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LARGE DEVIATION ESTIMATES FOR EXCEEDANCE TIMES OF PERPETUITY SEQUENCES AND THEIR DUAL PROCESSES[1]

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In a variety of problems in pure and applied probability, it is relevant to study the large exceedance probabilities of the perpetuity sequence $Y_n := B_1 + A_1 B_2 + \cdots + (A_1 \cdots A_{n-1}) B_n$, where $(A_i, B_i) \subset (0, \infty) \times \mathbb{R}$. Estimates for the stationary tail distribution of $\{Y_n\}$ have been developed in the seminal papers of Kesten [*Acta Math.* **131** (1973) 207–248] and Goldie [*Ann. Appl. Probab.* **1** (1991) 126–166]. Specifically, it is well known that if $M := \sup_n Y_n$, then $\mathbb{P}\{M > u\} \sim \mathcal{C}_M u^{-\xi}$ as $u \to \infty$. While much attention has been focused on extending such estimates to more general settings, little work has been devoted to understanding the path behavior of these processes. In this paper, we derive sharp asymptotic estimates for the normalized first passage time $T_u := (\log u)^{-1} \inf\{n : Y_n > u\}$. We begin by showing that, conditional on $\{T_u < \infty\}$, $T_u \to \rho$ as $u \to \infty$ for a certain positive constant ρ. We then provide a conditional central limit theorem for $\{T_u\}$, and study $\mathbb{P}\{T_u \in G\}$ as $u \to \infty$ for sets $G \subset [0, \infty)$. If $G \subset [0, \rho)$, then we show that $\mathbb{P}\{T_u \in G\} u^{I(G)} \to C(G)$ as $u \to \infty$ for a certain large deviation rate function I and constant $C(G)$. On the other hand, if $G \subset (\rho, \infty)$, then we show that the tail behavior is actually quite complex and different asymptotic regimes are possible. We conclude by extending our results to the corresponding forward process, understood in the sense of Letac [In *Random Matrices and Their Applications (Brunswick, Maine, 1984)* (1986) 263–273 Amer. Math. Soc.], namely to the reflected process $M_n^* := \max\{A_n M_{n-1}^* + B_n, 0\}$, $n \in \mathbb{Z}_+$. Using Siegmund duality, we relate the first passage times of $\{Y_n\}$ to the finite-time exceedance probabilities of $\{M_n^*\}$, yielding a new result concerning the convergence of $\{M_n^*\}$ to its stationary distribution.

CONTENTS

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1. Introduction. Since the pioneering work of Kesten (1973) and Vervaat (1979), there has been a continued interest in the probabilistic study of perpetuity sequences. Much of this interest has been driven by a wide variety of applications. Perpetuity sequences arise naturally in connection with the ARCH and GARCH financial time series models, the Asian option in discrete and continuous time and in insurance mathematics. [For a detailed description of these diverse applications, see, for example, Engle (1982), Bollerslev (1986), Mikosch (2003), Geman and Yor (1993), Carmona, Petit and Yor (2001), Paulsen (2002), Klüppelberg and Kostadinova (2008), and Collamore (2009).] From a theoretical perspective, perpetuity sequences also appear in connection with the weighted branching process and branching random walk. Indeed, utilizing an argument in Guivarc'h (1990) and Liu (2000), it is possible to relate the tail behavior of a perpetuity sequence to that of an associated nonhomogeneous recursion, leading to further applications, for example, to the Quicksort algorithm in computer science and to Mandelbrot cascades. [See Alsmeyer and Iksanov (2009), Buraczewski (2009), Guivarc'h (1990), Liu (2000), Buraczewski et al. (2014), and references therein.]

A central issue arising in all of these problems is the characterization of the tail behavior of the perpetuity sequence. Namely, letting $\{(A_i, B_i) : i \in \mathbb{Z}_+\}$ be an i.i.d. sequence of random variables taking values in $(0, \infty) \times \mathbb{R}$, and letting

$$Y_n = B_1 + A_1 B_2 + \cdots + (A_1 \cdots A_{n-1}) B_n, \qquad n = 1, 2, \ldots, \tag{1.1}$$

then it is of interest to consider

$$\mathbb{P}\{V > u\} \qquad \text{as } u \to \infty, \tag{1.2}$$

where, typically,

$$V := \lim_{n \to \infty} Y_n \quad \text{or} \quad V := \sup_n Y_n.$$

In either case, it is well known that under mild regularity conditions,

$$\mathbb{P}\{V > u\} \sim \mathscr{C} u^{-\xi} \qquad \text{as } u \to \infty \tag{1.3}$$

for certain positive constants $\mathscr{C}$ and ξ [cf. Kesten (1973), Goldie (1991)].

Much recent work has been devoted to showing that the estimate in (1.3) extends well beyond the setting of perpetuity sequences. Following Letac (1986), it is helpful to first observe that $\{Y_n\}$ can be identified as the *backward process* generated by the affine map $\Phi(x) = Ax + B$, where $(A, B) \stackrel{\mathscr{D}}{=} (A_1, B_1)$ (and $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution). More precisely, letting $\Phi_i(x) = A_i x + B_i$ for $i \in \mathbb{Z}_+$, then

$$Y_n = \Phi_1 \circ \cdots \circ \Phi_n(0), \qquad n = 1, 2, \ldots. \tag{1.4}$$

The limiting behavior of this sequence is, of course, the same as that of the corresponding *forward process*, namely

$$Y_n^* := \Phi_n \circ \cdots \circ \Phi_1(0), \qquad n = 1, 2, \ldots. \tag{1.5}$$

Then it is natural to consider more general random functions, including Markov-dependent sequences and random matrices. Extensions of this type can be also found, for example, in recent work of Alsmeyer (2003), Alsmeyer and Mentemeier (2012), Brofferio and Buraczewski (2015), Buraczewski et al. (2009), Collamore (2009), Collamore and Vidyashankar (2013a, 2013b), Guivarc'h and Le Page (2013a), Klüppelberg and Pergamenchtchikov (2004), Enriquez, Sabot and Zindy (2009), Mirek (2011) and Roitershtein (2007). We note that, for the process (1.5), recursions generated by random matrices were also considered in Kesten's (1973) original work. Moreover, some refined large deviation asymptotics for related recursive structures can be found in Buraczewski et al. (2013) and Buraczewski, Damek and Zienkiewicz (2015).

In contrast, very little is known concerning the path properties of perpetuity sequences. Two natural questions, well motivated by the theory of random walks, are the characterization of the distribution of the first passage time of the sequence in (1.4) and the convergence of the sequence in (1.5) to its stationary distribution. Indeed, these two questions are very much the same, since it is known by extensions of classical duality for random walks that

$$\mathbb{P}\{Y_k > u, \text{ some } k \le n\} = \mathbb{P}\{M_n^* > u\}, \tag{1.6}$$

where $\{M_n^*\}$ is defined as in (1.5), but with $\Phi(x)$ replaced with $\widetilde{\Phi}(x) := (Ax + B)^+$, and $Y_0 = 0 = M_0^*$. [Cf. Siegmund (1976), Asmussen and Sigman (1996), and the discussion in Section 2 below.] Thus, the *finite*-time exceedances of $\{M_n^*\}$ can be analyzed through the first passage times of $\{Y_n\}$, and vice versa.

The primary objective of this article is to study the asymptotic distribution of the scaled first passage time

$$T_u := \frac{1}{\log u} \inf\{n : Y_n > u\} \qquad \text{as } u \to \infty.$$

Motivated by the large deviation theory for random walks, developed in the classic papers of Donsker and Varadhan [cf. Varadhan (1984)], we study the asymptotic behavior of

$$\mathbb{P}\{T_u \in G\} \qquad \text{as } u \to \infty, \text{ where } G \subset [0, \infty).$$

We begin by showing that, conditional on $\{T_u < \infty\}$,

$$T_u \to \rho \qquad \text{in probability}$$

for some positive constant ρ, thus describing the "most likely" first passage time into the set (u, ∞). We then characterize the asymptotic distribution of $\{T_u\}$ on the respective time intervals $[0, \tau]$, where either $\tau < \rho$, $\tau = \rho$, or $\tau > \rho$, and the analysis on these various regions turns out to be quite different. On the first of these regions, we show that there exists a certain "rate function" $I : [0, \infty) \to [0, \infty)$ such that as $u \to \infty$,

$$\mathbb{P}\{T_u \le \tau\} \sim \frac{C(\tau)(\lambda(\alpha))^{-\Xi(u)}}{\sqrt{\log u}} u^{-I(\tau)} \qquad \text{for all } \tau < \rho, \tag{1.7}$$

where $C(\tau)$ is a positive constant which we explicitly identify and $\Xi(u) := \tau \log u - \lfloor \tau \log u \rfloor \in \{0, 1\}$. [For logarithmic large deviation asymptotics describing the decay parameter $I(\tau)$ for this case, see also Nyrhinen (2001).]

Next, we examine the behavior of $\{T_u\}$ around its central tendency, that is, around the constant ρ which describes the most likely exceedance time. To this end, we establish a conditional central limit theorem; namely, we show that conditional on $\{T_u < \infty\}$,

$$a\sqrt{\log u}(T_u - \rho) \Longrightarrow \mathscr{Z}, \tag{1.8}$$

where $\mathscr{Z}$ has the standard Normal distribution function (denoted Φ) and a is a certain constant. Thus, in particular, $\mathbb{P}\{T_u \in [0, \rho]\} \sim \Phi(0)\mathbb{P}\{T_u < \infty\}$, and hence as a direct analog of (1.7), we obtain that

$$\mathbb{P}\{T_u \in [0, \rho]\} \sim \frac{\mathscr{C}}{2} u^{-\xi} \qquad \text{as } u \to \infty, \tag{1.9}$$

where $(\mathscr{C}, \xi)$ is given as in (1.3). Both (1.7) and (1.8) are comparable to classical estimates for random walks, as we explain in Section 2.3 below.

Finally, we turn to a description of $\{T_u\}$ for large times, when $\tau > \rho$. Here we find that the behavior is quite complex, requiring new mathematical techniques. As we demonstrate, these asymptotics can be quite distinct from those expected from the large deviation theory of random walks, which, based on Arfwedson (1955), Asmussen [(2000), Chapter 4], and Collamore (1998), would suggest that

$$\mathbb{P}\{\tau \le T_u < \infty\} \sim \frac{C(\tau)}{\sqrt{\log u}} u^{-I(\tau)} \qquad \text{as } u \to \infty. \tag{1.10}$$

As we show, under certain conditions, the previous formula fails to hold and we obtain very different asymptotic behavior, not only for $\mathbb{P}\{\tau \le T_u < \infty\}$, but also for $\log \mathbb{P}\{\tau \le T_u < \infty\}$; thus, even the polynomial decay rate predicted by (1.10) need not hold, in general. Indeed, in Theorems 2.3 and 2.4 below, we provide asymptotic estimates showing that, under certain conditions,

$$\limsup_{u\to\infty} \log \mathbb{P}\{\tau \le T_u < \infty\} \le -I(\tau), \tag{1.11}$$

while under other conditions,

$$\liminf_{u\to\infty} \log \mathbb{P}\{\tau \le T_u < \infty\} > -I(\tau). \tag{1.12}$$

In this way, we exhibit an interesting asymmetry between the large-time behavior and the small-time behavior of $\{T_u\}$. The latter results are quite technical and show that for $\tau > \rho$, the story is very interesting, challenging and not fully understood.

We now turn to a more precise statement of our results. In the process, we also connect the convergence in (1.7) to that of the dual process of $\{Y_n\}$. The proofs are deferred to Sections 3–5, where we establish our main results for the three regimes ($\tau < \rho$, $\tau = \rho$, $\tau > \rho$) which we have just described.

2. Statement of results.

2.1. *A class of stochastic recursions*. Before stating our main results, we first introduce some notation related to our stochastic recursions and formulate a few of their basic properties. Let $\{(A_i, B_i) : i = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables taking values in $(0, \infty) \times \mathbb{R}$. Throughout the paper, we assume:

- $\mathbb{E}[\log A] \in (-\infty, 0)$ and $\mathbb{E}[\log^+ |B|] < \infty$.
- For every $x \in \mathbb{R}$, $\mathbb{P}\{Ax + B = x\} < 1$, which implies, in particular, that $\mathbb{P}\{B = 0\} < 1$.

We will be interested in the following two processes: the perpetuity sequence

$$Y_n := B_1 + \sum_{k=2}^{n} A_1 \cdots A_{k-1} B_k, \qquad n = 1, 2, \ldots, \ Y_0 = 0, \tag{2.1}$$

and, particularly, the process of partial maximums of this sequence, namely,

$$M_n := \max_{0\le k\le n} Y_k, \qquad n = 0, 1, \ldots. \tag{2.2}$$

These sequences represent the backward processes generated by the random mappings $\Phi_i(x) = A_i x + B_i$ and $\Phi_i(x) = (A_i x + B_i, 0)^+$, respectively. The corresponding forward processes [defined in (1.5)] are Markov chains satisfying the respective equations

$$\begin{aligned} Y_n^* &= A_n Y_{n-1}^* + B_n, \\ M_n^* &= \left(A_n M_{n-1}^* + B_n\right)^+. \end{aligned} \tag{2.3}$$

If $\mathbb{E}[\log A] < 0$ and $\mathbb{E}[\log^+ |B|] < \infty$, then it is well known that $\{Y_n\}$ converges pointwise to

$$Y = B_1 + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1} B_k,$$

while $\{M_n\}$ converges a.s. to

$$M = \sup_{n \geq 0} Y_n,$$

where Y and M are finite a.s. Then Y and M are called stationary solutions, since they satisfy the stochastic fixed point equations

$$Y \stackrel{\mathscr{D}}{=} AY + B, \qquad Y \text{ independent of } (A, B); \tag{2.4}$$

$$M \stackrel{\mathscr{D}}{=} (AM + B)^+, \qquad M \text{ independent of } (A, B). \tag{2.5}$$

In this paper, our objective will be to describe the path behavior of $\{Y_n\}$ and $\{M_n\}$, and, in this connection, it will be of interest to compare the limiting quantities we obtain to the tail behavior of Y and M. To this end, define the generating functions

$$\lambda(\alpha) = \mathbb{E}[A^\alpha], \qquad \Lambda(\alpha) = \log \lambda(\alpha), \qquad \alpha \in \mathbb{R};$$
$$\lambda_B(\alpha) = \mathbb{E}[|B|^\alpha], \qquad \Lambda_B(\alpha) = \log \lambda_B(\alpha), \qquad \alpha \in \mathbb{R}.$$

Note by the convexity of Λ and Λ_B that, if $\Lambda(\alpha) < \infty$ and $\Lambda_B(\alpha) < \infty$ for some $\alpha > 0$, then $\Lambda(\beta)$ and $\Lambda_B(\beta)$ are finite for every $\beta \in (0, \alpha)$. Moreover, these functions are infinitely differentiable on the interiors of their respective domains.

We will use some fundamental properties of the solutions to the stochastic equations (2.4). First, if $\Lambda(\alpha) < 0$ and $\Lambda_B(\alpha) < \infty$ for some $\alpha > 0$, then their αth moments must be finite, namely,

$$\mathbb{E}[|Y|^\alpha] < \infty \quad \text{and} \quad \mathbb{E}[M^\alpha] < \infty; \tag{2.6}$$

see Vervaat (1979). Next, to describe the tail behavior of Y and M, we focus on the nonzero solution, ξ, to the equation $\Lambda(\xi) = 0$. More precisely, assume that for some $\xi > 0$,

$$\Lambda(\xi) = 0, \qquad \Lambda'(\xi) < \infty \quad \text{and} \quad \Lambda_B(\xi) < \infty.$$

Then, if the random variable $\log A$ is nonarithmetic, it is well known that the tails of Y and M are regularly varying with index ξ; that is,

$$\begin{aligned} \mathbb{P}\{Y > u\} &\sim \mathscr{C}_Y u^{-\xi} \qquad \text{as } u \to \infty; \\ \mathbb{P}\{M > u\} &\sim \mathscr{C}_M u^{-\xi} \qquad \text{as } u \to \infty; \end{aligned} \tag{2.7}$$

see Goldie (1991). Various explicit expressions for the constants $\mathscr{C}_Y$ and $\mathscr{C}_M$ are also available; see Remark 2.2 below.

2.2. *Main results.* Let $\{Y_n\}$ denote the perpetuity sequence defined in (2.1), and let

$$T_u := \frac{1}{\log u} \inf\{n : Y_n > u\} \tag{2.8}$$

denote the scaled first passage time of $\{Y_n\}$ into the set (u, ∞). Then our primary objective is to study the asymptotic decay, as $u \to \infty$, of $\mathbb{P}\{T_u \in G\}$, where $G \subset [0, \infty)$. We will show that this probability decays at a polynomial rate, and provide sharp asymptotic estimates describing this rate of decay.

Set

$$\mu(\alpha) = \Lambda'(\alpha) \quad \text{and} \quad \sigma(\alpha) = \sqrt{\Lambda''(\alpha)}. \tag{2.9}$$

To characterize the behavior of $\{T_u\}$ as $u \to \infty$, it is helpful to first observe that, conditional on the event of ruin, the random variable T_u converges in probability to $\rho = (\mu(\xi))^{-1}$, where ξ is given as in (2.7). This constant ρ will play an important role in the sequel.

LEMMA 2.1. *Assume there exists a value $\xi > 0$ such that $\Lambda(\xi) = 0$, and suppose that Λ and Λ_B are finite in a neighborhood of ξ and the law of $\log A$ is non-lattice. Set $\rho = (\mu(\xi))^{-1}$. Then for any $\varepsilon > 0$,*

$$\mathbb{P}\{T_u \notin (\rho - \varepsilon, \rho + \varepsilon) | T_u < \infty\} \to 0 \qquad \text{as } u \to \infty. \tag{2.10}$$

Lemma 2.1 will follow as a direct consequence of a stronger result, Lemma 4.3, which will be proved in Section 4.

Turning now to our main results, we first introduce the rate function which we will use to describe the polynomial rates of decay. Recall that the convex conjugate (or Fenchel–Legendre transform) of the function Λ is defined by

$$\Lambda^*(x) = \sup_{\alpha \in \mathbb{R}} \{\alpha x - \Lambda(\alpha)\}, \qquad x \in \mathbb{R}.$$

Next, define

$$I(\tau) = \tau \Lambda^*\left(\frac{1}{\tau}\right), \qquad \tau > 0, \ I(0) = \infty. \tag{2.11}$$

This rate function appears in the large deviation study for random walks, and it is closely related to the "support function" in convex analysis, whose properties are well known [see Rockafellar (1970), Chapter 13]. Various convexity properties of the function $I(\cdot)$ itself (developed primarily for random walk in higher dimensions) can be found in Collamore (1998), Section 3. Note that if we set $\tau = (\mu(\alpha))^{-1}$ for some $\alpha \in \operatorname{dom}(\mu)$ (the domain of μ), then it follows that

$$I(\tau) = \alpha - \frac{\Lambda(\alpha)}{\mu(\alpha)}; \tag{2.12}$$

cf. Dembo and Zeitouni (1993), page 28.

We turn now to the characterization of $\mathbb{P}\{T_u \in [0, \tau]\}$ when $\tau < \rho$. Recall that the function Λ is differentiable on the interior of its domain. Moreover, if Λ is also essentially smooth (namely, if we further assume that $|\Lambda'(\alpha_i)| \uparrow \infty$ for any $\{\alpha_i\} \subset \operatorname{int}(\operatorname{dom} \Lambda)$ whose limit lies on the boundary of $\operatorname{dom} \Lambda$), then it is well known that Λ' maps $\mathbb{R}$ onto the entire real line. Thus, in this case, there always exists a point $\alpha(\tau)$ satisfying the equation

$$\mu\big(\alpha(\tau)\big) = \frac{1}{\tau}. \tag{2.13}$$

More generally, if τ^{-1} lies in the interior of the domain of Λ^*, then a solution $\alpha(\tau)$ exists in (2.13); cf. Ellis (1984), Theorem VI.5.7; Rockafellar (1970), Theorem 23.5.

Thus, the assumption of a solution to (2.13) is a very weak condition, which also appears to be necessary. In particular, when there fails to be a solution, one usually expects to obtain only logarithmic large deviation asymptotics rather than the sharp asymptotics which are the focus of this paper.

The most important solution to (2.13) appears, for our purposes, when we take $\tau = \rho$, where ρ is given as in the previous lemma. Then by definition of ρ, we have $\alpha(\rho) = \xi$. Then $\tau \in (0, \rho)$ if and only if $\alpha(\tau) > \xi$, which is the setting of our first main result.

THEOREM 2.1. *Let $\tau \in (0, \rho)$ and suppose that there exists a point $\alpha \equiv \alpha(\tau) \in \mathbb{R}$ such that (2.13) holds. Assume that Λ and Λ_B are finite in a neighborhood of α and the law of $\log A$ is non-lattice. Then*

$$\mathbb{P}\{T_u \le \tau\} = \frac{C(\tau)(\lambda(\alpha))^{-\Xi(u)}}{\sqrt{\log u}} u^{-I(\tau)}\big(1 + o(1)\big) \qquad \text{as } u \to \infty, \tag{2.14}$$

where $\Xi(u) := \tau \log u - \lfloor \tau \log u \rfloor$. Moreover,

$$\mathbb{P}\big\{T_u \le \tau - L_\tau(u)\big\} = o\left(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\right) \qquad \text{as } u \to \infty, \tag{2.15}$$

where $L_\tau(u) = \{c \log(\log u)\}/\log u$ and $c \ge \{2(\alpha + 1)\}/\Lambda(\alpha)$. The constant $C(\tau)$ is given by

$$C(\tau) = \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi\tau}} \lim_{n\to\infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}\big[M_n^\alpha\big] \in [0, \infty). \tag{2.16}$$

Moreover, if $\mathbb{P}\{A > 1, B > 0\} > 0$ then $C(\tau) > 0$.

