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MODERATE DEVIATION PRINCIPLES FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

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Moderate deviation principles for stochastic differential equations driven by a Poisson random measure (PRM) in finite and infinite dimensions are obtained. Proofs are based on a variational representation for expected values of positive functionals of a PRM.

1. Introduction. Large deviation principles for small noise diffusion equations have been extensively studied in the literature. Since the original work of Freidlin and Wentzell [27, 57], model assumptions have been significantly relaxed, and many extensions have been studied in both finite-dimensional and infinite-dimensional settings. In [9, 10] a general approach for studying large deviation problems in such settings was introduced that is based on a variational representation for expectations of positive functionals of an infinite-dimensional Brownian motion. This approach has been adopted for the study of large deviation problems for a broad range of stochastic partial differential equation based models, particularly those arising in stochastic fluid dynamics, and also for settings where the coefficients in the model have little regularity. We refer the reader to [11] for a partial list of references. Large deviation problems for finite-dimensional diffusions with jumps have been studied by several authors; see, for example, [25, 48]. In contrast, it is only recently that the analogous problems for infinite-dimensional stochastic differential equations (SDEs) have received attention [54, 56]. Budhiraja, Dupuis and Maroulas [11] derived a variational representation for expected values of positive functionals of a general Poisson random measure (or more generally, functions that depend both on a Poisson random measure and an infinite-dimensional Brownian motion). As in the Brownian motion case, the representation is motivated in part by applications to large deviation problems, and [11] illustrates how the representation can be applied in a simple finite-dimensional setting. In [8], Budhiraja,

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Dupuis and Maroulas used the representation to study large deviation properties of a family of infinite-dimensional SDE driven by a Poisson random measure (PRM).

The goal of the current work is to study moderate deviation problems for stochastic dynamical systems. In such a study, one is concerned with probabilities of deviations of a smaller order than in large deviation theory. Consider, for example, an independent and identically distributed (i.i.d.) sequence $\{Y_i\}_{i\geq 1}$ of $\mathbb{R}^d$-valued zero mean random variables with common probability law ρ. A large deviation principle (LDP) for $S_n = \sum_{i=1}^n Y_i$ will formally say that for $c > 0$,

$$\mathbb{P}(|S_n| > nc) \approx \exp\{-n \inf\{I(y) : |y| \geq c\}\},$$

where for $y \in \mathbb{R}^d$, $I(y) = \sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha, y\rangle - \log\int_{\mathbb{R}^d}\exp\langle\alpha, y\rangle\rho(dy)\}$. Now let $\{a_n\}$ be a positive sequence such that $a_n \uparrow \infty$ and $n^{-1/2}a_n \to 0$ as $n \to \infty$ (e.g., $a_n = n^{1/4}$). Then a moderate deviation principle (MDP) for S_n will say that

$$\mathbb{P}(|S_n| > n^{1/2}a_n c) \approx \exp\{-a_n^2 \inf\{I^0(y) : |y| \geq c\}\},$$

where $I^0(y) = \frac{1}{2}\langle y, \Sigma^{-1}y\rangle$ and $\Sigma = \mathrm{Cov}(Y)$. Thus the moderate deviation principle gives estimates on probabilities of deviations of order $n^{1/2}a_n$, which is of lower order than n and with a rate function that is a quadratic form. Since $a_n \to \infty$ as slowly as desired, moderate deviations bridge the gap between a central limit approximation and a large deviations approximation. Moderate deviation principles have been extensively studied in mathematical statistics. Early research considered the setting of i.i.d. sequences and arrays; see [1, 2, 28, 50, 51, 53, 55]. Empirical processes in general topological spaces have been studied in [3, 6, 7, 12, 16, 21, 45]. The setting of weakly dependent sequences was covered in [5, 13, 19, 20, 22, 32–36], and MDPs for occupation measures of Markov chains and general additive functionals of Markov chains were considered in [14, 15, 17, 18, 29, 31, 37, 58].

Moderate deviation principles for continuous time stochastic dynamical systems are less studied. The paper [46] considers a finite-dimensional two scale diffusion model under stochastic averaging. Additional results involving moderate deviations and the averaging principle were obtained in [38, 39, 47]. Hu and Shi [40] considered a certain diffusion process with Brownian potentials and derived moderate deviation estimates for its long-time behavior. Moderate deviation results in the context of statistical inference for finite-dimensional diffusions have been considered in [23, 30, 41]. None of the above results consider stochastic dynamical systems with jumps or infinite-dimensional models.

In this paper we study moderate deviation principles for finite- and infinite-dimensional SDEs with jumps. For simplicity we consider only settings where the noise is given in terms of a PRM and there is no Brownian component. However, as noted in Remark 2.9, the more general case, where both Poisson and Brownian noises are present, can be treated similarly. In finite dimensions, the basic stochastic dynamical system we study takes the form

$$X^\varepsilon(t) = x_0 + \int_0^t b(X^\varepsilon(s))\,ds + \int_{\mathbb{X}\times[0,t]} \varepsilon G(X^\varepsilon(s-), y) N^{\varepsilon^{-1}}(dy, ds).$$

Here $b:\mathbb{R}^d \to \mathbb{R}^d$ and $G:\mathbb{R}^d \times \mathbb{X} \to \mathbb{R}^d$ are suitable coefficients, and $N^{\varepsilon^{-1}}$ is a Poisson random measure on $\mathbb{X}_T = \mathbb{X} \times [0, T]$ with intensity measure $\varepsilon^{-1}\nu_T = \varepsilon^{-1}\nu \otimes \lambda_T$, where $\mathbb{X}$ is a locally compact Polish space, ν is a locally finite measure on $\mathbb{X}$, λ_T is Lebesgue measure on $[0, T]$ and $\varepsilon > 0$ is the scaling parameter. Under conditions, X^ε will converge in probability (in a suitable path space) to X^0 given as the solution of the ODE

$$\dot{X}^0(t) = b\big(X^0(t)\big) + \int_{\mathbb{X}} G\big(X^0(t), y\big)\nu(dy), \qquad X^0(0) = x_0.$$

The moderate deviations problem for $\{X^\varepsilon\}_{\varepsilon>0}$ corresponds to studying the asymptotics of

$$\big(\varepsilon/a^2(\varepsilon)\big)\log \mathbb{P}\big(Y^\varepsilon \in \cdot\big),$$

where $Y^\varepsilon = (X^\varepsilon - X^0)/a(\varepsilon)$ and $a(\varepsilon) \to 0$, $\varepsilon/a^2(\varepsilon) \to 0$ as $\varepsilon \to 0$. In this paper we establish a moderate deviation principle under suitable conditions on b and G. We in fact give a rather general sufficient condition for a moderate deviation principle to hold for systems driven by Poisson random measures; see Theorem 2.3. This sufficient condition covers many finite and infinite-dimensional models of interest. A typical infinite-dimensional model corresponds to the stochastic partial differential equation (SPDE)

$$\begin{aligned} dX^\varepsilon(u,t) &= \big(\mathcal{L}X^\varepsilon(u,t) + \beta\big(X^\varepsilon(u,t)\big)\big)\,dt \\ &\qquad + \varepsilon\int_{\mathbb{X}} G\big(X^\varepsilon(u,t-), u, y\big)N^{\varepsilon^{-1}}(dy, dt), \qquad (1.1)\\ X^\varepsilon(u,0) &= x(u), \qquad u \in O \subset \mathbb{R}^d, \end{aligned}$$

where $\mathcal{L}$ is a suitable differential operator, β is a function from $\mathbb{R}^d$ to $\mathbb{R}^d$, O is a bounded domain in $\mathbb{R}^d$ and the equation is considered with a suitable boundary condition on ∂O. Here $N^{\varepsilon^{-1}}$ is a PRM as above. The solution of such an SPDE has to be interpreted carefully, since typically solutions for which $\mathcal{L}X^\varepsilon(u,t)$ can be defined classically do not exist. We follow the framework of [44], where the solution space is described as the space of RCLL trajectories with values in the dual of a suitable nuclear space; see Section 2.4 for precise definitions. Roughly speaking, a nuclear space is given as an intersection of a countable collection of Hilbert spaces, where the different spaces may be viewed as "function spaces" with a varying degree of regularity. Since the action of the differential operator $\mathcal{L}$ on a function will typically produce a function with lesser regularity, this framework of nested Hilbert spaces enables one to efficiently investigate existence and uniqueness of solutions of SPDEs of the form of (1.1). Another common approach for studying equations of the form of (1.1) is through a mild solution formulation as in [52]. Although not investigated here, we expect that analogous results can be established using such a formulation.

Large and moderate deviation approximations can provide qualitative and quantitative information regarding complex stochastic models such as (1.1). For example, an equation studied in some detail at the end of this paper models the concentration of pollutants in a waterway. Depending on the event of interest, either the large and moderate deviation approximation could be appropriate, in which case one could use the rate function to identify the most likely interactions between the pollution source and the dynamics of the waterway that lead to a particular outcome, such as exceeding an allowed concentration. However, the rate function only gives an asymptotic approximation for probabilities of such outcomes, and the resulting error due to the use of this approximation cannot be eliminated.

An alternative is to use numerical schemes such as Monte Carlo, which have the property that if a large enough number of good quality samples can be generated, then an arbitrary level of accuracy can be achieved. While this may be true in principle, it is in practice difficult when considering events of small probability, since many samples are required for errors that are small relative to the quantity being computed. The issue is especially relevant for a problem modeled by an equation as complex as (1.1), since the generation of even a single sample could be relatively expensive. Hence an interesting potential use of the results of the present paper are in importance sampling and related accelerated Monte Carlo methods [4, 49]. If in fact the moderate deviation approximation is relevant, the relatively simple form of the corresponding rate function suggests that many of the constructions needed to implement an effective importance sampling scheme [26] would be simpler than in the corresponding large deviation context.

