \alpha_n)}(F)_t J_n^{(\beta_1, \ldots, \beta_n)}(G)_t | \mathcal{F}_0]$$
$$(6) \qquad = \mathbb{E}\Bigg[\int_0^t \int_0^{t_n-} \cdots \int_0^{t_2-} F(t_1, \ldots, t_n) G(t_1, \ldots, t_n) \, \mathrm{d}\langle X^{(\alpha_1)}, X^{(\beta_1)}\rangle_{t_1}$$
$$\cdots \mathrm{d}\langle X^{(\alpha_n)}, X^{(\beta_n)}\rangle_{t_n} \Big| \mathcal{F}_0\Bigg].$$

In the following definition, we introduce some spaces of iterated stochastic integrals.

DEFINITION 3.5. (i) Let $\mathcal{J}_{0,0}$ be the space of bounded $\mathcal{F}_0$-measurable martingales (the 0-fold elementary iterated integrals) and $\mathcal{J}_0$ the space of square integrable $\mathcal{F}_0$-measurable martingales (the 0-fold iterated integrals). To simplify the notation, we shall identify martingales $J_0(F_0)$ of $\mathcal{J}_{0,0}$ (resp., $\mathcal{J}_0$) with the bounded (resp., square integrable) $\mathcal{F}_0$-measurable random variable F_0.

(ii) Let $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in \Lambda$. By $\mathcal{J}_n^{(\alpha_1, \ldots, \alpha_n)}$ (resp., $\mathcal{J}_{n,0}^{(\alpha_1, \ldots, \alpha_n)}$) we denote the space of n-fold (resp., the linear hull of elementary) iterated stochastic integrals relative to the square integrable martingales $(X^{(\alpha_1)}, X^{(\alpha_2)}, \ldots, X^{(\alpha_n)})$.

(iii) For all $n \geq 1$, we introduce
$$\mathcal{J}_{n,0} := \mathrm{Span}\Bigg(\bigcup_{(\alpha_1, \ldots, \alpha_n) \in \Lambda^n} \mathcal{J}_{n,0}^{(\alpha_1, \ldots, \alpha_n)}\Bigg),$$
$$(7)$$
$$\mathcal{J}_n := \mathrm{cl}\Bigg(\mathrm{Span}\Bigg(\bigcup_{(\alpha_1, \ldots, \alpha_n) \in \Lambda^n} \mathcal{J}_n^{(\alpha_1, \ldots, \alpha_n)}\Bigg)\Bigg)_{\mathcal{H}^2},$$

and then we define

$$(8) \qquad \mathscr{J}_e := \mathrm{Span}\left(\bigcup_{n \geq 0} \mathscr{J}_{n,0}\right), \qquad \mathscr{J} := \mathrm{cl}\left(\mathrm{Span}\left(\bigcup_{n \geq 0} \mathscr{J}_n\right)\right)_{\mathscr{H}^2}.$$

We call $\mathscr{J}$ the space of iterated integrals *generated by* $\mathscr{X}$.

   (iv) By $\mathscr{J}_T$ we denote the linear subspace of $L^2(\mathbb{P})$ of terminal variables of iterated integrals from $\mathscr{J}$. The linear spaces $\mathscr{J}_{n,T}^{(\alpha_1,\ldots,\alpha_n)}$ and $\mathscr{J}_{n,T}$ of random variables in $L^2(\mathbb{P})$ are introduced analogously from the spaces of processes $\mathscr{J}_n^{(\alpha_1,\ldots,\alpha_n)}$ and $\mathscr{J}_n$, respectively, $n \geq 0$.

   Now we state the definition of the *chaotic representation property* on the space $L^2(\mathbb{P})$.

DEFINITION 3.6.   We say that $\mathscr{X} = \{X^{(\alpha)}, \alpha \in \Lambda\}$ possesses the *chaotic representation property* (CRP) on the Hilbert space $L^2(\mathbb{P}) = L^2(\Omega, \mathscr{F}, \mathbb{P})$ if the linear space $\mathscr{J}_T$ [cf. Definition 3.5(iii)] is equal to $L^2(\mathbb{P})$.

   We stress that, because the spaces $(L^2(\mathbb{P}), \|\cdot\|_2)$ and $(\mathscr{H}^2, \|\cdot\|_{\mathscr{H}^2})$ can be identified, we can equivalently claim that $\mathscr{X}$ possesses the CRP if $\mathscr{J} = \mathscr{H}^2$.

   Proposition 3.4(ii) yields that $\mathscr{J}_{n,T}$ $(n \geq 1)$ [resp., $\mathscr{J}_n$ $(n \geq 1)$] are pairwise orthogonal closed subspaces of $L^2(\mathbb{P})$ (resp., $\mathscr{H}^2$). Furthermore, it can easily be checked that $\mathscr{J}_{0,T}$ (resp., $\mathscr{J}_0$) is orthogonal to $\mathscr{J}_{n,T}$ (resp., $\mathscr{J}_n$) for all $n \geq 1$. This immediately leads to the following equivalent description of the CRP.

PROPOSITION 3.7.   (i) *It holds* $\mathscr{J}_T = \bigoplus_{n=0}^{\infty} \mathscr{J}_{n,T}$ *(resp.,* $\mathscr{J} = \bigoplus_{n=0}^{\infty} \mathscr{J}_n$*).*
   (ii) *The family* $\mathscr{X}$ *possesses the CRP if and only if* $L^2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathscr{J}_{n,T}$ *(resp.,* $\mathscr{H}^2 = \bigoplus_{n=0}^{\infty} \mathscr{J}_n$*).*

   Now we shortly discuss the relation between the CRP and the *predictable representation property* (PRP). We recall that a closed linear subspace $\mathscr{H}$ of $\mathscr{H}^2$ is a stable subspace of $\mathscr{H}^2$ if $1_A X^\tau$ belongs to $\mathscr{H}$, for every stopping time τ, $A \in \mathscr{F}_0$ and $X \in \mathscr{H}$. Let $\mathscr{X}$ be a subfamily of $\mathscr{H}^2$. The stable subspace generated by $\mathscr{X}$ is denoted by $\mathscr{L}^2(\mathscr{X})$ and is defined as the smallest stable subspace of $\mathscr{H}^2$ containing $\mathscr{X}$. Note that $\mathscr{L}^2(\mathscr{X})$ is the smallest stable subspace of $\mathscr{H}^2$ containing the set $\{H \cdot X, H \in \mathrm{L}^2(X), X \in \mathscr{X}\}$. Furthermore, we have $\mathscr{L}^2(\{1\}) = \{X \in \mathscr{H}^2, X_t \equiv X_0\} = \mathscr{J}_0$. For more details on the theory of stable subspaces of martingales, cf. Jacod [11], Chapter IV. We say that $\mathscr{X}$ possesses the PRP with respect to $\mathbb{F}$ if $\mathscr{L}^2(\mathscr{X} \cup \{1\}) = \mathscr{H}^2(\mathbb{F})$. We now assume that $\mathscr{X}$ possesses the CRP. Clearly the inclusion $\mathscr{J}_e \subseteq \mathscr{L}^2(\mathscr{X} \cup \{1\})$ holds. Indeed, an elementary iterated integral of order $n \geq 1$ can always be regarded as a stochastic integral with respect to an element of $\mathscr{X}$ (cf. Definition 3.1) and $\mathscr{J}_{0,0} \subseteq \mathscr{L}^2(\{1\})$. Using that $\mathscr{L}^2(\mathscr{X} \cup \{1\})$ is closed in $\mathscr{H}^2$ we obtain $\mathscr{H}^2(\mathbb{F}) = \mathscr{J} = \mathrm{cl}(\mathscr{J}_e)_{\mathscr{H}^2} \subseteq \mathscr{L}^2(\mathscr{X} \cup \{1\})$. Hence,

$\mathcal{L}^2(\mathcal{X} \cup \{1\}) = \mathcal{H}^2(\mathbb{F})$. Thus, we have shown that, for every family $\mathcal{X} \subseteq \mathcal{H}^2$ for which $\langle X, Y \rangle$ is deterministic, $X, Y \in \mathcal{X}$, the CRP implies the PRP.

The following technical lemma will be used to prove Theorem 3.9 below.

LEMMA 3.8. *Let* $X^{(\alpha_1)}, \ldots, X^{(\alpha_m)}; X^{(\beta_1)}, \ldots, X^{(\beta_m)}; X^{(\alpha_1^n)}, \ldots, X^{(\alpha_m^n)} \in \mathcal{X}$ *be such that for every* $k = 1, \ldots, m$, $X^{(\alpha_k^n)} \longrightarrow X^{(\alpha_k)}$ *in* $\mathcal{H}^2$ *as* $n \to +\infty$. *Then for any elementary function* $F = 1 \otimes (\bigotimes_{k=1}^m F_k)$ *of order* m *we have that*

$$
\begin{aligned}
&\left| \int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \left(\bigotimes_{k=1}^m F_k \right)(t_1, \ldots, t_m) \, \mathrm{d}\langle X^{(\alpha_1^n)}, X^{(\alpha_1^n)} \rangle_{t_1} \cdots \mathrm{d}\langle X^{(\alpha_m^n)}, X^{(\alpha_m^n)} \rangle_{t_m} \right. \\
(9) \quad &\left. - \int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \left(\bigotimes_{k=1}^m F_k \right)(t_1, \ldots, t_m) \, \mathrm{d}\langle X^{(\alpha_1)}, X^{(\alpha_1)} \rangle_{t_1} \cdots \mathrm{d}\langle X^{(\alpha_m)}, X^{(\alpha_m)} \rangle_{t_m} \right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \left(\bigotimes_{k=1}^m F_k \right)(t_1, \ldots, t_m) \, \mathrm{d}\langle X^{(\alpha_1^n)}, X^{(\beta_1)} \rangle_{t_1} \cdots \mathrm{d}\langle X^{(\alpha_m^n)}, X^{(\beta_m)} \rangle_{t_m} \right. \\
(10) \quad &\left. - \int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \left(\bigotimes_{k=1}^m F_k \right)(t_1, \ldots, t_m) \, \mathrm{d}\langle X^{(\alpha_1)}, X^{(\beta_1)} \rangle_{t_1} \cdots \mathrm{d}\langle X^{(\alpha_m)}, X^{(\beta_m)} \rangle_{t_m} \right|
\end{aligned}
$$

converge to zero uniformly in $t \in [0, T]$ *as* $n \to +\infty$.

PROOF. We verify only (9) because (10) easily follows from (9) using the polarization formula $\langle X, Y \rangle = \frac{1}{4}(\langle X + Y, X + Y \rangle - \langle X - Y, X - Y \rangle)$, $X, Y \in \mathcal{H}^2$, and the linearity of the Riemann–Stieltjes integral with respect to the integrator. Now we start with the proof of (9). Let $F = 1 \otimes F_1 \otimes \cdots \otimes F_m$ be such that $|F_k| \le c$ for $k = 1, \ldots, m$ with $c > 0$, and let $t \in [0, T]$. We introduce the abbreviation

$$
\begin{aligned}
&H_{t,m}^{(\alpha_1, \ldots, \alpha_m)} \\
(11) \quad &:= \int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \left(\bigotimes_{k=1}^m F_k \right)(t_1, \ldots, t_m) \, \mathrm{d}\langle X^{(\alpha_1)}, X^{(\alpha_1)} \rangle_{t_1} \cdots \mathrm{d}\langle X^{(\alpha_m)}, X^{(\alpha_m)} \rangle_{t_m}
\end{aligned}
$$

with the convention $H_{t,0} = 1$. Note that

$$
H_{t,m}^{(\alpha_1, \ldots, \alpha_m)} = \int_0^t F_m(u) H_{u,m-1}^{(\alpha_1, \ldots, \alpha_{m-1})} \, \mathrm{d}\langle X^{(\alpha_m)}, X^{(\alpha_m)} \rangle_u.
$$

Rewriting the left-hand side of (9) using (11) and observing that for all bounded measurable processes H, K and for $X, Y \in \mathcal{H}^2$ the equality

$$
H \cdot \langle X, X \rangle - K \cdot \langle Y, Y \rangle = (H - K) \cdot \langle X, Y \rangle + H \cdot \langle X, X - Y \rangle - K \cdot \langle Y, Y - X \rangle
$$

holds and that $|H \cdot A| \leq |H| \cdot \mathrm{Var}(A)$ for every $A \in \mathcal{V}$ and measurable process H, we get

$$
\sup_{t \in [0,T]} \left| \int_0^t F_m(t_m) H_{t_m-,m-1}^{(\alpha_1^n,\ldots,\alpha_{m-1}^n)} \, \mathrm{d}\langle X^{(\alpha_m^n)}, X^{(\alpha_m^n)} \rangle_{t_m} \right.
$$

$$
\left. - \int_0^t F_m(t_m) H_{t_m-,m-1}^{(\alpha_1,\ldots,\alpha_{m-1})} \, \mathrm{d}\langle X^{(\alpha_m)}, X^{(\alpha_m)} \rangle_{t_m} \right|
$$

$$
\leq c \sup_{t \in [0,T]} \left| H_{t,m-1}^{(\alpha_1^n,\ldots,\alpha_{m-1}^n)} - H_{t,m-1}^{(\alpha_1,\ldots,\alpha_{m-1})} \right| \left\| X^{(\alpha_m^n)} \right\|_{\mathcal{H}^2} \left\| X^{(\alpha_m)} \right\|_{\mathcal{H}^2}
$$

$$
+ c \sup_{t \in [0,T]} \left| H_{t,m-1}^{(\alpha_1^n,\ldots,\alpha_{m-1}^n)} \right| \left\| X^{(\alpha_m^n)} \right\|_{\mathcal{H}^2} \left\| X^{(\alpha_m^n)} - X^{(\alpha_m)} \right\|_{\mathcal{H}^2}
$$

$$
+ c \sup_{t \in [0,T]} \left| H_{t,m-1}^{(\alpha_1,\ldots,\alpha_{m-1})} \right| \left\| X^{(\alpha_m)} \right\|_{\mathcal{H}^2} \left\| X^{(\alpha_m)} - X^{(\alpha_m^n)} \right\|_{\mathcal{H}^2},
$$

where in the last passage we used that F_m is bounded, Kunita–Watanabe's inequality in the form of Meyer [15], Corollary II.22, the relation $\mathbb{E}[X_T^2] \geq \mathbb{E}[\langle X, X \rangle_T]$ for $X \in \mathcal{H}^2$, and the assumption that all the predictable covariations are deterministic. Because for $m = 1$ the previous inequality becomes

$$
\sup_{t \in [0,T]} \left| H_{t,1}^{(\alpha_1^n)} - H_{t,1}^{(\alpha_1)} \right|
$$

$$
\leq c \left\| X^{(\alpha_1^n)} - X^{(\alpha_1)} \right\|_{\mathcal{H}^2} \left\| X^{(\alpha_1^n)} \right\|_{\mathcal{H}^2} + c \left\| X^{(\alpha_1)} \right\|_{\mathcal{H}^2} \left\| X^{(\alpha_1^n)} - X^{(\alpha_1)} \right\|_{\mathcal{H}^2},
$$

and the right-hand side converges to zero as $n \to +\infty$, the uniform convergence of (9) to zero follows by induction. $\square$

Let $\mathcal{Z}$ be a subfamily of $\mathcal{X}$. We denote by $\mathcal{J}^{\mathcal{X}}$ and $\mathcal{J}^{\mathcal{Z}}$ the spaces of iterated integrals generated by $\mathcal{X}$ and $\mathcal{Z}$, respectively.