Note that (2.15) shows, heuristically, that the critical event $\{Y_n > u\}$ occurs near the end of the time interval $[0, \tau \log u]$.

Next, we turn to the behavior around the critical case, where we would take $\tau = \rho$ in the previous theorem. In the following result, we establish a more precise estimate, which yields a conditional central limit theorem for the normalized first passage time.

THEOREM 2.2. *Suppose that there exists a value $\xi > 0$ such that $\Lambda(\xi) = 0$. Also, assume that Λ and Λ_B are finite in a neighborhood of ξ and the law of* $\log A$ *is non-lattice. Then for any $y \in \mathbb{R}$,*

$$\mathbb{P}\{T_u \le \rho + a(\log u)^{-1/2} y\} = \mathscr{C}_M u^{-\xi} \Phi(y)\big(1 + o(1)\big) \qquad \text{as } u \to \infty, \tag{2.17}$$

where Φ is the standard Normal distribution function, $a = \rho^{3/2}\sigma(\xi)$, and where $\mathscr{C}_M \in [0, \infty)$ is given as in (2.7). Moreover, if $\mathbb{P}\{A > 1, B > 0\} > 0$, then $\mathscr{C}_M > 0$.

REMARK 2.1. It follows from (2.17) and (2.7) that

$$\mathbb{P}\{T_u \le \rho + a(\log u)^{-1/2} y | T_u < \infty\} = \Phi(y)\big(1 + o(1)\big); \tag{2.18}$$

consequently, conditional on $\{T_u < \infty\}$, we have that $a\sqrt{\log u}(T_u - \rho) \Longrightarrow \mathscr{Z}$, where $\mathscr{Z} \sim \text{Normal}(0, 1)$.

REMARK 2.2. Using Goldie's (1991) original characterization, the constant $\mathscr{C}_M$ in Theorem 2.2 can be expressed as

$$\mathscr{C}_M = \frac{1}{\xi \mu(\xi)} \mathbb{E}\big[\big((AM + B)^+\big)^\xi - (AM)^\xi\big]. \tag{2.19}$$

Recently, certain more explicit representation formulas have been derived for $\mathscr{C}_M$ and $\mathscr{C}_Y$ in (2.7); see Enriquez, Sabot and Zindy (2009) and Collamore and Vidyashankar (2013b). The main representation formula in Collamore and Vidyashankar (2013b) states that, under a weak continuity assumption on $\log A$,

$$\mathscr{C}_M = \frac{1}{\xi \mu(\xi) \mathbb{E}[\tau]} \mathbb{E}_\xi \left[\left(V_0 + \frac{B_1}{A_1} + \frac{B_2}{A_1 A_2} + \cdots \right)^\xi \mathbf{1}_{\{\tau = \infty\}} \right], \tag{2.20}$$

where $\mathbb{E}_\xi[\cdot]$ denotes expectation in the ξ-shifted measure (defined formally in Section 3 below), $\tau - 1$ is the first regeneration time of the forward process $\{M_n^*\}$ in (2.3), and M_0^* is chosen such that $M_0^* \overset{\mathscr{D}}{=} M_\tau^*$. Specifically, if $\mathbb{P}\{B < 0\} > 0$, then $\tau - 1$ can be taken to be the return time of $\{M_n^*\}$ to the origin. In particular, the positivity of $\mathscr{C}_M$ follows readily under these conditions from (2.20). Moreover, under the weaker requirements of Collamore and Vidyashankar (2013b), Theorem 2.2, together with the additional assumption that $\{M_n^*\}$ is ψ-irreducible (which is implicitly assumed in Section 9 of that article), one obtains (2.20) for the k-chain $\{M_{kn}^* : n = 1, 2, \ldots\}$, as well as the alternative representation

$$\mathscr{C}_M = \frac{1}{\xi \mu(\xi)} \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[M_n^\xi], \tag{2.21}$$

which is readily seen to have a closely related form to (2.16).

Finally, we turn to the case where $\tau > \rho$. Interestingly, in this case, we do *not* obtain a complete analog of Theorem 2.1. Indeed, counterexamples can be constructed where the asymptotics differ from those one might expect from the large deviation theory for random walks, as described in (1.10). For $\tau > \rho$, the condition that appears to lead to these counterexamples is that $\mathbb{E}[\log A] > \Lambda(\alpha(\tau))$. In this case, the true probability may decay at a *slower* polynomial rate than $I(\tau)$. More precisely, within a rather flexible class of processes with $\mathbb{E}[\log A] > \Lambda(\alpha(\tau))$, we have that

$$\mathbb{P}\{\tau \le T_u < \infty\} \ge \mathscr{D}_0 u^{-I(\tau)+\delta} \qquad \text{for sufficiently large } u. \tag{2.22}$$

On the other hand, under different hypotheses which, in particular, imply $\mathbb{E}[\log A] < \Lambda(\alpha(\tau))$, we have that

$$\mathbb{P}\{\tau \le T_u < \infty\} \le \frac{\mathscr{D}_1}{\sqrt{\log u}} u^{-I(\tau)} \qquad \text{for sufficiently large } u. \tag{2.23}$$

Thus, one cannot expect a direct analog of Theorems 2.1 and 2.2 here, and our next theorem provides, in effect, a source of counterexamples to the natural conjecture suggested by (1.10).

THEOREM 2.3. *Let $\tau \in (\rho, \infty)$, and suppose that there exists a point $\alpha \equiv \alpha(\tau) \in \mathrm{int}(\mathrm{dom}\,\Lambda)$ such that (2.13) holds and*

$$\mu(0) = \mathbb{E}[\log A] > \Lambda(\alpha). \tag{2.24}$$

Moreover, assume that $B = 1$ a.s. and the law of A has a strictly positive continuous density on $\mathbb{R}$. Then there exist positive constants $\mathscr{D}_0$ and δ such that, for sufficiently large u,

$$\mathbb{P}\{Y_{n_u - 1} \le u \text{ and } Y_{n_u} > u\} \ge \mathscr{D}_0 u^{-I(\tau)+\delta}, \qquad n_u = \lfloor \tau \log u \rfloor. \tag{2.25}$$

REMARK 2.3. Since the construction in the theorem is quite involved, we have restricted our attention to the case $B = 1$; however, the theorem can also be established under the weaker assumption that $B > 0$ a.s. For more details, see the discussion in Section 5.1 following the proof of the theorem.

While the previous theorem leads essentially to a negative conclusion, we also have the following complementary result.

THEOREM 2.4. *Let $\tau \in (\rho, \infty)$, and suppose that there exists a point $\alpha \equiv \alpha(\tau) \in \mathrm{int}(\mathrm{dom}\,\Lambda)$ such that (2.13) holds and*

$$\Lambda(\beta) < \Lambda(\alpha) \qquad \text{for some } \beta < \min\{1, \alpha\}. \tag{2.26}$$

Assume that $B > 0$ a.s. and $\lambda_B(-\alpha) < \infty$, and assume that the law of (A, B) has compact support and A has a bounded density. Then there exist finite constants $\mathscr{D}$ and U such that, for all $u \ge U$,

$$\mathbb{P}\{Y_{n_u + k - 1} \le u \text{ and } Y_{n_u + k} > u\} \le \frac{\mathscr{D}\varrho^k}{\sqrt{\log u}} u^{-I(\tau)}, \qquad n_u = \lfloor \tau \log u \rfloor, \tag{2.27}$$

where $\varrho := \lambda(\alpha) \in (0, 1)$ *and* k *is any nonnegative integer. Thus, for sufficiently large* u,

$$\mathbb{P}\{\tau \le T_u < \infty\} \le \frac{\mathscr{D}_1}{\sqrt{\log u}} u^{-I(\tau)} \tag{2.28}$$

for some positive constant $\mathscr{D}_1$.

REMARK 2.4. In these theorems, conditions (2.24) and (2.26) determine the relevant asymptotic regime. At first sight, it in not immediately clear that there are processes which satisfy these assumptions. In fact, such processes exist in abundance; see the discussion in Section 5 and, in particular, Lemma 5.1.

2.3. *A comparison with previous estimates for classical random walks.* Our motivation for considering $\mathbb{P}\{T_u \le \tau\}$ as $u \to \infty$ comes from classical estimates for random walks, to which we now compare the results of this paper.

For this purpose, let $\{X_i\} \subset \mathbb{R}$ be an i.i.d. sequence of random variables such that $\mathbb{E}[X_i] < 0$. With a slight abuse of notation, specifically for the discussion in this section set

$$S_n = X_1 + \cdots + X_n, \qquad n \in \mathbb{Z}_+; \qquad S_0 = 0;$$
$$\lambda(\alpha) = \mathbb{E}\bigl[e^{\alpha X_1}\bigr]; \quad \text{and} \quad \Lambda(\alpha) = \log \lambda(\alpha) \qquad \text{all } \alpha \in \mathbb{R}.$$

(If $X_i \equiv \log A_i$ for all i, then this notation agrees with that of the previous sections.) Also, let $\mu(\cdot)$, $\sigma(\cdot)$, and $I(\cdot)$ be defined as in (2.9) and (2.11), and let $\alpha(\tau)$ be defined as in (2.13). Finally, set

$$\mathscr{T}_u = \frac{1}{u} \inf\{n : S_n \ge u\}, \qquad u \ge 0.$$

Then, motivated by large deviation theory, it is of interest to consider $\mathbb{P}\{T_u \in G\}$ as $u \to \infty$ for various intervals $G \subset \mathbb{R}$.

Assume that the distribution of X_1 is non-lattice, and suppose that there exists a value $\xi > 0$ such that $\Lambda(\xi) = 0$. Set $\rho = (\mu(\xi))^{-1}$. Let $\tau < \rho$, and assume that $\alpha(\tau) \in \operatorname{int}(\operatorname{dom} \Lambda)$. Set $\alpha \equiv \alpha(\tau)$. Then it is shown in Lalley [(1984), Theorem 5] that

$$\mathbb{P}\{\mathscr{T}_u \le \tau\} = \frac{C_1(\tau)(\lambda(\alpha))^{-\Xi_1(u)}}{\alpha\sigma(\alpha)\sqrt{2\pi\tau u}} e^{-uI(\tau)}\bigl(1 + o(1)\bigr) \qquad \text{as } u \to \infty, \tag{2.29}$$

where $\Xi_1(u) =: \tau u - \lfloor \tau u \rfloor \in \{0, 1\}$. Similarly, if $\tau > \rho$, then

$$\mathbb{P}\bigl\{\mathscr{T}_u \in (\tau, \infty)\bigr\} = \frac{C_2(\tau)(\lambda(\alpha))^{\Xi_2(u)}}{\alpha\sigma(\alpha)\sqrt{2\pi\tau u}} e^{-uI(\tau)}\bigl(1 + o(1)\bigr) \qquad \text{as } u \to \infty, \tag{2.30}$$

where $\Xi_2(u) =: \lfloor \tau u + 1 \rfloor - \tau u \in \{0, 1\}$. In Lalley's theorem, the constants $C_1(\tau)$ and $C_2(\tau)$ are given (after an integration by parts in the numerator and an application of Wald's identity in the denominator) by the following expressions:

$$
\begin{aligned}
C_1(\tau) &= \frac{\lambda(\alpha)}{\lambda(\alpha) - 1} \frac{1 - \mathbb{E}_\alpha[e^{-\alpha S_{\mathscr{T}_0}}]}{\mathbb{E}[\mathscr{T}_0]} \quad \text{and} \\
C_2(\tau) &= \frac{1}{1 - \lambda(\alpha)} \frac{1 - \mathbb{E}_\alpha[e^{-\alpha S_{\mathscr{T}_0}}]}{\mathbb{E}[\mathscr{T}_0]},
\end{aligned}
\tag{2.31}
$$

where $\mathscr{T}_0 := \inf\{n \in \mathbb{Z}_+ : S_n \geq 0\}$.

Alternatively, the proof of our Theorem 2.1 could be repeated in this simplified setting to obtain (2.29), but with $C_1(\tau)$ replaced with

$$
C_1^*(\tau) := \lim_{n \to \infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}[\check{M}_n^\alpha] \qquad \text{where } \check{M}_n := \max_{0 \leq k \leq n} e^{\alpha S_k}. \tag{2.32}
$$

To see that these two expressions for the constant are consistent, observe by a change of measure argument that

$$
\frac{1}{\lambda^n(\alpha)} \mathbb{E}[\check{M}_n^\alpha] = \mathbb{E}_\alpha\Big[\max_{0 \leq k \leq n} e^{-\alpha(S_n - S_k)}\Big] = \mathbb{E}_\alpha[e^{-\alpha W_n}], \tag{2.33}
$$

where $\{W_n\}$ is the random walk reflected upon entering the *positive* half-line; that is, $W_n = (W_{n-1} + X_n) \wedge 0$ for $n \geq 1$ and $W_0 = 0$. Note that in the α-shifted measure, the Markov chain $\{W_n\}$ converges to a proper random variable, W, and by an application of the Athreya–Ney–Nummelin regeneration lemma [cf. Athreya and Ney (1978), Nummelin (1978)], we have that

$$
C_1^*(\tau) = \mathbb{E}_\alpha[e^{-\alpha W}] = \frac{1}{\mathbb{E}_\alpha[\mathscr{T}_0]} \mathbb{E}_\alpha\left[\sum_{i=1}^{\mathscr{T}_0} e^{-\alpha W_i}\right], \tag{2.34}
$$

where $\mathscr{T}_0 + 1$ is here identified as the first regeneration time of the process $\{W_n\}$. Moreover, since $W_n = S_n$ for all $n < \mathscr{T}_0$ and $W_{\mathscr{T}_0} = 0 = W_0$, we have that

$$
\begin{aligned}
\mathbb{E}_\alpha\left[\sum_{i=1}^{\mathscr{T}_0} e^{-\alpha W_i}\right] &= \mathbb{E}_\alpha\left[\sum_{i=0}^{\mathscr{T}_0 - 1} e^{-\alpha S_i}\right] = \mathbb{E}_\alpha\left[\sum_{i=0}^{\infty} e^{-\alpha S_i} \mathbf{1}_{\{\mathscr{T}_0 > i\}}\right] \\
&= \sum_{i=0}^{\infty} \{\mathbb{E}_\alpha[e^{-\alpha S_i}] - \mathbb{E}_\alpha[e^{-\alpha S_i} \mathbf{1}_{\{\mathscr{T}_0 \leq i\}}]\} \\
&= \sum_{i=0}^{\infty} \mathbb{E}_\alpha[e^{-\alpha S_i}] - \sum_{i=0}^{\infty} \mathbb{E}_\alpha[e^{-\alpha S_{\mathscr{T}_0}}] \mathbb{E}_\alpha[e^{-\alpha(S_{\mathscr{T}_0 + i} - S_{\mathscr{T}_0})}] \\
&= \sum_{i=0}^{\infty} (\lambda(\alpha))^{-i} \{1 - \mathbb{E}_\alpha[e^{-\alpha S_{\mathscr{T}_0}}]\},
\end{aligned}
\tag{2.35}
$$

where the last step was obtained by a further change of measure argument, applied to the first and third expectations of the next-to-last expression in (2.35). Substituting (2.35) into (2.34) yields that $C_1^*(\tau) = C_1(\tau)$, showing that our approach leads to the same constant as in Lalley (1984), Theorem 5. Conversely, if $\tau > \rho$, then our Theorem 2.3 shows that (2.30) does *not* hold for perpetuity sequences.

For the conditional central limit theorem developed in Theorem 2.2, a comparison can be made with a result of von Bahr (1974). Specifically, if the random walk $\{S_n\}$ is defined as above and if the parameter $\xi \in \operatorname{int}(\operatorname{dom} \Lambda)$ exists and X_1 is nonlattice, then it is known that for any $y \in \mathbb{R}$,

$$\mathbb{P}\{T_u \le \rho + a(\log u)^{-1/2} y\} = \mathscr{C}_M^* e^{-\xi u}\big(1 + o(1)\big)\Phi(y), \tag{2.36}$$

where $a = \rho^{3/2}\sigma(\xi)$ and $\rho \equiv 1/\mu(\xi)$, Φ is the standard Normal distribution function, and $\mathscr{C}_M^*$ is the Cramér–Lundberg constant, namely

$$\mathscr{C}_M^* = \frac{1}{\xi\mu(\xi)} \frac{1 - \mathbb{E}\Big[e^{\xi S_{\mathscr{T}_0^-}}\Big]}{\mathbb{E}[\mathscr{T}_0^-]}, \qquad \mathscr{T}_0^- = \inf\{n : S_n \le 0\};$$

cf. von Bahr (1974), Section 8; Siegmund (1975), Theorem 2 [and Iglehart (1972) for this representation for $\mathscr{C}_M^*$]. Also, the proof of our Theorem 2.2 could be repeated in this setting to obtain (2.36).

As with many results in large deviation theory, the estimates described above actually have a longer history from an applied perspective. Indeed, (2.29) and (2.30) are extensions of the classical Arfwedson approximations from collective risk theory, originally derived with the random walk $\{S_n\}$ replaced with a compound Poisson process; see Asmussen (2000), Chapter IV, Section 4c. Similarly, (2.36) is an extension of Segerdahl's conditional central limit theorem, known in collective risk theory for compound Poisson processes; see Asmussen (2000), Chapter IV, Section 4a.

Finally, we note that there are also higher-dimensional versions of (2.29) and (2.30) for random walks. In this context, it is natural to replace the region $[u, \infty)$ with a general open subset of $\mathbb{R}^d$ and to replace the one-dimensional random walk with a d-dimensional process (either i.i.d. random walk or a more general Gärtner–Ellis sequence), and to consider $\mathscr{T}_u := u^{-1} \inf\{n : S_n \in uA\}$ for any set $A \subset \mathbb{R}^d$. At this level of generality, one does not expect to obtain sharp asymptotics, but under natural conditions one can show weaker results; namely, for a certain large deviation rate function I_A,

$$\limsup_{u\to\infty} \frac{1}{u} \log \mathbb{P}\{T_u \in F\} \le -\inf_{\tau\in F} I_A(\tau), \tag{2.37}$$

for all closed sets $F \subset [0, \infty)$, and

$$\liminf_{u\to\infty} \frac{1}{u} \log \mathbb{P}\{T_u \in G\} \ge -\inf_{\tau\in G} I_A(\tau), \tag{2.38}$$

for all open sets $G \subset [0, \infty)$. Consequently, the probability law of $\{T_u\}$ satisfies the large deviation principle with a rate function I_A, and this function I_A reduces to our rate function I in the one-dimensional case. For further details, see Collamore (1998).

2.4. *Exceedance probabilities for forward recursive sequences.* We conclude this section by relating our previous results for perpetuities, obtained in Theorems 2.1 and 2.2, to the convergence of the corresponding forward sequence $\{M_n^*\}$, where $M_n^* := (A_n M_{n-1}^* + B_n)^+$, $n = 1, 2, \dots,$ and $M_0^* = 0$. Borrowing terminology from queuing theory, $\{M_n^*\}$ is called the "content process" corresponding to the "risk process"

$$U_n := \left(\frac{U_{n-1}}{A_n} - \frac{B_n}{A_n}\right)^+, \qquad n = 1, 2, \dots, \; U_0 = u. \tag{2.39}$$

Then $\{U_n\}$ and $\{M_n^*\}$ are dual processes in the sense of Siegmund (1976); see Asmussen and Sigman (1996), Example 6 (slightly modified). Following Asmussen and Sigman (1996), the finite-time ruin probability of $\{U_n\}$ may be equated to the finite-time exceedance probability of $\{M_n^*\}$; that is,

$$\Psi(u) := \mathbb{P}\{U_k \le 0, \text{some } k \le n | U_0 = u\} = \mathbb{P}\{M_n^* \ge u\}, \tag{2.40}$$

and a simple argument yields that $\Psi(u)$ also describes the finite-time ruin probability of $\{Y_n\}$, namely $\Psi(u) = \mathbb{P}\{Y_k \ge u, \text{some } k \le n\}$ [see Collamore (2009), Section 2.1]. Thus, it is natural to relate the ruin probabilities described in our previous theorems to the exceedance probabilities of $\{M_n^*\}$.

In fact, the equivalence in (2.40) can be obtained more directly in our problem. Indeed, since the finite-time distributions of the forward and backward sequences are the same, we immediately obtain that

$$\mathbb{P}\{M_n^* > u\} = \mathbb{P}\{M_n > u\} \equiv \mathbb{P}\{Y_k > u, \text{ some } k \le n\}. \tag{2.41}$$

Thus, in particular,

$$\mathbb{P}\{T_u \le \tau\} = \mathbb{P}\{M_{n_u}^* > u\}, \qquad n_u = \lfloor \tau \log u \rfloor. \tag{2.42}$$

These last two equations lead to the following theorem on the convergence of $\{M_n^*\}$ to its stationary distribution, obtained now as a direct consequence of Theorems 2.1 and 2.2.