We now make some comments on the technique of proof. As in [8], the starting point is the variational representation for expectations of positive functionals of a PRM from [11]. The usefulness of a variational representation in proving large deviation or moderate deviation type results lies in the fact that it allows one to bypass the traditional route of approximating the original sequence of solutions by discretizations; the latter approach is particularly cumbersome for SPDEs, and even more so for SPDEs driven by Poisson random measure. Moreover, the variational representation approach does not require proof of exponential tightness or other exponential probability estimates that are frequently some of the most technical parts of a traditional large deviations argument. A key step in our approach is to prove the tightness for controlled versions of the state processes, given that the costs for controls are suitably bounded. For example, to prove a moderate deviation principle for SPDEs of the form of (1.1), the tightness of the sequence of controlled processes $\bar{Y}^{\varepsilon,\varphi^\varepsilon}$ needs to be established, where

$$\bar{Y}^{\varepsilon,\varphi^\varepsilon} = \frac{1}{a(\varepsilon)}\big(\bar{X}^{\varepsilon,\varphi^\varepsilon} - X^0\big),$$

$$\begin{aligned} d\bar{X}^{\varepsilon,\varphi^\varepsilon}(u,t) = {} & \big(\mathcal{L}\bar{X}^{\varepsilon,\varphi^\varepsilon}(u,t) + b\big(\bar{X}^{\varepsilon,\varphi}(u,t)\big)\big)\,dt \\ & + \int_{\mathbb{X}} \varepsilon G\big(\bar{X}^{\varepsilon,\varphi^\varepsilon}(u,t-),y\big)N^{\varepsilon^{-1}\varphi^\varepsilon}(dy,dt), \end{aligned} \tag{1.2}$$

and the controls $\varphi^\varepsilon : \mathbb{X} \times [0, T] \to [0, \infty)$ are predictable processes satisfying $L_T(\varphi^\varepsilon) \le Ma^2(\varepsilon)$ for some constant M. Here L_T denotes the large deviation rate function associated with Poisson random measures [see (2.3)] and $N^{\varepsilon^{-1}\varphi^\varepsilon}$ is a controlled Poisson random measure, namely a counting process with intensity $\varepsilon^{-1}\varphi^\varepsilon(x, s)\nu_T(dx, ds)$; see (2.1) for a precise definition. By comparison (cf. [8]), to prove a large deviation principle for X^ε, the key step is proving the tightness of the controlled processes $\bar{X}^{\varepsilon,\varphi^\varepsilon}$ with the controls φ^ε satisfying $L_T(\varphi^\varepsilon) \le M$ for some constant M. The proof of this tightness property relies on the fact that the estimate $L_T(\varphi^\varepsilon) \le M$ implies tightness of φ^ε in a suitable space. Although in a moderate deviation problem, one has the stronger bound $L_T(\varphi^\varepsilon) \le Ma^2(\varepsilon)$ on the cost of controls, the mere tightness of φ^ε does not imply the tightness of $\bar{Y}^{\varepsilon,\varphi^\varepsilon}$. Instead one needs to study tightness properties of $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$. In general ψ^ε may not be in $L^2(\nu_T)$, and one of the challenges is to identify a space where suitable tightness properties of the centered and normalized controls $\{\psi^\varepsilon\}$ can be established. The key idea is to split ψ^ε into two terms, one of which lies in a closed ball in $L^2(\nu_T)$ (independent of ε), and the other approaches 0 in a suitable manner. Estimates on each of the two terms (see Lemma 3.2) are key ingredients in the proof and are used many times in this work, in particular to obtain uniform in ε moment estimates on centered and scaled processes of the form of (1.2).

The rest of the paper is organized as follows. Section 2.1 contains some background on PRMs and the variational representation from [11]. In Section 2.2 we present a general moderate deviation principle for measurable functionals of a PRM. Although this result concerns a large deviation principle with a certain speed, we refer to it as an MDP since its typical application is to the proof of moderate deviation type results. This general result covers many stochastic dynamical system models in finite and infinite dimensions. Indeed, by using the general theorem from Section 2.2, a moderate deviation principle for finite-dimensional SDEs driven by PRM is established in Section 2.3, and an infinite-dimensional model is considered in Section 2.4. Sections 3–5 are devoted to proofs. The result for the infinite-dimensional setting requires many assumptions on the model. In Section 6 we show that these assumptions are satisfied for an SPDE that has been proposed as a model for the spread of a pollutant with Poissonian point sources in a waterway.

Notation. The following notation is used. For a topological space $\mathcal{E}$, denote the corresponding Borel σ-field by $\mathcal{B}(\mathcal{E})$. We use the symbol $\Rightarrow$ to denote convergence in distribution. For a Polish space $\mathbb{X}$, denote by $C([0, T] : \mathbb{X})$ and $D([0, T] : \mathbb{X})$ the space of continuous functions and right continuous functions with left limits from $[0, T]$ to $\mathbb{X}$, endowed with the uniform and Skorokhod topology, respectively. For a metric space $\mathcal{E}$, denote by $M_b(\mathcal{E})$ and $C_b(\mathcal{E})$ the space of real, bounded $\mathcal{B}(\mathcal{E})/\mathcal{B}(\mathbb{R})$-measurable functions and real, bounded and continuous functions, respectively. For Banach spaces B_1, B_2, $L(B_1, B_2)$ will denote the space of bounded linear operators from B_1 to B_2. For a measure ν on $\mathcal{E}$ and a Hilbert space H, let $L^2(\mathcal{E}, \nu, H)$ denote the space of measurable functions f from $\mathcal{E}$ to H such that $\int_{\mathcal{E}} \|f(v)\|^2 \nu(dv) < \infty$, where $\|\cdot\|$ is the norm on H. When $H = \mathbb{R}$ and $\mathcal{E}$ is clear from the context, we write $L^2(\nu)$.

For a function $x:[0,T] \to \mathcal{E}$, we use the notation x_t and $x(t)$ interchangeably for the evaluation of x at $t \in [0,T]$. A similar convention will be followed for stochastic processes. We say a collection $\{X^\varepsilon\}$ of $\mathcal{E}$-valued random variables is tight if the distributions of X^ε are tight in $\mathcal{P}(\mathcal{E})$ (the space of probability measures on $\mathcal{E}$).

A function $I:\mathcal{E} \to [0,\infty]$ is called a rate function on $\mathcal{E}$ if for each $M < \infty$, the level set $\{x \in \mathcal{E} : I(x) \le M\}$ is a compact subset of $\mathcal{E}$.

Given a collection $\{b(\varepsilon)\}_{\varepsilon>0}$ of positive reals, a collection $\{X^\varepsilon\}_{\varepsilon>0}$ of $\mathcal{E}$-valued random variables is said to satisfy the Laplace principle upper bound (resp., lower bound) on $\mathcal{E}$ with speed $b(\varepsilon)$ and rate function I if for all $h \in C_b(\mathcal{E})$,

$$\limsup_{\varepsilon\to 0} b(\varepsilon) \log \mathbb{E}\left\{\exp\left[-\frac{1}{b(\varepsilon)} h(X^\varepsilon)\right]\right\} \le -\inf_{x\in\mathcal{E}}\{h(x) + I(x)\},$$

and, respectively,

$$\liminf_{\varepsilon\to 0} b(\varepsilon) \log \mathbb{E}\left\{\exp\left[-\frac{1}{b(\varepsilon)} h(X^\varepsilon)\right]\right\} \ge -\inf_{x\in\mathcal{E}}\{h(x) + I(x)\}.$$

The Laplace principle is said to hold for $\{X^\varepsilon\}$ with speed $b(\varepsilon)$ and rate function I if both the Laplace upper and lower bounds hold. It is well known that when $\mathcal{E}$ is a Polish space, the family $\{X^\varepsilon\}$ satisfies the Laplace principle upper (resp., lower) bound with a rate function I on $\mathcal{E}$ if and only if $\{X^\varepsilon\}$ satisfies the large deviation upper (resp., lower) bound for all closed sets (resp., open sets) with the rate function I. For a proof of this statement we refer to Section 1.2 of [25].

2. Preliminaries and main results.

2.1. *Poisson random measure and a variational representation.* Let $\mathbb{X}$ be a locally compact Polish space, and let $\mathcal{M}_{FC}(\mathbb{X})$ be the space of all measures ν on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ such that $\nu(K) < \infty$ for every compact $K \subset \mathbb{X}$. Endow $\mathcal{M}_{FC}(\mathbb{X})$ with the usual vague topology. This topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{X})$ is a Polish space [11]. Fix $T \in (0,\infty)$, and let $\mathbb{X}_T = \mathbb{X} \times [0,T]$. Fix a measure $\nu \in \mathcal{M}_{FC}(\mathbb{X})$, and let $\nu_T = \nu \otimes \lambda_T$, where λ_T is Lebesgue measure on $[0,T]$.

A Poisson random measure $\mathbf{n}$ on $\mathbb{X}_T$ with mean measure (or intensity measure) ν_T is a $\mathcal{M}_{FC}(\mathbb{X}_T)$-valued random variable such that for each $B \in \mathcal{B}(\mathbb{X}_T)$ with $\nu_T(B) < \infty$, $\mathbf{n}(B)$ is Poisson distributed with mean $\nu_T(B)$, and for disjoint $B_1, \dots, B_k \in \mathcal{B}(\mathbb{X}_T)$, $\mathbf{n}(B_1), \dots, \mathbf{n}(B_k)$ are mutually independent random variables; cf. [42]. Denote by $\mathbb{P}$ the measure induced by $\mathbf{n}$ on $(\mathcal{M}_{FC}(\mathbb{X}_T), \mathcal{B}(\mathcal{M}_{FC}(\mathbb{X}_T)))$. Then letting $\mathbb{M} = \mathcal{M}_{FC}(\mathbb{X}_T)$, $\mathbb{P}$ is the unique probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which the canonical map, $N:\mathbb{M} \to \mathbb{M}$, $N(m) \doteq m$, is a Poisson random measure with intensity measure ν_T. Also, for $\theta > 0$, $\mathbb{P}_\theta$ will denote a probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which N is a Poisson random measure with intensity $\theta\nu_T$. The corresponding expectation operators will be denoted by $\mathbb{E}$ and $\mathbb{E}_\theta$, respectively.