THEOREM 3.9. *If $\mathcal{X} \subseteq \mathrm{cl}(\mathrm{Span}(\mathcal{Z}))_{\mathcal{H}^2}$, then $\mathcal{J}^{\mathcal{Z}} = \mathcal{J}^{\mathcal{X}}$.*

PROOF. Because of $\mathcal{J}^{\mathcal{Z}} = \mathcal{J}^{\mathrm{Span}(\mathcal{Z})}$, without loss of generality we can assume that $\mathcal{Z}$ is a linear space. Then, for all $X^{(\alpha_1)}, \ldots, X^{(\alpha_m)} \in \mathcal{X}$, there exist $X^{(\alpha_1^n)}, \ldots, X^{(\alpha_m^n)} \in \mathcal{Z}$ such that $X^{(\alpha_k^n)} \longrightarrow X^{(\alpha_k)}$ in $\mathcal{H}^2$ as $n \to +\infty$, $k = 1, \ldots, m$. Let now $F = F_0 \otimes \cdots \otimes F_m$ be an elementary function. We show that $J_m^{(\alpha_1,\ldots,\alpha_m)}(F)$ belongs to $\mathcal{J}^{\mathcal{Z}}$. Let $J_m^{(\alpha_1^n,\ldots,\alpha_m^n)}(F)$ be the elementary iterated integral of F with respect to $(X^{(\alpha_1^n)}, \ldots, X^{(\alpha_m^n)})$. Because of (3), for all $t \in [0, T]$ we have

$$
\mathbb{E}\left[\left(J_m^{(\alpha_1,\ldots,\alpha_m)}(F)_t - J_m^{(\alpha_1^n,\ldots,\alpha_m^n)}(F)_t \right)^2 \right]
$$

$$
= \mathbb{E}\left[\left(J_m^{(\alpha_1,\ldots,\alpha_m)}(F)_t \right)^2 \right]
$$

$$+ \mathbb{E}\big[\big(J_m^{(\alpha_1^n,\dots,\alpha_m^n)}(F)_t\big)^2\big] - 2\,\mathbb{E}\big[J_m^{(\alpha_1,\dots,\alpha_m)}(F)_t J_m^{(\alpha_1^n,\dots,\alpha_m^n)}(F)_t\big]$$

$$= \mathbb{E}[F_0^2]\Bigg(\int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \Bigg(\bigotimes_{k=1}^m F_k\Bigg)^2 (t_1,\dots,t_m)\,\mathrm{d}\langle X^{(\alpha_1)}, X^{(\alpha_1)}\rangle_u$$

$$\cdots \mathrm{d}\langle X^{(\alpha_m)}, X^{(\alpha_m)}\rangle_u$$

$$+ \int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \Bigg(\bigotimes_{k=1}^m F_k\Bigg)^2 (t_1,\dots,t_m)\,\mathrm{d}\langle X^{(\alpha_1^n)}, X^{(\alpha_1^n)}\rangle_u$$

$$\cdots \mathrm{d}\langle X^{(\alpha_m^n)}, X^{(\alpha_m^n)}\rangle_u$$

$$- 2\int_0^t \int_0^{t_m-} \cdots \int_0^{t_2-} \Bigg(\bigotimes_{k=1}^m F_k\Bigg)^2 (t_1,\dots,t_m)\,\mathrm{d}\langle X^{(\alpha_1^n)}, X^{(\alpha_1)}\rangle_u$$

$$\cdots \mathrm{d}\langle X^{(\alpha_m^n)}, X^{(\alpha_m)}\rangle_u\Bigg),$$

which converges to zero because of Lemma 3.8. Since the space $\mathscr{J}^{\mathscr{Z}}$ is closed in $\mathscr{H}^2$, we conclude that $J_m^{(\alpha_1,\dots,\alpha_m)}(F) \in \mathscr{J}^{\mathscr{Z}}$. The result can be easily extended to arbitrary functions F in the Hilbert space $L^2(\overline{M}_T^{(m)}, m_{\mathbb{P}}^{(\alpha_1,\dots,\alpha_m)})$ by linearity and isometry of iterated integrals. Finally, using the definition of $\mathscr{J}^{\mathscr{X}}$, we get $\mathscr{J}^{\mathscr{X}} \subseteq \mathscr{J}^{\mathscr{Z}}$. The converse inclusion is clear because $\mathscr{Z} \subseteq \mathscr{X}$. $\quad\square$

Now we consider the case in which the martingales coming into play are orthogonal. We introduce the following notation: For every $\alpha_1,\dots,\alpha_n; \beta_1,\dots,\beta_m \in \Lambda$ we write $(\alpha_1,\dots,\alpha_n) \neq (\beta_1,\dots,\beta_m)$ if $n \neq m$ or if $n = m$ there exists $1 \leq \ell \leq n = m$ such that $\alpha_\ell \neq \beta_\ell$. The following proposition can be immediately deduced from Proposition 3.4(ii).

PROPOSITION 3.10. *Let $n \geq 1$ and $(\alpha_1,\dots,\alpha_n), (\beta_1,\dots,\beta_n) \in \Lambda^n$. Suppose that there exists some $i \in \{1,\dots,n\}$ such that the martingales $X^{(\alpha_i)}, X^{(\beta_i)}$ are orthogonal, that is, $X_0^{(\alpha_i)} X_0^{(\beta_i)} = 0$ and $\langle X^{(\alpha_i)}, X^{(\beta_i)}\rangle = 0$. Then $J_n^{(\alpha_1,\dots,\alpha_n)}(F)_t$ and $J_n^{(\beta_1,\dots,\beta_n)}(G)_t$ are orthogonal in $L^2(\mathbb{P})$, $t \in [0,T]$, for every F from the space $L^2(\overline{M}_T^{(n)}, m_{\mathbb{P}}^{(\alpha_1,\dots,\alpha_n)})$ and G from $L^2(\overline{M}_T^{(n)}, m_{\mathbb{P}}^{(\beta_1,\dots,\beta_n)})$.*

The following theorem will play an important role in the sequel.

THEOREM 3.11. *Let $\mathscr{X} := \{X^{(n)}, n \geq 1\} \subseteq \mathscr{H}^2$ be a family consisting of countably many mutually orthogonal martingales such that $\langle X^{(n)}, X^{(n)}\rangle$ is deter-*

ministic for all $n \geq 1$. Then the following identities hold:

$$\mathscr{I} = \mathscr{I}_0 \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\ldots,j_n) \in \mathbb{N}^n} \mathscr{I}_n^{(j_1,\ldots,j_n)},$$

(12)

$$\mathscr{I}_T = \mathscr{I}_{0,T} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\ldots,j_n) \in \mathbb{N}^n} \mathscr{I}_{n,T}^{(j_1,\ldots,j_n)}.$$

PROOF. We only verify the second relation. The space $\mathscr{I}_{n,T}^{(j_1,\ldots,j_n)}$ is closed in $L^2(\mathbb{P})$ for every fixed $(j_1,\ldots,j_n)$ and $n \geq 1$. Because of Proposition 3.10 and the mutual orthogonality of the martingales in $\mathscr{X}$, if $(j_1,\ldots,j_n) \neq (i_1,\ldots,i_n)$, then $\mathscr{I}_{n,T}^{(j_1,\ldots,j_n)}$ and $\mathscr{I}_{n,T}^{(i_1,\ldots,i_n)}$ are orthogonal in $L^2(\mathbb{P})$, $n \geq 1$. For every fixed $n \geq 1$, we put

$$\mathscr{C}^{(n)} := \bigoplus_{(j_1,\ldots,j_n) \in \mathbb{N}^n} \mathscr{I}_{n,T}^{(j_1,\ldots,j_n)}.$$

Then $\mathscr{C}^{(n)}$ is closed because it is an orthogonal sum of countably many mutually orthogonal closed subspaces of $L^2(\mathbb{P})$. Furthermore $\mathscr{C}^{(n)}$ contains $\mathscr{I}_{n,T}^{(j_1,\ldots,j_n)}$ for every $(j_1,\ldots,j_n) \in \mathbb{N}^n$ and hence also $\mathscr{I}_{n,T}$ (cf. Definition 3.5). Conversely, from the definition of $\mathscr{I}_{n,T}$ it is evident that the inclusion $\mathscr{C}^{(n)} \subseteq \mathscr{I}_{n,T}$ holds and, consequently, $\mathscr{C}^{(n)} = \mathscr{I}_{n,T}$, $n \geq 1$. The statement of the theorem follows now from Proposition 3.7. $\square$

As a consequence of Theorem 3.11 and Definition 3.6, if the family $\mathscr{X} := \{X^{(n)}, n \geq 1\}$ consisting of countably many orthogonal martingales possesses the CRP on $L^2(\mathbb{P})$, then the following orthogonal decompositions of $\mathscr{H}^2$ and $L^2(\mathbb{P})$ hold:

$$\mathscr{H}^2 = \mathscr{I}_0 \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\ldots,j_n) \in \mathbb{N}^n} \mathscr{I}_n^{(j_1,\ldots,j_n)},$$

(13)

$$L^2(\mathbb{P}) = \mathscr{I}_{0,T} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1,\ldots,j_n) \in \mathbb{N}^n} \mathscr{I}_{n,T}^{(j_1,\ldots,j_n)}.$$

4. Compensated-covariation stable families and iterated integrals. As in Section 3, we fix a time parameter $T > 0$, a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in [0,T]}$ satisfying the usual conditions and a family $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ contained in $\mathscr{H}^2(\mathbb{F})$ indexed on the set Λ. We recall that we always assume that $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic for all $\alpha, \beta \in \Lambda$ without explicit mention. In this section, we study the properties of iterated integrals generated by $\mathscr{X}$ under the further assumption that $\mathscr{X}$ is a *compensated-covariation stable* family of $\mathscr{H}^2$.

For $\alpha, \beta \in \Lambda$ we define the process

(14)
$$X^{(\alpha,\beta)} := \left[X^{(\alpha)}, X^{(\beta)}\right] - \left\langle X^{(\alpha)}, X^{(\beta)}\right\rangle$$

which we call *the compensated-covariation* process of $X^{(\alpha)}$ and $X^{(\beta)}$. The process $\left\langle X^{(\alpha)}, X^{(\beta)}\right\rangle$ being the compensator of $[X^{(\alpha)}, X^{(\beta)}]$, $X^{(\alpha,\beta)}$ is always a martingale and $X_0^{(\alpha,\beta)} = X_0^{(\alpha)} X_0^{(\beta)}$.

DEFINITION 4.1.   (i) We say that the family $\mathcal{X} := \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathcal{H}^2(\mathbb{F})$ is *compensated-covariation stable* if for all $\alpha, \beta \in \Lambda$ the compensated-covariation process $X^{(\alpha,\beta)}$ belongs to $\mathcal{X}$.

(ii) Let $\mathcal{X}$ be a compensated-covariation stable family and let $\alpha_1, \ldots, \alpha_m \in \Lambda$ with $m \geq 2$. The process $X^{(\alpha_1,\ldots,\alpha_m)}$ is defined recursively by

(15)
$$X^{(\alpha_1,\ldots,\alpha_m)} := \left[X^{(\alpha_1,\ldots,\alpha_{m-1})}, X^{(\alpha_m)}\right] - \left\langle X^{(\alpha_1,\ldots,\alpha_{m-1})}, X^{(\alpha_m)}\right\rangle.$$

If $\mathcal{X}$ is compensated-covariation stable, the process $X^{(\alpha_1,\ldots,\alpha_m)}$ belongs to $\mathcal{X}$ for every $\alpha_1, \ldots, \alpha_m$ in Λ and $X_0^{(\alpha_1,\ldots,\alpha_m)} = \prod_{i=1}^m X_0^{(\alpha_i)}$, $m \geq 2$.

We begin with the following proposition. For the notation, we refer to Section 3. Let $\mathcal{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ be a compensated-covariation stable family of $\mathcal{H}^2$. For any $\alpha \in \Lambda$, we introduce the notation $\tilde{X}^{(\alpha)} := X^{(\alpha)} - X_0^{(\alpha)}$. Note that $\tilde{X}^{(\alpha)} \in \mathcal{H}_0^2$, $\alpha \in \Lambda$.

PROPOSITION 4.2.   *The stochastic integral* $\tilde{X}_-^{(\alpha)} \cdot M$ *belongs to* $\bigoplus_{k=0}^{n+1} \mathcal{J}_{k,0}$, *for every* $\alpha \in \Lambda$, $M \in \mathcal{J}_{n,0}$ *and* $n \geq 0$.

PROOF.   The proof will be given by induction on the order n of the iterated integral $M \in \mathcal{J}_{n,0}$. If $n = 0$, that is, $M \in \mathcal{J}_{0,0}$, then $\tilde{X}_-^{(\alpha)} \cdot M \equiv 0 \in \mathcal{J}_{0,0} \subseteq \bigoplus_{k=0}^1 \mathcal{J}_{k,0}$. Let now $M \in \mathcal{J}_{1,0}$. By linearity it suffices to take $M = J_1^{(\alpha_1)}(F)$ where $F = F_0 \otimes F_1$ is an elementary function of order 1. Obviously, we have $\tilde{X}^{(\alpha)} = J_1^{(\alpha)}(1 \otimes 1)$ and $M = F_0(F_1 \cdot X^{(\alpha_1)})$ from which it follows

$$\tilde{X}_-^{(\alpha)} \cdot M_t = \int_0^t F_0 J_1^{(\alpha)}(1 \otimes 1)_{u-} F_1(u) \, dX_u^{(\alpha_1)}$$

$$= J_2^{(\alpha,\alpha_1)}(F_0 \otimes 1 \otimes F_1).$$

This shows that $\tilde{X}_-^{(\alpha)} \cdot M \in \mathcal{J}_{2,0} \subseteq \bigoplus_{k=0}^2 \mathcal{J}_{k,0}$. We now fix $n \geq 2$ and assume that the statement is satisfied for all $M \in \mathcal{J}_{n,0}$. For the induction step, it is enough to prove that $\tilde{X}_-^{(\alpha)} \cdot M \in \bigoplus_{k=0}^{n+2} \mathcal{J}_{k,0}$ for all M from $\mathcal{J}_{n+1,0}^{(\alpha_1,\ldots,\alpha_{n+1})}$ and $\alpha_1, \ldots, \alpha_{n+1} \in \Lambda$. To this end, let $M \in \mathcal{J}_{n+1,0}^{(\alpha_1,\ldots,\alpha_{n+1})}$ for some $\alpha_1, \ldots, \alpha_{n+1} \in \Lambda$. In view of the linearity of $\mathcal{J}_{n+2,0}$ and of the iterated integral, without loss of generality, we can assume that M is an *elementary* iterated integral with respect to

$(X^{(\alpha_1)}, X^{(\alpha_2)}, \ldots, X^{(\alpha_{n+1})})$, that is, M has the representation

$$M_t = J_{n+1}^{(\alpha_1,\ldots,\alpha_{n+1})}(F_0 \otimes F_1 \otimes \cdots \otimes F_{n+1})_t$$

(16)

$$= \int_0^t N_{u-} F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}, \qquad t \in [0, T],$$

where $N \in \mathscr{J}_{n,0}^{(\alpha_1,\ldots,\alpha_n)}$ can be written in the form

(17) $\quad N_t = J_n^{(\alpha_1,\ldots,\alpha_n)}(F_0 \otimes \cdots \otimes F_n)_t = \int_0^t R_{u-} F_n(u) \, dX_u^{(\alpha_n)}, \qquad t \in [0, T],$

with $R = J_{n-1}^{(\alpha_1,\ldots,\alpha_{n-1})}(F_0 \otimes \cdots \otimes F_{n-1}) \in \mathscr{J}_{n-1,0}^{(\alpha_1,\ldots,\alpha_{n-1})}$ and $F = F_0 \otimes \cdots \otimes F_{n+1}$ is an elementary function of order $n + 1$. Using partial integration for the product $\tilde{X}^{(\alpha)} N$ and the identities $N_0 = N_{0-} = 0$, $N_- \cdot \tilde{X}^{(\alpha)} = N_- \cdot X^{(\alpha)}$ and $[\tilde{X}^{(\alpha)}, N] = [X^{(\alpha)}, N]$ we get

$$\tilde{X}_-^{(\alpha)} \cdot M_t = \int_0^t \tilde{X}_{u-}^{(\alpha)} N_{u-} F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}$$

(18)
$$= \int_0^t \left(\int_0^{u-} N_{v-} \, dX_v^{(\alpha)} \right) F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}$$

(19)
$$+ \int_0^t \left(\int_0^{u-} \tilde{X}_{v-}^{(\alpha)} \, dN_v \right) F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}$$

(20)
$$+ \int_0^t \left[X^{(\alpha)}, N \right]_{u-} F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}.$$

The term (18) is equal to $J_{n+2}^{(\alpha_1,\ldots,\alpha_n,\alpha,\alpha_{n+1})}(F_0 \otimes \cdots \otimes F_n \otimes 1 \otimes F_{n+1})_t$ and hence this process belongs to $\mathscr{J}_{n+2,0}^{(\alpha_1,\ldots,\alpha_n,\alpha,\alpha_{n+1})} \subseteq \bigoplus_{k=0}^{n+2} \mathscr{J}_{k,0}$. The second term (19) belongs to $\bigoplus_{k=0}^{n+2} \mathscr{J}_{k,0}$ because, in view of the induction hypothesis, the integrand $\tilde{X}_-^{(\alpha)} \cdot N$ belongs to $\bigoplus_{k=0}^{n+1} \mathscr{J}_{k,0}$. Now we consider the third term (20). Using the representation (17), we can write

$$\left[X^{(\alpha)}, N \right]_t = \int_0^t R_{u-} F_n(u) \, d\left[X^{(\alpha_n)}, X^{(\alpha)} \right]_u,$$