THEOREM 2.5. *Let $\tau \in (0, \rho)$, and suppose that there exists a point $\alpha(\tau) \in \mathbb{R}$ such that (2.13) holds. Assume that Λ and Λ_B are finite in a neighborhood of $\alpha(\tau)$ and that the law of* $\log A$ *is nonlattice. Then for $n_u = \lfloor \tau \log u \rfloor$, we have*

$$\mathbb{P}\{M_{n_u}^* > u\} = \frac{C(\tau)(\lambda(\alpha))^{-\Xi(u)}}{\sqrt{\log u}} u^{-I(\tau)}\bigl(1 + o(1)\bigr) \qquad \textit{as } u \to \infty, \tag{2.43}$$

where $\Xi(u) := \tau \log u - \lfloor \tau \log u \rfloor$ *and* $C(\tau) < \infty$ *is given as in* (2.16). *The constant* $C(\tau)$ *is strictly positive if* $\mathbb{P}\{A > 1, B > 0\} > 0$.

Next, let ξ *and* $\mathscr{C}_M$ *be the constants appearing in* (2.7), *and suppose that* Λ *and* Λ_B *are finite in a neighborhood of* ξ *and that the law of* $\log A$ *is non-lattice. Let* $y \in \mathbb{R}$, *and set* $n_u = \lfloor \rho \log u + a(\log u)^{1/2} y \rfloor$, *where* $a = \rho^{3/2}\sigma(\xi)$. *Then*

$$\mathbb{P}\{M^*_{n_u} > u\} = \mathscr{C}_M u^{-\xi}\bigl(1 + o(1)\bigr)\Phi(y) \qquad \text{as } u \to \infty. \tag{2.44}$$

Moreover, $\mathscr{C}_M$ *is strictly positive if* $\mathbb{P}\{A > 1, B > 0\} > 0$.

3. Proof of Theorem 2.1. First, we introduce some further notation, as follows. Let

$$\Pi_n = A_1 \cdots A_n, \qquad n \in \mathbb{Z}_+;$$

$$S_n = \sum_{k=1}^{n} \log A_k = \log \Pi_n, \qquad n \in \mathbb{Z}_+;$$

$$\overline{Y}_n = \sum_{i=1}^{n} \Pi_{i-1}|B_i|, \qquad n \in \mathbb{Z}_+.$$

Also, let ν denote the probability law of $(\log A, B)$, and if $\lambda(\alpha) < \infty$, define

$$\nu_\alpha(E) = \int_E \frac{e^{\alpha x}}{\lambda(\alpha)}\, d\nu(x, y), \qquad E \in \mathscr{B}(\mathbb{R}^2), \tag{3.1}$$

where $\mathscr{B}(\mathbb{R}^2)$ denotes the Borel sets on $\mathbb{R}^2$. Let $\mathbb{E}_\alpha[\cdot]$ denote expectation with respect to the probability measure ν_α. Note that $\mu(\alpha) := \Lambda'(\alpha)$ and $\sigma^2(\alpha) := \Lambda''(\alpha)$ [defined previously in (2.9)] denote the mean and the variance, respectively, of the random variable $\log A$ with respect to the measure ν_α.

We start by establishing a variant of the exponential Chebyshev inequality from large deviation theory, commonly used in conjunction with Minkowski's inequality for perpetuity sequences (yielding estimates which are typically not very sharp). The next lemma will provide a sharper version of these estimates for our problem. Before stating this result, we recall that $\Lambda(\xi) = 0$, that is, ξ denotes the critical value that determines the decay rate of $\mathbb{P}\{M > u\}$ as $u \to \infty$. Thus, $\lambda(\alpha) \geq 1$ for $\alpha \geq \xi$.

LEMMA 3.1. *Let* $\alpha \geq \xi$, *and assume that* α *and* $\varepsilon > 0$ *have been chosen such that* $\Lambda(\alpha + \varepsilon) < \infty$ *and* $\Lambda_B(\alpha + \varepsilon) < \infty$. *Then*

$$\mathbb{P}\{\overline{Y}_n > u\} \leq \overline{C}_n \lambda^n(\alpha) u^{-(\alpha+\varepsilon)} \qquad \text{for all } u > 0, n \in \mathbb{Z}_+, \tag{3.2}$$

where

$$\overline{C}_n = bn(n-1)^{2(\alpha+\varepsilon)} \exp\bigl\{(n-1)\bigl(\varepsilon\mu(\alpha) + \varepsilon^2\sigma^2(\alpha)\bigr)\bigr\} \tag{3.3}$$

for $b = (\pi^2/6)^{\alpha+\varepsilon}\{\lambda_B(\alpha + \varepsilon)/\lambda(\alpha)\}$.

PROOF. From the elementary equality $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$, we obtain

$$\begin{aligned}\mathbb{P}\{\overline{Y}_n > u\} &\le \sum_{k=1}^{n} \mathbb{P}\left\{\Pi_{k-1}|B_k| > \frac{6u}{\pi^2 k^2}\right\} \\ &\le \sum_{k=1}^{n} \mathbb{E}\left[\Pi_{k-1}^{\alpha+\varepsilon}|B_k|^{\alpha+\varepsilon}\right]\left(\frac{\pi^2 k^2}{6u}\right)^{\alpha+\varepsilon}.\end{aligned} \tag{3.4}$$

Now by independence,

$$\mathbb{E}\left[\Pi_{k-1}^{\alpha+\varepsilon}|B_k|^{\alpha+\varepsilon}\right] = \left(\mathbb{E}\left[A^{\alpha+\varepsilon}\right]\right)^{k-1}\mathbb{E}\left[B^{\alpha+\varepsilon}\right] := \left(\lambda(\alpha+\varepsilon)\right)^{k-1}\lambda_B(\alpha+\varepsilon).$$

Moreover, since the generating function Λ is infinitely differentiable on the interior of its domain,

$$\lambda(\alpha+\varepsilon) = e^{\Lambda(\alpha+\varepsilon)} \le \exp\left\{\Lambda(\alpha) + \varepsilon\mu(\alpha) + \frac{\varepsilon^2 \mathfrak{m}}{2}\right\},$$

where $\mathfrak{m} := \sup\{\sigma^2(\beta) : \alpha \le \beta \le \alpha + \varepsilon\}$. Moreover, using the continuity of the function $\sigma^2(\cdot)$, we have that $\mathfrak{m}/2 \le \sigma^2(\alpha)$ when ε is sufficiently small. Hence, substituting the previous two equations into (3.4), we obtain that for sufficiently small ε,

$$\mathbb{P}\{\overline{Y}_n > u\} \le u^{-(\alpha+\varepsilon)} \sum_{k=1}^{n} G(k), \tag{3.5}$$

where

$$G(k) = \lambda(\alpha)^{k-1} \exp\left\{(k-1)\left(\varepsilon\mu(\alpha) + \varepsilon^2\sigma^2(\alpha)\right)\right\}\lambda_B(\alpha+\varepsilon)\left(\frac{\pi^2 k^2}{6}\right)^{\alpha+\varepsilon}.$$

Since $\lambda(\alpha) \ge 1$ and $\mu(\alpha) := \Lambda'(\alpha) \ge 0$, it follows that $G(k)$ is increasing in k. Hence, $\sum_{k=1}^{n} G(k) \le nG(n)$, and substituting this last estimate into (3.5) yields (3.2), as required. □

Next, define

$$\overline{T}_u = \frac{1}{\log u} \inf\{n : \overline{Y}_n > u\},$$

and note by definition that $\overline{T}_u \le T_u$ on $\{T_u < \infty\}$. Then as a simple consequence of the lemma, we obtain the following.

LEMMA 3.2. *Under the assumptions of Theorem 2.1*,

$$\mathbb{P}\left\{\overline{T}_u \le \tau - L_\tau(u)\right\} = o\left(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\right) \qquad \textit{as } u \to \infty, \tag{3.6}$$

for any $L_\tau(u) \ge \{c \log(\log u)\}/\log u$, *where* $c = \{2(\alpha+1)\}/\Lambda(\alpha)$.

PROOF. Set $\zeta_u = \lfloor \log u(\tau - L_\tau(u)) \rfloor$. Then it follows directly from the definitions that

$$\mathbb{P}\{\overline{T}_u \le \tau - L_\tau(u)\} = \mathbb{P}\{\overline{Y}_{\zeta_u} > u\}. \tag{3.7}$$

Now set $\alpha \equiv \alpha(\tau)$, where $\alpha(\tau)$ is defined as in (2.13). To apply the lemma, it is helpful to first observe [using (2.12)] that

$$\big(\lambda(\alpha)\big)^{\tau \log u} u^{-\alpha} = e^{-\log u(\alpha - \tau \Lambda(\alpha))} = u^{-I(\tau)}. \tag{3.8}$$

Hence

$$\big(\lambda(\alpha)\big)^{\zeta_u} u^{-\alpha} \le u^{-I(\tau)} \big(\lambda(\alpha)\big)^{-L_\tau(u) \log u}. \tag{3.9}$$

Next, choose $\varepsilon \equiv \varepsilon(u)$ such that $u^{-\varepsilon(u)} = (\log u)^{-1/2}$, which is achieved by setting

$$\varepsilon(u) = \frac{\log(\sqrt{\log u})}{\log u} \searrow 0, \qquad u \to \infty. \tag{3.10}$$

Then by (3.9), it is sufficient to show that

$$\overline{C}_{\zeta_u} \big(\lambda(\alpha)\big)^{-L_\tau(u) \log u} = o(1) \qquad \text{as } u \to \infty, \tag{3.11}$$

for $\overline{C}_{\zeta_u}$ defined as in (3.3). Observe that with the choice of $\varepsilon(u)$ given in (3.10) and the upper bound $(\zeta_u - 1) \le \tau \log u$, we obtain that

$$\exp\{(\zeta_u - 1)\big(\varepsilon(u)\mu(\alpha) + \varepsilon^2(u)\sigma^2(\alpha)\big)\} = O(\sqrt{\log u}) \qquad \text{as } u \to \infty;$$

hence

$$\overline{C}_{\zeta_u} = O\big((\log u)^{2(\alpha+\varepsilon)+3/2}\big) \qquad \text{as } u \to \infty. \tag{3.12}$$

Then (3.11) follows from (3.12), provided that we choose $L_\tau(u) \log u \ge c \log(\log u)$, where $c = 2(\alpha + 1)/\Lambda(\alpha)$. □

From the lemma, we see that the probability of ruin in the scaled time interval $[0, \tau - L_\tau(u)]$ is negligible, so we may concentrate on the critical interval $(\tau - L_\tau(u), \tau]$. In this region, we will argue that the process $\{\log Y_n \vee 0\}$ behaves similarly to a perturbed random walk when this process is large, that is, $\log Y_n$ can be approximated by $S_n + \boldsymbol{\varepsilon}_n$ for some perturbation term $\boldsymbol{\varepsilon}_n$ and $S_n := \sum_{i=1}^n \log A_i$. To analyze the behavior of the random walk $\{S_n\}$, the following uniform large deviation theorem, due to Petrov [(1965), Theorem 2], will play a key role.

THEOREM 3.1 (Petrov). *Let $a_0 = \sup_{\alpha \in \mathrm{dom}(\Lambda')} \Lambda'(\alpha)$. Suppose that c satisfies $\mathbb{E}[\log A] < c < a_0$, and suppose that $\delta(n)$ is an arbitrary function satisfying*

$\lim_{n\to\infty}\delta(n) = 0$. *Also, assume that the law of* $\log A$ *is non-lattice. Then with* α *chosen such that* $\Lambda'(\alpha) = c$, *we have that*

$$\mathbb{P}\{S_n > n(c+\gamma_n)\}$$

$$= \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi n}} \exp\left\{-n\big(\alpha(c+\gamma_n)\big) - \Lambda(\alpha) + \frac{\gamma_n^2}{2\sigma^2(\alpha)}\big(1+O(|\gamma_n|)\big)\right\} \times \big(1+o(1)\big) \tag{3.13}$$

as $n\to\infty$, *uniformly with respect to* c *and* γ_n *in the range*

$$\mathbb{E}[\log A] + \varepsilon \le c \le a_0 - \varepsilon \quad \textit{and} \quad |\gamma_n| \le \delta(n), \tag{3.14}$$

where $\varepsilon > 0$.

REMARK 3.1. In (3.14), we may have that $\sup\{\alpha : \alpha \in \operatorname{dom}(\Lambda)\} = \infty$ or $\mathbb{E}[\log A] = -\infty$. In these cases, the quantities $\infty - \varepsilon$ or $-\infty - \varepsilon$ should be interpreted as arbitrary positive, respectively negative, constants.

PROOF OF THEOREM 2.1. *Step* 1. Equation (2.15) was established in Lemma 3.2; thus, it is sufficient to show that

$$\mathbb{P}\{\tau - L_\tau(u) < T_u \le \tau\} = \frac{C(\tau)}{\sqrt{\log u}} u^{-I(\tau)}\big(1+o(1)\big) \qquad \text{as } u\to\infty, \tag{3.15}$$

for $L_\tau(u) = \{c\log(\log u)\}/\log u$, where $c = \{2(\alpha+1)\}/\Lambda(\alpha)$. Indeed, by Lemma 3.2,

$$\mathbb{P}\{T_u \le \tau - L_\tau(u)\} = o\left(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\right) \qquad \text{as } u\to\infty. \tag{3.16}$$

Set

$$\zeta_u = \lfloor \log u\big(\tau - L_\tau(u)\big)\rfloor \quad \text{and} \quad \tau_u = \lfloor \tau \log u\rfloor,$$

and define

$$\mathscr{M}_u = \max_{\zeta_u < n \le \tau_u} \{B_{\zeta_u+1} + A_{\zeta_u+1}B_{\zeta_u+2} + \cdots + (A_{\zeta_u+1}\cdots A_{n-1})B_n\} \vee 0.$$

Then on $\{\omega\in\Omega : \max_{\zeta_u<n\le\tau_u} Y_n(\omega) > Y_{\zeta_u}(\omega)\}$, we have

$$\max_{\zeta_u<n\le\tau_u} Y_n = Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u, \tag{3.17}$$

and our objective is to show that $\mathbb{P}\{\max_{\zeta_u<n\le\tau_u} Y_n > u\}$ decays at the rate specified on the right-hand side of (3.15).

Step 1a. We begin by analyzing the second term of the right-hand side of (3.17). Observe that

$$\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u\} = \int_{\mathbb{R}} \mathbb{P}\{\log \Pi_{\zeta_u} > \log u - s\}\, dF_u(s), \tag{3.18}$$

where F_u denotes the probability distribution function of $\log \mathscr{M}_u$. To evaluate this integral, note that $\log \Pi_{\zeta_u} := \sum_{k=1}^{\zeta_u} \log A_i := S_{\zeta_u}$, and thus

$$\log \Pi_{\zeta_u} > \log u - s \quad \Longleftrightarrow \quad \frac{S_{\zeta_u}}{\zeta_u} > \frac{\log u - s}{\zeta_u} =: \frac{1}{\tau} + \gamma_u. \tag{3.19}$$

Letting γ_u be defined as in this last equation and utilizing the definition of ζ_u, we then obtain

$$\zeta_u \gamma_u = \frac{L_\tau(u)}{\tau} \log u - s + \boldsymbol{\delta}_u \qquad \text{where } |\boldsymbol{\delta}_u| \le \frac{1}{\tau}. \tag{3.20}$$

Consequently,

$$\begin{aligned} \gamma_u &= \frac{1}{\zeta_u}\left(\frac{L_\tau(u)}{\tau} \log u - s + \boldsymbol{\delta}_u\right) \quad \text{and} \\ \zeta_u \gamma_u^2 &= \frac{1}{\zeta_u}\left(\frac{L_\tau(u)}{\tau} \log u - s + \boldsymbol{\delta}_u\right)^2. \end{aligned} \tag{3.21}$$

From these equations, it is apparent that $\gamma_u \to 0$ and $\zeta_u \gamma_u^2 \to 0$ as $u \to \infty$ and, moreover, this convergence is uniform in s provided that $s \in [-(\log u)^{1/3}, (\log u)^{1/3}]$.

Now set $\alpha \equiv \alpha(\tau)$ for the remainder of the proof. Then by applying Theorem 3.1, we obtain that

$$\mathbb{P}\{\log \Pi_{\zeta_u} > \log u - s\} = \frac{1}{\alpha \sigma(\alpha)\sqrt{2\pi\tau \log u}} u^{-\alpha} e^{\alpha s} (\lambda(\alpha))^{\zeta_u} (1 + o(1)) \qquad \text{as } u \to \infty, \tag{3.22}$$

uniformly in s such that $\log s \in [-(\log u)^{1/3}, (\log u)^{1/3}]$. Letting $\mathscr{G}_u = \{\omega \in \Omega : \log \mathscr{M}_u(\omega) \in [-(\log u)^{1/3}, (\log u)^{1/3}]\}$ and returning to (3.18), we then obtain

$$\mathbb{P}\{\Pi_{\zeta_u} \mathscr{M}_u > u, \mathscr{G}_u\} = \frac{1}{\alpha \sigma(\alpha)\sqrt{2\pi\tau \log u}} (\lambda(\alpha))^{\zeta_u} u^{-\alpha} \mathbb{E}[\mathscr{M}_u^\alpha \mathbf{1}_{\mathscr{G}_u}] (1 + o(1)) \qquad \text{as } u \to \infty. \tag{3.23}$$

Now recall [cf. (3.8)] that

$$(\lambda(\alpha))^{\tau \log u} u^{-\alpha} = u^{-I(\tau)}.$$

Moreover, since $\mathscr{M}_u \stackrel{\mathscr{D}}{=} \max\{Y_i : 0 \le i \le \tau_u - \zeta_u\} \equiv M_{\lfloor \tau_u - \zeta_u \rfloor}$, we have

$$\lim_{u \to \infty} \frac{1}{(\lambda(\alpha))^{\tau_u - \zeta_u}} \mathbb{E}[\mathscr{M}_u^\alpha \mathbf{1}_{\mathscr{G}_u}] = \lim_{n \to \infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}[M_n^\alpha \mathbf{1}_{\mathscr{H}_n}],$$

where $\mathscr{H}_n := \{\omega \in \Omega : \log(M_n(\omega)) \in [-e^{n/3c}, e^{n/3c}]\}$. [In the definition of $\mathscr{H}_n$, we have used that $\tau_u - \zeta_u \sim L_\tau(u) \log u = c \log(\log u)$.] Substituting these last two equations into (3.23) yields

$$\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_u\} = \frac{\hat{C}(\tau)(\lambda(\alpha))^{-\Xi(u)}}{\sqrt{\log u}} u^{-I(\tau)}\big(1+o(1)\big), \tag{3.24}$$

where $\Xi(u) := \tau\log u - \tau_u$ and

$$\hat{C}(\tau) = \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi\tau}} \lim_{n\to\infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}\big[M_n^\alpha \mathbf{1}_{\mathscr{H}_n}\big]. \tag{3.25}$$

To complete the proof, we now show that the restriction to the sets $\mathscr{G}_u$ and $\mathscr{H}_n$ can be removed on the left- and right-hand sides of (3.24), (3.25), and that the limit in n on the right-hand side of (3.25) exists and is both positive and finite. To this end, first observe by Chebyshev's inequality that

$$\begin{aligned}
&\mathbb{P}\big\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log\mathscr{M}_u < -(\log u)^{1/3}\big\} \\
&\quad\le \mathbb{P}\big\{S_{\zeta_u} > \log u + (\log u)^{1/3}\big\} \\
&\quad\le \exp\big\{-\alpha\big(\log u + (\log u)^{1/3}\big)\big\}\big(\lambda(\alpha)\big)^{\zeta_u} = o\bigg(\frac{1}{\sqrt{\log u}}\big(\lambda(\alpha)\big)^{\zeta_u} u^{-\alpha}\bigg),
\end{aligned} \tag{3.26}$$

since $\lim_{u\to\infty}\sqrt{\log u}\exp\{-\alpha(\log u)^{1/3}\} = 0$. This shows that the restriction to values $\{\omega\in\Omega : \log\mathscr{M}_u(\omega) \ge -(\log u)^{1/3}\}$ can now be removed on the left-hand side of (3.23), hence the left-hand side of (3.24).

Moreover, repeating the argument leading to (3.24), we find that $\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log\mathscr{M}_u > (\log u)^{1/3}\}$ is equal to the right-hand side of (3.24), but with $\mathbb{E}[M_n^\alpha\mathbf{1}_{\mathscr{H}_n}]$ replaced with

$$\mathbb{E}\big[M_n^\alpha \mathbf{1}_{\mathscr{H}_n'}\big] \qquad \text{where } \mathscr{H}_n' := \big\{\omega\in\Omega : \log M_n(\omega) > e^{n/3c}\big\}.$$

We claim that

$$\lim_{n\to\infty}\frac{1}{\lambda^n(\alpha)}\mathbb{E}\big[M_n^\alpha\mathbf{1}_{\mathscr{H}_n'}\big] = 0. \tag{3.27}$$

Set $\mathscr{H}_{n,k}' = \{\omega\in\Omega : \log M_n(\omega) - e^{n/3c} \in (k-1,k]\}$, $k = 1,2,\ldots$; thus $\bigcup_{k\in\mathbb{Z}_+}\mathscr{H}_{n,k} = \mathscr{H}_n'$. Then apply Lemma 3.1 to obtain that

$$\begin{aligned}
\frac{1}{\lambda^n(\alpha)}\sum_{k=1}^\infty \mathbb{E}\big[M_n^\alpha\mathbf{1}_{\mathscr{H}_{n,k}'}\big] &\le \frac{1}{\lambda^n(\alpha)}\sum_{k=1}^\infty e^{\alpha k}\exp\big(\alpha e^{n/3c}\big)\mathbb{P}\big\{\overline{Y}_n > e^k\exp\big(e^{n/3c}\big)\big\} \\
&= \overline{C}_n\exp\big(-\varepsilon e^{n/3c}\big)\sum_{k=1}^\infty e^{-\varepsilon k}
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Now choose $\varepsilon \equiv \varepsilon(n) = n^{-2}$. With this choice of $\varepsilon(n)$, note that $\overline{C}_n = O(n^{2\alpha+1})$ and $\sum_{k=1}^\infty e^{-\varepsilon k} = O(n^2)$. Then $\overline{C}_n\exp(-n^{-2}e^{n/3c}) \times n^2 \to 0$ as $n\to\infty$. Thus, we obtain (3.27).