Let $F \in M_b(\mathbb{M})$. We now present a representation from [11] for $-\log \mathbb{E}_\theta \times (\exp[-F(N)])$, in terms of a Poisson random measure constructed on a larger space. Let $\mathbb{Y} = \mathbb{X} \times [0, \infty)$ and $\mathbb{Y}_T = \mathbb{Y} \times [0, T]$. Let $\bar{\mathbb{M}} = \mathcal{M}_{FC}(\mathbb{Y}_T)$, and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ under which the canonical map $\bar{N} : \bar{\mathbb{M}} \to \bar{\mathbb{M}}$, $\bar{N}(m) \doteq m$ is a Poisson random measure with intensity measure $\bar{\nu}_T = \nu \otimes \lambda_\infty \otimes \lambda_T$, where λ_∞ is Lebesgue measure on $[0, \infty)$. The corresponding expectation operator will be denoted by $\bar{\mathbb{E}}$. Let $\mathcal{F}_t \doteq \sigma\{\bar{N}(A \times (0, s]) : 0 \le s \le t, A \in \mathcal{B}(\mathbb{Y})\}$ be the filtration generated by $\bar{N}$, and let $\bar{\mathcal{F}}_t$ denote the completion under $\bar{\mathbb{P}}$. We denote by $\bar{\mathcal{P}}$ the predictable σ-field on $[0, T] \times \bar{\mathbb{M}}$ with the filtration $\{\bar{\mathcal{F}}_t : 0 \le t \le T\}$ on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$. Let $\bar{\mathcal{A}}_+$ (resp., $\bar{\mathcal{A}}$) be the class of all $(\mathcal{B}(\mathbb{X}) \otimes \bar{\mathcal{P}})/\mathcal{B}[0, \infty)$ [resp., $(\mathcal{B}(\mathbb{X}) \otimes \bar{\mathcal{P}})/\mathcal{B}(\mathbb{R})$]-measurable maps from $\mathbb{X}_T \times \bar{\mathbb{M}}$ to $[0, \infty)$ (resp., $\mathbb{R}$). For $\varphi \in \bar{\mathcal{A}}_+$, define a counting process N^φ on $\mathbb{X}_T$ by

$$N^\varphi\big(U \times (0, t]\big) = \int_{U \times [0,\infty) \times [0,t]} 1_{[0,\varphi(x,s)]}(r) \bar{N}(dx\, dr\, ds), \qquad t \in [0, T], U \in \mathcal{B}(\mathbb{X}). \tag{2.1}$$

We think of N^φ as a controlled random measure, with φ selecting the intensity for the points at location x and time s in a possibly random but nonanticipating way. When $\varphi(x, s, \bar{m}) \equiv \theta \in (0, \infty)$, we write $N^\varphi = N^\theta$. Note that N^θ has the same distribution with respect to $\bar{\mathbb{P}}$ as N has with respect to $\mathbb{P}_\theta$.

Define $\ell : [0, \infty) \to [0, \infty)$ by

$$\ell(r) = r \log r - r + 1, \qquad r \in [0, \infty). \tag{2.2}$$

For any $\varphi \in \bar{\mathcal{A}}_+$ and $t \in [0, T]$, the quantity

$$L_t(\varphi) = \int_{\mathbb{X} \times [0,t]} \ell\big(\varphi(x, s, \omega)\big) \nu_T(dx\, ds) \tag{2.3}$$

is well defined as a $[0, \infty]$-valued random variable. Let $\{K_n \subset \mathbb{X}, n = 1, 2, \dots\}$ be an increasing sequence of compact sets such that $\bigcup_{n=1}^\infty K_n = \mathbb{X}$. For each n, let

$$\bar{\mathcal{A}}_{b,n} \doteq \big\{\varphi \in \bar{\mathcal{A}}_+ : \text{for all } (t, \omega) \in [0, T] \times \bar{\mathbb{M}}, n \ge \varphi(x, t, \omega) \ge 1/n \\ \text{if } x \in K_n \text{ and } \varphi(x, t, \omega) = 1 \text{ if } x \in K_n^c\big\},$$

and let $\bar{\mathcal{A}}_b = \bigcup_{n=1}^\infty \bar{\mathcal{A}}_{b,n}$. Considering φ as a control that perturbs jump rates away from 1 when $\varphi \neq 1$, we see that controls in $\bar{\mathcal{A}}_b$ are bounded and perturb only off a compact set, where the bounds and set can depend on φ.

The following is a representation formula proved in [11]. For the last equality in the theorem, see the proof of Theorem 2.4 in [8].

THEOREM 2.1. *Let $F \in M_b(\mathbb{M})$. Then for $\theta > 0$,*

$$-\log \mathbb{E}_\theta\big(e^{-F(N)}\big) = -\log \bar{\mathbb{E}}\big(e^{-F(N^\theta)}\big) = \inf_{\varphi \in \bar{\mathcal{A}}_+} \bar{\mathbb{E}}\big[\theta L_T(\varphi) + F\big(N^{\theta\varphi}\big)\big]$$

$$= \inf_{\varphi \in \bar{\mathcal{A}}_b} \bar{\mathbb{E}}\big[\theta L_T(\varphi) + F\big(N^{\theta\varphi}\big)\big].$$

2.2. *A general moderate deviation result.* For $\varepsilon > 0$, let $\mathcal{G}^\varepsilon$ be a measurable map from $\mathbb{M}$ to $\mathbb{U}$, where $\mathbb{U}$ is some Polish space. Let $a : \mathbb{R}_+ \to (0, 1)$ be such that as $\varepsilon \to 0$,

$$a(\varepsilon) \to 0 \quad \text{and} \quad b(\varepsilon) = \frac{\varepsilon}{a^2(\varepsilon)} \to 0. \tag{2.4}$$

In this section we will formulate a general sufficient condition for the collection $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$ to satisfy a large deviation principle with speed $b(\varepsilon)$ and a rate function that is given through a suitable quadratic form.

For $\varepsilon > 0$ and $M < \infty$, consider the spaces

$$\begin{aligned} \mathcal{S}^M_{+,\varepsilon} &\doteq \big\{\varphi : \mathbb{X} \times [0, T] \to \mathbb{R}_+ | L_T(\varphi) \le M a^2(\varepsilon)\big\}, \\ \mathcal{S}^M_\varepsilon &\doteq \big\{\psi : \mathbb{X} \times [0, T] \to \mathbb{R} | \psi = (\varphi - 1)/a(\varepsilon), \varphi \in \mathcal{S}^M_{+,\varepsilon}\big\}. \end{aligned} \tag{2.5}$$

We also let

$$\begin{aligned} \mathcal{U}^M_{+,\varepsilon} &\doteq \big\{\phi \in \bar{\mathcal{A}}_b : \phi(\cdot, \cdot, \omega) \in \mathcal{S}^M_{+,\varepsilon}, \bar{\mathbb{P}}\text{-a.s.}\big\}, \\ \mathcal{U}^M_\varepsilon &\doteq \big\{\phi \in \bar{\mathcal{A}} : \phi(\cdot, \cdot, \omega) \in \mathcal{S}^M_\varepsilon, \bar{\mathbb{P}}\text{-a.s.}\big\}. \end{aligned} \tag{2.6}$$

The norm in the Hilbert space $L^2(\nu_T)$ will be denoted by $\|\cdot\|_2$, and $B_2(r)$ will denote the closed ball of radius r in $L^2(\nu_T)$. Given a map $\mathcal{G}_0 : L^2(\nu_T) \to \mathbb{U}$ and $\eta \in \mathbb{U}$, let

$$\mathbb{S}_\eta \equiv \mathbb{S}_\eta[\mathcal{G}_0] = \big\{\psi \in L^2(\nu_T) : \eta = \mathcal{G}_0(\psi)\big\},$$

and define I by

$$I(\eta) = \inf_{\psi \in \mathbb{S}_\eta} \left[\frac{1}{2}\|\psi\|_2^2\right]. \tag{2.7}$$

Here we follow the convention that the infimum over an empty set is $+\infty$.

We now introduce a sufficient condition that ensures that I is a rate function, and the collection $Y^\varepsilon \equiv \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$ satisfies a LDP with speed $b(\varepsilon)$ and rate function I. A set $\{\psi^\varepsilon\} \subset \bar{\mathcal{A}}$ with the property that $\sup_{\varepsilon > 0} \|\psi^\varepsilon\|_2 \le M$ a.s. for some $M < \infty$ will be regarded as a collection of $B_2(M)$-valued random variables, where $B_2(M)$ is equipped with the weak topology on the Hilbert space $L^2(\nu_T)$. Since $B_2(M)$ is weakly compact, such a collection of random variables is automatically tight. Throughout this paper $B_2(M)$ will be regarded as the compact metric space obtained by equipping it with the weak topology on $L^2(\nu_T)$.

Suppose $\varphi \in \mathcal{S}^M_{+,\varepsilon}$, which we recall implies $L_T(\varphi) \le Ma(\varepsilon)^2$. Then as shown in Lemma 3.2 below, there exists $\kappa_2(1) \in (0,\infty)$ that is independent of ε and such that $\psi 1_{\{|\psi|<1/a(\varepsilon)\}} \in B_2((M\kappa_2(1))^{1/2})$, where $\psi = (\varphi - 1)/a(\varepsilon)$.

CONDITION 2.2. For a measurable map $\mathcal{G}_0 : L^2(\nu_T) \to \mathbb{U}$, the following two conditions hold:

(a) Given $M \in (0,\infty)$, suppose that $g^\varepsilon, g \in B_2(M)$ and $g^\varepsilon \to g$. Then

$$\mathcal{G}_0(g^\varepsilon) \to \mathcal{G}_0(g).$$

(b) Given $M \in (0,\infty)$, let $\{\varphi^\varepsilon\}_{\varepsilon>0}$ be such that for every $\varepsilon > 0$, $\varphi^\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ and for some $\beta \in (0,1]$, $\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}} \Rightarrow \psi$ in $B_2((M\kappa_2(1))^{1/2})$ where $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$. Then

$$\mathcal{G}^\varepsilon\bigl(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\bigr) \Rightarrow \mathcal{G}_0(\psi).$$

THEOREM 2.3. *Suppose that the functions $\mathcal{G}^\varepsilon$ and $\mathcal{G}_0$ satisfy Condition 2.2. Then I, defined by (2.7), is a rate function, and $\{Y^\varepsilon \equiv \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})\}$ satisfies a large deviation principle with speed $b(\varepsilon)$ and rate function I.*

In the proof of a result such as Theorem 2.3, one needs to prove convergence of the controlled processes. It would in fact be difficult to prove such convergence for controls of the general form $\varphi^\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$, $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$. A convenient aspect of the result we prove is that it is suffices to consider $\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}$ rather than ψ^ε. The truncated controls are much easier to work with since they takes values in the weakly compact ball of radius $(M\kappa_2(1))^{1/2}$ in $L^2(\nu_T)$.

In the next two sections we will present two applications. The first is to a general family of finite-dimensional SDE models driven by Poisson noise, and the second is to certain SPDE models with Poisson noise.