(21)
$$\langle X^{(\alpha)}, N \rangle_t = \int_0^t R_{u-} F_n(u) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_u$$

and hence, using the linearity of the stochastic integral and (15), we get

(22) $$\left[X^{(\alpha)}, N \right]_t - \langle X^{(\alpha)}, N \rangle_t = \int_0^t R_{u-} F_n(u) \, dX_u^{(\alpha_n,\alpha)}.$$

The third term (20) can be rewritten as

$$(20) = \int_0^t \left(\left[X^{(\alpha)}, N \right]_{u-} - \langle X^{(\alpha)}, N \rangle_{u-} \right) F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}$$
$$+ \int_0^t \langle X^{(\alpha)}, N \rangle_{u-} F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}$$

and inserting (21) and (22) in the previous equality we get that (20) is equal to

$$
\begin{aligned}
(23) \quad & \int_0^t \int_0^{u-} R_{v-} F_n(v) \, dX_v^{(\alpha_n, \alpha)} F_{n+1}(u) \, dX_u^{(\alpha_{n+1})} \\
& + \int_0^t \int_0^{u-} R_{v-} F_n(v) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_v F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}.
\end{aligned}
$$

Since $R \in \mathscr{J}_{n-1,0}$, F_n is bounded and, $\mathscr{X}$ being compensated-covariation stable, $X^{(\alpha_n, \alpha)} \in \mathscr{X}$, we can conclude that the right-hand side of (22) belongs to $\mathscr{J}_{n,0}$ and hence the first term in (23) is an element of $\mathscr{J}_{n+1,0} \subseteq \bigoplus_{k=0}^{n+2} \mathscr{J}_{k,0}$. Finally, for proving that the second term of (23) belongs to $\bigoplus_{k=0}^{n+2} \mathscr{J}_{k,0}$ we calculate its inner integral using partial integration:

$$
\begin{aligned}
& \int_0^t R_{v-} F_n(v) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_v \\
&\quad = R_t \int_0^t F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s - \int_0^t \int_0^{v-} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \, dR_v \\
&\qquad - \left[R, \int_0^{\cdot} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \right]_t \\
&\quad = R_t \int_0^t F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s - \int_0^t \int_0^{v-} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \, dR_v \\
&\qquad - \int_0^t \left(\Delta \int_0^{\cdot} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \right)_v dR_v \\
&\quad = R_t \int_0^t F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s - \int_0^t \int_0^{v} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \, dR_v,
\end{aligned}
$$

where in the last but one equality we have used Jacod and Shiryaev [12], Proposition I.4.49b). Substituting this in the second term of (23), we get

$$
\begin{aligned}
(24) \quad & \int_0^t \int_0^{u-} R_{v-} F_n(v) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_v F_{n+1}(u) \, dX_u^{(\alpha_{n+1})} \\
&= \int_0^t R_{u-} \int_0^{u-} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s F_{n+1}(u) \, dX_u^{(\alpha_{n+1})} \\
&\quad - \int_0^t \int_0^{u-} \int_0^{v} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \, dR_v F_{n+1}(u) \, dX_u^{(\alpha_{n+1})}.
\end{aligned}
$$

The first summand on the right-hand side belongs to $\mathscr{J}_{n,0}$ because $R \in \mathscr{J}_{n-1,0}$ and the function $\widetilde{F}_{n+1}$ defined by $\widetilde{F}_{n+1}(u) := \int_0^{u-} F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s F_{n+1}(u)$ is bounded. Similarly, we obtain from $R = J_{n-1}^{(\alpha_1, \ldots, \alpha_{n-1})}(F_0 \otimes \cdots \otimes F_{n-1})$ and the fact that the function F with $F(v) := \int_0^v F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s$ is bounded

$$
\begin{aligned}
\int_0^u \int_0^v F_n(s) \, d\langle X^{(\alpha_n)}, X^{(\alpha)} \rangle_s \, dR_v &= \int_0^u F(v) \, dR_v \\
&= J_{n-1}^{(\alpha_1, \ldots, \alpha_{n-1})}(F_0 \otimes \cdots \otimes (F F_{n-1}))_u
\end{aligned}
$$

and hence the second integral of the right-hand side of (24) is equal to

$$\int_0^t J_{n-1}^{(\alpha_1,\dots,\alpha_{n-1})}\big(F_0 \otimes \cdots \otimes (FF_{n-1})\big)_{u-} F_{n+1}(u)\,dX_u^{(\alpha_{n+1})}$$

$$= J_n^{(\alpha_1,\dots,\alpha_{n-1},\alpha_{n+1})}\big(F_0 \otimes \cdots \otimes (FF_{n-1}) \otimes F_{n+1}\big)_t$$

which belongs to $\mathscr{J}_{n,0} \subseteq \bigoplus_{k=0}^{n+2} \mathscr{J}_{k,0}$. This proves the proposition. $\square$

Now we come to the main result of this section. Recall that $\tilde{X}^{(\alpha)} := X^{(\alpha)} - X_0^{(\alpha)}$, $\alpha \in \Lambda$.

THEOREM 4.3. *Let* $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathscr{H}^2(\mathbb{F})$ *be a compensated-covariation stable family. Then the stochastic integral* $(\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)}) \cdot M$ *belongs to* $\bigoplus_{k=0}^{n+m} \mathscr{J}_{k,0}$, *for all* $m \geq 0$, $\alpha_1,\dots,\alpha_m \in \Lambda$ *and* $M \in \bigoplus_{k=0}^n \mathscr{J}_{k,0}$, *for every* $n \geq 0$.

PROOF. The proof will be given by induction on m. If $m = 0$, because $\prod_{i=1}^0 \tilde{X}^{(\alpha_i)} := 1$ by convention, then the claim is evident because $M = 1 \cdot M$. We now assume that the statement of the theorem holds for $m \geq 1$ and $M \in \bigoplus_{k=0}^n \mathscr{J}_{k,0}$, for any $n \geq 0$, and prove it for $m+1$. Let $\alpha_1,\dots,\alpha_{m+1}$ be given. Setting $\widetilde{M} := \tilde{X}_-^{(\alpha_{m+1})} \cdot M$, we can calculate

$$\left(\prod_{i=1}^{m+1} \tilde{X}_-^{(\alpha_i)}\right) \cdot M = \left(\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)}\right) \cdot \big(\tilde{X}_-^{(\alpha_{m+1})} \cdot M\big) = \left(\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)}\right) \cdot \widetilde{M}.$$

From Proposition 4.2, we obtain that $\widetilde{M} \in \bigoplus_{k=0}^{n+1} \mathscr{J}_{k,0}$. The induction hypothesis yields that the right-hand side belongs to $\bigoplus_{k=0}^{(n+1)+m} \mathscr{J}_{k,0} = \bigoplus_{k=0}^{n+(m+1)} \mathscr{J}_{k,0}$. This proves the induction step and hence the proof of the theorem is complete. $\square$

An immediate consequence of Theorem 4.3 is the following corollary.

COROLLARY 4.4. *Let* $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathscr{H}^2(\mathbb{F})$ *be a compensated-covariation stable family. Then the process* $F_0(\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)}) \cdot X^{(\alpha)}$ *belongs to* $\bigoplus_{k=0}^{n+1} \mathscr{J}_{k,0}$ *for all bounded* $\mathcal{F}_0$-*measurable* F_0 *and* $\alpha, \alpha_1,\dots,\alpha_m \in \Lambda$, $m \geq 1$.

PROOF. We have $(F_0 \prod_{i=1}^m \tilde{X}_-^{(\alpha_i)}) \cdot X^{(\alpha)} = (\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)}) \cdot (F_0 \tilde{X}^{(\alpha)})$ and we can apply Theorem 4.3 to $M = J_1^{(\alpha)}(F_0 \otimes 1) = F_0 \tilde{X}^{(\alpha)}$. $\square$

We conclude this section with the next corollary which shows that if $\mathscr{X} = \{X^{(\alpha)}, \alpha \in \Lambda\}$ is a subfamily of $\mathscr{H}^2$ satisfying the assumptions of Theorem 4.3 and another technical condition, then the random variable $\tilde{X}_t^{(\alpha)}$ has finite absolute moments of every order for all $\alpha \in \Lambda$ and $t \in [0, T)$.

COROLLARY 4.5. *Let $\mathscr{X} = \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathscr{H}^2$ be a compensated-covaria- tion stable family. If there exists $\beta \in \Lambda$ such that $\langle X^{(\beta)}, X^{(\beta)} \rangle_t < \langle X^{(\beta)}, X^{(\beta)} \rangle_T$ for all $t < T$ then, for every $\alpha \in \Lambda$ and $t < T$, $\tilde{X}_t^{(\alpha)}$ has finite absolute moments of arbitrary order.*

PROOF. From Corollary 4.4, we have $X := (\tilde{X}_-^{(\alpha)})^m \cdot X^{(\beta)} \in \bigoplus_{k=0}^{m+1} \mathscr{J}_{k,0}$. This implies that X belongs to $\mathscr{H}^2$ and $\langle X, X \rangle_T = \int_0^T |\tilde{X}_{u-}^{(\alpha)}|^{2m} \, \mathrm{d}\langle X^{(\beta)}, X^{(\beta)} \rangle_u$, there- fore

$$\int_0^T \mathbb{E}[|\tilde{X}_{u-}^{(\alpha)}|^{2m}] \, \mathrm{d}\langle X^{(\beta)}, X^{(\beta)} \rangle_u = \mathbb{E}[\langle X, X \rangle_T] < +\infty.$$

Using the martingale property of $(\tilde{X}^{(\alpha)}, \mathbb{F})$ and Jensen's inequality for conditional expectations, for every $t < u$, we can estimate

$$|\tilde{X}_t^{(\alpha)}|^{2m} = |\mathbb{E}[\tilde{X}_u^{(\alpha)} | \mathscr{F}_t]|^{2m} = |\mathbb{E}[\mathbb{E}[\tilde{X}_u^{(\alpha)} | \mathscr{F}_{u-}] | \mathscr{F}_t]|^{2m}$$
$$= |\mathbb{E}[\tilde{X}_{u-}^{(\alpha)} | \mathscr{F}_t]|^{2m} \leq \mathbb{E}[|\tilde{X}_{u-}^{(\alpha)}|^{2m} | \mathscr{F}_t]$$

which yields $\mathbb{E}[|\tilde{X}_t^{(\alpha)}|^{2m}] \leq \mathbb{E}[|\tilde{X}_{u-}^{(\alpha)}|^{2m}]$, $0 \leq t < u$. Hence, we obtain

$$(\langle X^{(\beta)}, X^{(\beta)} \rangle_T - \langle X^{(\beta)}, X^{(\beta)} \rangle_t) \mathbb{E}[|\tilde{X}_t^{(\alpha)}|^{2m}] \leq \int_{(t,T]} \mathbb{E}[|\tilde{X}_{u-}^{(\alpha)}|^{2m}] \, \mathrm{d}\langle X^{(\beta)}, X^{(\beta)} \rangle_u$$
$$< +\infty.$$

Let β be chosen such that $\langle X^{(\beta)}, X^{(\beta)} \rangle_t < \langle X^{(\beta)}, X^{(\beta)} \rangle_T$ for all $t < T$. Then the above inequality yields $\mathbb{E}[|\tilde{X}_t^{(\alpha)}|^{2m}] < +\infty$, $t < T$, which proves the claim. $\square$

5. The chaotic representation property.

In this section, we shall give suf- ficient conditions for a subfamily $\mathscr{X}$ of $\mathscr{H}^2$ to possess the CRP. For this pur- pose, it will be useful to work with families of martingales in $\mathscr{H}^2$ which are stable under stopping with respect to deterministic stopping times. For a given $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathscr{H}^2$ and a collection $\mathscr{S}$ of *finite-valued* stopping times, we define the family $\mathscr{X}^{\mathscr{S}}$ by

(25) $$\mathscr{X}^{\mathscr{S}} := \{X^{(\alpha)^{\tau}}, \tau \in \mathscr{S}, \alpha \in \Lambda\},$$

where the superscript τ denotes the operation of stopping at $\tau \in \mathscr{S}$. It is clear that $\mathscr{X}^{\mathscr{S}} \subseteq \mathscr{H}^2$. The following lemma states a condition on $\mathscr{S}$ ensuring that $\mathscr{X}^{\mathscr{S}}$ is a compensated-covariation stable family whenever $\mathscr{X}$ is one. We will be particularly interested in the case $\mathscr{S} = \mathbb{R}_+$. Using the properties of the brackets $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ the proof is straightforward and therefore omitted.

LEMMA 5.1. *If $\mathscr{S}$ is a minimum-stable family of finite-valued stopping times and $\mathscr{X} \subseteq \mathscr{H}^2$ is compensated-covariation stable, then $\mathscr{X}^{\mathscr{S}}$ is compensated- covariation stable, too.*

Now we come to a useful representation formula for products of elements of a compensated-covariation stable family $\mathscr{X}$. Its proof is given by induction using integration by parts. See Di Tella and Engelbert [6], Proposition 3.3, where $X_0 = 0$ for every $X \in \mathscr{X}$ was additionally assumed. However, using our convention $X_{0-} = 0$ and the definition (1) of the quadratic covariation $[X, Y]$ (including the jump $\Delta X_0 \Delta Y_0 = X_0 Y_0$ at time zero), in case of a general compensated covariation stable family $\mathscr{X} \subseteq \mathscr{H}^2$, the reader may notice that the formula and its proof are completely the same.

PROPOSITION 5.2. *Let $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ be a compensated-covariation stable family of $\mathscr{H}^2$. For every $m \geq 1$ and $\alpha_1, \ldots, \alpha_m \in \Lambda$, we have*

$$\prod_{i=1}^{m} X^{(\alpha_i)} = \sum_{i=1}^{m} \sum_{1 \leq j_1 < \cdots < j_i \leq m} \left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m} X_{-}^{(\alpha_k)} \right) \cdot X^{(\alpha_{j_1}, \ldots, \alpha_{j_i})}$$

(26)
$$+ \sum_{p=0}^{m-2} \sum_{i=p+2}^{m} \sum_{1 \leq j_1 < \cdots < j_i \leq m} \left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m} X_{-}^{(\alpha_k)} \prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})} \right)$$

$$\cdot \left\langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \right\rangle.$$

Note that the initial value of the left-hand side of (26) is given by the initial value of the first term on the right-hand side for $i = m$.

A simplification of formula (26) can be obtained by assuming that the family $\mathscr{X}$ consists of *quasi-left continuous* martingales. Indeed, in this case one can choose continuous versions of the processes $\langle X^{(\alpha)}, X^{(\beta)} \rangle$, $\alpha, \beta \in \Lambda$ (cf. Jacod and Shiryaev [12], Theorem I.4.2), and so all the terms appearing in the second sum on the right-hand side of (26) vanish for $p \neq 0$.

Let $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ be a subfamily of $\mathscr{H}^2$. The filtration $\mathbb{F}^{\mathscr{X}} := (\mathscr{F}_t^{\mathscr{X}})_{t \in [0,T]}$ is defined as the smallest filtration satisfying the usual conditions and with respect to which each process in $\mathscr{X}$ is adapted. In the remaining part of the present paper, we shall consider $\mathscr{X}$ as a subfamily of $\mathscr{H}^2(\mathbb{F}^{\mathscr{X}})$ on the probability space $(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$.

We define the family $\mathscr{K}$ by

(27)
$$\mathscr{K} := \left\{ \prod_{i=1}^{m} X_{t_i}^{(\alpha_i)}, \alpha_i \in \Lambda, t_i \in [0, T], i = 1, \ldots, m; m \geq 0 \right\}$$

which is the family of monomials formed by products of elements of $\mathscr{X}$ at different times. Obviously, $\sigma(\mathscr{K})$ augmented by the $\mathbb{P}$-null sets of $\mathscr{F}_T^{\mathscr{X}}$ equals $\mathscr{F}_T^{\mathscr{X}}$. We make the following assumption:

ASSUMPTION 5.3. The family $\mathscr{K}$ defined in (27) is contained in the space $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ and is total (i.e., its linear hull is dense) in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$.