From (3.27), we conclude that the restrictions on large values can be removed in (3.24) and (3.25) [i.e., the restriction that $\log \mathscr{M}_u \le (\log u)^{1/3}$ in (3.24), and the restriction that $M_n \le e^{n/3c}$ in (3.25)]. Moreover, by a trivial calculation, the restriction to values $M_n \ge -e^{n/3c}$ can also be removed in (3.25). Consequently, we conclude that (3.24) and (3.25) hold *without* including the term $\mathscr{G}_u$ in (3.24), or the term $\mathbf{1}_{\mathscr{H}_n}$ in (3.25).

Step 1b. Finally, to establish (2.14), recall that $\max_{\zeta_u < n \le \tau_u} Y_n = Y_{\zeta_u} + \Pi_{\zeta_u} M_u$; cf. (3.17). Now we have just shown that

$$\mathbb{P}\{\Pi_{\zeta_u} \mathscr{M}_u > u\} = \frac{\hat{C}(\tau)(\lambda(\alpha))^{-\Xi(u)}}{\sqrt{\log u}} u^{-I(\tau)}\bigl(1 + o(1)\bigr), \tag{3.28}$$

where

$$C(\tau) = \frac{1}{\alpha\sigma(\alpha)\sqrt{2\pi\tau}} \lim_{n\to\infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}[M_n^\alpha]. \tag{3.29}$$

Moreover, by another application of Lemma 3.2, we obtain that

$$\mathbb{P}\{|Y_{\zeta_u}| > u\} = o\biggl(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\biggr) \qquad \text{as } u \to \infty. \tag{3.30}$$

Note that (3.30) implies the existence of a function $\Delta(u) \downarrow 0$ such that

$$\mathbb{P}\bigl\{|Y_{\zeta_u}| > \Delta(u)u\bigr\} = o\biggl(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\biggr) \qquad \text{as } u \to \infty. \tag{3.31}$$

Moreover, on the one hand,

$$\begin{aligned}
&\mathbb{P}\{Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u > u\} \\
&\quad = \mathbb{P}\bigl\{Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u > u, |Y_{\zeta_u}| \le \Delta(u)u\bigr\} \\
&\qquad + \mathbb{P}\bigl\{Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u > u, |Y_{\zeta_u}| > \Delta(u)u\bigr\} \\
&\quad \le \mathbb{P}\bigl\{\Pi_{\zeta_u}\mathscr{M}_u > \bigl(1 - \Delta(u)\bigr)u\bigr\} + \mathbb{P}\bigl\{|Y_{\zeta_u}| > \Delta(u)u\bigr\};
\end{aligned}$$

while on the other hand,

$$\begin{aligned}
&\mathbb{P}\bigl\{\Pi_{\zeta_u}\mathscr{M}_u > \bigl(1 + \Delta(u)\bigr)u\bigr\} \\
&\quad = \mathbb{P}\bigl\{\Pi_{\zeta_u}\mathscr{M}_u > \bigl(1 + \Delta(u)\bigr)u, |Y_{\zeta_u}| \le \Delta(u)u\bigr\} \\
&\qquad + \mathbb{P}\bigl\{\Pi_{\zeta_u}\mathscr{M}_u > \bigl(1 + \Delta(u)\bigr)u, |Y_{\zeta_u}| > \Delta(u)u\bigr\} \\
&\quad \le \mathbb{P}\{Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u > u\} + \mathbb{P}\bigl\{|Y_{\zeta_u}| > \Delta(u)u\bigr\}.
\end{aligned}$$

Thus, in view of (3.31),

$$\begin{aligned}
&\mathbb{P}\bigl\{\Pi_{\zeta_u}\mathscr{M}_u > \bigl(1 + \Delta(u)\bigr)u\bigr\} - o\biggl(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\biggr) \\
&\quad \le \mathbb{P}\{Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u > u\} \le \mathbb{P}\bigl\{\Pi_{\zeta_u}\mathscr{M}_u > \bigl(1 - \Delta(u)\bigr)u\bigr\} + o\biggl(\frac{u^{-I(\tau)}}{\sqrt{\log u}}\biggr).
\end{aligned} \tag{3.32}$$

Now apply (3.28) to the left- and right-hand sides of this equation. This yields that

$$\mathbb{P}\{Y_{\zeta_u} + \Pi_{\zeta_u}\mathscr{M}_u > u\} \sim \mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u\} \qquad \text{as } u \to \infty.$$

Hence, the required result follows from (3.28) and (3.16).

Step 2. It remains to show that this constant $C(\tau)$ is positive and finite, and that the limit in this equation actually exists.

Step 2a. First, we prove existence of the limit. For this purpose, we utilize the α-shifted measure defined previously in (3.1). Namely, observe that by (2.1) and (2.2)

$$\begin{aligned}\frac{1}{\lambda^n(\alpha)}\mathbb{E}[M_n^\alpha] &= \mathbb{E}_\alpha\Big[\Big(\max_{0\le k\le n} Y_k\Big)^\alpha \Pi_n^{-\alpha}\Big] \\ &= \mathbb{E}_\alpha\left[\left(\max_{1\le k\le n}\sum_{j=1}^k \widetilde{B}_j(\widetilde{A}_{j+1}\cdots\widetilde{A}_n)\vee 0\right)^\alpha\right],\end{aligned}$$

where $\widetilde{A}_j := 1/A_j$ and $\widetilde{B}_j := B_j/A_j$ for all j. By exchanging indices in this last expression, where we let $j \mapsto n+1-j$ in the expectation on the right-hand side, we then obtain

$$\frac{1}{\lambda^n(\alpha)}\mathbb{E}[M_n^\alpha] = \mathbb{E}_\alpha\left[\left(\max_{1\le k\le n}\sum_{j=k}^n (\widetilde{A}_1\cdots\widetilde{A}_{j-1})\widetilde{B}_j \vee 0\right)^\alpha\right]. \tag{3.33}$$

Note that in this expression, the pair $(\widetilde{A}, \widetilde{B})$ satisfies the following moment conditions:

$$\begin{aligned}&\mathbb{E}_\alpha[\log\widetilde{A}] = -\mathbb{E}_\alpha[\log A] = -\frac{1}{\lambda(\alpha)}\mathbb{E}[A^\alpha \log A] < 0;\\ &\mathbb{E}_\alpha[\widetilde{A}^\alpha] = \frac{1}{\lambda(\alpha)} < 1 \quad \text{and} \quad \mathbb{E}_\alpha[|\widetilde{B}|^\alpha] = \frac{1}{\lambda(\alpha)}\mathbb{E}[|B|^\alpha] < \infty.\end{aligned} \tag{3.34}$$

To further analyze the limit in (3.33) as $n \to \infty$, we first show the following.

ASSERTION. *Let $s_n = \sum_{j=1}^n d_j$ be an absolutely convergent series. Then the sequence*

$$m_n = \max\{d_n, d_{n-1} + d_n, \ldots, d_1 + \cdots + d_n\}$$

converges.

PROOF. It is sufficient to prove that m_n is a Cauchy sequence. Fix $\varepsilon > 0$. Since the series is absolutely convergent, there exists N such that $\sum_{j>N}|d_j| < \varepsilon$. Note

$$m_N = \max\{d_N, d_{N-1} + d_N, \ldots, d_1 + \cdots + d_N\},$$

and for any $p > N$,

$$\begin{aligned}m_p = \max\{&d_p, d_{p-1} + d_p, \ldots, d_{N+1} + \cdots + d_p,\\ &\ldots, d_1 + \cdots + d_N + d_{N+1} + \cdots + d_p\}.\end{aligned}$$

Note that m_p contains all of the factors that appear in m_N, but they are modified by adding $d_{N+1} + \cdots + d_p$ (which is at most ε in absolute value). Moreover, m_p contains $N - p$ additional terms, but all of them are bounded, in absolute value, by ε. Therefore $|m_N - m_p| < \varepsilon$ and m_n is convergent. $\square$

Now, in view of (3.34), the perpetuity

$$\widetilde{Y}_n = \sum_{j=1}^{n} \widetilde{A}_1 \cdots \widetilde{A}_{j-1} \widetilde{B}_j$$

converges $\mathbb{P}_\alpha$-a.s. Hence, by the last assertion,

$$X_n = \max_{1 \le k \le n} \sum_{j=k}^{n} (\widetilde{A}_1 \cdots \widetilde{A}_{j-1}) \widetilde{B}_j \vee 0$$

also converges $\mathbb{P}_\alpha$-a.s. Set $X = \lim_{n\to\infty} X_n$. Now X_n can be dominated by $R = \sum_{j=1}^{\infty} \widetilde{A}_1 \cdots \widetilde{A}_{j-1} |\widetilde{B}_j|$; and in view of (3.34) and (2.6), we have that $\mathbb{E}[R^\alpha] < \infty$. Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}[M_n^\alpha] = \lim_{n\to\infty} \mathbb{E}_\alpha[X_n^\alpha] = \mathbb{E}_\alpha[X^\alpha],$$

and this last expectation is finite. This proves the existence of the limit.

Step 2b. Finally, we prove that this limit is strictly positive. To this end, consider $\widetilde{Y}_n \vee 0$ as $n \to \infty$ [which we recognize as a single term in the maximum on the right-hand side of (3.33)]. Clearly, $\widetilde{Y}_n \le X_n$. Furthermore, $\widetilde{Y}_n$ converges to $\widetilde{Y}$ with $\mathbb{E}[|\widetilde{Y}|^\alpha] < \infty$, and

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\lambda^n(\alpha)} \mathbb{E}\big[(\max\{0, Y_n\})^\alpha\big] &= \lim_{n\to\infty} \mathbb{E}_\alpha\big[(\max\{0, \widetilde{Y}_n\})^\alpha\big] \\ &= \mathbb{E}_\alpha\big[(\max\{0, \widetilde{Y}\})^\alpha\big]. \end{aligned} \tag{3.35}$$

Also, observe that

$$\mathbb{E}_\alpha\big[(\max\{0, \widetilde{Y}\})^\alpha\big] \le \mathbb{E}_\alpha[X^\alpha].$$

We claim that if $\mathbb{P}\{A > 1, B > 0\} > 0$, then this last expectation is strictly positive. Let $\widetilde{\pi}$ denote the probability law of $\widetilde{Y}$, and assume the assertion to be false. Then $\widetilde{Y} \le 0$ $\mathbb{P}_\alpha$-a.s.; that is, $\operatorname{supp}(\widetilde{\pi}) \subset (-\infty, 0]$. Notice that $\operatorname{supp}(\widetilde{\pi})$ must be ν_α-invariant a.s. under the action of $(\widetilde{A}, \widetilde{B})$. Also note that $\mathbb{P}\{A > 1, B > 0\} > 0$ implies that $\mathbb{P}_\alpha\{\widetilde{A} < 1, \widetilde{B} > 0\} > 0$. Let $x_0 = \sup\{x : x \in \operatorname{supp}(\widetilde{\pi})\}$. Then $x_0 \le 0$, but taking a pair $(\widetilde{A}, \widetilde{B})$ such that $\widetilde{A} < 1$, $\widetilde{B} > 0$, we obtain that $\widetilde{A}x_0 + \widetilde{B} > x_0$, and we are led to a contradiction.

This shows that the constant $C(\tau)$ in (3.29) must be positive, thereby completing the proof of the theorem. $\square$

4. Proof of Theorem 2.2.

4.1. *Preliminary considerations.* As in the previous section, define $\overline{T}_u = (\log u)^{-1} \inf\{n : \overline{Y}_n > u\}$, where $\overline{Y}_n = \sum_{i=1}^n \Pi_{i-1}|B_i|$. First we establish an analog of Lemma 3.2 for the case $\tau = \rho$.

LEMMA 4.1. *Assume that $\Lambda(\xi+\eta) < \infty$ and $\Lambda_B(\xi+\eta) < \infty$ for some $\eta > 0$. Then there exists a finite constant $\overline{D}$ and positive constant $\delta \equiv \delta(\eta)$ such that for all $u \geq 0$,*

$$\mathbb{P}\{\overline{T}_u \leq \rho - L_\rho(u)\} \leq \overline{D} u^{-\xi} (\log u)^{-\delta}, \tag{4.1}$$

where $L_\rho(u) = b\sqrt{\{\log(\log u)\}/\log u}$ for any constant $b > \rho\{2(\xi+1) + \rho\sigma^2(\xi)\}$.

PROOF. Let $\zeta_u = \lfloor \log u(\rho - L_\rho(u)) \rfloor$, then by definition

$$\mathbb{P}\{\overline{T}_u \leq \rho - L_\rho(u)\} = \mathbb{P}\{\overline{Y}_{\zeta_u} > u\}. \tag{4.2}$$

Now apply Lemma 3.1 with $\alpha \equiv \xi$. Since $\Lambda(\xi) = 0$, it suffices to show that for some $\varepsilon \equiv \varepsilon(u)$,

$$\overline{C}_{\zeta_u} u^{-\varepsilon(u)} \leq \overline{D}(\log u)^{-\delta}. \tag{4.3}$$

Let

$$\varepsilon(u) = \left(\frac{\log(\log u)}{\log u}\right)^{1/2}. \tag{4.4}$$

To analyze $\overline{C}_{\zeta_u}$, first note by (2.13) and the definition of ρ that $\Lambda'(\xi) = \rho^{-1}$. Hence, for some finite constant D,

$$\overline{C}_{\zeta_u} \leq D(\log u)^{2(\xi+\varepsilon(u))+1} \exp\left\{\log u(\rho - L_\rho(u))\left(\frac{\varepsilon(u)}{\rho} + \varepsilon^2(u)\sigma^2(\xi)\right)\right\}.$$

Thus, for sufficiently large u,

$$\begin{aligned} \overline{C}_{\zeta_u} u^{-\varepsilon(u)} \leq D \exp\bigg\{ & 2(\xi+1)\log(\log u) - \frac{1}{\rho}\log u(L_\rho(u)\varepsilon(u)) \\ & + \rho \log u(\varepsilon^2(u)\sigma^2(\xi))\bigg\}. \end{aligned} \tag{4.5}$$

Substituting the definitions of $L_\rho(u)$ and $\varepsilon(u)$ into this last equation yields

$$\begin{aligned} \overline{C}_{\zeta_u} u^{-\varepsilon(u)} &\leq D \exp\left\{2(\xi+1)\log(\log u) - \frac{b}{\rho}\log(\log u) + \rho\sigma^2(\xi)\log(\log u)\right\} \\ &= D(\log u)^{-\delta}, \end{aligned} \tag{4.6}$$

where $\delta > 0$ whenever $b > \rho\{2(\xi+1) + \rho\sigma^2(\xi)\}$. Thus, we obtain (4.1) for sufficiently large u (with $\overline{D} = D$), and hence, with another choice $\overline{D} \geq D$, we obtain this equation for all $u \geq 0$. □

In the proofs below, it will be useful to observe that an analog of Lemma 4.1 also holds for the right tail of the hitting time of $\{\overline{Y}_n\}$ to the level u. To this end, set

$$\overline{Y}^n = \sum_{k=n+1}^{\infty} \Pi_{k-1}|B_k|, \qquad i = 1, 2, \dots, \tag{4.7}$$
$$\overline{T}^u = (\log u)^{-1} \sup\{n \in \mathbb{Z}_+ : \overline{Y}^n > u\}.$$

LEMMA 4.2. *Assume that $\Lambda(\xi+\eta) < \infty$ and $\Lambda_B(\xi+\eta) < \infty$ for some $\eta > 0$. Then there are constants $C, \delta, b > 0$ such that for every $u > e$,*

$$\mathbb{P}\{\overline{T}^u \geq \rho + L_\rho(u)\} \leq C u^{-\xi} (\log u)^{-\delta}, \tag{4.8}$$

where $L_\rho(u) = b\sqrt{\{\log(\log u)\}/\log u}$.

PROOF. Since $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$, it follows that for some $\varepsilon > 0$ (possibly dependent on k and u),

$$\begin{aligned}\mathbb{P}\{\overline{Y}^n > u\} &\leq \sum_{k=n+1}^{\infty} \mathbb{P}\left\{\Pi_{k-1}|B_k| > \frac{6u}{\pi^2(k-n)^2}\right\} \\ &\leq \sum_{k=n+1}^{\infty} \mathbb{E}\left[\Pi_{k-1}^{\xi-\varepsilon}|B_k|^{\xi-\varepsilon}\right]\left(\frac{\pi^2(k-n)^2}{6u}\right)^{\xi-\varepsilon}.\end{aligned} \tag{4.9}$$

Note by independence that

$$\mathbb{E}\left[\Pi_{k-1}^{\xi-\varepsilon}|B_k|^{\xi-\varepsilon}\right] = \left(\mathbb{E}\left[A^{\xi-\varepsilon}\right]\right)^{k-1}\mathbb{E}\left[|B|^{\xi-\varepsilon}\right] := \left(\lambda(\xi-\varepsilon)\right)^{k-1}\lambda_B(\xi-\varepsilon).$$

Moreover, since $\Lambda(\xi) = 0$, $\mu(\xi) = \rho^{-1}$, and Λ is infinitely differentiable on the interior of its domain,

$$\lambda(\xi-\varepsilon) = e^{\Lambda(\xi-\varepsilon)} \leq \exp\left\{-\frac{\varepsilon}{\rho} + \frac{\varepsilon^2 \mathfrak{l}}{2}\right\},$$

where $\mathfrak{l} := \sup\{\sigma^2(\alpha) : \xi - \varepsilon \leq \alpha \leq \xi\}$. Then using the continuity of $\sigma^2(\cdot)$, we have that for sufficiently small ε, $\mathfrak{l}/2 \leq \sigma^2(\xi)$. Hence, substituting the last two equations into (4.9) yields

$$\begin{aligned}&\mathbb{P}\{\overline{Y}^n > u\} \\ &\quad \leq \left(\frac{\pi^2}{6}\right)^{\xi} u^{-\xi} \sum_{j=1}^{\infty} j^{2\xi} u^{\varepsilon} \exp\left\{(n+j-1)\left(-\frac{\varepsilon}{\rho} + \varepsilon^2\sigma^2(\xi)\right)\right\}\lambda_B(\xi-\varepsilon).\end{aligned} \tag{4.10}$$

Now specialize to the case where $n \geq \log u(\rho + L_\rho(u))$. Then with $\varepsilon \equiv \varepsilon(j) \equiv \varepsilon(j,u)$, we obtain

$$\begin{aligned}\mathbb{P}\{\overline{Y}^n > u\} \leq \frac{\pi^2}{6} u^{-\xi} \sum_{j=1}^{\infty} j^{2\xi} \exp\biggl\{&-\frac{\varepsilon(j) L_\rho(u) \log u}{\rho} - (j-1)\frac{\varepsilon(j)}{\rho} \\ &+ (n+j-1)\varepsilon^2(j)\sigma^2(\xi)\biggr\}\lambda_B(\xi - \varepsilon(j)).\end{aligned} \tag{4.11}$$

Now choose

$$\varepsilon(j) = \gamma \frac{L_\rho(u)\log u + (j-1)}{\rho\sigma^2(\xi)(n+j-1)},$$

where γ is a positive constant. Since this expression remains bounded as $u \to \infty$ (uniformly in $j \geq 1$), the constant γ can be chosen such that $\varepsilon(j)$ is arbitrarily small. Then for $n(u) = \lfloor \log u(\rho + L_\rho(u)) \rfloor$, $b \geq \rho$, and $\gamma_1 = \gamma - \gamma^2$, we obtain by (4.11) that

$$\begin{aligned}\mathbb{P}\{\overline{Y}^{n(u)} > u\} &\leq C u^{-\xi} \sum_{j=1}^{\infty} j^{2\xi} \exp\biggl\{-\frac{\gamma_1 (L_\rho(u)\log u + j - 1)^2}{4\rho^2\sigma^2(\xi)(n(u)+j-1)}\biggr\} \\ &\leq C u^{-\xi}\Bigl((n(u))^{2\xi+1} \exp\{-\gamma_1 b \log(\log u)/16\rho^2\sigma^2(\xi)\} \\ &\qquad + \sum_{j \geq n(u)+1} j^{2\xi} \exp\{-\gamma_1 (j-1)/8\rho^2\sigma^2(\xi)\}\Bigr) \\ &\leq C u^{-\xi} (\log u)^{2\xi + 1 - \gamma_1 b/16\rho^2\sigma^2(\xi)},\end{aligned} \tag{4.12}$$

since for $j \leq n(u)$ we have

$$\frac{(L_\rho(u)\log u + j - 1)^2}{4\rho^2\sigma(n(u)+j-1)} \geq \frac{b^2 \log(\log u)}{8\rho^2(\rho+b)\sigma} \geq \frac{b\log(\log u)}{16\rho^2\sigma}.$$

Thus, (4.8) follows from (4.12) upon choosing $b \geq \max\{\rho, 16\sigma^2(\xi)\rho^2(2\xi + 1)/\gamma_1\}$. $\square$

From the previous lemma, we draw two conclusions. First, we observe that this lemma combined with Lemma 4.1 may be used to prove a strengthening of Lemma 2.1, thus establishing a conditional law of large numbers for the scaled first passage time of $\{Y_n\}$ to level u.