2.3. *Finite-dimensional SDEs*. In this section we study SDEs of the form

$$X^\varepsilon(t) = x_0 + \int_0^t b\bigl(X^\varepsilon(s)\bigr)\,ds + \int_{\mathbb{X}\times[0,t]} \varepsilon G\bigl(X^\varepsilon(s-), y\bigr) N^{\varepsilon^{-1}}(dy, ds), \tag{2.8}$$

where the coefficients b and G satisfy the following condition.

CONDITION 2.4. The functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \times \mathbb{X} \to \mathbb{R}^d$ are measurable and satisfy:

(a) for some $L_b \in (0,\infty)$

$$\bigl|b(x) - b(x')\bigr| \le L_b|x - x'|, \qquad x, x' \in \mathbb{R}^d,$$

(b) for some $L_G \in L^1(\nu) \cap L^2(\nu)$

$$\bigl|G(x,y) - G(x',y)\bigr| \le L_G(y)|x - x'|, \qquad x, x' \in \mathbb{R}^d, y \in \mathbb{X},$$

(c) for some $M_G \in L^1(\nu) \cap L^2(\nu)$

$$\big|G(x,y)\big| \le M_G(y)\big(1+|x|\big), \qquad x \in \mathbb{R}^d, y \in \mathbb{X}.$$

The following result follows by standard arguments; see Theorem IV.9.1 of [42].

THEOREM 2.5. *Fix $x_0 \in \mathbb{R}^d$, and assume Condition 2.4. The following conclusions hold*:

(a) *For each $\varepsilon > 0$, there is a measurable map $\bar{\mathcal{G}}^\varepsilon : \mathbb{M} \to D([0,T]:\mathbb{R}^d)$ such that for any probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ on which is given a Poisson random measure $\mathbf{n}_\varepsilon$ on $\mathbb{X}_T$ with intensity measure $\varepsilon^{-1}\nu_T$, $\tilde{X}^\varepsilon = \bar{\mathcal{G}}^\varepsilon(\varepsilon \mathbf{n}_\varepsilon)$ is a $\tilde{\mathcal{F}}_t = \sigma\{\mathbf{n}_\varepsilon(B \times [0,s]), s \le t, B \in \mathcal{B}(\mathbb{X})\}$ adapted process that is the unique solution of the stochastic integral equation*

$$\tilde{X}^\varepsilon(t) = x_0 + \int_0^t b\big(\tilde{X}^\varepsilon(s)\big)\,ds + \int_{\mathbb{X}\times[0,t]} \varepsilon G\big(\tilde{X}^\varepsilon(s-), y\big)\mathbf{n}_\varepsilon(dy, ds), \qquad t \in [0,T]. \tag{2.9}$$

In particular $X^\varepsilon = \bar{\mathcal{G}}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$ is the unique solution of (2.8).

(b) *There is a unique X^0 in $C([0,T]:\mathbb{R}^d)$ that solves the equation*

$$X^0(t) = x_0 + \int_0^t b\big(X^0(s)\big)\,ds + \int_{\mathbb{X}\times[0,t]} G\big(X^0(s), y\big)\nu(dy)\,ds, \qquad t \in [0,T]. \tag{2.10}$$

We now state a LDP for $\{Y^\varepsilon\}$, where

$$Y^\varepsilon \equiv \frac{1}{a(\varepsilon)}\big(X^\varepsilon - X^0\big), \tag{2.11}$$

and $a(\varepsilon)$ is as in (2.4). For this we will need the following additional condition on the coefficients. Let

$$m_T = \sup_{0\le t\le T}\big|X^0(t)\big|. \tag{2.12}$$

For a differentiable function $f : \mathbb{R}^d \to \mathbb{R}^d$, let $Df(x) = (\partial f_i(x)/\partial x_j)_{i,j}$, $x \in \mathbb{R}^d$, and let $|Df|_{\mathrm{op}}$ denote the operator norm of the matrix Df. For $\delta > 0$, we define a class of functions

$$\mathcal{H}^\delta \doteq \Big\{h : \mathbb{X} \to \mathbb{R} : \forall \Gamma \in \mathcal{B}(\mathbb{X}) \text{ with } \nu(\Gamma) < \infty, \int_\Gamma \exp\big(\delta\big|h(y)\big|\big)\nu(dy) < \infty\Big\}. \tag{2.13}$$

CONDITION 2.6. (a) The functions L_G and M_G are in the class $\mathcal{H}^\delta$ for some $\delta > 0$.

(b) For every $y \in \mathbb{X}$, the maps $x \mapsto b(x)$ and $x \mapsto G(x, y)$ are differentiable. For some $L_{Db} \in (0, \infty)$

$$\left|Db(x) - Db(x')\right|_{\mathrm{op}} \le L_{Db}|x - x'|, \qquad x, x' \in \mathbb{R}^d$$

and for some $L_{DG} \in L^1(\nu)$,

$$\left|D_x G(x, y) - D_x G(x', y)\right|_{\mathrm{op}} \le L_{DG}(y)|x - x'|, \qquad x, x' \in \mathbb{R}^d, y \in \mathbb{X}.$$

With $m_T < \infty$ as in (2.12)

$$\sup_{\{x \in \mathbb{R}^d : |x| \le m_T\}} \int_{\mathbb{X}} \left|D_x G(x, y)\right|_{\mathrm{op}} \nu(dy) < \infty.$$

The following result gives a moderate deviation principle for the finite-dimensional SDEs (2.8).

THEOREM 2.7. *Suppose that Conditions 2.4 and 2.6 hold. Then $\{Y^\varepsilon\}$ satisfies a large deviation principle in $D([0, T] : \mathbb{R}^d)$ with speed $b(\varepsilon)$ and the rate function given by*

$$\bar{I}(\eta) = \inf_{\psi} \left\{ \frac{1}{2} \|\psi\|_2^2 \right\},$$

where the infimum is taken over all $\psi \in L^2(\nu_T)$ such that

$$\begin{aligned} \eta(t) = {} & \int_0^t \left[Db(X^0(s))\right]\eta(s)\, ds + \int_{\mathbb{X}\times[0,t]} \left[D_x G(X^0(s), y)\right]\eta(s)\nu(dy)\, ds \\ & + \int_{\mathbb{X}\times[0,t]} \psi(y, s) G(X^0(s), y)\nu(dy)\, ds, \qquad t \in [0, T]. \end{aligned} \tag{2.14}$$

Note that for each $\psi \in L^2(\nu_T)$, (2.14) has a unique solution $\eta \in C([0, T] : \mathbb{R}^d)$. In particular, $\bar{I}(\eta) = \infty$ for all $\eta \in D([0, T] : \mathbb{R}^d) \setminus C([0, T] : \mathbb{R}^d)$.

The following theorem gives an alternative expression for the rate function. From Condition 2.4(c) it follows that $y \mapsto G_i(X^0(s), y)$ is in $L^2(\nu)$ for all $s \in [0, T]$ and $i = 1, \ldots, d$, where $G = (G_1, \ldots, G_d)'$. For $i = 1, \ldots, d$, let $e_i : \mathbb{X} \times [0, T] \to \mathbb{R}$ be measurable functions such that for each $s \in [0, T]$, $\{e_i(\cdot, s)\}_{i=1}^d$ is an orthonormal collection in $L^2(\nu)$, and the linear span of the collection is same as that of $\{G_i(X^0(s), \cdot)\}_{i=1}^d$. Define $A : [0, T] \to \mathbb{R}^{d\times d}$ such that for each $s \in [0, T]$,

$$A_{ij}(s) = \left\langle G_i(X^0(s), \cdot), e_j(s, \cdot)\right\rangle_{L^2(\nu)}, \qquad i, j = 1, \ldots, d.$$

For $\eta \in D([0, T] : \mathbb{R}^d)$, let

$$I(\eta) = \inf_u \frac{1}{2} \int_0^T \left|u(s)\right|^2 ds,$$

where the infimum is taken over all $u \in L^2([0,T]:\mathbb{R}^d)$ such that

$$\eta(t) = \int_0^t [Db(X^0(s)) + G_1(X^0(s))]\eta(s)\,ds + \int_0^t A(s)u(s)\,ds, \qquad t \in [0,T] \tag{2.15}$$

and $G_1(x) = \int_{\mathbb{X}} D_x G(x,y)\nu(dy)$.

THEOREM 2.8. *Under the conditions of Theorem 2.7, $I = \bar{I}$.*

REMARK 2.9. (1) Theorem 2.8 in particular says that the rate function for $\{Y^\varepsilon\}$ is the same as that associated with the large deviation principle with speed ε for the Gaussian process

$$dZ^\varepsilon(t) = A_1(t)Z^\varepsilon(t)\,dt + \sqrt{\varepsilon}A(t)\,dW(t), \qquad Z^\varepsilon(0) = x_0,$$

where W is a standard d-dimensional Brownian motion and $A_1(t) = Db(X^0(t)) + G_1(X^0(t))$.

(2) One can similarly establish moderate deviations results for systems that have both Poisson and Brownian noise. In particular the following result holds. Suppose $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a Lipschitz continuous function, and X^ε solves the integral equation

$$\begin{aligned} X^\varepsilon(t) = x_0 &+ \int_0^t b(X^\varepsilon(s))\,ds + \sqrt{\varepsilon}\int_{[0,t]} \sigma(X^\varepsilon(s))\,dW(s) \\ &+ \int_{\mathbb{X}\times[0,t]} \varepsilon G(X^\varepsilon(s-),y)\,dN^{\varepsilon^{-1}}. \end{aligned}$$

Then under Conditions 2.4 and 2.6, $\{Y^\varepsilon\}$ defined as in (2.11) satisfies a large deviation principle in $D([0,T]:\mathbb{R}^d)$ with speed $b(\varepsilon)$ and the rate function given by

$$\bar{I}(\eta) = \inf_{\psi,u} \frac{1}{2}\left\{\|\psi\|_2^2 + \int_{[0,T]} |u(s)|^2\,ds\right\},$$

where the infimum is taken over all $(\psi,u) \in L^2(\nu_T) \times L^2([0,T]:\mathbb{R}^d)$ such that

$$\begin{aligned} \eta(t) = \int_0^t [Db(X^0(s))]\eta(s)\,ds &+ \int_0^t \sigma(X^0(s))u(s)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} [D_x G(X^0(s),y)]\eta(s)\nu(dy)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} \psi(y,s)G(X^0(s),y)\nu(dy)\,ds. \end{aligned}$$

Also, the rate function can be simplified as in Theorem 2.8.