If $\mathscr{X} = \{X^{(\alpha)}, \alpha \in \Lambda\}$ satisfies Assumption 5.3, then $X_t^{(\alpha)}$ admits finite moments of every order for all $t \in [0, T]$ and $\alpha \in \Lambda$, that is,

$$(28) \qquad \mathbb{E}[|X_t^{(\alpha)}|^m] < +\infty, \qquad t \in [0, T], \alpha \in \Lambda, m \geq 1.$$

Sufficient conditions on the family $\mathscr{X}$ for $\mathscr{K}$ to fulfil Assumption 5.3 are extensively studied in the literature. The following well-known result, being useful for many applications, exploits the existence of finite exponential moments. For an elementary proof cf. Di Tella and Engelbert [6], Theorem A.4.

THEOREM 5.4.   *If for every $\alpha \in \Lambda$ and $t \geq 0$ there exists a constant $c_\alpha(t) > 0$ such that the expectation $\mathbb{E}[\exp(c_\alpha(t)|X_t^{(\alpha)}|)]$ is finite, then $\mathscr{K}$ satisfies Assumption 5.3.*

Now we state two technical lemmas which will be needed in the proof of Proposition 5.7 below.

LEMMA 5.5.   *Let A be a deterministic process of finite variation, $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ a family of martingales contained in $\mathscr{H}^2$ satisfying (28) and $p, q \geq 0$. We define the processes K by*

$$(29) \qquad K := \prod_{j=1}^{q} \Delta X^{(\beta_j)} \prod_{i=1}^{p} X_-^{(\alpha_i)}, \qquad \alpha_i, \beta_j \in \Lambda, i = 1, \ldots, p; j = 1, \ldots, q.$$

Then the process $K \cdot A$ is of integrable variation.

PROOF.   Obviously, $|\operatorname{Var}(K \cdot A_T)| \leq \sup_{0 \leq t \leq T} |K_t| \operatorname{Var}(A)_T$. We will show below that $\sup_{0 \leq t \leq T} |K_t|$ is integrable and, because $\operatorname{Var}(A)_T$ is deterministic and finite, this yields that $\operatorname{Var}(K \cdot A_T)$ is integrable and hence the claim. For proving that $\sup_{0 \leq t \leq T} |K_t|$ is integrable, we estimate

$$
\begin{aligned}
\sup_{0 \leq t \leq T} |K_t| &\leq \prod_{j=1}^{q} 2 \sup_{0 \leq t \leq T} |X_t^{(\beta_j)}| \prod_{i=1}^{p} \sup_{0 \leq t \leq T} |X_t^{(\alpha_i)}| \\
(30) \qquad &\leq 2^{q-1} \left(\prod_{j=1}^{q} \sup_{0 \leq t \leq T} |X_t^{(\beta_j)}|^2 + \prod_{i=1}^{p} \sup_{0 \leq t \leq T} |X_t^{(\alpha_i)}|^2 \right) \\
&\leq 2^{q-1} \left(q^{-1} \sum_{j=1}^{q} \sup_{0 \leq t \leq T} |X_t^{(\beta_j)}|^{2q} + p^{-1} \sum_{i=1}^{p} \sup_{0 \leq t \leq T} |X_t^{(\alpha_i)}|^{2p} \right),
\end{aligned}
$$

where we have used the inequality $\prod_{k=1}^{m} a_k \leq m^{-1} \sum_{k=1}^{m} a_k^m$ for all nonnegative numbers $a_1, \ldots, a_m$ and $m \in \mathbb{N}$. Now from (28) it follows that $|X^{(\alpha)}|^m$ is a nonnegative submartingale for all $\alpha \in \Lambda$ and $m \in \mathbb{N}$. Using Doob's inequality, we can conclude that the right-hand side of (30) is integrable which completes the proof. $\square$

LEMMA 5.6. *Let $\mathscr{X} = \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathscr{H}^2$ be a family of martingales which satisfies* (28); *$q, r \geq 1$ be fixed and $\xi \in L^2(\mathbb{P})$ be such that, for some $\alpha_1, \ldots, \alpha_{q+r} \in \Lambda$, ξ is orthogonal in $L^2(\mathbb{P})$ to $\prod_{k=1}^{q+r} X_{t_k}^{(\alpha_k)}$ for every $t_1, \ldots, t_{q+r} \in [0, T]$. Then ξ is orthogonal to $\prod_{k=1}^q X_{t-}^{(\alpha_k)} \prod_{j=q+1}^{q+r} X_t^{(\alpha_j)}$ for every $t \in [0, T]$.*

PROOF. Fixing $t \in [0, T]$, similar as in the proof of Lemma 5.5 above we can show that

$$|\xi| \prod_{k=1}^q \sup_{0 \leq u \leq T} |X_u^{(\alpha_k)}| \prod_{j=q+1}^{q+r} |X_t^{(\alpha_j)}|$$

is integrable. Choosing t_n such that $t_n < t$ for all $n \in \mathbb{N}$ and $t_n \to t$ as $n \to \infty$, we obtain, for all $n \in \mathbb{N}$, $\mathbb{E}[\xi \prod_{k=1}^q X_{t_n}^{(\alpha_k)} \prod_{j=q+1}^{q+r} X_t^{(\alpha_j)}] = 0$ in view of the assumption and letting $n \to \infty$ the claim follows from Lebesgue's theorem on dominated convergence. $\square$

Let $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ be a compensated-covariation stable family of $\mathscr{H}^2$ such that, for every $\alpha, \beta \in \Lambda$, the process $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic. We introduce the following systems:

$$\mathscr{R} := \left\{ F_0 \left(\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)} \right) \cdot X^{(\alpha)}, \alpha, \alpha_1, \ldots, \alpha_m \in \Lambda, m \geq 0, F_0 \in \mathscr{J}_{0,0} \right\} \cup \mathscr{J}_{0,0},$$

(31)

$$\mathscr{R}_T := \left\{ F_0 \left(\prod_{i=1}^m \tilde{X}_-^{(\alpha_i)} \right) \cdot X_T^{(\alpha)}, \alpha, \alpha_1, \ldots, \alpha_m \in \Lambda, m \geq 0, F_0 \in \mathscr{J}_{0,0} \right\} \cup \mathscr{J}_{0,0}.$$

We stress that $\mathscr{R}$ is contained in $\mathscr{H}^2$ and $\mathscr{R}_T$ in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ and that $\mathscr{X} \subseteq \text{cl}(\text{Span}(\mathscr{R}))_{\mathscr{H}^2}$, hence $\mathscr{X}$ and $\mathscr{R}$ generate the same filtration.

The next elementary identity will be useful in the proof of Proposition 5.7 below. For real numbers $a_r, b_r, r = 1, \ldots, m$, we have:

$$(32) \qquad \prod_{r=1}^m (a_r + b_r) = \sum_{r=0}^m \sum_{1 \leq q_1 < \cdots < q_r \leq m} \prod_{\substack{k=1 \\ k \neq q_1, \ldots, q_r}}^m a_k \prod_{\ell=1}^r b_{q_\ell}.$$

The following proposition being used for the proof of Theorem 5.8, is of interest in its own right.

PROPOSITION 5.7. *Suppose that $\mathscr{X}$ is stable under stopping with respect to deterministic stopping times and that Assumption 5.3 holds. Then $\mathscr{R}$ is total in $\mathscr{H}^2$ and $\mathscr{R}_T$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$.*

PROOF.   It is sufficient to verify the second claim. We are going to show that any $\xi \in L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ which is orthogonal to $\mathscr{R}_T$ is orthogonal in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ to $\prod_{i=1}^m X_T^{(\alpha_i)}$ for every $\alpha_1, \ldots, \alpha_m \in \Lambda$ and $m \geq 0$. The stability under stopping with respect to deterministic stopping times of the family $\mathscr{X}$ then yields that ξ is also orthogonal to $\mathscr{K}$, where $\mathscr{K}$ is given in (27). But by Assumption 5.3 the system $\mathscr{K}$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ and therefore ξ is evanescent. This implies that $\mathscr{R}_T \subseteq L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ is a total subset and therefore the claim of the proposition.

Let $\xi \in L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ be orthogonal to $\mathscr{R}_T$. For verifying that ξ is orthogonal to $\prod_{i=1}^m X_T^{(\alpha_i)}$ for every $\alpha_1, \ldots, \alpha_m \in \Lambda$ and $m \in \mathbb{N} \cup \{0\}$, we proceed by strong induction on m. For $m = 0$, we notice that $\prod_{i=1}^0 X_T^{(\alpha_i)} = 1 \in \mathscr{R}_T$ and hence by assumption ξ is orthogonal to $\prod_{i=1}^0 X_T^{(\alpha_i)}$. Let us now assume that for some fixed $m \geq 1$

$$(33) \qquad \mathbb{E}\left[\xi \prod_{i=1}^n X_T^{(\alpha_i)}\right] = 0, \qquad \alpha_1, \ldots, \alpha_n \in \Lambda, n \leq m.$$

From (33) and the property that $\mathscr{X}$ is stable under stopping with respect to deterministic stopping times, it easily follows that

$$(34) \quad \mathbb{E}\left[\xi \prod_{i=1}^n X_{t_i}^{(\alpha_i)}\right] = 0, \qquad t_1, \ldots, t_n \in [0, T], \alpha_1, \ldots, \alpha_n \in \Lambda, n \leq m.$$

We now show that $\mathbb{E}[\xi \prod_{i=1}^{m+1} X_T^{(\alpha_i)}] = 0$ for all $\alpha_1, \ldots, \alpha_{m+1} \in \Lambda$. Representing $\prod_{i=1}^{m+1} X_T^{(\alpha_i)}$ by the product formula (26), we get

$$
\mathbb{E}\left[\xi \prod_{i=1}^{m+1} X_T^{(\alpha_i)}\right]
$$

$$
(35) \quad = \sum_{i=1}^{m+1} \sum_{1 \leq j_1 < \cdots < j_i \leq m+1} \mathbb{E}\left[\xi\left(\left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)}\right) \cdot X_T^{(\alpha_{j_1}, \ldots, \alpha_{j_i})}\right)\right]
$$

$$
+ \sum_{p=0}^{m-1} \sum_{i=p+2}^{m+1} \sum_{1 \leq j_1 < \cdots < j_i \leq m+1} \mathbb{E}\left[\xi\left(\left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \prod_{\ell=i-p+1}^i \Delta X^{(\alpha_{j_\ell})}\right)\right.\right.
$$

$$
\left.\left. \cdot \langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})}\rangle_T\right)\right].
$$

We now analyse the first summand on the right-hand side of (35). Using the decomposition $X^{(\alpha_k)} = \tilde{X}^{(\alpha_k)} + X_0^{(\alpha_k)}$ and the identity (32) with $a_r = \tilde{X}_-^{(\alpha_r)}$ and

$b_r = X_0^{(\alpha_r)}$, $r = 1, \ldots, m + 1$, from the definition of $\mathscr{R}_T$ it can easily be verified that it holds

$$\left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \right) \cdot X_T^{(\alpha_{j_1}, \ldots, \alpha_{j_i})} \in \mathrm{cl}(\mathrm{Span}(\mathscr{R}_T))_{L^2(\mathbb{P})}.$$

By assumption we have that ξ is orthogonal to $\mathscr{R}_T$ and therefore also to $\mathrm{cl}(\mathrm{Span}(\mathscr{R}_T))_{L^2(\mathbb{P})}$. (Note that $X_t^{(\alpha)}$ admits finite moments of every order for all $t \in [0, T]$ and $\alpha \in \Lambda$, cf. (28).) Hence, the first summand in (35) vanishes and we get

$$\mathbb{E}\left[\xi \prod_{i=1}^{m+1} X_T^{(\alpha_i)} \right]$$

$$= \sum_{p=0}^{m-1} \sum_{i=p+2}^{m+1} \sum_{1 \leq j_1 < \cdots < j_i \leq m+1} \mathbb{E}\left[\xi \left(\left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})} \right) \right. \right.$$

$$\left. \left. \cdot \langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \rangle_T \right) \right].$$

The processes $X^{(\alpha_{j_{i-p}})}$ and $X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}$ belong to $\mathscr{X}$ for every i and p, because the family $\mathscr{X}$ is compensated-covariation stable. By assumption, the processes $\langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \rangle$ are deterministic. Lemma 5.5 implies that $(\prod_{\substack{k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})}) \cdot \langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \rangle$ are processes of integrable variation. Thus, we can apply Fubini's theorem and for every summand we get

$$\mathbb{E}\left[\xi \left(\left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})} \right) \cdot \langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \rangle_T \right) \right]$$

(36)

$$= \mathbb{E}\left[\xi \left(\prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})} \right)_\cdot \right] \cdot \langle X^{(\alpha_{j_1}, \ldots, \alpha_{j_{i-p-1}})}, X^{(\alpha_{j_{i-p}})} \rangle_T.$$

We consider the generic element

$$K := \prod_{\substack{k=1 \\ k \neq j_1, \ldots, j_i}}^{m+1} X_-^{(\alpha_k)} \prod_{\ell=i-p+1}^{i} \Delta X^{(\alpha_{j_\ell})}.$$

After expanding the product we observe that K_t is equal to a finite sum of terms of type

$$\prod_{k=1}^{q} X_{t-}^{(\alpha_{i_k})} \prod_{j=q+1}^{q+r} X_t^{(\alpha_{i_j})},$$

with $1 \leq i_1, \ldots, i_{q+r} \leq m+1$ pairwise different, $q+r \leq m-1$. From the induction hypothesis, (34) and Lemma 5.6 we now obtain $\mathbb{E}[\xi K_t] = 0$ for all $t \in [0, T]$. Thus every summand (36) vanishes, therefore $\mathbb{E}[\xi \prod_{i=1}^{m+1} X_T^{(\alpha_i)}] = 0$ and the proof of the induction step is finished. Consequently, ξ is orthogonal to $\prod_{i=1}^{m} X_T^{(\alpha_i)}$ for every $m \geq 0$ and the proof is complete. $\square$

Now, from Proposition 5.7 and Theorem 4.3 we can deduce the main result of this paper.

THEOREM 5.8. *Let $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\} \subseteq \mathscr{H}^2(\mathbb{F}^{\mathscr{X}})$ be a compensated-covariation stable family such that $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic for all $\alpha, \beta \in \Lambda$. Suppose moreover that Assumption 5.3 is satisfied. Then $\mathscr{X}$ possesses the CRP on $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$.*

PROOF. We set $\mathscr{S} := \mathbb{R}_+$ and introduce the family $\mathscr{Z} := \mathscr{X}^{\mathscr{S}}$ as in (25). Because of Lemma 5.1, $\mathscr{Z}$ is a compensated-covariation stable family of $\mathscr{H}^2$. Starting from $\mathscr{Z}$ we define the family $\mathscr{R}_T$ as in (31). Clearly $\mathscr{Z}$ satisfies all the assumptions of Proposition 5.7 and therefore $\mathscr{R}_T$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{Z}}, \mathbb{P})$. On the other side, Corollary 4.4 yields that the family $\mathscr{R}_T$ is contained in the closed linear space $\mathscr{J}_T^{\mathscr{Z}}$ of the terminal variables of the iterated integrals generated by $\mathscr{Z}$ and hence $\mathscr{J}_T^{\mathscr{Z}} = L^2(\Omega, \mathscr{F}_T^{\mathscr{Z}}, \mathbb{P})$. Furthermore, the identity $\mathbb{F}^{\mathscr{Z}} = \mathbb{F}^{\mathscr{X}}$ holds. Therefore $\mathscr{Z}$ possesses the CRP on $L^2(\Omega, \mathscr{F}_T^{\mathscr{Z}}, \mathbb{P})$. To show that $\mathscr{X}$ possesses the CRP on $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$, we only need to show that the space $\mathscr{J}^{\mathscr{X}}$ of the iterated integrals generated by $\mathscr{X}$ contains the space $\mathscr{J}^{\mathscr{Z}}$ of those generated by $\mathscr{Z}$. But this is obvious because for every $X \in \mathscr{Z}$ there exist $\alpha \in \Lambda$ and $u \in \mathbb{R}_+$ such that $X = X^{(\alpha)^u}$, where the superscript u denotes the operation of stopping at the deterministic time u. Clearly, the identities $X = (X^{(\alpha)} - X_0^{(\alpha)})^u + X_0^{(\alpha)} = J_1^{(\alpha)}(1 \otimes 1_{[0,u]}) + X_0^{(\alpha)}$ hold. This means that $\mathscr{Z} \subseteq \mathscr{J}^{\mathscr{X}}$ and hence $\mathscr{J}^{\mathscr{Z}} \subseteq \mathscr{J}^{\mathscr{X}}$ implying the CRP for $\mathscr{X}$. $\square$

Applications of Theorem 5.8 to Lévy processes will be given in Section 6 below. Further applications and examples will be provided in the concluding Section 7.