LEMMA 4.3. *Let $L_\rho(u)$ be given as in Lemma 4.1, and assume that Λ and Λ_B are finite in a neighborhood of ξ and the law of* $\log A$ *is nonlattice. Then*

$$\lim_{u\to\infty} \mathbb{P}\{|T_u - \rho| \geq L_\rho(u) | T_u < \infty\} = 0. \tag{4.13}$$

PROOF. Note $\overline{Y}_n \geq Y_n$, for all n, implying that $\{\overline{T}_u \leq \rho - L_\rho(u)\} \supset \{T_u \leq \rho - L_\rho(u)\}$. Consequently, it follows by Lemma 4.1 that

$$\mathbb{P}\{T_u \leq \rho - L_\rho(u) | T_u < \infty\} = o(1) \qquad \text{as } u \to \infty. \tag{4.14}$$

Next, set $n_u = \lceil \log u (\rho + L_\rho(u)) \rceil$ and define $R_n = M - M_n$. Observe that $R_n \leq \overline{Y}^n$, for all n. Hence, by Lemma 4.1,

$$\mathbb{P}\{R_{n_u} > u\} = o(u^{-\xi}) \qquad \text{as } u \to \infty. \tag{4.15}$$

Thus, by repeating the argument following (3.28) above, we obtain that the tail decay of $M = M_{n_u} + R_{n_u}$ is dominated by the larger of the tails of M_{n_u} and R_{n_u}, respectively, which must necessarily be the tail of M_{n_u}; that is,

$$\lim_{u \to \infty} u^\xi \mathbb{P}\{M > u\} = \lim_{u \to \infty} u^\xi \mathbb{P}\{M_{n_u} > u\}. \tag{4.16}$$

Since $\{M_{n_u} > u\} \subset \{M > u\}$, it follows that

$$\mathbb{P}\{T_u \geq \rho + L_\rho(u) | T_u < \infty\} = 1 - \frac{\mathbb{P}\{M_{n_u} > u\}}{\mathbb{P}\{M > u\}} = o(1) \qquad \text{as } u \to \infty, \tag{4.17}$$

as required. □

From the perspective of our main theorems, a more important consequence to be drawn from Lemma 4.1 is the convergence of a certain measure H to Lebesgue measure. We will establish this convergence in the assertion given in the proof of Theorem 2.2.

4.2. *The conditional central limit theorem for the critical case.*

PROOF OF THEOREM 2.2. *Step* 1. Let $L_\rho(u) = b\sqrt{\{\log(\log u)\}/\log u}$, where $b > \rho\{2(\xi + 1) + \rho\sigma^2(\xi)/2\}$. Then by Lemma 4.1,

$$\mathbb{P}\{\overline{T}_u \leq \rho - L_\rho(u)\} = o(u^{-\xi}) \qquad \text{as } u \to \infty. \tag{4.18}$$

Now let $y \in \mathbb{R}$ be given, and set

$$\zeta_u = \lfloor \log u (\rho - L_\rho(u)) \rfloor, \qquad \rho_u = \lfloor \rho \log u + (\rho^{3/2}\sigma(\xi)\sqrt{\log u}) y \rfloor, \qquad y \in \mathbb{R},$$

and

$$\mathscr{M}_u = \max_{\zeta_u < n \leq \rho_u} \{B_{\zeta_u+1} + A_{\zeta_u+1}B_{\zeta_u+2} + \cdots + (A_{\zeta_u+1} \cdots A_{n-1})B_n\} \vee 0. \tag{4.19}$$

Then arguing as in the proof of Theorem 2.1 [specifically, by repeating the argument following (3.28)], we obtain by (4.18) that

$$\mathbb{P}\{T_u \leq \rho\} = \mathbb{P}\{\Pi_{\zeta_u} \mathscr{M}_u > u\}(1 + o(1)) \qquad \text{as } u \to \infty. \tag{4.20}$$

To analyze the right-hand side of this equation, we begin by observing, as in the proof of Theorem 2.1, that

$$\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u\} = \int_{\mathbb{R}} \mathbb{P}\{\log \Pi_{\zeta_u} > \log u - s\}\, dF_u(s), \tag{4.21}$$

where F_u denotes the probability distribution function of $\log \mathscr{M}_u$. Then apply Petrov's theorem to handle the probability on the right-hand side.

First, observe [cf. (3.19), (3.21)] that

$$\log \Pi_{\zeta_u} > \log u - s \quad \Longleftrightarrow \quad \frac{S_{\zeta_u}}{\zeta_u} > \frac{\log u - s}{\zeta_u} := \frac{1}{\rho} + \gamma_u,$$

where, for a deterministic function $\boldsymbol{\delta}_u$ with $|\boldsymbol{\delta}_u| \le 1/\rho$, we have

$$\begin{aligned}\gamma_u &= \frac{1}{\zeta_u}\left(\frac{L_\rho(u)}{\rho}\log u - s + \boldsymbol{\delta}_u\right) \quad \text{and} \\ \zeta_u \gamma_u^2 &= \frac{1}{\zeta_u}\left(\frac{L_\rho(u)}{\rho}\log u - s + \boldsymbol{\delta}_u\right)^2.\end{aligned} \tag{4.22}$$

Now let $\Delta > 0$ and consider $\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_u\}$, where

$$\begin{aligned}\mathscr{G}_u &= \{\omega \in \Omega : \log \mathscr{M}_u(\omega) \in [0, D(u) + (y + \Delta)\sqrt{\zeta_u}\sigma(\xi)]\}, \\ D(u) &= \frac{L_\rho(u)}{\rho}\log u + \boldsymbol{\delta}_u.\end{aligned}$$

Note that when $s \in \mathscr{H}_u := [0, D(u) + (y + \Delta)\sqrt{\zeta_u}\sigma(\xi)]$ [corresponding to the event $\mathscr{G}_u$ occurring in (4.21)], we have by elementary calculations that $\gamma_u \to 0$ and $\zeta_u\gamma_u^3 \to 0$ as $u \to \infty$, uniformly for $s \in \mathscr{H}_u$. However, we do not have that $\zeta_u\gamma_u^2 \to 0$ as $u \to \infty$. Thus, focusing on the exponential term in Petrov's theorem, we see that the first- *and* second-order terms must be retained in the expansion (in contrast to the proof of Theorem 2.1, where it was sufficient to analyze the first-order term), while the third-order term may again be neglected. Consequently, by Petrov's Theorem 3.1, we obtain that

$$\begin{aligned}&\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_u\} \\ &\quad = \frac{1}{\xi\sigma(\xi)\sqrt{2\pi\zeta_u}} \int_0^{D(u)+(y+\Delta)\sqrt{\zeta_u}\sigma(\xi)} g(u, s)\, dF_u(s)\big(1 + o(1)\big),\end{aligned} \tag{4.23}$$

where

$$g(u, s) = u^{-\xi} e^{\xi s} \exp\left\{-\frac{1}{2\sigma^2(\xi)\zeta_u}\big(D(u) - s\big)^2\right\}. \tag{4.24}$$

Next, introduce the transformation

$$\mathbb{T}_u(s) = \frac{1}{\sigma(\xi)\sqrt{\zeta_u}}\big(s - D(u)\big), \tag{4.25}$$

and let $G_u(E) = F_u(\mathbb{T}_u^{-1}(E))$, for all $E \in \mathscr{B}(\mathbb{R})$. Then after a change of variables [Billingsley (1986), page 219], we obtain that

$$\begin{aligned}
&\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_u\} \\
&\quad = \frac{u^{-\xi}}{\xi\sigma(\xi)\sqrt{2\pi\zeta_u}} \int_{-D(u)/\sigma(\xi)\sqrt{\zeta_u}}^{y+\Delta} e^{\xi\mathbb{T}_u^{-1}(z)} e^{-z^2/2}\, dG_u(z)\big(1+o(1)\big) \qquad (4.26)\\
&\quad = \frac{u^{-\xi}}{\sqrt{2\pi}} \int_{-D(u)/\sigma(\xi)\sqrt{\zeta_u}}^{y+\Delta} e^{-z^2/2}\, dH_u(z)\big(1+o(1)\big)
\end{aligned}$$

as $u \to \infty$, where for any set $E \in \mathscr{B}(\mathbb{R})$,

$$\begin{aligned}
H_u(E) &= \frac{1}{\xi\sigma(\xi)\sqrt{\zeta_u}} \int_E e^{\xi\mathbb{T}_u^{-1}(z)}\, dG_u(z) \\
&= \frac{1}{\xi\sigma(\xi)\sqrt{\zeta_u}} \int_{\mathbb{T}_u^{-1}(E)} e^{\xi s}\, dF_u(s).
\end{aligned} \qquad (4.27)$$

Step 2. Our strategy now is to characterize the measure H_u on $(-\infty, y+\Delta]$, and then to handle the remaining part (namely, $\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_u^c\}$) by a separate argument. This remaining part will turn out to be asymptotically negligible. To characterize the measure H_u on $(-\infty, y+\Delta]$, we establish the following.

ASSERTION. (i) *Let $\mathscr{C}_M$ denote the constant appearing in* (2.7), *let l denote Lebesgue measure on $\mathbb{R}$, and let $\mathbf{C}_K$ denote the collection of continuous functions on $(-\infty, 0)$ with compact support. Then for any $f \in \mathbf{C}_K$,*

$$\lim_{u\to\infty} \int_{-\infty}^{y} f(s)\, dH_u(s) = \mathscr{C}_M \int_{-\infty}^{y} f(s)\, dl(s). \qquad (4.28)$$

(ii) *There is a constant $\overline{\mathscr{C}}$ such that for every $-\infty < v < w < \infty$ and every $u \ge 0$,*

$$H_u(v, w) \le \overline{\mathscr{C}}(w - v) + \frac{K}{\sqrt{\zeta_u}} \qquad \textit{for some } K < \infty. \qquad (4.29)$$

PROOF. We begin with the proof of (ii). Let $-\infty < v < w < \infty$, and set $v^*(u) = \mathbb{T}_u^{-1}(v)$, $w^*(u) = \mathbb{T}_u^{-1}(w)$, and let $\overline{F}_u(z) = 1 - F_u(-\infty, z)$, $z \in \mathbb{R}$. Then from (4.27) and an integration by parts,

$$\begin{aligned}
H_u(v, w) &= -\frac{1}{\xi\sigma(\xi)\sqrt{\zeta_u}}\big\{e^{\xi w^*(u)}\overline{F}_u\big(w^*(u)\big) - e^{\xi v^*(u)}\overline{F}_u\big(v^*(u)\big)\big\} \\
&\quad + \frac{1}{\sigma(\xi)\sqrt{\zeta_u}} \int_{v^*(u)}^{w^*(u)} e^{\xi s}\overline{F}_u(s)\, ds.
\end{aligned} \qquad (4.30)$$

To analyze the first term on the right-hand side of this equation, observe that the distribution of $\mathscr{M}_u$ is stochastically dominated by M. Moreover, as we have observed in Section 2,

$$\lim_{u\to\infty} u^{\xi}\mathbb{P}\{M > u\} = \mathscr{C}_M. \tag{4.31}$$

Consequently, there exists a finite positive constant $\overline{\mathscr{C}}$ such that

$$\mathbb{P}\{M > u\} \le \overline{\mathscr{C}} u^{-\xi} \qquad \text{for all } u \ge 0. \tag{4.32}$$

From (4.32), we obtain that $e^{\xi s}\overline{F}_u(s) \le \overline{\mathscr{C}}$ for all $s \in \mathbb{R}$. Hence we conclude that the first term on the right-hand side of (4.30) is dominated by $O(1/\sqrt{\zeta_u})$ as $u \to \infty$, where $\zeta_u \to \infty$, independent of the choice of v and w.

For the second term on the right-hand side of this equation, first note that by inverting the function $\mathbb{T}_u$ defined in (4.25), we obtain

$$\mathbb{T}_u^{-1}(t) = D(u) + t\sigma(\xi)\sqrt{\zeta_u}, \qquad t \in \mathbb{R}. \tag{4.33}$$

Hence,

$$w^*(u) - v^*(u) := \mathbb{T}_u^{-1}(w) - \mathbb{T}_u^{-1}(v) = \sigma(\xi)\sqrt{\zeta_u}(w - v),$$

and consequently,

$$\begin{aligned} \frac{1}{\sigma(\xi)\sqrt{\zeta_u}} \int_{v^*(u)}^{w^*(u)} e^{\xi s}\overline{F}_u(s)\,ds &\le \frac{\overline{\mathscr{C}}}{\sigma(\xi)\sqrt{\zeta_u}}\big(w^*(u) - v^*(u)\big) \\ &= \overline{\mathscr{C}}(w - v). \end{aligned} \tag{4.34}$$

Thus, we have established (ii).

Turning to the proof of (i), now suppose that $-\infty < \mathfrak{a} \le v < w \le \mathfrak{b} < y$. We begin by observing that

$$\lim_{u\to\infty} e^{\xi s}\overline{F}_u(s) = \mathscr{C}_M \qquad \text{uniformly for } s \in \mathbb{T}_u^{-1}\big([\mathfrak{a}, \mathfrak{b}]\big). \tag{4.35}$$

To establish this claim, first recall that F_u is the distribution function of $\log \mathscr{M}_u$, where $\mathscr{M}_u$ was defined in (4.19). Set

$$\begin{aligned} m_u = \rho_u - \zeta_u &:= \big\lfloor \rho \log u + \big(\rho^{3/2}\sigma(\xi)\sqrt{\log u}\big)y\big\rfloor - \big\lfloor \log u\big(\rho - L_\rho(u)\big)\big\rfloor \\ &= L_\rho(u)\log u + \big(\rho^{3/2}\sigma(\xi)\sqrt{\log u}\big)y + \boldsymbol{\delta}_u^*, \qquad \big|\boldsymbol{\delta}_u^*\big| \le 1. \end{aligned} \tag{4.36}$$

Now to prove (4.35), we use Lemma 4.1. Namely, we show that for some finite constant $\overline{D}$ and some positive constant δ,

$$e^{\xi s}\mathbb{P}\big\{\overline{Y}^{m_u} > e^s\big\} \le \overline{D}s^{-\delta}, \qquad s \in \mathbb{T}_u^{-1}\big([\mathfrak{a}, \mathfrak{b}]\big), \tag{4.37}$$

where $\overline{Y}^n$ was defined in (4.7).

Before establishing (4.37), we first observe that (4.37) implies (4.35). For this purpose, let $\{M_n\}$ and $\{R_n\}$ be defined as in the proof of Lemma 4.3, and observe that these definitions imply that $R_{m_u} \le \overline{Y}^{m_u}$ and $M = M_{m_u} + R_{m_u}$. Therefore, assuming that (4.37) holds, we have that

$$e^{\xi s}\mathbb{P}\{R_{m_u} > e^s\} \le \overline{D}s^{-\delta}. \tag{4.38}$$

Arguing as in the proof of Lemma 4.3, we then conclude that

$$\lim_{s\to\infty} e^{\xi s}\mathbb{P}\{M_{m_u} > e^s\} = \lim_{s\to\infty} e^{\xi s}\mathbb{P}\{M > e^s\} = \mathscr{C}_M. \tag{4.39}$$

Moreover, as a straightforward consequence of the definitions, we see that $\log M_{m_u}$ is equal in distribution to $\log \mathscr{M}_u$, which has the distribution F_u. Since $\mathbb{T}^{-1}(s)$ converges uniformly to infinity on $[\mathfrak{a}, \mathfrak{b}]$, (4.35) follows.

To establish (4.37), in view of Lemma 4.1 it is enough to observe that

$$m_u > (\rho + L_\rho(e^s))s \qquad \text{for all } s \in \mathbb{T}_u^{-1}([\mathfrak{a}, \mathfrak{b}]).$$

[We will apply Lemma 4.1 with e^s in place of u.] Now the largest possible value obtained by ρs for $s \in \mathbb{T}_u^{-1}([\mathfrak{a}, \mathfrak{b}])$ is given by

$$\begin{aligned} &\rho\mathbb{T}_u^{-1}(\mathfrak{b}) = L_\rho(u)\log u + \rho\mathfrak{b}\sigma(\xi)\sqrt{\zeta_u} + \rho\boldsymbol{\delta}_u; \\ &\text{and recall} \quad m_u = L_\rho(u)\log u + \big(\rho^{3/2}\sigma(\xi)\sqrt{\log u}\big)y + \boldsymbol{\delta}_u^*, \end{aligned} \tag{4.40}$$

where $|\rho\boldsymbol{\delta}_u|, |\boldsymbol{\delta}_u^*| \le 1$. Hence, we need to show that for sufficiently large u,

$$L_\rho(e^s)s + \rho\mathfrak{b}\sigma(\xi)\sqrt{\zeta_u} + (\rho+1) < \big(\rho^{3/2}\sigma(\xi)\sqrt{\log u}\big)y,$$

or

$$\begin{aligned} &L_\rho(e^s)s + \mathfrak{b}\sigma(\xi)\rho^{3/2}\big(\sqrt{\zeta_u/\rho} - \sqrt{\log u}\big) + (\rho+1) \\ &\quad < \big(\rho^{3/2}\sigma(\xi)\sqrt{\log u}\big)(y - \mathfrak{b}) \qquad \text{where } y > \mathfrak{b}. \end{aligned} \tag{4.41}$$

But for sufficiently large u, we see from the first equation in (4.40) that [as $L_\rho(u) = b\sqrt{\{\log(\log u)\}/\log u}$ and $s \le \mathbb{T}^{-1}(\mathfrak{b})$], we have that

$$s \le \text{const.} \cdot \sqrt{\log(\log u)}\sqrt{\log u} \qquad \text{while } L_\rho(e^s) = b\sqrt{\log s/s}.$$

Hence,

$$L_\rho(e^s)s = b\sqrt{s\log s} = o(\sqrt{\zeta_u}) \qquad \text{as } u \to \infty,$$

and (4.41) follows. Thus, we have established (4.37), and consequently (4.35).

Now returning to (4.30) and focusing on the second term on the right-hand side of this equation, observe by the uniform convergence in (4.35) that

$$\begin{aligned} \lim_{u\to\infty} \frac{1}{\sigma(\xi)\sqrt{\zeta_u}} \int_{v^*(u)}^{w^*(u)} e^{\xi s}\overline{F}_u(s)\,ds &= \lim_{u\to\infty} \frac{\mathscr{C}_M(w^*(u) - v^*(u))}{\sigma(\xi)\sqrt{\zeta_u}} \\ &= \mathscr{C}_M(w - v), \end{aligned} \tag{4.42}$$

where the last step follows as in (4.34). Since the first term on the right of (4.30) is $O(1/\sqrt{\zeta_u})$, as we have shown in the proof of (i), we conclude that

$$\lim_{u\to\infty} H(u,v) = \mathscr{C}_M(w-v), \qquad -\infty < \mathfrak{a} \le v < w \le \mathfrak{b} < 0. \tag{4.43}$$

Then taking Riemann sums, we obtain that for any $f \in \mathbf{C}_K$,

$$\lim_{u\to\infty} \int_{-\infty}^{y} f(s)\, dH_u(s) = \mathscr{C}_M \int_{-\infty}^{y} dl(s), \tag{4.44}$$

as required. □

Step 3. Now returning to the proof of the main theorem, we split our interval into three parts,

$$\left[-\frac{D(u)}{\sqrt{\zeta_u}\sigma(\xi)}, -J\right], \qquad [-J, y-\Delta], \qquad [y-\Delta, y+\Delta] \qquad \text{where } 0 < J < \infty,$$

and observe by part (i) of the previous assertion that for any J,

$$\lim_{u\to\infty} \int_{-J}^{-\Delta} e^{-z^2/2}\, dH_u(s) = \mathscr{C}_M \int_{-J}^{-\Delta} e^{-z^2/2}\, dl(s),$$

while by part (ii) of the assertion,

$$\int_{-D(u)/(\sigma(\xi)\sqrt{\zeta_u})}^{-J} e^{-z^2/2}\, dH_u(s) \le \overline{\mathscr{C}} \int_{-\infty}^{-J} e^{-z^2/2}\, dl(s) \le \overline{\mathscr{C}} e^{-J^2/2},$$

$$\int_{-\Delta}^{\Delta} e^{-z^2/2}\, dH_u(s) \le \overline{\mathscr{C}} \int_{-\Delta}^{\Delta} e^{-z^2/2}\, dl(s) \le 2\overline{\mathscr{C}}\Delta.$$

Letting $J \to \infty$ yields

$$\begin{aligned}\lim_{u\to\infty} u^{\xi}\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_u\} &= \frac{\mathscr{C}_M}{\sqrt{2\pi}} \int_{-\infty}^{y+\Delta} e^{-z^2/2}\, dl(z) \\ &= \Phi(y) + o(1) \qquad \text{as } \Delta \to 0.\end{aligned} \tag{4.45}$$

Step 4. It remains to show that the restriction to $\mathscr{G}_u$ can be removed on the left-hand side of this last equation.