2.4. *Infinite-dimensional SDEs*. The equation considered here has been studied in [44] where general sufficient conditions for strong existence and pathwise uniqueness of solutions are identified. The solutions in general will be distribution valued, and a precise formulation of the solution space is given in terms of countable Hilbertian nuclear spaces; cf. [44]. Recall that a separable Fréchet space Φ is called a *countable Hilbertian nuclear space* (CHNS) if its topology is given by an increasing sequence $\|\cdot\|_n$, $n \in \mathbb{N}_0$, of compatible Hilbertian norms, and if for each $n \in \mathbb{N}_0$ there exists $m > n$ such that the canonical injection from Φ_m into Φ_n is Hilbert–Schmidt. Here Φ_k, for each $k \in \mathbb{N}_0$, is the completion of Φ with respect to $\|\cdot\|_k$.

Identify Φ_0' with Φ_0 using Riesz's representation theorem, and denote the space of bounded linear functionals on Φ_n by Φ_{-n}. This space has a natural inner product (and norm) which we denote by $\langle\cdot,\cdot\rangle_{-n}$ (resp., $\|\cdot\|_{-n}$), $n \in \mathbb{N}_0$, such that $\{\Phi_{-n}\}_{n\in\mathbb{N}_0}$ is a sequence of increasing Hilbert spaces and the topological dual of Φ, denoted as Φ', equals $\bigcup_{n=0}^{\infty} \Phi_{-n}$; see Theorem 1.3.1 of [44]. Solutions of the SPDE considered in this section will have sample paths in $D([0,T]:\Phi_{-n})$ for some finite value of n.

We will assume that there is a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset \Phi$ such that $\{\phi_j\}$ is a complete orthonormal system (CONS) in Φ_0 and is a complete orthogonal system (COS) in each $\Phi_n, n \in \mathbb{Z}$. Then $\{\phi_j^n\} = \{\phi_j \|\phi_j\|_n^{-1}\}$ is a CONS in Φ_n for each $n \in \mathbb{Z}$. It is easily seen that, for each $r > 0$, $\eta \in \Phi_{-r}$ and $\phi \in \Phi_r$, $\eta[\phi]$ can be expressed as

$$\eta[\phi] = \sum_{j=1}^{\infty} \langle \eta, \phi_j\rangle_{-r} \langle \phi, \phi_j\rangle_r. \tag{2.16}$$

We refer the reader to Example 1.3.2 of [44] for a canonical example of such a countable Hilbertian nuclear space (CHNS) defined using a closed densely defined self-adjoint operator on Φ_0. A similar example is considered in Section 6. The SPDE we consider takes the form

$$X^{\varepsilon}(t) = x_0 + \int_0^t b\big(X^{\varepsilon}(s)\big)\,ds + \int_{\mathbb{X}\times[0,t]} \varepsilon G\big(X^{\varepsilon}(s-), y\big) N^{\varepsilon^{-1}}(dy, ds), \tag{2.17}$$

where the coefficients b and G satisfy Condition 2.11 below; cf. [44], Chapter 6. A precise definition of a solution to (2.17) is as follows.

DEFINITION 2.10. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space on which is given a Poisson random measure $\mathbf{n}_\varepsilon$ on $\mathbb{X}_T$ with intensity measure $\varepsilon^{-1}\nu_T$. Fix $r \in \mathbb{N}_0$, and suppose that $x_0 \in \Phi_{-r}$. A stochastic process $\{X_t^\varepsilon\}_{t\in[0,T]}$ defined on $\tilde{\Omega}$ is said to be a Φ_{-r}-valued strong solution to the SDE (2.17) with $N^{\varepsilon^{-1}}$ replaced by $\mathbf{n}_\varepsilon$ and initial value x_0, if the following hold:

(a) X_t^ε is a Φ_{-r}-valued $\tilde{\mathcal{F}}_t$-measurable random variable for all $t \in [0,T]$, where $\tilde{\mathcal{F}}_t = \sigma\{\mathbf{n}_\varepsilon(B \times [0,s]), s \le t, B \in \mathcal{B}(\mathbb{X})\}$.

(b) $X^\varepsilon \in D([0,T]:\Phi_{-r})$ a.s.

(c) The map $(s,\omega) \mapsto b(X_s^\varepsilon(\omega))$ is measurable from $[0,T] \times \Omega$ to Φ_{-r}, and the map $(s,\omega,y) \mapsto G(s, X_{s-}^\varepsilon(\omega), y)$ is $(\tilde{\mathcal{P}} \times \mathcal{B}(\mathbb{X}))/\mathcal{B}(\Phi_{-r})$ measurable, where $\tilde{\mathcal{P}}$ is the predictable σ-field corresponding to the filtration $\{\tilde{\mathcal{F}}_t\}$. Furthermore,

$$\tilde{E}\int_0^T \int_{\mathbb{X}} \|G(s, X_s^\varepsilon, v)\|_{-r}^2 \nu(dv)\, ds < \infty$$

and

$$\tilde{E}\int_0^T \|b(X_s^\varepsilon)\|_{-r}^2\, ds < \infty.$$

(d) For all $t \in [0,T]$, almost all $\omega \in \tilde{\Omega}$ and all $\phi \in \Phi$,

$$\begin{aligned} X_t^\varepsilon[\phi] &= x_0[\phi] + \int_0^t b(X_s^\varepsilon)[\phi]\, ds \\ &\quad + \varepsilon \int_{\mathbb{X}\times[0,t]} G(s, X_{s-}^\varepsilon, y)[\phi]\mathbf{n}_\varepsilon(dy, ds). \end{aligned} \tag{2.18}$$

We now present a condition from [44] that ensures unique solvability of (2.17). Let $\theta_p : \Phi_{-p} \to \Phi_p$ be the isometry such that for all $j \in \mathbb{N}$, $\theta_p(\phi_j^{-p}) = \phi_j^p$. It is easy to check that for all $p \in \mathbb{N}$, $\theta_p(\Phi) \subseteq \Phi$; see Remark 6.1.1 of [44].

CONDITION 2.11. For some $p, q \in \mathbb{N}$ with $q > p$ for which the embedding of Φ_{-p} to Φ_{-q} is Hilbert–Schmidt, the following hold:

(a) (Continuity) $b : \Phi' \to \Phi'$ is such that it is a continuous function from Φ_{-p} to Φ_{-q}. G is a map from $\Phi' \times \mathbb{X}$ to Φ' such that for each $u \in \Phi_{-p}$, $G(u, \cdot) \in L^2(\mathbb{X}, \nu, \Phi_{-p})$, and the mapping $\Phi_{-p} \ni u \mapsto G(u,\cdot) \in L^2(\mathbb{X}, \nu, \Phi_{-p})$ is continuous.

(b) There exist $M_b \in (0,\infty)$ and $M_G \in L^1(\nu) \cap L^2(\nu)$ such that

$$\|b(u)\|_{-q} \le M_b(1 + \|u\|_{-p}), \qquad \|G(u,y)\|_{-p} \le M_G(y)(1 + \|u\|_{-p}), \qquad u \in \Phi_{-p}, y \in \mathbb{X}.$$

(c) For some $C_b \in (0,\infty)$ and all $\phi \in \Phi$,

$$2b(\phi)[\theta_p \phi] \le C_b(1 + \|\phi\|_{-p}^2).$$

(d) For some $L_b \in (0,\infty)$,

$$\langle u - u', b(u) - b(u')\rangle_{-q} \le L_b \|u - u'\|_{-q}^2, \qquad u, u' \in \Phi_{-p}.$$

(e) For some $L_G \in L^1(\nu) \cap L^2(\nu)$,

$$\|G(u,y) - G(u',y)\|_{-q} \le L_G(y)\|u - u'\|_{-q}, \qquad u, u' \in \Phi_{-p}, y \in \mathbb{X}.$$

The following unique solvability result follows from [44]. For part (b), see Theorem 3.7 in [8].

THEOREM 2.12. *Fix $x_0 \in \Phi_{-p}$, and assume Condition 2.11. The following conclusions hold*:

(a) *For each $\varepsilon > 0$, there is a measurable map $\bar{\mathcal{G}}^\varepsilon : \mathbb{M} \to D([0,T] : \Phi_{-q})$ such that for any probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and Poisson random measure $\mathbf{n}_\varepsilon$ as in Definition 2.10, $\tilde{X}^\varepsilon = \bar{\mathcal{G}}^\varepsilon(\varepsilon \mathbf{n}_\varepsilon)$ is the unique Φ_{-q}-valued strong solution of (2.17) with $N^{\varepsilon^{-1}}$ replaced with $\mathbf{n}_\varepsilon$. Furthermore, for every $t \in [0,T]$, $\tilde{X}^\varepsilon_t \in \Phi_{-p}$ and $\tilde{E} \sup_{0 \le t \le T} \|\tilde{X}^\varepsilon_t\|^2_{-p} < \infty$. In particular, $X^\varepsilon = \bar{\mathcal{G}}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$ satisfies, for every $\phi \in \Phi$,*

$$\begin{aligned} X^\varepsilon_t[\phi] = X_0[\phi] + \int_0^t b(X^\varepsilon_s)[\phi]\,ds \\ + \varepsilon \int_{[0,t]\times\mathbb{X}} G(X^\varepsilon_{s-}, y)[\phi] N^{\varepsilon^{-1}}(dy, ds). \end{aligned} \tag{2.19}$$

(b) *The integral equation*

$$X^0(t) = x_0 + \int_0^t b(X^0(s))\,ds + \int_{[0,t]\times\mathbb{X}} G(X^0(s), y)\nu(dy)\,ds \tag{2.20}$$

has a unique Φ_{-q}-valued continuous solution. That is, there is a unique $X^0 \in C([0,T], \Phi_{-q})$ such that for all $t \in [0,T]$ and all $\phi \in \Phi$,

$$X^0_t[\phi] = X_0[\phi] + \int_0^t b(X^0_s)[\phi]\,ds + \int_{[0,t]\times\mathbb{X}} G(X^0_s, y)[\phi]\nu(dy)\,ds. \tag{2.21}$$

Furthermore, $X^0_t \in \Phi_{-p}$ for all $t \in [0,T]$ and

$$m_T = \sup_{0\le t\le T} \|X^0_t\|_{-p} < \infty. \tag{2.22}$$

As before, we are interested in a LDP for $\{Y^\varepsilon\}$, where

$$Y^\varepsilon \equiv \frac{1}{a(\varepsilon)}(X^\varepsilon - X^0),$$

and $a(\varepsilon)$ is as in (2.4). For this we will need some additional conditions on the coefficients. Recall the definition of Fréchet derivative of a real valued function on a Hilbert space (see Chapter II.5 of [24]), which characterizes the derivative as a bounded linear functional on the Hilbert space. For the remainder of this section we consider a fixed p and q that satisfy Condition 2.11.