6. The CRP for Lévy processes. In this section, given a Lévy Process L on a fixed time horizon $[0, T]$, $T > 0$, we construct families of martingales possessing

the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. We start with a short introduction to Lévy processes and Poisson random measures.

A càdlàg process L on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that $L_0 = 0$ is called a Lévy process if it is stochastically continuous and has homogeneous and independent increments. Let L be a Lévy process. By $\mathbb{F}^L = (\mathscr{F}_t^L)_{t \in [0, T]}$ we denote the natural filtration of L, that is, the smallest filtration satisfying the usual conditions such that L is adapted. From now on, we restrict ourselves to the probability space $(\Omega, \mathscr{F}_T^L, \mathbb{P})$ and the filtration $\mathbb{F}^L$. Because $L_0 = 0$, $\mathscr{F}_0^L$ is trivial.

On $(E, \mathscr{B}(E)) := ([0, T] \times \mathbb{R}, \mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R}))$, where $\mathscr{B}(\cdot)$ denotes the Borel σ-algebra, we introduce the random measure M (cf. Jacod and Shiryaev [12], Definition II.1.3) by

$$\mathrm{M}(\omega, A) := \sum_{s \geq 0} 1_{\{\Delta L_s(\omega) \neq 0\}} 1_A(s, \Delta L_s(\omega)), \qquad \omega \in \Omega, A \in \mathscr{B}(E).$$

We call M the jump measure of L. It is known that M is a homogeneous Poisson random measure relative to the filtration $\mathbb{F}^L$, that is, an integer-valued random measure (cf. Jacod and Shiryaev [12], Definition II.1.13) such that (i) $\mathbb{E}[\mathrm{M}(A)] = (\lambda_+ \otimes \nu)(A)$ for every $A \in \mathscr{B}(E)$, where λ_+ is the Lebesgue measure on $[0, T]$ and ν is a σ-finite measure on $\mathbb{R}$; (ii) for all $s \geq 0$ and $A \in \mathscr{B}(E)$ such that $A \subseteq (s, T] \times \mathbb{R}$ the random variable $\mathrm{M}(A)$ is independent of $\mathscr{F}_s^L$ (cf. Jacod and Shiryaev [12], Definition II.1.20). The σ-finite measure ν is the Lévy measure of L, which satisfies $\nu(\{0\}) = 0$ and $x^2 \wedge 1 \in L^1(\nu)$. We put $\mathrm{m} := \lambda_+ \otimes \nu$. Now we introduce the compensated Poisson random measure associated with the jump measure of L. The system $\mathscr{E} := \{A \in \mathscr{B}(E) : \mathrm{m}(A) < +\infty\}$ is a ring of Borel subsets of E. For every $A \in \mathscr{E}$ we define $\overline{\mathrm{M}}(A) := \mathrm{M}(A) - \mathrm{m}(A)$. The family $\overline{\mathrm{M}} := \{\overline{\mathrm{M}}(A), A \in \mathscr{E}\}$ is an *elementary orthogonal random measure* (cf. Gihman and Skorohod [8], IV, Section 4), that is, for every $A, B \in \mathscr{E}$, $\overline{\mathrm{M}}(A)$ belongs to the space $L^2(\mathbb{P})$; $\mathbb{E}[\overline{\mathrm{M}}(A)\overline{\mathrm{M}}(B)] = \mathrm{m}(A \cap B)$ and if, moreover, $A \cap B = \varnothing$, then it follows $\overline{\mathrm{M}}(A \cup B) = \overline{\mathrm{M}}(A) + \overline{\mathrm{M}}(B)$. We call $\overline{\mathrm{M}}$ the *compensated Poisson random measure* (associated with M).

Next, we briefly recall the stochastic integral with respect to the jump measure M and the associated compensated Poisson random measure $\overline{\mathrm{M}}$ for measurable functions f on $(E, \mathscr{B}(E))$. First, we set $\mathrm{m}(f) := \int_E f(t, x)\mathrm{m}(\mathrm{d}t, \mathrm{d}x)$ if the integral on the right-hand side exists. If $f \geq 0$, $\mathrm{m}(f)$ always exists. Analogously, we can define the integral of f with respect to M ω-*wise*. In case that $\int_E |f(t, x)|\mathrm{M}(\mathrm{d}t, \mathrm{d}x) < +\infty$ a.s. holds, we put $\mathrm{M}(f) := \int_E f(t, x)\mathrm{M}(\mathrm{d}t, \mathrm{d}x)$ a.s. and call $\mathrm{M}(f)$ the *stochastic integral* of f with respect to M. From Kallenberg [14], Lemma 12.13, we know that $\mathrm{M}(f)$ exists and is finite a.s. if and only if $\mathrm{m}(|f| \wedge 1) < \infty$. For $f \in L^1(\mathrm{m})$ we have $\mathbb{E}[\mathrm{M}(f)] = \mathrm{m}(f)$.

The stochastic integral with respect to $\overline{\mathrm{M}}$ for deterministic functions in $L^2(\mathrm{m})$ is defined as in Gihman and Skorohod [8], IV, Section 4, for a general elementary orthogonal random measure, and we do not repeat the definition in detail. We only recall that in a first step the stochastic integral with respect to $\overline{\mathrm{M}}$ is defined for

simple functions in $L^2(\mathrm{m})$ and is then extended to arbitrary functions in $L^2(\mathrm{m})$ by isometry using the denseness of the simple functions: There exists a unique isometric mapping on $L^2(\mathrm{m})$ into $L^2(\mathbb{P})$, again denoted by $\overline{\mathrm{M}}$, such that $\overline{\mathrm{M}}(1_B) = \overline{\mathrm{M}}(B)$, $B \in \mathscr{E}$. If $f \in L^2(\mathrm{m})$, then $\overline{\mathrm{M}}(f)$ is called the stochastic integral of f with respect to the compensated Poisson random measure $\overline{\mathrm{M}}$. The proof of the following proposition is left to the reader.

PROPOSITION 6.1. *If $f \in L^1(\mathrm{m}) \cap L^2(\mathrm{m})$, then $\overline{\mathrm{M}}(f) = \mathrm{M}(f) - \mathrm{m}(f)$.*

The Itô–Lévy decomposition of L asserts that there exists a Wiener process relative to $\mathbb{F}$ with *variance function* $\sigma^2(t) := \sigma^2 t$, say W^σ, such that the following decomposition holds: For all $t \in [0, T]$, a.s.,

$$(37) \qquad L_t = \beta t + \mathrm{W}_t^\sigma + \mathrm{M}(1_{[0,t] \times \{|x| > 1\}} x) + \overline{\mathrm{M}}(1_{[0,t] \times \{|x| \leq 1\}} x),$$

where $\beta \in \mathbb{R}$, cf. Kallenberg [14], Theorem 15.4. We call the triplet (β, σ^2, ν) the characteristics of L and the process W^σ the Gaussian part of L. For a Lévy process L with Gaussian part W^σ and Lévy measure ν, we introduce the measure μ by

$$(38) \qquad \mu := \sigma^2 \delta_0 + \nu,$$

where δ_0 denotes the Dirac measure in the origin. Since $\nu(\{0\}) = 0$, without loss of generality we can assume that $f(0) = 0$ for every $f \in L^p(\nu)$, $p \in [1, +\infty]$. With this convention, $L^p(\nu)$ is a subspace of $L^p(\mu)$. For any $f \in L^2(\mu)$ we introduce the martingale $X^{(f)} = (X_t^{(f)})_{t \geq 0}$ by

$$(39) \qquad X_t^{(f)} = f(0) \mathrm{W}_t^\sigma + \overline{\mathrm{M}}(1_{[0,t]} 1_{\mathbb{R} \setminus \{0\}} f), \qquad t \in [0, T].$$

We stress that for $f = 1_{\{0\}}$ we get $X^{(f)} = \mathrm{W}^\sigma$ as a special case.

For a measure ϱ, we use the notation $\varrho(f) := \int_{\mathbb{R}} f(x) \varrho(\mathrm{d}x)$ if the integral on the right exists.

THEOREM 6.2. *For every $f \in L^2(\mu)$ the process $X^{(f)}$ defined by (39) has the following properties*:

   (i) $(X^{(f)}, \mathbb{F}^L)$ *is a Lévy process and* $\mathbb{E}[(X_t^{(f)})^2] = t\mu(f^2) < +\infty$.
  (ii) $X^{(f)} \in \mathscr{H}^2(\mathbb{F}^L)$ *and* $\langle X^{(f)}, X^{(g)} \rangle_t = t\mu(fg)$, $f, g \in L^2(\mu)$.
 (iii) $\Delta X^{(f)} = f(\Delta L) 1_{\{\Delta L \neq 0\}}$ *a.s. and* $X^{(f)}$ *is locally bounded if f is bounded.*
 (iv) $X^{(f)} = 0$ *a.s. if and only if $f = 0$ μ-a.e.*
  (v) $X^{(f)}$ *and* $X^{(g)}$ *are orthogonal if and only if $f, g \in L^2(\mu)$ are orthogonal.*

Compensated-covariation stable families. As a preliminary step, given a Lévy process L with characteristics (β, σ^2, ν), our aim is to construct

compensated-covariation stable families of $\mathbb{F}^L$-martingales possessing the CRP on the space $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$.

Let $(L, \mathbb{F}^L)$ be a Lévy process with Gaussian part W^σ and jump measure M; $\mathbb{F}^{W^\sigma} = (\mathscr{F}_t^{W^\sigma})_{t \in [0,T]}$ denotes the completion in $\mathscr{F}_T^L$ of the filtration generated by W^σ. For every $t \in [0, T]$, we introduce the σ-algebra

$$\mathscr{F}_t^M := \sigma(\{M(A), A \in \mathscr{B}(E), A \subseteq [0, t] \times \mathbb{R}\}) \vee \mathscr{N}(\mathbb{P}),$$

where $\mathscr{N}(\mathbb{P})$ denotes the system of the $\mathbb{P}$-null sets of $\mathscr{F}_T^L$, and put $\mathbb{F}^M = (\mathscr{F}_t^M)_{t \in [0,T]}$. It is easy to see that $\mathbb{F}^L = \mathbb{F}^{W^\sigma} \vee \mathbb{F}^M$.

We shall consider systems $\mathscr{C}$ of real functions with the following properties:

ASSUMPTION 6.3. (i) $\mathscr{C} \subseteq L^1(\mu) \cap L^2(\mu)$; (ii) $\mathscr{C}$ is total in $L^2(\mu)$; (iii) $\mathscr{C}$ is stable under multiplication and $1_{\mathbb{R}\setminus\{0\}}f \in \mathscr{C}$ whenever $f \in \mathscr{C}$; (iv) $\mathscr{C}$ is a system of bounded functions.

We observe that a system $\mathscr{C}$ satisfying Assumption 6.3 always exists: Obviously, we can choose

$$\mathscr{C} := \{f = c1_{\{0\}} + 1_{(a,b]}, a, b \in \mathbb{R} : a < b, 0 \notin [a, b]; c \in \mathbb{R}\} \cup \{0\}$$

as an example.

For a system $\mathscr{C}$ satisfying Assumption 6.3, we introduce the set $\tilde{\mathscr{C}}$ of all $\tilde{f} := 1_{\mathbb{R}\setminus\{0\}}f$, $f \in \mathscr{C}$. Recalling the convention above, we observe that $\tilde{\mathscr{C}}$ is a subset of $L^1(\nu) \cap L^2(\nu)$, is total in $L^2(\nu)$ and is stable under multiplication. We also define the family

$$(40) \qquad \mathscr{X}_\mathscr{C} := \{X^{(f)}, f \in \mathscr{C}\},$$

where the martingales $X^{(f)}$ are introduced in (39). Then the following proposition holds.

PROPOSITION 6.4. *The family $\mathscr{X}_\mathscr{C}$ is a compensated-covariation stable family of $\mathbb{F}^L$-martingales in $\mathscr{H}^2(\mathbb{F}^L)$. Moreover $\mathscr{X}_\mathscr{C}$ generates the filtration $\mathbb{F}^L$; $\mathbb{E}[\exp(\lambda|X_t|)] < +\infty$ for every $X \in \mathscr{X}_\mathscr{C}$, $\lambda > 0$, $t \in [0, T]$, and $\langle X, Y \rangle$ is deterministic for every $X, Y \in \mathscr{X}_\mathscr{C}$.*

PROOF. It is clear that $\mathscr{X}_\mathscr{C} \subseteq \mathscr{H}^2(\mathbb{F}^L)$. Now we show that $\mathscr{F}_T^{\mathscr{X}_\mathscr{C}} = \mathscr{F}_T^L$ $(= \mathscr{F}_T^{W^\sigma} \vee \mathscr{F}_T^M)$. By assumption we have $f \in L^1(\mu) \cap L^2(\mu)$ for $f \in \mathscr{C}$. From this, it follows that $\tilde{f} := 1_{\mathbb{R}\setminus\{0\}}f$ belongs to $L^1(\nu) \cap L^2(\nu)$ and an application of Proposition 6.1 yields $X_t^{(f)} = f(0)W_t^\sigma + M(1_{[0,t]}\tilde{f}) - t\nu(\tilde{f})$, $t \geq 0$. We set $\mathscr{G} := \sigma(\{M(1_{[0,t]}\tilde{f}), t \in [0, T], f \in \tilde{\mathscr{C}}\}) \vee \mathscr{N}(\mathbb{P})$. Recall that $\tilde{\mathscr{C}}$ is total in $L^2(\nu)$. It is evident that $\mathscr{F}_T^{\mathscr{X}_\mathscr{C}} = \mathscr{F}_T^{W^\sigma} \vee \mathscr{G}$ and therefore it is enough to verify that $\mathscr{G} = \mathscr{F}_T^M$. Recalling the definition of $\mathscr{F}_T^M$ we easily obtain that $M(1_{[0,\cdot]}g)$ is

$\mathscr{F}_T^{\mathrm{M}}$-measurable for every nonnegative measurable function g. Since $\tilde{\mathscr{C}} \subseteq L^1(\nu)$, we can write $\mathrm{M}(1_{[0,t]}\tilde{f}) = \mathrm{M}(1_{[0,t]}\tilde{f}^+) - \mathrm{M}(1_{[0,t]}\tilde{f}^-)$ a.s. for every $\tilde{f} \in \tilde{\mathscr{C}}$ and $t \geq 0$, which is $\mathscr{F}_T^{\mathrm{M}}$-measurable. This yields the inclusion $\mathscr{G} \vee \mathscr{N}(\mathbb{P}) \subseteq \mathscr{F}_T^{\mathrm{M}}$. Conversely, let $B_n \subseteq \mathbb{R}$ be such that $B_n \uparrow \mathbb{R}$ and $\nu(B_n) < +\infty$ for all $n \geq 1$. Using the monotone class theorem, we deduce that $\mathrm{M}(1_{[0,t] \times B_n} g)$ is $\mathscr{G} \vee \mathscr{N}(\mathbb{P})$-measurable for every bounded measurable function g on $E = [0, T] \times \mathbb{R}$ and hence for $g = 1_A$ with $A \in \mathscr{B}(E)$. Finally, letting $n \to \infty$, we observe that $\mathrm{M}(([0,t] \times \mathbb{R}) \cap A)$ is $\mathscr{G} \vee \mathscr{N}(\mathbb{P})$-measurable for all $A \in \mathscr{B}(E)$, proving the inclusion $\mathscr{F}_T^{\mathrm{M}} \subseteq \mathscr{G} \vee \mathscr{N}(\mathbb{P})$. Next, we show that the family $\mathscr{K}$ of monomials generated by $\mathscr{X}_{\mathscr{C}}$ is total in $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}_{\mathscr{C}}}, \mathbb{P})$ and hence, from the previous step, in $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. Indeed, the Lévy measure of $X^{(f)}$ is $\nu^{\tilde{f}}$, where $\nu^{\tilde{f}}(\{0\}) := 0$ and $\nu^{\tilde{f}}(B) := \nu \circ \tilde{f}^{-1}(B)$, $B \in \mathscr{B}(\mathbb{R} \setminus \{0\})$. Because each function $\tilde{f}$ in $\tilde{\mathscr{C}}$ is bounded, $\nu^{\tilde{f}}$ has bounded support. From Sato [19], Lemmas 25.6 and 25.7, we can deduce that $X_t^{(f)}$ has finite exponential moments of every order for all $t \in [0, T]$ and $f \in \mathscr{C}$. Now the claim follows from Theorem 5.4. From Theorem 6.2, it is clear that for all $f, g \in \mathscr{C}$ the brackets $\langle X^{(f)}, X^{(g)} \rangle$ are deterministic. It remains to show that $\mathscr{X}_{\mathscr{C}}$ is compensated-covariation stable. Let $f, g \in \mathscr{C}$ and define $h := fg1_{\mathbb{R}\setminus\{0\}}$. We notice that h again belongs to $\mathscr{C}$. Using (14), (1), Theorem 6.2(ii), (iii) and Proposition 6.1, we can compute

$$
\begin{aligned}
X_t^{(f,g)} &:= \big[X^{(f)}, X^{(g)}\big]_t - \big\langle X^{(f)}, X^{(g)} \big\rangle_t \\
&= f(0)g(0)\sigma^2 t + \sum_{0 \leq s \leq t} \tilde{f}(\Delta L_s)\tilde{g}(\Delta L_s)1_{\{\Delta L_s \neq 0\}} - \mu(fg)t \\
&= \sum_{0 \leq s \leq t} h(\Delta L_s)1_{\{\Delta L_s \neq 0\}} - \nu(h)t \\
&= \overline{\mathrm{M}}(1_{[0,t]}h) = X_t^{(h)}, \qquad t \in [0, T], \text{ a.s.}
\end{aligned}
$$

Hence, $X^{(f,g)}$ belongs to $\mathscr{X}_{\mathscr{C}}$ proving that $\mathscr{X}_{\mathscr{C}}$ is a compensated-covariation stable family. $\square$

As a consequence of Proposition 6.4 and of Theorem 5.8, we get the following result.