Step 4a. We begin by removing the restriction that $\log \mathscr{M}_u \le D(u) + (y+\Delta)\sqrt{\zeta_u}\sigma(\xi)$. To this end, letting $\mathbb{T}_u$ be defined as in (4.25), note that it is sufficient to show that for any $\Delta > 0$,

$$\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log \mathscr{M}_u > \mathbb{T}_u^{-1}(y+\Delta)\} = o(1)u^{-\xi} \qquad \text{as } u \to \infty. \tag{4.46}$$

Let $\Delta > 0$ be given, and set $\mathscr{G}_{u,k} = \{\omega \in \Omega : \log \mathscr{M}_u(\omega) - \mathbb{T}_u^{-1}(y+\Delta) \in (k-1, k]\}$, $k = 1, 2, \ldots$. Then

$$\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log \mathscr{M}_u > \mathbb{T}_u^{-1}(y+\Delta)\} = \bigcup_{k\in\mathbb{Z}_+} \{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_{u,k}\}.$$

Moreover,

$$\begin{aligned}\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_{u,k}\} &\le \mathbb{P}\{\Pi_{\zeta_u} > ue^{-(\mathbb{T}_u^{-1}(y+\Delta)+k)}\}\mathbb{P}\{\mathscr{G}_{u,k}\} \\ &\le u^{-\xi}e^{\xi\mathbb{T}_u^{-1}(y+\Delta)}e^{\xi k}\mathbb{P}\{\mathscr{G}_{u,k}\}\end{aligned} \tag{4.47}$$

by Chebyshev's inequality. In addition, by applying Lemma 3.1 with $\varepsilon \equiv \varepsilon(u) = (\log u)^{-1/3}$, we obtain that

$$\begin{aligned}\mathbb{P}\{\mathscr{G}_{u,k}\} &= \mathbb{P}\{\mathscr{M}_u > e^{\mathbb{T}_u^{-1}(y+\Delta)}e^{k-1}\} \\ &\le \overline{C}(u)e^{-(\xi+\varepsilon(u))\mathbb{T}_u^{-1}(y+\Delta)}e^{-(\xi+\varepsilon(u))(k-1)},\end{aligned} \tag{4.48}$$

where $\overline{C}(u)$ corresponds to the quantity $\overline{C}_n$ appearing in the statement of Lemma 3.1. To identify the growth rate of this function as $u \to \infty$, note that $\mathscr{M}_u \stackrel{\mathscr{D}}{=} \{Y_i : 0 \le i \le \rho_u - \zeta_u\}$, where [from the definitions following (4.18)] we have that

$$m_u := \rho_u - \zeta_u = L_\rho(u)\log u + \left(\rho^{3/2}\sigma(\xi)\sqrt{\log u}\right)y + \boldsymbol{\delta}_u^* \qquad \text{for } |\boldsymbol{\delta}_u^*| \le 1;$$

cf. (4.36). Now in Lemma 3.1, we must replace the parameter n with m_u. Since $\mu(\xi)=\rho^{-1}$, we then obtain that for some positive constant K,

$$\overline{C}(u) \le K\left(L_\rho(u)\log u\right)^{2(\xi+1)}\exp\left\{\varepsilon(u)\left(\frac{L_\rho(u)}{\rho}\log u + \left(\sigma(\xi)\sqrt{\rho\log u}\right)y\right)\right\}.$$

To obtain this expression, note that the term "$(n-1)\varepsilon^2\sigma^2(\alpha)$" of Lemma 3.1 is negligible, since m_u grows at rate $\sqrt{\log(\log u)}\sqrt{\log u}$, while $\varepsilon(u)$ is chosen such that

$$\varepsilon(u) \sim (\log u)^{-1/3} \qquad \text{as } u \to \infty.$$

Moreover from (4.33),

$$\mathbb{T}_u^{-1}(y+\Delta) = \frac{L_\rho(u)}{\rho}\log u + (y+\Delta)\sigma(\xi)\sqrt{\zeta_u} + \boldsymbol{\delta}_u.$$

Combining these last two equations yields

$$\overline{C}(u)e^{-\varepsilon(u)\mathbb{T}_u^{-1}(y+\Delta)} \le K'\left(L_\rho(u)\log u\right)^{2(\xi+1)}e^{-\varepsilon(u)g(u)},$$

where

$$g(u) = \Delta\sigma(\xi)\sqrt{\zeta_u} + y\sigma(\xi)(\sqrt{\zeta_u} - \sqrt{\rho\log u}).$$

Now

$$\sqrt{\rho\log u} - \sqrt{\zeta_u} \sim \sqrt{L_\rho(u)\log u} \sim \text{const.}\cdot\sqrt{\log(\log u)\sqrt{\log u}} \qquad \text{as } u \to \infty,$$

and, as $\varepsilon(u) \sim (\log u)^{-1/3}$ as $u \to \infty$, it follows that

$$-\varepsilon(u)(\sqrt{\zeta_u} - \sqrt{\rho \log u}) \le \text{const}.$$

Since

$$\begin{aligned}\big(L_\rho(u)\log u\big)^{2(\xi+1)} e^{-\varepsilon(u)\Delta\sigma(\xi)\sqrt{\zeta_u}} &\le c_2\big(\log(\log u)\log u\big)^{\xi+1/2} e^{-c_1\Delta\sigma(\xi)(\log u)^{1/6}} \\ &\le c_3 e^{-\Delta\sigma(\xi)(\log u)^{1/8}},\end{aligned}$$

we have

$$\overline{C}(u)e^{-\varepsilon(u)\mathbb{T}_u^{-1}(y+\Delta)} = O\big(\exp\{-\Delta\sigma(\xi)(\log u)^{1/8}\}\big) \qquad \text{as } u \to \infty. \tag{4.49}$$

Then by (4.48),

$$\mathbb{P}\{\mathscr{G}_{u,k}\} = O\big(\exp\{-\Delta\sigma(\xi)(\log u)^{1/8}\}\big)e^{-\xi\mathbb{T}_u^{-1}(y+\Delta)}e^{-(\xi+\varepsilon(u))(k-1)} \qquad \text{as } u \to \infty. \tag{4.50}$$

Substituting this equation into (4.47), we conclude that

$$\begin{aligned}&\mathbb{P}\big\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log\mathscr{M}_u > \mathbb{T}_u^{-1}(y+\Delta)\big\} \\ &\quad = \sum_{k=1}^{\infty}\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \mathscr{G}_{u,k}\} \\ &\quad = O\big(\exp\{-\Delta\sigma(\xi)(\log u)^{1/8}\}\big)u^{-\xi}\sum_{k=0}^{\infty}e^{-k\varepsilon(u)} \\ &\quad = O\big(\exp\{-\Delta\sigma(\xi)(\log u)^{1/8}\}\big)u^{-\xi}(\log u)^{1/3} = o(1)u^{-\xi} \\ &\qquad\qquad \text{as } u \to \infty,\end{aligned} \tag{4.51}$$

which establishes (4.46).

Step 4b. Finally, observe by Chebyshev's inequality followed by a Taylor expansion that

$$\begin{aligned}&\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log\mathscr{M}_u < 0\} \\ &\quad \le \mathbb{P}\{\Pi_{\zeta_u} > u\} \le u^{-\xi-\varepsilon(u)}\exp\left\{\zeta_u\left(\frac{\varepsilon(u)}{\rho} + \varepsilon^2(u)\sigma^2(\xi)\right)\right\}.\end{aligned} \tag{4.52}$$

Next, recall that $\zeta_u := \lfloor \log u(\rho - L_\rho(u))\rfloor$; thus,

$$u^{-\varepsilon(u)}e^{\varepsilon(u)\zeta_u/\rho} \le \exp\left\{-\varepsilon(u)\left(\frac{L_\rho(u)}{\rho}\right)\log u + 1\right\}.$$

Now choose $\varepsilon(u) = (\log u)^{-1/2}$. Then on the right-hand side of the previous equation, the exponential term tends to $-\infty$ as $u \to \infty$. Moreover, with this choice of $\varepsilon(u)$, we also have that $\zeta_u\varepsilon^2(u)$ is bounded as $u \to \infty$. Thus, we conclude that

$$\mathbb{P}\{\Pi_{\zeta_u}\mathscr{M}_u > u, \log\mathscr{M}_u < 0\} = o(1)u^{-\xi} \qquad \text{as } u \to \infty, \tag{4.53}$$

as required.

Step 5. It remains to prove that the constant $\mathscr{C}_M$ is strictly positive. To this end, let Y be defined as in Section 2.1, let π denote the probability law of Y, and let $\mathscr{C}_Y$ be given as in (2.7). Clearly $\mathscr{C}_Y \le \mathscr{C}_M$. Thus, it is sufficient to analyze the case where $\mathscr{C}_Y = 0$. Then, by a result of Guivarc'h and Le Page (2013b), the support of π is unbounded from below.

We claim that Collamore and Vidyashankar [(2013b), Remark 2.3 and Section 9] can be applied to obtain the representation formula (2.20); hence $\mathscr{C}_M$ is strictly positive. For this purpose, we need to show that Lemma 5.1(iii) of that paper holds without the continuity assumption given there [namely, condition (H_0) of that paper]. Let the forward process $\{M_n^*\}$ be defined as in (2.3), and let P denote its transition kernel. Then P satisfies a minorization condition, namely,

$$P(x, E) \ge \mathbf{1}_{\mathbb{C}}(x)\eta(E), \qquad x \in \mathbb{R}, E \in \mathscr{B}(\mathbb{R}), \tag{4.54}$$

where $\mathbb{C} = \{0\}$ and η denotes the probability law of B^+. Note that this minorization condition is nontrivial, since $\operatorname{supp}(\pi) \cap (-\infty, 0] \neq \varnothing$ implies that $Y_n^* = A_n Y_{n-1}^* + B_n$ hits $(-\infty, 0]$ with positive probability; and hence, $\{M_n^*\}$ hits $\{0\}$ with positive probability, as these two processes agree up until the first time that $Y_n^* \le 0$. Thus, in particular, $\{M_n^*\}$ is ψ-irreducible [Nummelin (1984), Remark 2.1].

Moreover, if B^+ has a density on some subinterval of $(0, \infty)$, then the set $[0, K]$ is petite. Indeed, letting π^+ denote the stationary measure of $\{M_n^*\}$, then $\operatorname{supp}(\pi^+) \supset \operatorname{supp}(\eta)$ (the support of B^+), implying that $\operatorname{supp}(\pi^+)$ is of second category. Hence, since $[0, K]$ is a compact set and $\{M_n^*\}$ is a weak Feller chain, we may apply Remark 2.7(i) of Nummelin and Tuominen (1982) to conclude that $[0, K]$ is petite.

But if B^+ does not have a density on some subinterval of $(0, \infty)$, then we can dominate $\{M_n^*\}$ from below by the process $\{\widetilde{M}_n^*\}$, where

$$\widetilde{M}_n^* = \left(A_n \widetilde{M}_{n-1}^* + \widetilde{B}_n\right)^+, \qquad \widetilde{B}_n = B_n + \zeta_n,$$

where $\{\zeta_n\}$ is an i.i.d. sequence, independent of $\{(A_n, B_n)\}$, such that ζ has a smooth density supported on the interval $(-\delta, 0)$ for some $\delta > 0$. The process $\{\widetilde{M}_n^*\}$ is regularly varying at infinity with parameter ξ; that is, it satisfies (2.7) and the above argument can be applied to conclude that the corresponding constant $\mathscr{C}_{\widetilde{M}}$ is positive. Since $\{M_n^*\}$ dominates the process $\{\widetilde{M}_n^*\}$, we conclude that $\mathscr{C}_M$ is also strictly positive. □

5. Proof of Theorems 2.3 and 2.4.

5.1. *Proof of Theorem* 2.3. Before we proceed with the proof of Theorem 2.3, it is worthwhile to observe that there exists a measure satisfying the hypotheses of this theorem and, in particular, (2.24).

LEMMA 5.1. *There exists a measure satisfying the assumptions of Theorem 2.3.*

PROOF. Take an arbitrary measure ν with continuous non-vanishing density h and such that, if A has law ν, then $\mathbb{E}[\log A] < 0$ and $\mathbb{E}[A^{\alpha+\varepsilon}] < 1$ for some $\alpha > 1$ [e.g., one could choose $h(a) = C\eta^{-1}(1 + Ca)^{-\alpha-2\varepsilon-1}$ with C sufficiently large and $\eta = \int_0^\infty (1+u)^{-\alpha-2\varepsilon-1}\,du$]. Define the family of probability measures

$$\nu_t(da) = th(ta)\,da, \qquad t > 0.$$

Let $\Lambda_t(\beta) = \log \mathbb{E}_{\nu_t}[A^\beta]$ denote the cumulant generating function. Then

$$\Lambda_t(\alpha) = \log\left(\int_{\mathbb{R}} a^\alpha h(ta)t\,da\right) = -\alpha \log t + \Lambda(\alpha)$$

and

$$\mu_t(0) := \Lambda_t'(0) = \int_{\mathbb{R}} \log a h(ta)t\,da = -\log t + \mu(0).$$

Hence,

$$\mu_t(0) - \Lambda_t(\alpha) = (\alpha - 1)\log t + \mu(0) - \Lambda(\alpha),$$

and choosing t appropriately large, we have that $\mu_t(0) > \Lambda_t(\alpha)$. Thus, the measure ν_t satisfies the hypotheses of Theorem 2.3. □

The proof of Theorem 2.3 will now be based on the following two lemmas. In these lemmas, we study the joint distribution of $\Pi_n := A_1 \cdots A_n$ and $Y_n := 1 + A_1 + \cdots + (A_1 \cdots A_{n-1})$ as $n \to \infty$.

LEMMA 5.2. *Let $\beta \in \operatorname{int}(\operatorname{dom}\Lambda)$ be chosen such that $\Lambda(\beta) < \infty$, and set $\tau_\beta = (\mu(\beta))^{-1}$ and $k_u = \lfloor \log u/\mu(\beta)\rfloor$. Then there are positive constants D_0, γ, a, b with $\gamma b < 1$, such that for sufficiently large u,*

$$\mathbb{P}\{\gamma u \le \Pi_{k_u-1} \le \gamma a u, Y_{k_u} \le \gamma b u\} \ge \frac{D_0}{\sqrt{\log u}} u^{-I(\tau_\beta)}. \tag{5.1}$$

LEMMA 5.3. *Assume $\mathbb{E}[|\log A|^3] < \infty$, and set $\varepsilon(m) = e^{(m-1)\mu(0)}$. Then there are positive constants D_1, c, d such that for sufficiently large m,*

$$\mathbb{P}\{\varepsilon(m) \le \Pi_{m-1} \le c\varepsilon(m), Y_m \le d\} \ge \frac{D_1}{\sqrt{m}}. \tag{5.2}$$

Heuristically, Lemmas 5.2 and 5.3 can be understood as follows. Since $\mu(\beta) = \mathbb{E}_\beta[\log A]$, it follows that in the β-shifted measure, the random walk

$$S_n = \log \Pi_n := \sum_{i=1}^n \log A_i, \qquad n = 1, 2, \ldots$$

will reach the boundary at level $\log u$ at approximately the time $\tau_\beta \log u$, i.e., roughly at time k_u. Hence, it follows from standard large deviation arguments [based on the Bahadur–Rao approximation, cf. Dembo and Zeitouni (1993)] that

$$\mathbb{P}\{\log u + \log \gamma \le S_{k_u} \le \log u + \log(\gamma a)\} \sim \frac{D_0(\tau)}{\sqrt{\log u}} u^{-I(\tau_\beta)} \tag{5.3}$$

as $u \to \infty$. [In particular, (5.3) can be concluded from Petrov's Theorem 3.1, stated above.] Hence, (5.1) states that on $\{\log u + \log \gamma \le S_{k_u} \le \log u + \log(\gamma a)\}$, we have with high probability that $\log Y_{k_u} \le \log u + \log(\gamma b)$ for some positive constant b. The latter event can be expected, since $Y_n := 1 + A_1 + \cdots + (A_1 \cdots A_{n-1})$, and, in the β-shifted measure, this process will grow as $u \to \infty$ since $\mu(\beta) > 0$. Roughly speaking, $\{Y_n\}$ will then be dominated by its last term, namely Π_{n-1}, where $\log \Pi_{n-1} = S_{n-1}$.

In a similar way, Lemma 5.3 can be viewed, roughly speaking, as a consequence of the Berry–Esséen theorem, which studies the asymptotic behavior of $\{S_n\}$ around its central tendency, namely around $n\mu(0) = n\mathbb{E}[\log A]$. Then the Berry–Esséen theorem yields the estimate (5.2), but without the "$Y_m \le d$" term on the left-hand side. Note that when $\{S_n\}$ follows a trajectory which is close it its mean trajectory, one expects $\Pi_n \downarrow 0$ and $n \to \infty$, and hence $\{Y_n\}$ to be convergent.

Nonetheless, the formal proofs of Lemmas 5.2 and 5.3 are quite technical, and hence postponed to the end of this section. Before turning to these rigorous proofs, we first show how our main result may be deduced from these two lemmas.

PROOF OF THEOREM 2.3. Let $\alpha \equiv \alpha(\tau)$ [where $\alpha(\tau)$ is given as in (2.13)], and let $0 < \alpha < \beta < \xi$ be chosen such that

$$\Lambda'(\beta) = \frac{\Lambda'(\alpha)}{p}.$$

Later, we will choose β close to α and p close to one, but the precise choices of these constants will be fixed only at the end of the proof. Note that since $\mu(\cdot) := \Lambda'(\cdot)$,

$$\tau = \frac{1}{\mu(\alpha)} = \frac{1}{p\mu(\beta)} := \frac{\tau_\beta}{p}.$$

For $q = 1 - p$, now define

$$n_u = \lfloor \tau \log u \rfloor, \qquad k_u = \lfloor p n_u \rfloor, \qquad m_u = n_u - k_u.$$

We begin by writing

$$Y_{n_u} = Y_{k_u} + \Pi_{k_u - 1} A_{k_u} Y'_{m_u}, \tag{5.4}$$

where

$$Y'_{m_u} = 1 + A'_1 + \cdots + A'_1 \cdots A'_{m_u - 1} \qquad \text{for } A'_i = A_{n_u + i}.$$

Note that Y'_{m_u} is independent of Y_{k_u}, Π_{k_u-1} and A_{k_u}. Let

$$\Pi'_{m_u-1} = A'_1 \cdots A'_{m_u-1}, \qquad \boldsymbol{\varepsilon}_u = e^{(m_u-1)\mu(0)}$$

and

$$\Omega_u = \{\gamma u \le \Pi_{k_u-1} \le \gamma a u, Y_{k_u} \le b\gamma u, \boldsymbol{\varepsilon}_u \le \Pi'_{m_u-1} \le c\boldsymbol{\varepsilon}_u, Y'_{m_u} \le d\},$$

where the constants a, b, c, d are given as in Lemmas 5.2 and 5.3. Applying Lemmas 5.2 and 5.3, we then conclude that there exists a constant D such that, for sufficiently large u,

$$\mathbb{P}\{\Omega_u\} \ge \frac{D}{\sqrt{m_u}\sqrt{\log u}} u^{-I(\tau_\beta)}. \tag{5.5}$$

Next, observe

$$\begin{aligned}
&\mathbb{P}\{Y_{n_u-1} \le u \text{ and } Y_{n_u} > u\} \\
&\quad = \mathbb{P}\{Y_{k_u} + \Pi_{k_u-1}A_{k_u}Y'_{m_u-1} \le u \text{ and } Y_{k_u} + \Pi_{k_u-1}A_{k_u}Y'_{m_u} > u\} \\
&\quad \ge \mathbb{P}\left\{\left(\frac{u - Y_{k_u}}{\Pi_{k_u-1}Y'_{m_u}} < A_{k_u} \le \frac{u - Y_{k_u}}{\Pi_{k_u-1}Y'_{m_u-1}}\right) \cap \Omega_u\right\} \\
&\quad = \mathbb{P}\left\{\left(C_- \le \frac{u - Y_{k_u}}{\Pi_{k_u-1}Y'_{m_u}} < A_{k_u} \le \frac{u - Y_{k_u}}{\Pi_{k_u-1}Y'_{m_u-1}} \le C_+\right) \cap \Omega_u\right\}
\end{aligned} \tag{5.6}$$

for certain constants C_+ and C_-, since on the set Ω_u we have

$$C_+ := \frac{1}{\gamma} \ge \frac{u}{\Pi_{k_u-1}Y'_{m_u-1}} \quad \text{and} \quad C_- := \frac{1-\gamma b}{\gamma a d} \le \frac{u - Y_{k_u-1}}{\Pi_{k_u-1}Y'_{m_u}}.$$

Notice that on the set Ω_u, we have that for sufficiently large u,

$$\frac{u - Y_{k_u}}{\Pi_{k_u-1}Y'_{m_u-1}} - \frac{u - Y_{k_u}}{\Pi_{k_u-1}Y'_{m_u}} \ge \frac{(u - Y_{k_u})\Pi'_{m_u-1}}{\Pi_{k_u-1}d^2/2} \ge \frac{1-\gamma b}{\gamma a d^2/2} \cdot \boldsymbol{\varepsilon}_u := \gamma^* \boldsymbol{\varepsilon}_u.$$

Therefore, for sufficiently large u, A_{k_u} must belong to a random interval of length at least $\gamma^* \boldsymbol{\varepsilon}_u$.