CONDITION 2.13. There exists a positive integer $q_1 > q$ such that the canonical mapping of Φ_{-q} to Φ_{-q_1} is Hilbert–Schmidt, and the following hold:

(a) For every $\phi \in \Phi$, the Fréchet derivative of the map $\Phi_{-q} \ni v \mapsto b(v)[\phi]$ from Φ_{-q} to $\mathbb{R}$ exists and is denoted by $D(b(\cdot)[\phi])$. For each $\phi \in \Phi$, there exists $L_{Db}(\phi) \in (0, \infty)$ such that

$$\|D(b(u)[\phi]) - D(b(u')[\phi])\|_{\mathrm{op}} \le L_{Db}(\phi)\|u - u'\|_{-q}, \qquad u, u' \in \Phi_{-p}.$$

Here $\|\cdot\|_{\mathrm{op}}$ is the operator norm in $L(\Phi_{-q}, \mathbb{R})$.

(b) Recall that $\phi_k^{q_1} \doteq \phi_k \|\phi_k\|_{q_1}^{-1}$. Then for every $\eta \in \Phi_{-q}$,

$$\sup_{\{v \in \Phi_{-p} : \|v\|_{-p} \le m_T\}} \sum_{k=1}^{\infty} |D(b(v)[\phi_k^{q_1}])[\eta]|^2 \equiv M_2(\eta) < \infty.$$

This means that $A_v(\eta) : \Phi \to \mathbb{R}$ defined by $A_v(\eta)[\phi] = D(b(v)[\phi])[\eta]$ extends to a bounded linear map from Φ_{q_1} to $\mathbb{R}$ (i.e., an element of Φ_{-q_1}). For all $v \in \Phi_{-p}$ such that $\|v\|_{-p} \le m_T$, $\eta \mapsto A_v(\eta)$ is a continuous map from Φ_{-q} to Φ_{-q_1}, and there exist $M_A, L_A, C_A \in (0, \infty)$ such that

$$\sup_{\{v \in \Phi_{-p} : \|v\|_{-p} \le m_T\}} \|A_v(\eta)\|_{-q_1} \le M_A(1 + \|\eta\|_{-q}) \qquad \text{for all } \eta \in \Phi_{-q}, \tag{2.23}$$

$$\sup_{\{v \in \Phi_{-p} : \|v\|_{-p} \le m_T\}} \langle \eta, A_v(\eta)\rangle_{-q_1} \le L_A \|\eta\|_{-q_1}^2 \qquad \text{for all } \eta \in \Phi_{-q}, \tag{2.24}$$

$$\begin{aligned} &\sup_{\{v \in \Phi_{-p} : \|v\|_{-p} \le m_T\}} 2A_v(\phi + \zeta)[\theta_q \phi] \le C_A(\|\zeta\|_{-p} + \|\phi\|_{-q})\|\phi\|_{-q} \\ &\qquad\qquad \text{for all } \phi \in \Phi, \zeta \in \Phi_{-p}, \end{aligned} \tag{2.25}$$

where θ_q was defined just before Condition 2.11.

(c) For every $\phi \in \Phi_{q_1}$, $y \in \mathbb{X}$, the Fréchet derivative of $G(\cdot, y)[\phi] : \Phi_{-q_1} \to \mathbb{R}$, denoted as $D_x(G(\cdot, y)[\phi])$, exists. The map $\Phi_{-p} \ni u \to D_x(G(u, y)[\phi]) \in L(\Phi_{-q_1}, \mathbb{R})$ is Lipschitz continuous: for each $\phi \in \Phi_{q_1}$ there exists $L_{DG}(\phi, \cdot) \in L^1(\nu)$ such that

$$\begin{aligned} &\|D_x G(u, y)[\phi] - D_x G(u', y)[\phi]\|_{\mathrm{op}, -q_1} \le L_{DG}(\phi, y)\|u - u'\|_{-q}, \\ &\qquad\qquad u, u' \in \Phi_{-p}, y \in \mathbb{X}. \end{aligned}$$

There exists $M_{DG} : \Phi_{-p} \times \mathbb{X} \to \mathbb{R}_+$ such that

$$\begin{aligned} &\|D_x(G(u, y)[\phi])\|_{\mathrm{op}, -q_1} \le M_{DG}(u, y)\|\phi\|_{q_1}, \\ &\qquad\qquad u \in \Phi_{-p}, \phi \in \Phi_{q_1}, y \in \mathbb{X}, \\ &\|D_x(G(u, y)[\phi])\|_{\mathrm{op}, -q} \le M_{DG}(u, y)\|\phi\|_q, \qquad u \in \Phi_{-p}, \phi \in \Phi_q, y \in \mathbb{X} \end{aligned} \tag{2.26}$$

and

$$M_{DG}^* \doteq \sup_{\{u \in \Phi_{-p} : \|u\|_{-p} \le m_T\}} \int_{\mathbb{X}} \max\{M_{DG}(u, y), M_{DG}^2(u, y)\} \nu(dy) < \infty.$$

Here $\|\cdot\|_{\mathrm{op}, -q_1}$ (resp., $\|\cdot\|_{\mathrm{op}, -q}$) is the operator norm in $L(\Phi_{-q_1}, \mathbb{R})$ [resp., $L(\Phi_{-q}, \mathbb{R})$].

(d) Let M_G, L_G be as in Condition 2.11. For some $\delta > 0$, the functions M_G^2 and L_G are in $\mathcal{H}^\delta$ defined by (2.13).

Theorem 5.1 shows that under Conditions 2.11 and 2.13, for every $\psi \in L^2(\nu_T)$ there is a unique $\eta \in C([0,T], \Phi_{-q_1})$ that solves

$$\begin{aligned}\eta(t) = &\int_0^t Db(X^0(s))\eta(s)\,ds + \int_{\mathbb{X}\times[0,t]} D_x G(X^0(s), y)\eta(s)\nu(dy)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} G(X^0(s), y)\psi(y,s)\nu(dy)\,ds,\end{aligned}\tag{2.27}$$

in the sense that for every $\phi \in \Phi$,

$$\begin{aligned}\eta(t)[\phi] = &\int_0^t D(b(X^0(s))[\phi])\eta(s)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} D_x(G(X^0(s), y)[\phi])\eta(s)\nu(dy)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} G(X^0(s), y)[\phi]\psi(y,s)\nu(dy)\,ds.\end{aligned}\tag{2.28}$$

The following is the main result of this section.

THEOREM 2.14. *Suppose Conditions 2.11 and 2.13 hold. Then $\{Y^\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle in $D([0,T], \Phi_{-q_1})$ with speed $b(\varepsilon)$ and rate function I given by*

$$I(\eta) = \inf_{\psi}\left\{\frac{1}{2}\|\psi\|_2^2\right\},$$

where the infimum is taken over all $\psi \in L^2(\nu_T)$ such that (η, ψ) satisfy (2.27).

In Section 6 we will provide an example taken from [44] where Conditions 2.11 and 2.13 hold.

REMARK 2.15. The reader will note that for the finite-dimensional problem, the main exponential integrability condition on L_G and M_G given in the first part of Condition 2.6, essentially requires that the moment generating function for the jump distribution, when pushed through the coefficient G, be finite in a neighborhood of the origin. This is a natural condition for moderate deviations, and indeed one expects that weaker conditions would be required for the moderate deviation result than for the corresponding large deviation result. In contrast, the assumption made in the infinite-dimensional case [Condition 2.13(d)] is the analogous condition, but for M_G^2 rather than M_G. It seems possible that one could relax the condition also in this case, but the proof under the current more restrictive condition is already rather detailed and technical, and so we did not attempt this weakening.

3. Proof of Theorem 2.3. The following inequalities will be used several times. Recall the function $\ell(r) \doteq r\log r - r + 1$.

LEMMA 3.1. (a) *For $a, b \in (0,\infty)$ and $\sigma \in [1,\infty)$, $ab \le e^{\sigma a} + \frac{1}{\sigma}\ell(b)$.*

(b) *For every $\beta > 0$, there exist $\kappa_1(\beta), \kappa_1'(\beta) \in (0,\infty)$ such that $\kappa_1(\beta), \kappa_1'(\beta) \to 0$ as $\beta \to \infty$, and*

$$|x-1| \le \kappa_1(\beta)\ell(x) \qquad \text{for } |x-1| \ge \beta, x \ge 0 \quad \text{and}$$
$$x \le \kappa_1'(\beta)\ell(x) \qquad \text{for } x \ge \beta > 1.$$

(c) *There is a nondecreasing function $\kappa_2 : (0,\infty) \to (0,\infty)$ such that, for each $\beta > 0$,*

$$|x-1|^2 \le \kappa_2(\beta)\ell(x) \qquad \text{for } |x-1| \le \beta, x \ge 0.$$

(d) *There exists $\kappa_3 \in (0,\infty)$ such that*

$$\ell(x) \le \kappa_3|x-1|^2, \qquad \bigl|\ell(x) - (x-1)^2/2\bigr| \le \kappa_3|x-1|^3 \qquad \text{for all } x \ge 0.$$

The following result is immediate from Lemma 3.1.

LEMMA 3.2. *Suppose $\varphi \in \mathcal{S}^M_{+,\varepsilon}$ for some $M < \infty$, where $\mathcal{S}^M_{+,\varepsilon}$ is defined in (2.5). Let $\psi = \frac{\varphi - 1}{a(\varepsilon)}$. Then:*

(a) $\int_{\mathbb{X}\times[0,T]} |\psi| 1_{\{|\psi| \ge \beta/a(\varepsilon)\}}\, d\nu_T \le Ma(\varepsilon)\kappa_1(\beta)$ *for all $\beta > 0$,*

(b) $\int_{\mathbb{X}\times[0,T]} \varphi 1_{\{\varphi \ge \beta\}}\, d\nu_T \le Ma^2(\varepsilon)\kappa_1'(\beta)$ *for all $\beta > 1$,*

(c) $\int_{\mathbb{X}\times[0,T]} |\psi|^2 1_{\{|\psi| \le \beta/a(\varepsilon)\}}\, d\nu_T \le M\kappa_2(\beta)$ *for all $\beta > 0$,*

where κ_1, κ_1' and κ_2 are as in Lemma 3.1.