PROPOSITION 6.5. *Let* $(L, \mathbb{F}^L)$ *be a Lévy process with characteristics* (β, σ^2, ν), μ *be the measure defined in* (38) *and* $\mathscr{C} \subseteq L^2(\mu)$ *satisfy Assumption* 6.3. *Then the family* $\mathscr{X}_{\mathscr{C}}$ *defined in* (40) *possesses the CRP on* $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$.

PROOF. From Proposition 6.4 and Theorem 5.4, it follows that the family $\mathscr{X}_{\mathscr{C}}$ satisfies all the assumptions of Theorem 5.8 and therefore it possesses the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. $\square$

General families of martingales with the CRP.   Let $(L, \mathbb{F}^L)$ be a Lévy process with characteristic triplet (β, σ^2, ν) and let μ be as in (38). With a system $\mathscr{T} \subseteq L^2(\mu)$, we associate the family $\mathscr{X}_{\mathscr{T}}$ by

$$(41) \qquad\qquad \mathscr{X}_{\mathscr{T}} := \{X^{(f)}, f \in \mathscr{T}\}.$$

In this subsection, we give necessary and sufficient conditions on $\mathscr{T}$ for $\mathscr{X}_{\mathscr{T}}$ to possess the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$.

We stress that in general the family $\mathscr{X}_{\mathscr{T}}$ need not satisfy all the assumptions of Theorem 5.8. In particular, the family $\mathscr{X}_{\mathscr{T}}$ need not be compensated-covariation stable or possess exponential moments.

THEOREM 6.6.   *Let $\mathscr{T}$ be a system of functions in $L^2(\mu)$, where μ is defined in* (38). *The family $\mathscr{X}_{\mathscr{T}}$ defined in* (41) *possesses the CRP with respect to $\mathbb{F}^L$ if and only if $\mathscr{T}$ is total in $L^2(\mu)$.*

PROOF.   First, we assume that the family $\mathscr{X}_{\mathscr{T}}$ possesses the CRP and show that $\mathscr{T}$ is total in $L^2(\mu)$. We choose a function $h \in L^2(\mu)$ which is orthogonal to $\mathscr{T}$ in $L^2(\mu)$. By Theorem 6.2(v), the martingale $X^{(h)} \in \mathscr{H}^2$ associated with h is orthogonal to $\mathscr{X}_{\mathscr{T}}$. For an elementary iterated integral

$$J_n^{(f_1, \ldots, f_n)}(F)_t := \int_0^t J_{n-1}^{(f_1, \ldots, f_{n-1})}(F_0 \otimes \cdots \otimes F_{n-1})_{u-} F_n(u)\, dX_u^{(f_n)},$$

$t \in [0, T], n \geq 1$, with respect to $(X^{(f_1)}, X^{(f_2)}, \ldots, X^{(f_n)})$, $f_k \in \mathscr{T}$, $k = 1, \ldots, n$, where $F = F_0 \otimes \cdots \otimes F_n$ is an elementary function (see Definition 3.1), for all $t \in [0, T]$ we obtain

$$\langle J_n^{(f_1, \ldots, f_n)}(F), X^{(h)} \rangle_t$$
$$= \int_0^t J_{n-1}^{(f_1, \ldots, f_{n-1})}(F_0 \otimes \cdots \otimes F_{n-1})_{u-} F_n(u)\, d\langle X^{(f_n)}, X^{(h)} \rangle_u = 0.$$

Hence, the martingales $X^{(h)}$ and $J_n^{(f_1, \ldots, f_n)}(F)$ are orthogonal. It is clear that $X^{(h)}$ is also orthogonal to $\mathscr{J}_0$. This implies that the terminal value $X_T^{(h)}$ is orthogonal to $J_n^{(f_1, \ldots, f_n)}(F)_T$ and also orthogonal to $\mathscr{J}_{0,T}$ in $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. Recalling the construction of the space $\mathscr{J}_T^{\mathscr{X}_{\mathscr{T}}}$ of the iterated integrals generated by $\mathscr{X}_{\mathscr{T}}$ [cf. Definition 3.5 and Proposition 3.7(i)], we observe that the system of elementary iterated integrals of order n $(n \geq 0)$ is total in $\mathscr{J}_T^{\mathscr{X}_{\mathscr{T}}}$. Consequently, $X_T^{(h)}$ is orthogonal to $\mathscr{J}_T^{\mathscr{X}_{\mathscr{T}}}$. By definition of the CRP, $\mathscr{J}_T^{\mathscr{X}_{\mathscr{T}}} = L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ and hence $X_T^{(h)} = 0$. From this, we deduce that the martingale $X^{(h)}$ is indistinguishable from the null-process and by Theorem 6.2(iv) it follows $h = 0$ μ-a.e. This proves that $\mathscr{T}$ is total in $L^2(\mu)$. Conversely, we now assume that $\mathscr{T}$ is total in $L^2(\mu)$ and show that $\mathscr{X}_{\mathscr{T}}$ has the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. For this purpose, we consider a system $\mathscr{C}$

satisfying Assumption 6.3. From Proposition 6.5, we know that $\mathscr{X}_{\mathscr{C}}$ has the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. We denote by $\mathscr{J}^{\mathscr{X}_{\mathscr{C}}}$ the space of iterated integrals generated by $\mathscr{X}_{\mathscr{C}}$. It is enough to prove that $\mathscr{J}^{\mathscr{X}_{\mathscr{C}}} = \mathscr{J}^{\mathscr{X}_{\mathscr{T}}}$. Because $\mathscr{T}$ is total in $L^2(\mu)$ and the mapping $f \mapsto X^{(f)}$ is linear and isometric from $L^2(T\mu)$ into $\mathscr{H}^2$ (see Theorem 6.2), we immediately establish $\mathrm{cl}(\mathrm{Span}(\mathscr{X}_{\mathscr{T}}))_{\mathscr{H}^2} = \mathscr{X}_{L^2(\mu)}$ and hence the inclusion $\mathscr{X}_{\mathscr{C}} \subseteq \mathrm{cl}(\mathrm{Span}(\mathscr{X}_{\mathscr{T}}))_{\mathscr{H}^2}$ holds. Using Theorem 3.9 for $\mathscr{X} = \mathscr{X}_{\mathscr{T}}$ we conclude $\mathscr{J}^{\mathscr{X}_{\mathscr{C}}} = \mathscr{J}^{\mathscr{X}_{\mathscr{T}}}$, proving the claim. This completes the proof of the theorem. $\square$

We remark that if L is a square integrable Lévy process, or equivalently the function x belongs to $L^2(\nu)$, we can choose the total system $\mathscr{T} \subseteq L^2(\mu)$ in such a way that the function $h := 1_{\{0\}} + x$ belongs to $\mathscr{T}$. In this case, we have that $X^{(h)} = \overline{L}$, where $\overline{L}_t := L_t - \mathbb{E}[L_t]$, $t \in [0, T]$. In other words, the Lévy process $\overline{L}$ can be included in the family $\mathscr{X}_{\mathscr{T}}$.

An important question is in which cases it is possible to choose a family $\mathscr{X}_{\mathscr{T}}$ consisting of finitely many martingales and possessing the CRP. The next corollary explains that this is possible only in a rather few cases.

COROLLARY 6.7. *Let $(L, \mathbb{F}^L)$ be a Lévy process with characteristics (β, σ^2, ν). The following statements are equivalent*: (i) *There exists a finite family $\mathscr{X}_{\mathscr{T}}$ possessing the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$*; (ii) $L^2(\mu)$ *is finite-dimensional*; (iii) ν *has finite support*.

The situation discussed in Corollary 6.7 occurs if L is a *simple* Lévy process, that is, it is of the form $L_t := \mathrm{W}_t^\sigma + \alpha_1 N_t^1 + \cdots + \alpha_m N_t^m$, $t \geq 0$, where $(\mathrm{W}^\sigma, \mathbb{F}^L)$ is a Brownian motion with variance function $\langle \mathrm{W}^\sigma, \mathrm{W}^\sigma \rangle_t = \sigma^2 t$; $(N^j, \mathbb{F}^L)$ a homogeneous Poisson process with parameter $\gamma_j > 0$, $j = 1, \ldots, m$, and $(N^1, \ldots, N^m)$ is an independent vector of processes; $\alpha_1, \ldots, \alpha_m$ are real numbers.

If $\mathscr{T} \subseteq L^2(\mu)$ is a *complete orthogonal system*, say $\mathscr{T} := \{f_n, n \geq 1\}$ [note that $L^2(\mu)$ is a separable Hilbert space], then the associated family $\mathscr{X}_{\mathscr{T}}$ consists of countably many mutually orthogonal martingales [cf. Theorem 6.2(v)] and Theorem 6.6, Theorem 3.11 and (13) yield the following theorem. Note that $\mathscr{J}_0 = \mathscr{J}_{0,T} = \mathbb{R}$ because the σ-field $\mathscr{F}_0^{\mathscr{X}_{\mathscr{T}}} = \mathscr{F}_0^L$ is $\mathbb{P}$-trivial.

THEOREM 6.8. *Let $(L, \mathbb{F}^L)$ be a Lévy process with characteristics (β, σ^2, ν) and $\mathscr{T} = \{f_n, n \geq 1\}$ be a complete orthogonal system in $L^2(\mu)$ where μ is as in (38). Then the associated family $\mathscr{X} = \mathscr{X}_{\mathscr{T}}$ has the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ and the following decompositions hold*:

(42)
$$\mathscr{H}^2(\mathbb{F}^{\mathscr{X}}) = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1, \ldots, j_n) \in \mathbb{N}^n} \mathscr{J}_n^{(f_{j_1}, \ldots, f_{j_n})},$$

$$L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P}) = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{(j_1, \ldots, j_n) \in \mathbb{N}^n} \mathscr{J}_{n,T}^{(f_{j_1}, \ldots, f_{j_n})},$$

where, for $f_1, \ldots, f_n \in \mathcal{T}$, $\mathcal{J}_n^{(f_1, \ldots, f_n)}$ denotes the linear space of n-fold iterated integrals with respect to $(X^{(f_1)}, \ldots, X^{(f_n)})$; $n \geq 1$.

Multiple Itô integrals and iterated integrals. The aim of this subsection is to establish the relation between *multiple* Itô integrals introduced in Itô [10] and *iterated* integrals introduced in Section 3.

Let L be a Lévy process with characteristics (β, σ^2, ν). The measure μ is defined in (38) while the martingale $X^{(f)}$ for $f \in L^2(\mu)$ in (39) above.

We introduce the measures ζ and η by

$$\zeta(B) := \int_B x^2 \nu(dx), \quad B \in \mathscr{B}(\mathbb{R}); \qquad \eta = \sigma^2 \delta_0 + \zeta.$$

For any square integrable function G on $([0, T] \times \mathbb{R})^n$ with respect to the measure $(\lambda_+ \otimes \eta)^n$ (recall that λ_+ denotes the Lebesgue measure on $\mathbb{R}_+$) the *multiple* Itô integral $I_n(G)$ of order n is defined as follows: For $n = 1$ and $G_1 \in L^2(\lambda_+ \otimes \eta)$, we put

$$I_1(G_1)_t := G_1(\cdot, 0) \cdot W_t^\sigma + \overline{M}(1_{[0,t]} x G_1), \qquad t \in [0, T].$$

Now, if $G = 1_{A_1} \otimes \cdots \otimes 1_{A_n}$ with $A_1, \ldots, A_n \in \mathscr{B}([0, T] \times \mathbb{R})$ pairwise disjoint and $(\lambda_+ \otimes \eta)(A_i) < +\infty$, $i = 1, \ldots, n$, then we define

$$I_n(G)_t := I_1(1_{A_1})_t \cdots I_1(1_{A_n})_t, \qquad t \in [0, T].$$

The system of all these functions G is total in the L^2-space over $([0, T] \times \mathbb{R})^n$ equipped with the measure $(\lambda_+ \otimes \eta)^n$ (cf. Itô [9], Theorem 2.1). The mapping $G \mapsto I_n(G)_t$ can now be extended to $L^2(([0, T] \times \mathbb{R})^n, (\lambda_+ \otimes \eta)^n)$ by linearity and continuity (see Itô [10] for more details).

Let $f_1, \ldots, f_n$ from $L^2(\mu)$ be normalized. Note that then the measures $m^{(f_i)}$ on $[0, T]$ associated with $\langle X^{(f_i)}, X^{(f_i)} \rangle$ are equal to λ_+, $i = 1, \ldots, n$, and therefore the space $L^2(\Omega \times [0, T]^n, m_{\mathbb{P}}^{(f_1, \ldots, f_n)})$ is just $L^2(\Omega \times [0, T]^n, \mathbb{P} \otimes \lambda_+^n)$. For any function F from $L^2([0, T]^n) := L^2([0, T]^n, \lambda_+^n)$ the function $1 \otimes F$ belongs to the space $L^2(\Omega \times [0, T]^n, \mathbb{P} \otimes \lambda_+^n)$ and above we have introduced the *iterated integral* $J_n^{(f_1, \ldots, f_n)}(1 \otimes F) = (J_n^{(f_1, \ldots, f_n)}(1 \otimes F)_t)_{t \in [0, T]}$ with respect to the martingales $(X^{(f_1)}, \ldots, X^{(f_n)})$ (cf. Definition 3.3).

With a real function g defined on $\mathbb{R}$, we associate the function $\hat{g}$ defining

$$(43) \qquad \hat{g}(x) = \begin{cases} xg(x), & \text{if } x \neq 0, \\ g(0), & \text{if } x = 0. \end{cases}$$

If $g \in L^2(\eta)$, then the associated function $\hat{g}$ belongs to $L^2(\mu)$ and conversely. Obviously, $\|g\|_{L^2(\eta)} = \|\hat{g}\|_{L^2(\mu)}$. Now we consider normalized $g_1, \ldots, g_n \in L^2(\eta)$ and put $f_i := \hat{g}_i$, $i = 1, \ldots, n$. Then the functions $f_1, \ldots, f_n \in L^2(\mu)$ are normalized, too. Furthermore, we choose $F \in L^2([0, T]^n)$. It is clear that the function $G = F g_1 \otimes \cdots \otimes g_n$ belongs to $L^2((\lambda_+ \otimes \eta)^n)$ and hence the multiple integral

$I_n(G)$ is well-defined. In the next proposition, we denote the set of all permutations of $\{1, \ldots, n\}$ by Π_n and its generic element by $\pi = (i_1, \ldots, i_n)$. The mapping σ_π on $[0, T]^n$ is defined as $\sigma_\pi(t_1, \ldots, t_n) = (t_{i_1}, \ldots, t_{i_n})$.