Let I_{ε_u} be an arbitrary interval of length $\gamma^* \boldsymbol{\varepsilon}_u$. Since the density of A is bounded from below on the interval $[C_+, C_-]$ by some constant δ, we have

$$\begin{aligned}
\mathbb{P}\{Y_{n_u-1} \le u \text{ and } Y_{n_u} > u\} &\ge \inf_{I_{\boldsymbol{\varepsilon}_u} \subset [C_-, C_+]} \mathbb{P}\{(A_{k_u} \in I_{\boldsymbol{\varepsilon}_u}) \cap \Omega_u\} \\
&\ge \inf_{I_{\boldsymbol{\varepsilon}_u} \subset [C_-, C_+]} \mathbb{P}\{A \in I_{\boldsymbol{\varepsilon}_u}\}\mathbb{P}\{\Omega_u\} \\
&\ge \frac{\delta \boldsymbol{\varepsilon}_u}{\sqrt{m_u}} \frac{D}{\sqrt{\log u}} u^{-I(\tau_\beta)}.
\end{aligned} \tag{5.7}$$

We emphasize that in this computation, Ω_u has been defined so that A_{k_u} is independent of Ω_u. We now elaborate on the last term. Our objective is to compare the decay rate $\boldsymbol{\varepsilon}_u u^{-I(\tau_\beta)}$ to the "expected" decay rate governed by $u^{-I(\tau)}$. To this end, note by an application of (2.12) [cf. (3.8)] that

$$u^{I(\tau)-I(\tau_\beta)}\boldsymbol{\varepsilon}_u = \exp\{\alpha \log u - \Lambda(\alpha)n_u^* - \beta \log u + \Lambda(\beta)k_u^* + m_u^*\mu(0)\}, \tag{5.8}$$

where $n_u^* = \tau \log u$, $k_u^* = pn_u^*(= \tau_\beta \log u)$ and $m_u^* = n_u^* - k_u^*$. Now estimate the exponent in (5.8) from below. Recalling that $\mu(\alpha) = \tau^{-1}$, we obtain

$$\begin{aligned}
&\alpha \log u - \Lambda(\alpha)n_u^* - \beta \log u + \Lambda(\beta)k_u^* + m_u^*\mu(0)\\
&\quad = n_u^*\{\mu(\alpha)(\alpha - \beta) + p(\Lambda(\beta) - \Lambda(\alpha)) + q(\mu(0) - \Lambda(\alpha))\}\\
&\quad = p(\Lambda(\beta) - \Lambda(\alpha) - \mu(\alpha)(\beta - \alpha)) + q\mu(\alpha)(\alpha - \beta) + q(\mu(0) - \Lambda(\alpha))\\
&\quad \geq q(\mu(\alpha)(\alpha - \beta) + \mu(0) - \Lambda(\alpha)),
\end{aligned} \tag{5.9}$$

since $\Lambda(\beta) - \Lambda(\alpha) - \mu(\alpha)(\beta - \alpha) = \Lambda''(\theta) > 0$, for some $\theta \in [\alpha, \beta]$.

Since we are assuming that $\mu(0) > \Lambda(\alpha)$, we see that when β is close to α, the last expression in (5.9) is strictly positive. Thus, $\boldsymbol{\varepsilon}_u u^{-I(\tau_\beta)}$ decays at a slower polynomial rate than $u^{-I(\tau)}$. Hence, the required result follows from (5.7). □

We now return to the proofs of Lemmas 5.2 and 5.3.

PROOF OF LEMMA 5.2. *Step* 1. By Theorem 3.1, there exists a constant C_0 such that

$$\mathbb{P}\{\gamma u \leq \Pi_{k_u-1} \leq \gamma a u\} = \frac{C_0}{\sqrt{\log u}}\gamma^{-\beta}u^{-I(\tau_\beta)}(1 + o(1)) \qquad \text{as } u \to \infty. \tag{5.10}$$

The main step is to prove that there exist positive constants ε and C_1 and a constant $0 < \delta < 1$ such that, for sufficiently large u,

$$\begin{aligned}
&\mathbb{P}\left\{\gamma u \leq \Pi_{k_u-1}, \Pi_{k_u-i-1} > \frac{\gamma b u}{2i^2}\right\}\\
&\qquad \leq \frac{C_1}{\sqrt{\log u}}(\gamma^{-\beta}b^{-\varepsilon}\delta^i)u^{-I(\tau_\beta)} \qquad \text{for } i = 1, \dots, k_u - 1.
\end{aligned} \tag{5.11}$$

The final result follows easily from (5.10) and (5.11), since then we obtain

$$\begin{aligned}
&\mathbb{P}\{\gamma u \leq \Pi_{k_u-1} \leq \gamma a u, Y_{k_u} \leq \gamma b u\}\\
&\quad \geq \mathbb{P}\left\{\gamma u \leq \Pi_{k_u-1} \leq \gamma a u, \Pi_{k_u-i-1} \leq \frac{\gamma b u}{2i^2} \text{ for all } i \in \{1, \dots, k_u - 1\}\right\}\\
&\quad \geq \mathbb{P}\{\gamma u \leq \Pi_{k_u-1} \leq \gamma a u\} - \sum_i \mathbb{P}\left\{\gamma u \leq \Pi_{k_u-1} \leq \gamma a u, \Pi_{k_u-i-1} > \frac{\gamma b u}{2i^2}\right\}\\
&\quad \geq \left(C_0(1 + o(1)) - \frac{b^{-\varepsilon}C_1}{1 - \delta}\right)\frac{\gamma^{-\beta}}{\sqrt{\log u}}u^{-I(\tau_\beta)} \qquad \text{as } u \to \infty.
\end{aligned} \tag{5.12}$$

The result then follows after choosing the constant b sufficiently large.

Step 2. We now prove (5.11). For this purpose, we consider two cases, namely the case $i \le K \log k_u$ and then the case $i > K \log k_u$, where K is a large positive constant and $k_u = \lfloor \tau_\beta \log u \rfloor$.

Case 1: First, assume that $i \le K \log k_u$, and suppose that a constant L has been chosen such that

$$-L\beta - \Lambda(\beta)K + \tfrac{1}{2} \le -\eta < 0. \tag{5.13}$$

Clearly,

$$\begin{aligned} &\mathbb{P}\left\{\gamma u \le \Pi_{k_u-1}, \Pi_{k_u-i-1} \ge \frac{\gamma bu}{i^2}\right\} \\ &\quad \le \mathbb{P}\{\Pi_{k_u-i-1} \ge \gamma bu \cdot k_u^L\} \\ &\qquad + \mathbb{P}\left\{\gamma u \le \Pi_{k_u-1}, \frac{\gamma bu}{i^2} \le \Pi_{k_u-i-1} \le \gamma bu \cdot k_u^L\right\}. \end{aligned} \tag{5.14}$$

Now the first term on the right-hand side is asymptotically negligible, since it follows by an application of Chebyshev's inequality that for some finite constant C_2,

$$\begin{aligned} \mathbb{P}\{\Pi_{k_u-i-1} \ge \gamma bu \cdot k_u^L\} &\le (\gamma bu)^{-\beta} k_u^{-L\beta} \big(\lambda(\beta)\big)^{k_u-i-1} \\ &\le \frac{C_2}{\sqrt{\log u}} u^{-I(\tau_\beta)} \big(k_u^{-L\beta+1/2} e^{-i\Lambda(\beta)}\big) \gamma^{-\beta}, \end{aligned} \tag{5.15}$$

and by (5.13) and $\Lambda(\beta) < 0$, we have that for all $i \le K \log k_u$,

$$k_u^{-L\beta+1/2} e^{-i\Lambda(\beta)} \le e^{-\eta \log k_u} \searrow 0 \qquad \text{as } u \to \infty.$$

Thus, it is sufficient to focus on the second term on the right-hand side of (5.14). To this end, first note that

$$\begin{aligned} &\mathbb{P}\left\{\gamma u \le \Pi_{k_u-1}, \frac{\gamma bu}{i^2} \le \Pi_{k_u-i-1} \le \gamma bu \cdot k_u^L\right\} \\ &\quad \le \sum_{0 \le l \le \log(i^2 k_u^L)} \mathbb{P}\left\{\frac{\gamma bu}{i^2} \cdot e^l \le \Pi_{k_u-i-1} < \frac{\gamma bu}{i^2} \cdot e^{l+1}\right\} \mathbb{P}\left\{\Pi_i \ge \frac{i^2}{be^{l+1}}\right\} \\ &\quad \le \sum_{0 \le l \le (L+1)\log k_u} \mathbb{P}\left\{\Pi_{k_u-i-1} \ge \frac{\gamma bu}{i^2} \cdot e^l\right\} \mathbb{P}\left\{\Pi_i \ge \frac{i^2}{be^{l+1}}\right\} \end{aligned} \tag{5.16}$$

for sufficiently large u. The strategy is then to estimate the first term on the right-hand side by Petrov's theorem, and to estimate the second term using Chebyshev's inequality. Using that $S_n := \log \Pi_n$, we see that the first term can be written as

$$\mathbb{P}\left\{S_{k_u-i-1} \ge \log\left(\frac{\gamma bu}{i^2}\right) + l\right\},$$

where $k_u := \lfloor \tau \log u \rfloor \Longrightarrow \{\log(\gamma bu/i^2) + l\}/(k_u - i - 1) \sim \tau^{-1}$ as $u \to \infty$. Note that since $i \le K \log k_u$, the conditions of Theorem 3.1 are easily verified. Then by an application of Petrov's Theorem 3.1, we obtain that

$$\begin{aligned}\mathbb{P}\left\{S_{k_u-i-1} \ge \log\left(\frac{\gamma bu}{i^2}\right) + l\right\} \\ = \frac{1}{\beta\sigma(\beta)\sqrt{2\pi k_u}}\left(\frac{\gamma b e^l}{i^2}\right)^{-\beta} u^{-\beta} e^{(k_u-i)\Lambda(\beta)}\big(1+o(1)\big)\end{aligned} \tag{5.17}$$

as $u \to \infty$, uniformly for $i \le K \log u$. Recalling that $u^{-\beta}\exp\{\tau_\beta \log u \cdot \Lambda(\beta)\} = u^{-I(\tau_\beta)}$ [cf. (3.8)], we then obtain that for some finite constant C_3,

$$\mathbb{P}\left\{\Pi_{k_u-i-1} \ge \frac{\gamma bu}{i^2}\cdot e^l\right\} \le \frac{C_3}{\sqrt{\log u}} u^{-I(\tau_\beta)} \cdot i^{2\beta}\gamma^{-\beta} b^{-\beta} e^{-l\beta} e^{-i\Lambda(\beta)}, \qquad u \ge \text{ some } U_0. \tag{5.18}$$

Moreover, by Chebyshev's inequality, we have that for $\varepsilon > 0$ sufficiently small

$$\mathbb{P}\left\{\Pi_i \ge \frac{i^2}{be^{l+1}}\right\} \le i^{-2(\beta-\varepsilon)} b^{\beta-\varepsilon} e^{(l+1)(\beta-\varepsilon)} e^{i\Lambda(\beta-\varepsilon)}, \tag{5.19}$$

since $i^{2(\beta-\varepsilon)} \ge 1$. Next, observe that $\Lambda(\beta) - \Lambda(\beta-\varepsilon) = \varepsilon\Lambda'(\bar\beta) > 0$ for some $\bar\beta \in (\beta-\varepsilon, \beta)$, where positivity of Λ' follows from the convexity of Λ. Hence, we obtain from the previous two equations that, for some positive constant C_4 and sufficiently large u,

$$\begin{aligned}\sum_{0\le l\le (L+1)\log k_u} \mathbb{P}\left\{\Pi_{k_u-i-1} \ge \frac{\gamma bu}{i^2}\cdot e^l\right\}\mathbb{P}\left\{\Pi_i \ge \frac{i^2}{be^{l+1}}\right\} \\ \le \frac{C_4}{\sqrt{\log u}}(\gamma^{-\beta} b^{-\varepsilon}) u^{-I(\tau_\beta)} i^{2\varepsilon} e^{-i\varepsilon\Lambda'(\bar\beta)} \sum_{l=0}^{\infty} e^{-\varepsilon l},\end{aligned} \tag{5.20}$$

which yields (5.11).

Case 2: Now suppose that $i > K \log k_u$. Then by Chebyshev's inequality,

$$\begin{aligned}&\mathbb{P}\left\{\gamma u \le \Pi_{k_u-1}, \Pi_{k_u-i-1} > \frac{\gamma bu}{2i^2}\right\} \\ &\quad\le \sum_{l=0}^{\infty} \mathbb{P}\left\{\frac{\gamma bu}{i^2}\cdot e^l \le \Pi_{k_u-i-1} < \frac{\gamma bu}{i^2}\cdot e^{l+1}\right\}\mathbb{P}\left\{\Pi_i \ge \frac{i^2}{be^{l+1}}\right\} \\ &\quad\le \sum_{l=0}^{\infty}\left(\frac{\gamma bu}{i^2}\cdot e^l\right)^{-\beta} e^{(k_u-i-1)\Lambda(\beta)} \cdot \left(\frac{i^2}{be^{l+1}}\right)^{-(\beta-\varepsilon)} e^{i\Lambda(\beta-\varepsilon)} \\ &\quad\le \frac{1}{\lambda(\beta)}(\gamma^{-\beta} b^{-\varepsilon}) u^{-I(\tau_\beta)}\big(i^{2\varepsilon} e^{-i\varepsilon\Lambda'(\bar\beta)}\big)\sum_{l=0}^{\infty} e^{-l\varepsilon},\end{aligned} \tag{5.21}$$

where $\bar{\beta}$ is given as in (5.20). Hence, using that $(\log u)^{-1/2} \sim \sqrt{\tau}\exp\{\log k_u/2\} \le \sqrt{\tau}e^{i/K}$ for $i > K\log u$, we then obtain that for some positive constant C_5,

$$\begin{aligned}(5.22)\qquad &\mathbb{P}\left\{\gamma u \le \Pi_{k_u-1}, \Pi_{k_u-i-1} > \frac{\gamma b u}{2i^2}\right\}\\ &\quad\le \frac{C_5}{\sqrt{\log u}}(\gamma^{-\beta}b^{-\varepsilon})u^{-I(\tau_\beta)}\big(i^{2\varepsilon}e^{-i\varepsilon\Lambda'(\bar{\beta})}e^{i/K}\big),\end{aligned}$$

which establishes (5.11) upon choosing K sufficiently large. □

PROOF OF LEMMA 5.3. *Step* 1. With a slight abuse of notation, we will write ε_m in place of $\varepsilon(m)$ throughout the proof.

First, we prove that there exist finite constants c and $\mathfrak{M}$ such that

$$(5.23)\qquad \mathbb{P}\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m\} \ge \frac{\log c}{2\sigma\sqrt{2\pi}}\frac{1}{\sqrt{m}} \qquad \text{for all } m \ge \mathfrak{M},$$

where $\sigma^2 = \operatorname{Var}(\log A)$.

Since $\mu(0) = \mathbb{E}[X_1]$, it follows by the Berry–Esséen theorem [Petrov (1995), Theorem 5.5] that for all m,

$$(5.24)\quad \sup_x\left|\mathbb{P}\left\{\frac{S_m - m\mu(0)}{\sigma\sqrt{m}} < x\right\} - \Phi(x)\right| \le \frac{\mathscr{A}\mathbb{E}[|X_1 - \mathbb{E}X_1|^3]}{\sigma^3}\frac{1}{\sqrt{m}} := \frac{\rho}{\sqrt{m}},$$

where $S_m := \sum_{j=1}^m \log A_i$ and Φ denotes the normal distribution function, and where $\mathscr{A}$ is a universal constant. Hence, for any $c > 1$,

$$(5.25)\qquad \mathbb{P}\{0 \le S_m - m\mu(0) \le \log c\} \ge \left(\Phi\left(\frac{\log c}{\sigma\sqrt{m}}\right) - \Phi(0)\right) - \frac{2\rho}{\sqrt{m}}.$$

Now it follows from the definitions that $\log\{\Pi_{m-1}/\varepsilon_m\} = S_{m-1} - (m-1)\mu(0)$. Thus, from the previous equation we obtain that

$$\begin{aligned}(5.26)\qquad &\mathbb{P}\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m\}\\ &\quad\ge \frac{\log c}{\sigma\sqrt{2\pi}}\frac{1}{\sqrt{m-1}}e^{-(1/2)(\log c/(\sigma\sqrt{m-1}))^2} - \frac{2\rho}{\sqrt{m-1}}.\end{aligned}$$

Then choosing c sufficiently large yields (5.23).

Step 2. Next we show that for sufficiently large m,

$$(5.27)\qquad \mathbb{P}\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m, Y_m > d\} \le \frac{\mathscr{B}}{d^\theta\sqrt{m}},$$

where $\mathscr{B}$ and θ are finite positive constants. Noting that

$$\mathbb{P}\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m, Y_m \le d\} \le \sum_j \mathbb{P}\left\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m, \Pi_j > \frac{d}{2j^2}\right\},$$

we then divide the sum on the right-hand side into two parts.

Let $p < 1$, and suppose that $\theta > 0$ has been chosen such that $\Lambda(\theta) < 0$. Then, on the one hand,

$$\sum_{j>pm} \mathbb{P}\left\{\Pi_j > \frac{d}{2j^2}\right\} \le \sum_{j>pm} \frac{(2j^2)^\theta}{d^\theta} e^{j\Lambda(\theta)} = o\left(\frac{1}{\sqrt{m}}\right) \tag{5.28}$$

as $m \to \infty$. On the other hand, we also have

$$\begin{aligned} &\sum_{j\le pm} \mathbb{P}\left\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m, \Pi_j > \frac{d}{2j^2}\right\} \\ &\quad\le \sum_{j\le pm}\sum_{k\ge 0} \mathbb{P}\left\{\frac{d}{2j^2}\cdot e^k \le \Pi_j < \frac{d}{2j^2}\cdot e^{k+1} \text{ and } \varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m\right\} \\ &\quad\le \sum_{j\le pm}\sum_{k\ge 0} \mathbb{P}\left\{\Pi_j \ge \frac{d}{2j^2}\cdot e^k\right\}\mathbb{P}\left\{\frac{2\varepsilon_m j^2}{de^{k+1}} < \Pi_{m-j} \le \frac{2c\varepsilon_m j^2}{de^k}\right\}. \end{aligned} \tag{5.29}$$

To estimate the last quantity on the right-hand side, apply once again the Berry–Esséen theorem, noting that $j \le pm \Longrightarrow m - j > (1-p)m$. This yields (after a short computation) that

$$\begin{aligned} &\mathbb{P}\left\{\frac{2\varepsilon_m j^2}{de^{k+1}} < \Pi_{m-j} \le \frac{2c\varepsilon_m j^2}{de^k}\right\} \\ &\quad\le \frac{1}{\sqrt{2\pi}}\int_{C_{j,k}/(\sigma\sqrt{m-j})}^{(C_{j,k}+\log c+1)/(\sigma\sqrt{m-j})} e^{-x^2/2}\,dx + \frac{2\rho}{\sqrt{m}} \\ &\quad\le \frac{\mathscr{B}'}{\sqrt{m}}, \end{aligned} \tag{5.30}$$

where $C_{j,k} = j\mu(0) + \log(2j^2) - \log d - k - 1$ and $\mathscr{B}'$ is a finite constant. Note that this integral is taken over an interval of length $(\log c + 1)/(\sigma\sqrt{m-j})$. Consequently, with θ chosen as before and $\mathscr{B}''$ a positive constant, we obtain that for sufficiently large m,

$$\begin{aligned} \sum_{j\le pm} \mathbb{P}\left\{\varepsilon_m \le \Pi_{m-1} \le c\varepsilon_m, \Pi_j > \frac{d}{2j^2}\right\} &\le \sum_{j\le pm}\sum_{k\ge 0} \frac{(2j^2)^\theta}{d^\theta e^{\theta k}} e^{j\Lambda(\theta)} \frac{\mathscr{B}''}{\sqrt{m}} \\ &\le \frac{\mathscr{B}}{d^\theta\sqrt{m}} \end{aligned} \tag{5.31}$$

for some positive constant $\mathscr{B}$, as required.

Step 3. Finally, observe that if d is chosen sufficiently large in the previous equation, then the decay in (5.23) dominates the decay in (5.31). Consequently, the required result follows from (5.23) and (5.27). □

Finally, we remark that Theorem 2.3 also holds with $B > 0$ a.s. (but not necessarily constant, as was assumed in the previous proofs). However, in this case, the proofs become noticeably more technical. Thus, in order to emphasize the main ideas in the proofs, we have restricted our attention to the case $B = 1$.

To prove Theorem 2.3 for $B > 0$ a.s., we would need Lemma 5.2 at the required level of generality, and also Lemma 5.3 slightly modified. Namely, in place of Lemma 5.3 we would need the following result, which can be proved by analogous arguments.

LEMMA 5.4. *Assume* $\mathbb{E}[|\log A|^3] < \infty$, *and set* $\varepsilon(m) = e^{(m-1)\mu(0)}$. *Then there exist positive constants* $\widetilde{D}_1, \widetilde{c}, \widetilde{d}, \mathfrak{a}, \mathfrak{b}$ *such that*

$$\mathbb{P}\{\varepsilon(m) \le \Pi_{m-1} \le \widetilde{c}\varepsilon(m), Y_m \le \widetilde{d}, \mathfrak{a} < B_1 < \mathfrak{b}\} \ge \frac{\widetilde{D}_1}{\sqrt{m}} \tag{5.32}$$

for sufficiently large m.