The property that I defined in (2.7) is a rate function is immediate on observing that Condition 2.2(a) says that $\Gamma_K = \{\mathcal{G}_0(g) : g \in B_2(K)\}$ is compact for all $K < \infty$, and therefore for every $M < \infty$, $\{\eta \in \mathbb{U} : I(\eta) \le M\} = \bigcap_{n\ge 1}\Gamma_{2M+1/n}$ is compact as well.

To prove Theorem 2.3 it suffices to show that the Laplace principle lower and upper bounds hold for all $F \in C_b(\mathbb{U})$. Let $\mathcal{G}^\varepsilon$ be as in the statement of Theorem 2.3. Then it follows from Theorem 2.1 with $\theta = \varepsilon^{-1}$ and $F(\cdot)$ there replaced by $F \circ \mathcal{G}^\varepsilon(\varepsilon\cdot)/b(\varepsilon)$ that

$$-b(\varepsilon)\log \bar{\mathbb{E}}\bigl[e^{-F(Y^\varepsilon)/b(\varepsilon)}\bigr] = \inf_{\varphi\in\bar{\mathcal{A}}_b} \bar{\mathbb{E}}\bigl[b(\varepsilon)\varepsilon^{-1}L_T(\varphi) + F \circ \mathcal{G}^\varepsilon\bigl(\varepsilon N^{\varepsilon^{-1}\varphi}\bigr)\bigr]. \tag{3.1}$$

We first prove the lower bound

$$\liminf_{\varepsilon\to 0} -b(\varepsilon)\log\bar{\mathbb{E}}\bigl[e^{-F(Y^\varepsilon)/b(\varepsilon)}\bigr] \ge \inf_{\eta\in\mathbb{U}}\bigl[I(\eta) + F(\eta)\bigr]. \tag{3.2}$$

For $\varepsilon \in (0,1)$, choose $\tilde{\varphi}^\varepsilon \in \bar{\mathcal{A}}_b$ such that

$$-b(\varepsilon)\log \bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big] \ge \bar{\mathbb{E}}\big[b(\varepsilon)\varepsilon^{-1}L_T(\tilde{\varphi}^\varepsilon) + F\circ \mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\tilde{\varphi}^\varepsilon}\big)\big] - \varepsilon. \tag{3.3}$$

Since $\|F\|_\infty \equiv \sup_{x\in\mathbb{U}}|F(x)| < \infty$, we have for all $\varepsilon \in (0,1)$ that

$$\tilde{M} \doteq \big(2\|F\|_\infty + 1\big) \ge \bar{\mathbb{E}}\big[b(\varepsilon)\varepsilon^{-1}L_T(\tilde{\varphi}^\varepsilon)\big]. \tag{3.4}$$

Fix $\delta > 0$, and define

$$\tau^\varepsilon = \inf\big\{t\in[0,T] : b(\varepsilon)\varepsilon^{-1}L_t(\tilde{\varphi}^\varepsilon) > 2\tilde{M}\|F\|_\infty/\delta\big\} \wedge T.$$

Let

$$\varphi^\varepsilon(y,s) = \tilde{\varphi}^\varepsilon(y,s)1_{\{s\le\tau^\varepsilon\}} + 1_{\{s>\tau^\varepsilon\}}, \qquad (y,s)\in\mathbb{X}\times[0,T].$$

Observe that $\varphi^\varepsilon \in \bar{\mathcal{A}}_b$ and $b(\varepsilon)\varepsilon^{-1}L_T(\varphi^\varepsilon) \le M \doteq 2\tilde{M}\|F\|_\infty/\delta$. Also,

$$\begin{aligned}
\bar{\mathbb{P}}\{\varphi^\varepsilon \ne \tilde{\varphi}^\varepsilon\} &\le \bar{\mathbb{P}}\big\{b(\varepsilon)\varepsilon^{-1}L_T(\tilde{\varphi}^\varepsilon) > M\big\}\\
&\le \bar{\mathbb{E}}\big[b(\varepsilon)\varepsilon^{-1}L_T(\tilde{\varphi}^\varepsilon)\big]/M\\
&\le \frac{\delta}{2\|F\|_\infty},
\end{aligned} \tag{3.5}$$

where the last inequality holds by (3.4). For $(y,s)\in\mathbb{X}\times[0,T]$, define

$$\tilde{\psi}^\varepsilon(y,s) \equiv \frac{\tilde{\varphi}^\varepsilon(y,s)-1}{a(\varepsilon)}, \qquad \psi^\varepsilon(y,s) \equiv \frac{\varphi^\varepsilon(y,s)-1}{a(\varepsilon)} = \tilde{\psi}^\varepsilon(y,s)1_{\{s\le\tau^\varepsilon\}}.$$

Fix $\beta \in (0,1]$, and let $B_\varepsilon = \beta/a(\varepsilon)$. Applying Lemma 3.1(d), Lemma 3.2(c), using $\kappa_2(1) \ge \kappa_2(\beta)$ and (3.3), we have that

$$\begin{aligned}
&-b(\varepsilon)\log \bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big]\\
&\quad\ge \bar{\mathbb{E}}\left[\frac{b(\varepsilon)}{\varepsilon}\int_{\mathbb{X}\times[0,T]}\ell(\tilde{\varphi}^\varepsilon)\,d\nu_T + F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\tilde{\varphi}^\varepsilon}\big)\right] - \varepsilon\\
&\quad\ge \bar{\mathbb{E}}\left[\frac{b(\varepsilon)}{\varepsilon}\int_{\mathbb{X}\times[0,T]}\ell(\varphi^\varepsilon)1_{\{|\psi^\varepsilon|\le B_\varepsilon\}}\,d\nu_T + F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\tilde{\varphi}^\varepsilon}\big)\right] - \varepsilon\\
&\quad\ge \bar{\mathbb{E}}\left[\frac{1}{2}\int_{\mathbb{X}\times[0,T]}\big((\psi^\varepsilon)^2 - \kappa_3 a(\varepsilon)|\psi^\varepsilon|^3\big)1_{\{|\psi^\varepsilon|\le B_\varepsilon\}}\,d\nu_T + F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big)\right]\\
&\qquad + \bar{\mathbb{E}}\big[F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\tilde{\varphi}^\varepsilon}\big) - F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big)\big] - \varepsilon\\
&\quad\ge \bar{\mathbb{E}}\left[\frac{1}{2}\int_{\mathbb{X}\times[0,T]}(\psi^\varepsilon)^2 1_{\{|\psi^\varepsilon|\le B_\varepsilon\}}\,d\nu_T + F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big)\right]\\
&\qquad - \delta - \varepsilon - \frac{1}{2}\beta\kappa_3 M\kappa_2(1),
\end{aligned} \tag{3.6}$$

where the last inequality follows from (3.5) on noting that

$$\big|\bar{\mathbb{E}}\big(F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\tilde{\varphi}^\varepsilon}\big) - F\circ\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big)\big)\big| \le 2\|F\|_\infty\bar{\mathbb{P}}\{\varphi^\varepsilon\ne\tilde{\varphi}^\varepsilon\} \le \delta.$$

By the weak compactness of $B_2(r)$ and again using the monotonicity of $\kappa_2(\beta)$, $\{\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}\}$ is a tight family of $B_2((M\kappa_2(1))^{1/2})$-valued random variables. Let ψ be a limit point along a subsequence which we index once more by ε. By a standard argument by contradiction it suffices to prove (3.2) along this subsequence. From Condition 2.2(b), along this subsequence $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon})$ converges in distribution to $\eta = \mathcal{G}_0(\psi)$. Hence taking limits in (3.6) along this subsequence, we have

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} -b(\varepsilon)\log\bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big]\\
&\qquad\ge \bar{\mathbb{E}}\left[\frac{1}{2}\int_{\mathbb{X}\times[0,T]}\psi^2\,d\nu_T + F(\eta)\right] - \delta - \frac{\beta}{2}\kappa_3 M\kappa_2(1)\\
&\qquad\ge \bar{\mathbb{E}}\big[I(\eta)+F(\eta)\big] - \delta - \frac{1}{2}\beta\kappa_3 M\kappa_2(1)\\
&\qquad\ge \inf_{\eta\in\mathbb{U}}\big[I(\eta)+F(\eta)\big] - \delta - \frac{1}{2}\beta\kappa_3 M\kappa_2(1),
\end{aligned}
$$

where the first line is from Fatou's lemma, and the second uses the definition of I in (2.7). Sending δ and β to 0 we get (3.2).