PROPOSITION 6.9. *Let $F \in L^2([0, T]^n)$, $g_1, \ldots, g_n \in L^2(\eta)$ be normalized and define $f_i := \hat{g}_i$, $i = 1, \ldots, n$, as in* (43). *Then the following relation between the multiple Itô integral and the iterated integral holds*:

$$I_n(Fg_1 \otimes \cdots \otimes g_n)_t = \sum_{\pi=(i_1,\ldots,i_n)\in\Pi_n} J_n^{(f_{i_1},\ldots,f_{i_n})}\big(1 \otimes (F \circ \sigma_\pi)\big)_t, \qquad t \in [0, T].$$

In particular, recalling the notation $M_t^{(n)}$ of (4),

$$I_n(F 1_{M_t^{(n)}} g_1 \otimes \cdots \otimes g_n)_t = J_n^{(f_1,\ldots,f_n)}(1 \otimes F)_t, \qquad t \in [0, T].$$

PROOF. Let $F = F_1 \otimes \cdots \otimes F_n$ be such that $F_i = 1_{B_i}$, $i = 1, \ldots, n$; $B_1, \ldots, B_n \in \mathscr{B}([0, T])$ and pairwise disjoint. The normalized functions $g_1, \ldots, g_n$ are chosen such that $g_i = c_i 1_{C_i}$, $C_i \in \mathscr{B}(\mathbb{R})$, $0 < \eta(C_i) < +\infty$, $c_i > 0$, $i = 1, \ldots, n$. Using the definition of Itô's multiple integral and of $X^{(f_i)}$ [cf. (39)] with $f_i := \hat{g}_i$ we obtain

$$I_n(Fg_1 \otimes \cdots \otimes g_n)_t = \prod_{i=1}^n \int_0^t F_i(u) \, dX_u^{(f_i)}, \qquad t \in [0, T].$$

Because $B_1, \ldots, B_n$ are pairwise disjoint, setting $Z^i := \int_0^{\cdot} F_i(u) \, dX_u^{(f_i)}$, it follows $[Z^i, Z^j] = 0$, $i \neq j$, $i, j = 1, \ldots, n$. Therefore, an application of Itô's formula yields

$$\prod_{i=1}^n \int_0^t F_i(u) \, dX_u^{(f_i)} = \sum_{i=1}^n \int_0^t \prod_{\substack{k=1 \\ k \neq i}}^n \int_0^{u-} F_k(s) \, dX_s^{(f_k)} F_i(u) \, dX_u^{(f_i)}, \qquad t \in [0, T].$$

By an induction argument, we can now derive

$$(44) \quad I_n(Fg_1 \otimes \cdots \otimes g_n)_t = \sum_{\pi=(i_1,\ldots,i_n)\in\Pi_n} J_n^{(f_{i_1},\ldots,f_{i_n})}(F \circ \sigma_\pi)_t, \qquad t \in [0, T].$$

The set of functions F considered in the previous step is total in $L^2([0, T]^n)$ (cf., e.g., Itô [9], Theorem 2.1). The right- and left-hand sides of identity (44) are linear and continuous in F on $L^2([0, T]^n)$. Hence (44) is valid for all $F \in L^2([0, T]^n)$. Now we fix $F \in L^2([0, T]^n)$. From the previous step, we know that (44) is valid for all $g_1, \ldots, g_n$ chosen as in the first step of the proof. Clearly, the set of all normalized $g_1 = c_1 1_{C_1}$ with $C_1 \in \mathscr{B}(\mathbb{R})$ and $0 < \eta(C_1) < +\infty$, $c_1 > 0$, is total in $L^2(\eta)$. The right- and left-hand sides of identity (44) being linear and continuous in g_1 on $L^2(\eta)$, identity (44) remains valid for all $g_1 \in L^2(\eta)$. Repeating the argument

for all $i = 2, \ldots, n$ yields that (44) is valid for all $g_1, \ldots, g_n \in L^2(\eta)$. The proof of the proposition is now complete. $\square$

By $\mathscr{H}_{n,T}$ we denote the closed linear subspace of $L^2(\mathbb{P}) := L^2(\Omega, \mathcal{F}_T^L, \mathbb{P})$ consisting of the terminal values $I_n(G)_T$ of all the multiple Itô integrals of order n. Itô's chaos expansion (see Itô [10], Theorem 2) states the following decomposition of $L^2(\mathbb{P})$:

$$(45) \qquad L^2(\mathbb{P}) = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}_{n,T}.$$

The decomposition (45) can be deduced as a consequence of Theorem 6.6. Indeed, the linear space $\mathscr{J}_T$ of terminal values of all iterated integrals with respect to $\mathscr{X}_{\mathscr{T}} := \{X^{(f)} : f \in \mathscr{T}\}$, where $\mathscr{T} \subseteq L^2(\mu)$ is a total set, is dense in $L^2(\mathbb{P})$. But from Proposition 6.9, we deduce $\mathscr{J}_T \subseteq \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}_n(T)$.

We conclude this subsection deriving Theorem 6.6 from the Itô chaos decomposition. If $(g_k)_{k \geq 1}$ is a complete orthonormal system in the space $L^2(\eta)$, $(g_{k_1,\ldots,k_n})_{(k_1,\ldots,k_n) \in \mathbb{N}^n} := (g_{k_1} \otimes \cdots \otimes g_{k_n})_{(k_1,\ldots,k_n) \in \mathbb{N}^n}$ is a complete orthonormal system in $L^2(\mathbb{R}^n, \eta^n)$. For any function $G \in L^2(([0,T] \times \mathbb{R})^n, (\lambda_+ \otimes \eta)^n)$ and $(t_1, \ldots, t_n) \in [0,T]^n$ we introduce the function $G_{t_1,\ldots,t_n}$ by $G_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = G((t_1, x_1), \ldots, (t_n, x_n))$, where $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Clearly, the set of all $(t_1, \ldots, t_n) \in [0,T]^n$ such that $G_{t_1,\ldots,t_n}$ does not belong to $L^2(\mathbb{R}^n, \eta^n)$ is measurable and has zero λ_+^n-measure. Hence, without loss of generality, we can put $G_{t_1,\ldots,t_n} \equiv 0$ for such points. Consequently, $G_{t_1,\ldots,t_n}$ belongs to $L^2(\mathbb{R}^n, \eta^n)$ for all $(t_1, \ldots, t_n)$ from $[0,T]^n$. Developing $G_{t_1,\ldots,t_n}$ as Fourier series gives $G_{t_1,\ldots,t_n} = \sum_{(k_1,\ldots,k_n) \in \mathbb{N}^n} c(t_1, \ldots, t_n; k_1, \ldots, k_n) g_{k_1,\ldots,k_n}$ in $L^2(\mathbb{R}^n, \eta^n)$ where the function $c(t_1, \ldots, t_n; k_1, \ldots, k_n) = (G_{t_1,\ldots,t_n}, g_{k_1,\ldots,k_n})_{L^2(\mathbb{R}^n, \eta^n)}$ is measurable in $(t_1, \ldots, t_n) \in [0,T]^n$. Now we verify that

$$(46) \qquad G = \sum_{(k_1,\ldots,k_n) \in \mathbb{N}^n} c(\cdot, k_1, \ldots, k_n) g_{k_1,\ldots,k_n}$$

(convergence in $L^2(([0,T] \times \mathbb{R})^n, (\lambda_+ \otimes \eta)^n)$). We already know that the series converges to $G_{t_1,\ldots,t_n}$ in $L^2(\mathbb{R}^n, \eta^n)$ and hence in η^n-measure for every $(t_1, \ldots, t_n) \in [0,T]^n$. Therefore, the series converges to G in $(\lambda_+ \otimes \eta)^n$-measure. We introduce the notation $\mathbb{N}_m^n := \mathbb{N}^n \setminus \{1, \ldots, m\}^n$. For proving the claim we show that $\sum_{(k_1,\ldots,k_n) \in \mathbb{N}_m^n} c(\cdot, k_1, \ldots, k_n) g_{k_1,\ldots,k_n}$ converges in the Hilbert space $L^2(([0,T] \times \mathbb{R})^n, (\lambda_+ \otimes \eta)^n)$ to zero as $m \to \infty$. From the orthonormality of $(g_{k_1,\ldots,k_n})_{(k_1,\ldots,k_n) \in \mathbb{N}^n}$ we have

$$\left\| \sum_{(k_1,\ldots,k_n) \in \mathbb{N}_m^n} c(\cdot, k_1, \ldots, k_n) g_{k_1,\ldots,k_n} \right\|_{L^2((\lambda_+ \otimes \eta)^n)}^2$$

$$= \int_{[0,T]^n} \sum_{(k_1,\ldots,k_n) \in \mathbb{N}_m^n} c^2(t_1, \ldots, t_n; k_1, \ldots, k_n) \, d\lambda_+^n(t_1, \ldots, t_n),$$

the right-hand side converging to zero as $m \to \infty$ because

$$\sum_{(k_1,\ldots,k_n)\in\mathbb{N}_m^n} c^2(t_1,\ldots,t_n;k_1,\ldots,k_n) \to 0 \qquad \text{as } m \to \infty$$

for every $(t_1,\ldots,t_n) \in [0,T]^n$ [note that $c(t_1,\ldots,t_n;k_1,\ldots,k_n)$ are Fourier coefficients] and is bounded by $\sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} c^2(t_1,\ldots,t_n;k_1,\ldots,k_n)$ being an integrable function:

$$\int_{[0,T]} \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} c^2(t_1,\ldots,t_n;k_1,\ldots,k_n)\, d\lambda_+^n(t_1,\ldots,t_n) = \|G\|_{L^2((\lambda_+\otimes\eta)^n)}^2.$$

Hence, the Fourier expansion (46) of G is verified. As a result, using the continuity of I_n and Proposition 6.9 above, we get

$$I_n(G)_t = \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} I_n\big(c(\cdot,k_1,\ldots,k_n)g_{k_1,\ldots,k_n}\big)_t$$

$$= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \sum_{\pi=(i_1,\ldots,i_n)\in\Pi_n} J_n^{(f_{k_{i_1}},\ldots,f_{k_{i_n}})}\big(c(\cdot,k_1,\ldots,k_n)\circ\sigma_\pi\big)_t.$$

Applying Itô's chaos expansion, from this we get $L^2(\mathbb{P}) = \mathbb{R}\otimes\bigoplus_{n=1}^\infty \mathscr{J}_{n,T}$ where $\mathscr{J}_{n,T}$ is the closed linear space of iterated integrals of order n with respect to the *orthogonal* family $\{X^{(f_k)} : k \in \mathbb{N}\}$ of martingales (cf. Definition 3.5) and $f_k := \hat{g}_k, k \in \mathbb{N}$ [cf. (43)]. Clearly, $(f_k)_{k\in\mathbb{N}}$ is a complete orthonormal system of $L^2(\mu)$. Conversely, we can also start from a complete orthonormal system $(f_k)_{k\in\mathbb{N}}$ of $L^2(\mu)$ and construct the complete orthonormal system $(g_k)_{k\in\mathbb{N}}$ of $L^2(\eta)$ from it.

7. Applications. In this last section, we shall provide a few applications of the main theorem of the present paper, Theorem 5.8 above. This will illustrate, in particular, that several important results on the CRP which have been known before are an immediate consequence of Theorem 5.8. However, there will also be discussed new examples which are beyond the scope of known results. We start with families of continuous local martingales which are pairwise Gausssian and state the result in a general form which to our knowledge has not been established before. We proceed with families of compensated Poisson processes. Then we pass on to concrete applications of Section 6 on the CRP of families of martingales related with Lévy processes: Teugels martingales, families of martingales constructed from Hermite polynomials and from Haar systems.

Gaussian families. On a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ and with respect to a filtration $\mathbb{F}$ satisfying the usual conditions, we consider a family $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ of continuous local martingales. We shall assume that $\mathscr{X}$ is pairwise Gaussian, that is, that the pair $(X^{(\alpha)}, X^{(\beta)})$ of processes is Gausssian for every $\alpha, \beta \in \Lambda$.

THEOREM 7.1. *The family $\mathscr{X}$ possesses the CRP relative to $\mathbb{F}^{\mathscr{X}}$.*

PROOF. First, we notice that $\mathbb{E}[X_t^{(\alpha)}]$ is continuous and hence bounded in $t \in [0, T]$ (use characteristic functions to verify this). Applying Fernique's theorem to the centred Gaussian random variable $(X_t^{(\alpha)} - \mathbb{E}[X_t^{(\alpha)}])_{t \in [0,T]}$ with values in the space $C([0, T])$ of continuous real functions on $[0, T]$, we see that $\sup_{t \in [0,T]} |X_t^{(\alpha)} - \mathbb{E}[X_t^{(\alpha)}]|$ and hence $\sup_{t \in [0,T]} |X_t^{(\alpha)}|$ is integrable. From this, we can conclude that every continuous Gaussian local martingale $X^{(\alpha)}$ is actually a Gaussian martingale and hence a process with independent increments.[3] This yields that $\langle X^{(\alpha)}, X^{(\alpha)} \rangle$ is deterministic. Applying this observation to $X^{(\alpha)} + X^{(\beta)}$ and $X^{(\alpha)} - X^{(\beta)}$ for $\alpha \neq \beta$, which are again Gaussian, by the polarization formula we obtain that $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ is deterministic. Obviously, $X_t^{(\alpha)}$ has finite exponential moments of arbitrary order for every $t \geq 0$ and $\alpha \in \Lambda$. Finally, the compensated-covariation process $[X^{(\alpha)}, X^{(\beta)}] - \langle X^{(\alpha)}, X^{(\beta)} \rangle$ equals 0 because the martingales $X^{(\alpha)}$ and $X^{(\beta)}$ are continuous [cf. (1)] for every $\alpha, \beta \in \Lambda$. Thus $\mathscr{X} \cup \{0\}$ is compensated-covariation stable. Now, upon using Theorem 5.4, Theorem 5.8 yields that $\mathscr{X} \cup \{0\}$, and therefore also $\mathscr{X}$, possesses the CRP. $\square$

If moreover $\mathscr{X}$ is a countable family of orthogonal martingales, say $\mathscr{X} := \{X^{(n)}, n \geq 1\}$, then it possesses the CRP on $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ and (13) holds.

As a special case, we get that a Brownian motion W possesses the CRP relative to its natural filtration $\mathbb{F}^W$. This is the well-known result of Itô [9] about the CRP of the Wiener process.

Poisson families. Let $\overline{N} := N - a(\cdot)$ be a compensated Poisson process with *continuous* intensity function $a(\cdot)$ (cf. Jacod and Shiryaev [12], Definition I.3.26). Clearly, $\overline{N}$ is a square integrable martingale and we can calculate

$$[\overline{N}, \overline{N}] - \langle \overline{N}, \overline{N} \rangle = \sum_{0 \leq s \leq \cdot} (\Delta N_s)^2 - a(\cdot) = \sum_{0 \leq s \leq \cdot} \Delta N_s - a(\cdot) = \overline{N}.$$

This observation leads to the following theorem.

THEOREM 7.2. *Let $\mathscr{X} := \{X^{(\alpha)}, \alpha \in \Lambda\}$ be a family of compensated Poisson processes on the probability space $(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ and with respect to the filtration $\mathbb{F}^{\mathscr{X}}$. If the family $\mathscr{X}$ is pairwise independent, then $\mathscr{X}$ possesses the CRP relative to $\mathbb{F}^{\mathscr{X}}$.*

PROOF. If $\alpha \neq \beta$, then $X^{(\alpha)}$ is independent of $X^{(\beta)}$ which implies that $[X^{(\alpha)}, X^{(\beta)}] = 0$ and hence also $\langle X^{(\alpha)}, X^{(\beta)} \rangle = 0$. As shown above, we have

---

[3]The authors are indebted to M. Urusov for pointing out the proof of this fact using Fernique's theorem.

$[X^{(\alpha)}, X^{(\alpha)}] - \langle X^{(\alpha)}, X^{(\alpha)} \rangle = X^{(\alpha)}$ and hence $\mathscr{X} \cup \{0\}$ is compensated-covariation stable. The predictable covariations $\langle X^{(\alpha)}, X^{(\beta)} \rangle$ are deterministic for all α and β from Λ. The exponential moments of $X_t^{(\alpha)}$ are finite for every $t \geq 0$ and $\alpha \in \Lambda$. Upon using Theorem 5.4, we see that Theorem 5.8 can be applied and we conclude that $\mathscr{X}$ has the CRP on $L^2(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$. $\square$

If $\mathscr{X}$ is, moreover, a countable family, say $\mathscr{X} := \{X^{(n)}, n \geq 1\}$, then (13) holds.