We now show by example the typical difficulty that one encounters when B is allowed to be random. In the proof of Lemma 5.3, we need to estimate

$$\mathbb{P}\left\{\gamma u \le \Pi_{k_u}, \frac{\gamma b u}{i^2} e^l \le \Pi_{k_u - i} \max(1, B_{k(u)-i+1}) \le \frac{\gamma b u}{i^2} \cdot e^{l+1}\right\}.$$

This estimate is obtained by considering $\Pi_i' := \Pi_{k_u} / \Pi_{k(u)-i}$, and we need to have bounds on the two independent random variables, $\Pi_{k(u)-i}$ and Π_i'. For this purpose, we essentially need to eliminate the $A_{k(u)-i+1}$ and $B_{k(u)-i+1}$ terms, estimating the above probability by

$$\sum_{j,r} \mathbb{P}\left\{\gamma u \le \Pi_{k_u - i} \Pi_{i-1}' e^{j+1}, \frac{\gamma b u}{i^2} e^{-r-1} e^l \le \Pi_{k_u - i} \le \frac{\gamma b u}{i^2} e^{-r} e^{l+1},\right.$$
$$\left. e^j \le \max(1, A_{k_u - i+1}) \le e^{j+1}, e^r \le \max(1, B_{k_u - i+1}) \le e^{r+1}\right\},$$

and then using that $\mathbb{E}[A^\alpha + |B|^\alpha] < \infty$, to sum over all j and r. We omit the details, which are straightforward but technical.

5.2. *Proof of Theorem* 2.4. Let $b \ge 1$ and $B \in (0, b)$. Then Theorem 2.4 is a consequence of the following.

LEMMA 5.5. *Assume that the hypotheses of Theorem* 2.4 *are satisfied. Then there exists a constant* $\theta \in (0, 1)$ *and* $\mathscr{D}' < \infty$ *such that for every* $\varepsilon \in (0, 1/2)$ *and* u *sufficiently large*,

$$\begin{aligned} &\mathbb{P}\left\{Y_{n_u+k-1} \in \left((1-\varepsilon)u, \left(1 - \frac{\varepsilon}{2}\right)u\right), \Pi_{n_u+k-1} > \frac{\varepsilon}{2b} u\right\} \\ &\qquad \le \varepsilon^{1-\theta} \frac{\mathscr{D}' \lambda^k(\alpha)}{\sqrt{\log u}} u^{-I(\tau)}, \end{aligned} \tag{5.33}$$

where $n_u := \lfloor \tau \log u \rfloor$ and k is any non-negative integer, and the above result holds uniformly in k.

Before presenting the proof of the lemma, we first show how it may be applied to establish the theorem.

PROOF OF THEOREM 2.4. The proof is a simple consequence of the lemma. Set $r_u = n_u + k - 1$. Since

$$(0, u] = \bigcup_{i \geq 0} \left(\left(1 - \frac{1}{2^i}\right)u, \left(1 - \frac{1}{2^{i+1}}\right)u \right],$$

and since $Y_{r_u+1} = Y_{r_u} + \Pi_{r_u} B_{r_u+1}$, it follows that

$$\begin{aligned}
&\mathbb{P}\{Y_{r_u} \leq u \text{ and } Y_{r_u+1} > u\} \\
&\quad \leq \sum_{i \geq 0} \mathbb{P}\left\{ Y_{r_u} \in \left(\left(1 - \frac{1}{2^i}\right)u, \left(1 - \frac{1}{2^{i+1}}\right)u \right], \Pi_{r_u} B_{r_u+1} > \frac{u}{2^{i+1}} \right\} \\
&\quad \leq \sum_{i \geq 0} \mathbb{P}\left\{ Y_{r_u} \in \left(\left(1 - \frac{1}{2^i}\right)u, \left(1 - \frac{1}{2^{i+1}}\right)u \right], \Pi_{r_u} > \frac{u}{2^{i+1} b} \right\} \\
&\quad \leq \sum_{i \geq 0} \left(\frac{1}{2^{1-\theta}} \right)^i \frac{\mathscr{D}' \lambda^k(\alpha)}{\sqrt{\log u}} u^{-I(\tau)} \\
&\quad \leq \frac{\mathscr{D} \lambda^k(\alpha)}{\sqrt{\log u}} u^{-I(\tau)},
\end{aligned}$$

as required. □

PROOF OF LEMMA 5.5. We begin by establishing the following result.

ASSERTION. *For any $c > 1$ and $\varepsilon \in (0, 1/2)$, there exist positive constants $\theta \in (0, 1)$ and $\mathscr{D}' < \infty$ such that, for some finite constant U,*

$$\mathbb{P}\{c^{-1} u \leq Y_{n_{u+k-1}} \leq cu, \Pi_{n_u+k-1} > \varepsilon u\} \leq \varepsilon^{-\theta} \frac{\mathscr{D}' \lambda^k(\alpha)}{\sqrt{\log u}} u^{-I(\tau)} \tag{5.34}$$

all $k \geq 0, u \geq U$.

PROOF. Fix k, and set $r_u = n_u + k - 1$. Then define the set of indices

$$\mathscr{W}_j^u = \{i : i < r_u \text{ and } (cu)e^{-j} \leq \Pi_i B_{i+1} \leq (cu)e^{-j+1}\}.$$

Now suppose that $c^{-1} u \leq Y_{r_u} \leq cu$. Then we claim that for some j, the number of elements in the set $\mathscr{W}_j^u$ must be greater than $e^j/(10c^2 j^2)$. Indeed, if this were not the case, then (setting $\Pi_0 = 1$) we would have

$$\begin{aligned} Y_{r_u} &:= \sum_{i=0}^{r_u-1} \Pi_i B_{i+1} \le \sum_{j=1}^{\infty} \sum_{i \in \mathscr{W}_j^u} \Pi_i B_{i+1} \\ &\quad (\text{since } Y_{r_u} \le cu \Longrightarrow \Pi_i B_{i+1} \le cu \text{ for all } i \le r_u) \\ &\le \sum_{j=1}^{\infty} \frac{e^j}{10c^2 j^2} \cdot \frac{ecu}{e^j} \le \frac{e}{10} \cdot \frac{\pi^2}{6} \cdot \frac{u}{c} < \frac{u}{c}, \end{aligned}$$

a contradiction.

We now focus on the event $\{c^{-1}u \le Y_{r_u} \le cu\}$, which appears on the left-hand side of (5.34). Let $\mathscr{K}^u$ denote the following set of indices:

$$\mathscr{K}^u = \left\{ (j, m_1, m_2) : j \ge 1, 1 \le m_1 < r_u, m_1 + \frac{e^j}{10c^2 j^2} < m_2 < r_u \right\}.$$

Recall that for some j, $\mathscr{W}_j^u$ contains at least $e^j/(10c^2 j^2)$ members. This means that the first and last occurrences of the event described in $\mathscr{W}_j^u$ must be separated by a distance of at least $e^j/(10c^2 j^2)$; that is, there must exist values m_1 and m_2 such that

$$(cu)e^{-j} \le \Pi_{m_i} B_{m_i+1} \le (cu)e^{-j+1}, \qquad i = 1, 2 \quad \text{and} \quad m_2 - m_1 > \frac{e^j}{10c^2 j^2}.$$

Consequently,

$$\begin{aligned} &\mathbb{P}\{c^{-1}u \le Y_{r_u} \le cu, \Pi_{r_u} > \varepsilon u\} \\ &\quad \le \sum_{(j,m_1,m_2) \in \mathscr{K}^u} \mathbb{P}\{(cu)e^{-j} \le \Pi_{m_i} B_{m_i+1} \le (cu)e^{-j+1}, \\ &\qquad i = 1, 2; \Pi_{r_u} > \varepsilon u\} \\ &\quad \le \sum_{(j,m_1,m_2) \in \mathscr{K}^u} \mathbb{P}\left\{ \Pi_{m_1} \ge \frac{cu}{b} e^{-j} \right\} \mathbb{P}\left\{ \Pi_{m_2-m_1} B_1^{-1} \ge \frac{1}{b} e^{-1} \right\} \\ &\qquad \times \mathbb{P}\left\{ \Pi_{r_u-m_2} B_1^{-1} > \frac{\varepsilon}{bc} e^{j-1} \right\} \\ &\quad := \sum_{(j,m_1,m_2) \in \mathscr{K}^u} P_1^u P_2^u P_3^u, \end{aligned} \tag{5.35}$$

where P_1^u, P_2^u, P_3^u denote, respectively, the three probabilities appearing in the previous expression on the right-hand side.

While we will ultimately need a sharper estimate, we first estimate these probabilities via Chebyshev's inequality by choosing parameters $\beta_1 \in (0, \alpha)$ and

$\beta_2 \in (0, \alpha \wedge 1)$ such that

$$\rho_1 := \frac{\lambda(\beta_1)}{\lambda(\alpha)} < 1 \quad \text{and} \quad \rho_2 := \frac{\lambda(\beta_2)}{\lambda(\alpha)} < 1.$$

Note that the parameter β_2 exists due to the assumption (2.26). Applying Chebyshev's inequality with the parameter α to the probability P_1^u, with parameter β_1 to the probability P_2^u, and with parameter β_2 to the probability P_3^u, we obtain by (5.35) that

$$\begin{aligned} &\mathbb{P}\{c^{-1}u \le Y_{r_u} \le cu, \Pi_{r_u} > \varepsilon u\} \\ &\quad\le C_1 \varepsilon^{-\beta_2} \sum_{(j,m_1,m_2)\in\mathscr{K}^u} e^{j(\alpha-\beta_2)}\big(\lambda^{m_1}(\alpha)\lambda^{m_2-m_1}(\beta_1)\lambda^{r_u-m_2}(\beta_2)\big)u^{-\alpha} \qquad (5.36)\\ &\quad= \big(C_1\varepsilon^{-\beta_2}\big)\lambda^k(\alpha)u^{-I(\tau)} \sum_{(j,m_1,m_2)\in\mathscr{K}^u} e^{j(\alpha-\beta_2)}\rho_1^{m_2-m_1}\rho_2^{r_u-m_2} \end{aligned}$$

for some constant $C_1 < \infty$, where we have used the assumption that $\lambda_B(-\alpha) < \infty$.

Next, fix $t > 0$ and divide the set $\mathscr{K}^u$ into four subsets, as follows:

$$\begin{aligned} &\mathscr{K}_1^u = \{(j, m_1, m_2) \in \mathscr{K}^u : e^j > tr_u\}; \\ &\mathscr{K}_2^u = \{(j, m_1, m_2) \in \mathscr{K}^u : e^j \le tr_u, m_2 < r_u - r_u^{1/4}\}; \\ &\mathscr{K}_3^u = \{(j, m_1, m_2) \in \mathscr{K}^u : e^j \le tr_u, m_1 < r_u - 2r_u^{1/4}, m_2 \ge r_u - r_u^{1/4}\}; \\ &\mathscr{K}_4^u = \{(j, m_1, m_2) \in \mathscr{K}^u : e^j \le tr_u, m_1 \ge r_u - 2r_u^{1/4}\}. \end{aligned}$$

We now study (5.36) by calculating the sum on the right-hand side separately over the respective sets $\mathscr{K}_i^u$, $i = 1, \dots, 4$.

Case 1: First, we estimate the sum over $\mathscr{K}_1^u$. Since $\mathscr{K}_1^u \subset \mathscr{K}^u$, we have $m_2 - m_1 > e^j/(10c^2 j^2) \ge Le^{j/2}$ for some constant $L > 0$. Thus, for some positive constant L_1,

$$\begin{aligned} &\sum_{(j,m_1,m_2)\in\mathscr{K}_1^u} P_1^u P_2^u P_3^u \\ &\qquad\le \big(C_1\varepsilon^{-\beta_2}\big)\lambda^k(\alpha)u^{-I(\tau)} \sum_{(j,m_1,m_2)\in\mathscr{K}_1^u} e^{j(\alpha-\beta_2)}e^{-L_1 e^j}\rho_2^{r_u-m_2} \qquad (5.37)\\ &\qquad= o\bigg(\frac{\lambda^k(\alpha)}{\sqrt{\log u}}u^{-I(\tau)}\bigg) \qquad \text{as } u \to \infty \end{aligned}$$

when t is sufficiently large. The last step follows since $\rho_2 < 1$ and, by definition, the set $\mathscr{K}^u$ contains at most $r_u := \lfloor \tau \log u \rfloor + k$ members, while the subset $\mathscr{K}_u^1$ contains only those members where $e^j > tr_u$ (so that in (5.35)), the sum over j is finite and dominated by its initial term, that is, $\sum_{\mathscr{K}_1^u} e^{j(\alpha-\beta_2)}e^{-L_1 e^j} \le C_2[e^{j(\alpha-\beta_2)}e^{-L_1 e^j}]_{\{j=\log(tr_u)\}} \downarrow 0$ as $u \to \infty$).

Case 2: Next, consider the sum over $\mathscr{K}_2^u$. In this case, $r_u - m_2 > r_u^{1/4}$ and so in (5.35)

$$\begin{aligned}
&\sum_{(j,m_1,m_2)\in\mathscr{K}_2^u} P_1^u P_2^u P_3^u \\
&\quad\le \bigl(C_1\varepsilon^{-\beta_2}\bigr)\lambda^k(\alpha)u^{-I(\tau)} \sum_{(j,m_1,m_2)\in\mathscr{K}_2^u} (tr_u)^{\alpha-\beta_2}\rho_1^{m_2-m_1}e^{-L_2 n^{1/4}} \qquad (5.38)\\
&\quad= o\biggl(\frac{\lambda^k(\alpha)}{\sqrt{\log u}}u^{-I(\tau)}\biggr) \qquad \text{as } u\to\infty
\end{aligned}$$

for L_2 a positive constant and $\rho_1 < 1$, where we have again used that $\mathscr{K}^u$ contains at most $r_u := \lfloor \tau \log u\rfloor + k$ members, and on the subset $\mathscr{K}_2^u$, we have $e^j \le tr_u$.

Case 3: For the sum over $\mathscr{K}_3^u$, we can follow the same argument as in Case 2. In (5.35), we now utilize that $m_2 - m_1 > r_u^{1/4}$ and observe that $\rho_2 < 1$ (rather than observing that $r_u - m_2 > r_u^{1/4}$ and $\rho_1 < 1$). Hence, in either case, we have that

$$\rho_1^{m_2-m_1}\rho_2^{r_u-m_2} \le e^{-L_2 n^{1/4}},$$

and (5.35) can be applied to deduce the same estimate as in (5.38).

Case 4: Finally, we estimate the sum over $\mathscr{K}_4^u$. This estimate requires a more intricate calculation than (5.35), relying now on Petrov's Theorem 3.1.

Since $m_1 \ge r_u - 2r_u^{1/4}$, we may apply Theorem 3.1 to obtain that, uniformly in $m_1 \in [r_u - 2r_u^{1/4}, r_u]$,

$$P_1^u := \mathbb{P}\biggl\{\Pi_{m_1} \ge \frac{cu}{b}e^{-j}\biggr\} \le \frac{C_3 e^{\alpha j}}{\sqrt{m_1}}\lambda^{m_1}(\alpha)u^{-\alpha}, \qquad u \ge U_0, \qquad (5.39)$$

independent of k, where C_3 are U_0 are finite positive constants. Thus, repeating the calculation in (5.36), but using this estimate for P_1^u in place of the previous estimate (which was based on Chebyshev's inequality), we obtain that

$$\begin{aligned}
&\sum_{(j,m_1,m_2)\in\mathscr{K}_4^u} P_1^u P_2^u P_3^u \qquad (5.40)\\
&\quad\le \bigl(C_4\varepsilon^{-\beta_2}\bigr)\frac{\lambda^k(\alpha)}{\sqrt{\log u}}u^{-I(\tau)} \sum_{(j,m_1,m_2)\in\mathscr{K}_4^u} e^{j(\alpha-\beta_2)}\rho_1^{m_2-m_1}\rho_2^{r_u-m_2}
\end{aligned}$$

for some finite constant C_4 and u sufficiently large. To complete the proof, it is sufficient to justify that the last sum is bounded. For this purpose, first recall that since $\mathscr{K}_4^u \subset \mathscr{K}^u$, then as argued in Case 1, we have that $m_2 - m_1 > Le^{j/2}$ for some $L > 0$. Hence, for some positive constant L_3,

$$\begin{aligned}
&\sum_{(j,m_1,m_2)\in\mathscr{K}_4^u} e^{j(\alpha-\beta_2)}\rho_1^{m_2-m_1}\rho_2^{r_u-m_2} \\
&\quad\le \sum_{(j,m_1,m_2)\in\mathscr{K}_4^u} e^{j(\alpha-\beta_2)}e^{-L_1e^{j/2}}\rho_1^{(m_2-m_1)/2}\rho_2^{r_u-m_2} \qquad (5.41)\\
&\quad\le \Big(\sum_j e^{j(\alpha-\beta_2)}e^{-L_1e^{j/2}}\Big)\Big(\sum_{m_1<m_2}\rho_1^{(m_2-m_1)/2}\Big)\Big(\sum_{m_2<r_u}\rho_2^{r_u-m_2}\Big) < \infty,
\end{aligned}$$

since $\rho_1 < 1$ and $\rho_2 < 1$. Combining the estimates in steps 1–4, we obtain (5.34), as required. □

Returning now to the proof of the lemma, set

$$J_\varepsilon = \Big((1-\varepsilon)u, \Big(1-\frac{\varepsilon}{2}\Big)u\Big), \qquad \varepsilon > 0;$$

$$Y'_n = B_2 + \sum_{i=3}^{n+1}(A_2\cdots A_{i-1})B_i, \qquad n = 1,2,\dots;$$

$$\Pi'_n = \prod_{i=2}^{n+1} A_i, \qquad n = 1,2,\dots.$$

Then for all n, $(Y_n, \Pi_n) \stackrel{\mathscr{D}}{=} (Y'_n, \Pi'_n)$ and $Y_n = B_1 + A_1 Y'_{n-1}$.

Suppose that the constant a has been chosen such that w.p.1, the support of the law of A_1 is contained in the interval $[1/a, a]$. Setting $r_u = n_u + k - 1$, we then obtain

$$\begin{aligned}
&\mathbb{P}\Big\{Y_{r_u} \in J_\varepsilon, \Pi_{r_u} > \frac{\varepsilon}{2b}u\Big\} \\
&\quad\le \mathbb{P}\Big\{B_1 + A_1Y'_{r_u-1} \in J_\varepsilon, \Pi'_{r_u-1} > \Big(\frac{\varepsilon}{2ab}\Big)u\Big\} \qquad (5.42)\\
&\quad\le \mathbb{P}\Big\{A_1 \in \frac{1}{Y'_{r_u-1}}\Big((1-\varepsilon)u - b, \Big(1-\frac{\varepsilon}{2}\Big)u\Big), \Pi'_{r_u-1} > \Big(\frac{\varepsilon}{2ab}\Big)u\Big\},
\end{aligned}$$

where A_1 is independent of $(Y'_{r_u-1}, \Pi'_{r_u-1})$. Moreover, since $a^{-1} \le A_1 \le a$, we also have when $Y_{r_u} \in J_\varepsilon$ that

$$\begin{aligned}
Y'_{r_u-1} &\in \frac{1}{A_1}\Big((1-\varepsilon)u - b, \Big(1-\frac{\varepsilon}{2}\Big)u\Big) \subset \Big(\frac{(1-\varepsilon)u-b}{a}, a\Big(1-\frac{\varepsilon}{2}\Big)u\Big) \qquad (5.43)\\
&\subset \Big(\frac{u}{2a}, au\Big)
\end{aligned}$$

for sufficiently large u, independent of k. Then for fixed $Y'_{r_u-1} \in (u/2a, au)$, an easy calculation shows that the length of the interval

$$\frac{1}{Y'_{r_u-1}}\left((1-\varepsilon)u - b, \left(1-\frac{\varepsilon}{2}\right)u\right)$$

is bounded above by $d\varepsilon$ for some positive constant d. Hence, returning to (5.42), we obtain that

$$\begin{aligned}
\mathbb{P}\left\{Y_{r_u} \in J_\varepsilon, \Pi_{r_u} > \frac{\varepsilon}{2b}u\right\} &\le \int_{u/2a}^{au} \mathbb{P}\left\{A_1 \in \frac{1}{s}\left((1-\varepsilon)u - b, \left(1-\frac{\varepsilon}{2}\right)u\right)\right\} \\
&\quad \times \mathbb{P}\left\{\Pi'_{r_u-1} > \left(\frac{\varepsilon}{2ab}\right)u, Y'_{r_u-1} \in ds\right\} \\
&\le d\varepsilon \mathbb{P}\left\{\frac{u}{2a} \le Y_{r_u-1} \le au, \Pi_{r_u-1} > \left(\frac{\varepsilon}{2ab}\right)u\right\} \\
&\le d\varepsilon \mathbb{P}\left\{\frac{u}{a} \le Y_{r_u-1} \le au, \Pi_{r_u-1} > \varepsilon^* u\right\}
\end{aligned} \tag{5.44}$$

for certain positive constants ε^* and u. Applying (5.34) to the last quantity on the right-hand side yields (5.33), as required. □

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