REMARK 7.3.   We notice that the case of Gaussian and Poisson families can be unified in the following way. Let $\mathscr{X} = \mathscr{Y} \cup \mathscr{Z}$ be a family of square integrable martingales on the probability space $(\Omega, \mathscr{F}_T^{\mathscr{X}}, \mathbb{P})$ and with respect to the filtration $\mathbb{F}^{\mathscr{X}}$. Suppose that $\mathscr{Y}$ is a pairwise Gaussian family of continuous martingales and $\mathscr{Z}$ a pairwise independent family of compensated Poisson processes. Then $\mathscr{X}$ possesses the CRP. For the proof, upon recalling the arguments from above, we have only to remark that $[Y, Z] = 0$ and hence $\langle Y, Z \rangle = 0$ for all $Y \in \mathscr{Y}$ and $Z \in \mathscr{Z}$.

Teugels martingales.   This example shows that under certain restrictions on the Lévy measure ν, Teugels martingales can be introduced as a family possessing the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. Teugels martingales were considered in Nualart and Schoutens [16] where it was proven that the orthogonalized Teugels martingales possess the CRP. We are going to obtain the CRP for the orthogonalized Teugels martingales as an application of the results of the present paper.

We fix a time horizon $[0, T]$, $T > 0$. Let $(L, \mathbb{F}^L)$ be a Lévy process with characteristic triplet (β, σ^2, ν). We require that there exist two constants $\varepsilon, \lambda > 0$ such that the function $x \mapsto e^{(\lambda/2)|x|} 1_{\{|x| > \varepsilon\}}$ is in $L^2(\nu)$. From this assumption, it follows that the function $x \mapsto x^n$ belongs to $L^1(\nu) \cap L^2(\nu)$ for every $n \geq 2$ and the identity $x \mapsto x$ is in $L^2(\nu)$. Moreover, the system $\{x \mapsto x^n, n \geq m\}$ is total in $L^2(\nu)$ for every $m \geq 1$. We set $h_1(x) = 1_{\{0\}} + x$ and, for $n \geq 2$, $h_n(x) = x^n$, $x \in \mathbb{R}$. Because of $h_n \in L^2(\mu)$ with $\mu := \sigma^2 \delta_0 + \nu$, $n \geq 1$, we can introduce the martingales $X^{(h_n)} = (X_t^{(h_n)})_{t \in [0, T]}$ as in (39). With $\mathscr{T} := \{h_n, n \geq 1\} \subseteq L^2(\mu)$, we associate $\mathscr{X}_{\mathscr{T}} := \{X^{(h_n)}, n \geq 1\}$ which turns out to be the family of *Teugels martingales*. From Theorem 6.6, $\mathscr{X}_{\mathscr{T}}$ has the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$. We stress that, because of the assumptions on the Lévy measure, the family of Teugels martingales is also compensated-covariation stable and satisfies the assumptions of Theorem 5.8. Let $\mathscr{P}$ be the system of polynomials obtained by the Gram–Schmidt orthogonalization of $\mathscr{T}$ in $L^2(\mu)$. The associated family $\mathscr{X}_{\mathscr{P}}$ of martingales is the one of the *orthogonalized* Teugels martingales: It possesses the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ and the decomposition (42) of $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ holds.

Hermite polynomials.   We consider a Lévy process $(L, \mathbb{F}^L)$ with characteristic triplet (β, σ^2, ν), where ν is of the form

$$(47) \qquad\qquad d\nu(x) = h(x)\, dx, \qquad h(x) > 0, x \in \mathbb{R}.$$

An important class of Lévy processes with Lévy measure as in (47) is, for example, the class of α-stable processes (see, e.g., Sato [19], Chapter 3). We begin with the case of the Cauchy process. A Cauchy process $(L, \mathbb{F}^L)$ is a purely non-Gaussian Lévy process with characteristic triplet $(0, 0, \nu)$ and $d\nu(x) := x^{-2}\,dx$. For a Cauchy process no moment exists and therefore Teugels martingales cannot be introduced. We choose the function g as $g(x) := x \exp(-x^2/2)$, $x \in \mathbb{R}$, and define the finite measure $d\nu^g(x) := g^2(x)\,d\nu(x) = \exp(-x^2)\,dx$. Let $(H_n)_{n=0,1,\ldots}$ be the sequence of normalized Hermite polynomials. The sequence $(H_n)_{n \geq 0}$ is an orthonormal basis for $L^2(\nu^g)$. Therefore $\mathscr{T} = \{C_n, n \geq 0\} \subseteq L^2(\nu)$, where $C_n := gH_n$, $n \geq 0$, is a complete orthonormal system in $L^2(\nu)$. Moreover, each C_n is a bounded function. In view of Theorem 6.2 and Theorem 6.8, the family $\mathscr{X}_{\mathscr{T}} := \{X^{(C_n)}, n \geq 0\}$ is a family of locally bounded orthogonal martingales with the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ for which the decomposition (42) of $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ holds. Note that the system $\mathscr{T}$ is not stable under multiplication, because the system of Hermite polynomials is not. Therefore, the family $\mathscr{X}_{\mathscr{T}}$ is not compensated-covariation stable. For the general case of a Lévy process with characteristic triplet (β, σ^2, ν) where ν is as in (47), we define $g \in L^2(\nu)$ by $g(x) := (h(x)^{-1/2}) \exp(-x^2/2)$, $x \in \mathbb{R}$, and introduce the system $\mathscr{T} := \{P_n, n \geq 0\}$, where $P_n := g(0)H_n(0)1_{\{0\}} + 1_{\mathbb{R}\setminus\{0\}} gH_n$, $n \geq 0$, $(H_n)_{n \geq 0}$ being the system of Hermite polynomials. The system $\mathscr{T}$ is an orthogonal basis for $L^2(\mu)$ and therefore the associated family $\mathscr{X}_{\mathscr{T}} := \{X^{(P_n)}, n \geq 0\}$ is an orthogonal family of martingales possessing the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ for which the decomposition (42) of $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ holds. In the general case, we cannot expect that $\mathscr{X}_{\mathscr{T}}$ consists of locally bounded martingales. Furthermore, $\mathscr{X}_{\mathscr{T}}$ is not compensated-covariation stable. Note that this variety of examples is beyond earlier known results and techniques on the CRP.

Haar wavelet. We consider a Lévy process $(L, \mathbb{F}^L)$ with characteristics (β, σ^2, ν) and Lévy measure as in (47). Let λ be the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and $\psi \in L^2(\lambda)$. The function ψ is called a *wavelet* if the system of functions $\{\psi_{jk} : \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), x \in \mathbb{R}, j, k \in \mathbb{Z}\}$ is a complete orthonormal system of $L^2(\lambda)$. An example of a wavelet, known as *Haar wavelet*, is the function defined by $\psi(x) := 1$, if $x \in [0, 1/2)$; $\psi(x) := -1$ if $x \in [1/2, 1)$ and $\psi(x) := 0$ otherwise. The system $\{\psi_{jk}, j, k \in \mathbb{Z}\}$ generated by ψ is the *Haar basis* (cf. Wojtaszczyk [23], 1). The system given by $\mathscr{T} := \{h^{-1/2}(0)\psi_{jk}(0)1_{\{0\}} + 1_{\mathbb{R}\setminus\{0\}}h^{-1/2}\psi_{jk}, j, k \in \mathbb{Z}\}$ is a complete orthogonal system in $L^2(\mu)$. The associated family $\mathscr{X}_{\mathscr{T}} := \{X^{(f)}, f \in \mathscr{T}\}$ of orthogonal martingales possesses the CRP on $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ and the decomposition (42) of $L^2(\Omega, \mathscr{F}_T^L, \mathbb{P})$ holds. This method, leading to interesting families of martingales with the CRP not considered previously, can also be extended to Lévy processes with arbitrary Lévy measure ν.

Conclusions. We conclude with a short discussion concerning the relevance of the approach and the results given in the present paper.

The first two examples about families of Gaussian continuous local martingales and pairwise independent Poisson processes are general formulations of existing results for Wiener and Poisson processes. Being immediate and straightforward applications of Theorem 5.8, these examples demonstrate the power and flexibility of our approach to the chaotic representation property.

The most innovative examples of the present paper are those about the construction of families of martingales associated with a Lévy process which do possess the CRP. Beside the classical case of Brownian motion and compensated Poisson process, only one example, the family of Teugels martingales, has been investigated before by Nualart and Schoutens in [16]. However, this example requires the quite strong assumption on the underlying Lévy process L that L_t $(t > 0)$ possesses a finite exponential moment of some order $\lambda > 0$.

As we have seen in Section 7, the case of Teugels martingales can be directly deduced from the general approach given in Section 5 and applied to Lévy processes in Section 6. It should be emphasized that the family of Teugels martingales (if it can be constructed) is only *one example* of a great variety of families of martingales of a Lévy process possessing the CRP. In addition, Theorem 6.6 (as a consequence of Theorem 5.8) allows us to construct families of martingales possessing the CRP on $\mathbb{F}^L$ for an *arbitrary* Lévy process L, *without any assumption* on the Lévy measure ν. This is outside the scope of other techniques and, in particular, those of Nualart and Schoutens [16]. Applications are illustrated in the examples *Hermite Polynomials* and *Haar Wavelet*, where we investigated Lévy processes for which the Lévy measure ν only is equivalent to the Lebesgue measure on $\mathbb{R}$. This is a rather wide class of Lévy processes, containing also processes, as the Cauchy process, which do not have any finite moments. For such cases it is impossible to follow the techniques of Nualart and Schoutens [16].

The assumption of the equivalence of ν to the Lebesgue measure is not important at all. As proved in Theorem 6.6, a necessary and sufficient condition for a family $\mathscr{X}_{\mathscr{T}}$ to possess the CRP with respect to $\mathbb{F}^L$ is that the system $\mathscr{T} \subseteq L^2(\mu)$ is total, $\mu := \sigma^2 \delta_0 + \nu$, without further assumptions on ν. In particular, such a family of martingales can be associated with any complete orthonormal system $\mathscr{T} = \{f_n : n \geq 1\}$ of $L^2(\mu)$. This result is probably one of the most important of the present paper because it allows to construct numerous families of martingales possessing the CRP starting from the characteristics (β, σ^2, ν) of a given Lévy process L. We have only to look for adequate complete orthonormal systems $\mathscr{T}$ in $L^2(\mu)$. If moreover the Lévy measure possesses further properties, we can use them to construct special families of martingales with the CRP as we have seen above.

This large degree of freedom in the construction of families with the CRP can play an important role in applications, for example, for the problem of the completion of financial markets driven by geometric Lévy processes. In Corcuera, Nualart

and Schoutens [4], a Lévy market driven by a geometric Lévy process L has been considered and, under the assumptions of [16] on the Lévy measure ν, the market is completed involving *compensated power-jump assets*, that is, the Teugels martingales. In this context, the Teugels martingales have the interpretation of price processes associated with certain contingent claims that, if included in the market, make it complete, in the sense that any contingent claim of the market can be approximated (*hedged*) trading in the stock and the Teugels martingales. However, by the results of the present paper it becomes clear that for *any* geometric Lévy market there are many alternative systems of martingales which can serve for its completion and the question arises which are the "most adequate" systems from a theoretical and/or practical point of view.

In Sections 3, 4 and 5, one of the most important assumptions on the family $\mathscr{X}$ is that $\langle X, Y \rangle$ is deterministic, $X, Y \in \mathscr{X}$. This hypothesis made possible the definition of iterated integrals ensuring natural isometry properties. On the other side, it is an important premise for the proof of Proposition 5.7 and hence Theorem 5.8. A major extension of the approach would be to allow predictable covariations which are random. To explore adequate conditions should be the subject of future research.

Finally, we notice that the *compensated-covariation stability* is not a necessary condition for the CRP as we have seen for Lévy processes in Section 6 (see Theorem 6.6). However, in the general part of the paper (Sections 4 and 5) this property plays an important role (see, e.g., Proposition 4.2 and Proposition 5.2). To further exploit the relevance of the compensated-covariation, stability for the CRP remains an open task for future research.

REFERENCES

[1] AZÉMA, J. (1985). Sur les fermés aléatoires. In *Séminaire de Probabilités*, *XIX*, 1983/84. *Lecture Notes in Math*. **1123** 397–495. Springer, Berlin. MR0889496

[2] AZÉMA, J. and YOR, M. (1989). Étude d'une martingale remarquable. In *Séminaire de Probabilités XXIII. Lecture Notes in Math*. **1372** 88–130. Springer, Berlin.

[3] CAMERON, R. H. and MARTIN, W. T. (1947). The orthogonal development of non-linear functionals in series of Fourier–Hermite functionals. *Ann. of Math*. (2) **48** 385–392. MR0020230

[4] CORCUERA, J. M., NUALART, D. and SCHOUTENS, W. (2005). Completion of a Lévy market by power-jump assets. *Finance Stoch*. **9** 109–127. MR2210930

[5] DI TELLA, P. (2013). On the predictable representation property of martingales associated with Lévy processes. PhD thesis, Univ. Jena. Available at htpp://www.db-thueringen.de/servlets/DocumentServlet?id=23407.

[6] DI TELLA, P. and ENGELBERT, H.-J. (2015). The predictable representation property of compensated-covariation stable families of martingales. *Theory Probab. Appl*. **60** 99–130.

[7] ÉMERY, M. (1989). On the Azéma martingales. In *Séminaire de Probabilités*, *XXIII. Lecture Notes in Math.* **1372** 66–87. Springer, Berlin. MR1022899

[8] GIHMAN, I. I. and SKOROHOD, A. V. (1974). *The Theory of Stochastic Processes*: *Vol.* 1. *Die Grundlehren der Mathematischen Wissenschaften* **210**. Springer, Berlin. MR0346882

[9] ITÔ, K. (1951). Multiple Wiener integral. *J. Math. Soc. Japan* **3** 157–169. MR0044064

[10] ITÔ, K. (1956). Spectral type of the shift transformation of differential processes with stationary increments. *Trans. Amer. Math. Soc.* **81** 253–263. MR0077017

[11] JACOD, J. (1979). *Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Math.* **714**. Springer, Berlin. MR0542115

[12] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[13] KAKUTANI, S. (1950). Determination of the spectrum of the flow of Brownian motion. *Proc. Natl. Acad. Sci. USA* **36** 319–323. MR0035924

[14] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

[15] MEYER, P. A. (1976). Un cours sur les intégrales stochastiques. In *Séminaire de Probabilités*, *X* (*Seconde Partie*: *Théorie des Intégrales Stochastiques*, *Univ. Strasbourg*, *Strasbourg*, *Année Universitaire* 1974/1975) *Lecture Notes in Math.* **511** 245–400. Springer, Berlin. MR0501332

[16] NUALART, D. and SCHOUTENS, W. (2000). Chaotic and predictable representations for Lévy processes. *Stochastic Process. Appl.* **90** 109–122. MR1787127

[17] PECCATI, G. and TAQQU, M. S. (2011). *Wiener Chaos*: *Moments*, *Cumulants and Diagrams*. *Bocconi & Springer Series* **1**. Bocconi Univ. Press, Springer, Milan. MR2791919

[18] PROTTER, P. E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **21**. Springer, Berlin. MR2273672

[19] SATO, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics* **68**. Cambridge Univ. Press, Cambridge. MR1739520

[20] SEGALL, A. and KAILATH, T. (1976). Orthogonal functionals of independent-increment processes. *IEEE Trans. Inform. Theory* **IT-22** 287–298. MR0413257

[21] SOLÉ, J. L., UTZET, F. and VIVES, J. (2007). Chaos expansions and Malliavin calculus for Lévy processes. In *Stochastic Analysis and Applications. Abel Symp.* **2** 595–612. Springer, Berlin. MR2397807

[22] WIENER, N. (1938). The homogeneous chaos. *Amer. J. Math.* **60** 897–936. MR1507356

[23] WOJTASZCZYK, P. (1997). *A Mathematical Introduction to Wavelets. London Mathematical Society Student Texts* **37**. Cambridge Univ. Press, Cambridge. MR1436437

INSTITUT FÜR STOCHASTIK
TECHNISCHE UNIVERSITÄT DRESDEN
ZELLESCHER WEG 12–14, 01069 DRESDEN
GERMANY
E-MAIL: Paolo.Di_Tella@tu-dresden.de

INSTITUT FÜR MATHEMATIK
FRIEDRICH-SCHILLER-UNIVERSITÄT JENA
ERNST-ABBE-PLATZ 2, 07743 JENA
GERMANY
E-MAIL: hans-juergen.engelbert@uni-jena.de