To complete the proof we now show the upper bound

$$
\limsup_{\varepsilon\to 0} -b(\varepsilon)\log\bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big] \le \inf_{\eta\in\mathbb{U}}\big[I(\eta)+F(\eta)\big]. \tag{3.7}
$$

Fix $\delta > 0$. Then there exists $\eta\in\mathbb{U}$ such that

$$
I(\eta)+F(\eta) \le \inf_{\eta\in\mathbb{U}}\big[I(\eta)+F(\eta)\big] + \delta/2. \tag{3.8}
$$

Choose $\psi\in L^2(\nu_T)$ such that

$$
\frac{1}{2}\int_{\mathbb{X}\times[0,T]}|\psi|^2\,d\nu_T \le I(\eta)+\delta/2, \tag{3.9}
$$

where $\eta = \mathcal{G}_0(\psi)$. For $\beta\in(0,1]$ define

$$
\psi^\varepsilon = \psi 1_{\{|\psi|\le\beta/a(\varepsilon)\}}, \qquad \varphi^\varepsilon = 1 + a(\varepsilon)\psi^\varepsilon.
$$

From Lemma 3.1(d), for every $\varepsilon > 0$,

$$
\begin{aligned}
\int_{\mathbb{X}\times[0,T]}\ell(\varphi^\varepsilon)\,d\nu_T &\le \kappa_3\int_{\mathbb{X}\times[0,T]}(\varphi^\varepsilon-1)^2\,d\nu_T\\
&= a^2(\varepsilon)\kappa_3\int_{\mathbb{X}\times[0,T]}|\psi^\varepsilon|^2\,d\nu_T\\
&\le a^2(\varepsilon)M,
\end{aligned}
$$

where $M = \kappa_3\int_{\mathbb{X}\times[0,T]}|\psi|^2\,d\nu_T$. Thus $\varphi^\varepsilon\in\mathcal{U}^M_{+,\varepsilon}$ for all $\varepsilon > 0$. Also

$$
\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}} = \psi 1_{\{|\psi|\le\beta/a(\varepsilon)\}}
$$

which converges to ψ in $L^2(\nu_T)$ [and hence in $B_2((M\kappa_2(1))^{1/2})$] as $\varepsilon \to 0$. Thus from Condition 2.2(b),

$$\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big) \Rightarrow \mathcal{G}_0(\psi). \tag{3.10}$$

Finally, from (3.1), Lemma 3.1(d) and using $b(\varepsilon)\varepsilon^{-1} = 1/a(\varepsilon)^2$,

$$\begin{aligned}
&-b(\varepsilon)\log \bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big] \\
&\quad\le b(\varepsilon)\varepsilon^{-1} L_T\big(\varphi^\varepsilon\big) + \bar{\mathbb{E}} F\circ \mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big) \\
&\quad\le \frac{1}{2}\int_{\mathbb{X}\times[0,T]} \big|\psi^\varepsilon\big|^2\, d\nu_T + \kappa_3 \int_{\mathbb{X}\times[0,T]} a(\varepsilon)\big|\psi^\varepsilon\big|^3\, d\nu_T + \bar{\mathbb{E}} F\circ \mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big) \\
&\quad\le \frac{1}{2}(1+2\kappa_3\beta)\int_{\mathbb{X}\times[0,T]} |\psi|^2\, d\nu_T + \bar{\mathbb{E}} F\circ \mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big).
\end{aligned}$$

Taking limits as $\varepsilon \to 0$ and using (3.10), we have

$$\limsup_{\varepsilon\to 0} -b(\varepsilon)\log \bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big] \le \frac{1}{2}(1+2\kappa_3\beta)\int |\psi|^2\, d\nu_T + F(\eta).$$

Sending $\beta \to 0$ gives

$$\begin{aligned}
\limsup_{\varepsilon\to 0} -b(\varepsilon)\log \bar{\mathbb{E}}\big[e^{-F(Y^\varepsilon)/b(\varepsilon)}\big] &\le \frac{1}{2}\int |\psi|^2\, d\nu_T + F(\eta) \\
&\le I(\eta) + F(\eta) + \delta/2 \\
&\le \inf_{\eta\in\mathbb{U}}\big[I(\eta)+F(\eta)\big] + \delta,
\end{aligned}$$

where the second inequality is from (3.9), and the last inequality follows from (3.8). Since $\delta > 0$ is arbitrary, this completes the proof of (3.7).

4. Proofs for the finite-dimensional problem (Theorem 2.7). From Theorem 2.5 we see that there exists a measurable map $\bar{\mathcal{G}}^\varepsilon : \mathbb{M} \to D([0,T]:\mathbb{R}^d)$ such that $X^\varepsilon \equiv \bar{\mathcal{G}}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$, and hence there is a map $\mathcal{G}^\varepsilon$ such that $Y^\varepsilon \equiv \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$. Define $\mathcal{G}_0 : L^2(\nu_T) \to C([0,T]:\mathbb{R}^d)$ by

$$\mathcal{G}_0(\psi) = \eta \qquad \text{if for } \psi \in L^2(\nu_T), \eta \text{ solves (2.14)}. \tag{4.1}$$

In order to prove the theorem, we will verify that Condition 2.2 holds with these choices of $\mathcal{G}^\varepsilon$ and $\mathcal{G}_0$.

We begin by verifying part (a) of the condition.

LEMMA 4.1. *Suppose Conditions 2.4 and 2.6 hold. Fix $M \in (0,\infty)$ and $g^\varepsilon, g \in B_2(M)$ such that $g^\varepsilon \to g$. Let $\mathcal{G}_0$ be as defined in (4.1). Then $\mathcal{G}_0(g^\varepsilon) \to \mathcal{G}_0(g)$.*

PROOF. Note that from Condition 2.4(c), $(y,s) \mapsto G(X^0(s), y)$ is in $L^2(\nu_T)$. Thus, since $g^\varepsilon \to g$, we have for every $t \in [0,T]$,

$$\begin{aligned} &\int_{\mathbb{X}\times[0,t]} g^\varepsilon(y,s) G(X^0(s), y)\nu(dy)\,ds \\ &\qquad \to \int_{\mathbb{X}\times[0,t]} g(y,s) G(X^0(s), y)\nu(dy)\,ds. \end{aligned} \tag{4.2}$$

We argue that the convergence is in fact uniform in t. Note that for $0 \le s \le t \le T$,

$$\begin{aligned} &\left|\int_{\mathbb{X}\times[s,t]} g^\varepsilon(y,u) G(X^0(u), y)\nu(dy)\,du\right| \\ &\qquad \le \Bigl(1 + \sup_{0\le u\le T} |X^0(u)|\Bigr) \int_{\mathbb{X}\times[s,t]} M_G(y)|g^\varepsilon(y,u)|\nu(dy)\,du \\ &\qquad \le \Bigl(1 + \sup_{0\le u\le T} |X^0(u)|\Bigr)|t-s|^{1/2} M \|M_G\|_2, \end{aligned}$$

where abusing notation we have denoted the norm in $L^2(\nu)$ as $\|\cdot\|_2$ as well. This implies equicontinuity, and hence the convergence in (4.2) is uniform in $t \in [0,T]$. The conclusion of the lemma now follows by an application of Gronwall's inequality. $\square$

In order to verify part (b) of Condition 2.2, we first prove some a priori estimates. Recall the spaces $\mathcal{H}^\delta$ introduced in (2.13) and $\mathcal{S}^M_{+,\varepsilon}$ in (2.5).

LEMMA 4.2. *Let $h \in \mathcal{H}^\delta$ for some $\delta > 0$ and $c : [0,\infty) \to [0,\infty)$ be such that $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Assume that $\nu(|h| > m) < \infty$ for some $m > 0$. Then for any $p < \infty$,*

$$\int_{\mathbb{X}} |h(y)|^p e^{\delta h(y)/2} 1_{\{|h|>1/c(\varepsilon)\}} \nu(dy) \to 0 \qquad \text{as } \varepsilon \to 0.$$

PROOF. The result follows by noting that for every $p, \delta > 0$, there exists $c(p,\delta) < \infty$ such that $|h(y)|^p e^{\delta h(y)/2} \le c(p,\delta) e^{\delta h(y)}$. $\square$

LEMMA 4.3. *Let $h \in L^2(\nu) \cap \mathcal{H}^\delta$ for some $\delta > 0$, and let I be a measurable subset of $[0,T]$. Let $\zeta, M \in (0,\infty)$. Then there exist maps $\vartheta, \rho, \theta : (0,\infty) \to (0,\infty)$ such that $\vartheta(u) \to 0$ as $u \to \infty$, $\theta(u) \to 0$ as $u \to 0$ and for all $\varepsilon, \beta \in (0,\infty)$,*

$$\sup_{\psi \in \mathcal{S}^M_\varepsilon} \int_{\mathbb{X}\times I} |h(y)\psi(y,s)| 1_{\{|\psi| \ge \beta/a(\varepsilon)\}} \nu(dy)\,ds \le \vartheta(\beta) a^{1-\zeta}(\varepsilon) + (1 + |I|)\theta(\varepsilon)$$

and

$$\sup_{\psi \in \mathcal{S}^M_\varepsilon} \int_{\mathbb{X}\times I} |h(y)\psi(y,s)| \nu(dy)\,ds \le \rho(\beta)|I|^{1/2} + \vartheta(\beta) a^{1-\zeta}(\varepsilon) + (1 + |I|)\theta(\varepsilon).$$

PROOF. Let $\psi \in \mathcal{S}_\varepsilon^M$ and $\beta \in (0,\infty)$. Then

$$\begin{aligned}\int_{\mathbb{X}\times I}\big|h(y)\psi(y,s)\big|\nu(dy)\,ds \le \int_{\mathbb{X}\times I}\big|h(y)\psi(y,s)\big|1_{\{|\psi|<\beta/a(\varepsilon)\}}\nu(dy)\,ds \\ + \int_{\mathbb{X}\times I}\big|h(y)\psi(y,s)\big|1_{\{|\psi|\ge\beta/a(\varepsilon)\}}\nu(dy)\,ds.\end{aligned} \tag{4.3}$$

Let $\varphi = 1 + a(\varepsilon)\psi$, and note that $\varphi \in \mathcal{S}_{+,\varepsilon}^M$; see (2.5). For the second term in (4.3), an application of Lemma 3.2(a) gives

$$\begin{aligned}&\int_{\mathbb{X}\times I}\big|h(y)\psi(y,s)\big|1_{\{|\psi|\ge\beta/a(\varepsilon)\}}\nu(dy)\,ds\\ &\quad= \int_{\mathbb{X}\times I}\big|h(y)\psi(y,s)\big|1_{\{|\psi|\ge\beta/a(\varepsilon)\}}1_{\{|h|\le 1/a^\zeta(\varepsilon)\}}\nu(dy)\,ds\\ &\qquad+ \int_{\mathbb{X}\times I}\big|h(y)\psi(y,s)\big|1_{\{|\psi|\ge\beta/a(\varepsilon)\}}1_{\{|h|> 1/a^\zeta(\varepsilon)\}}\nu(dy)\,ds\\ &\quad\le M\kappa_1(\beta)a^{1-\zeta}(\varepsilon) + \frac{1}{a(\varepsilon)}\int_{\mathbb{X}\times I}\big|h(y)\big|1_{\{|h|>1/a^\zeta(\varepsilon)\}}\nu(dy)\,ds\\ &\qquad+ \frac{1}{a(\varepsilon)}\int_{\mathbb{X}\times I}\big|h(y)\big|\varphi(y,s)1_{\{|h|>1/a^\zeta(\varepsilon)\}}\nu(dy)\,ds\\ &\quad\le M\kappa_1(\beta)a^{1-\zeta}(\varepsilon) + |I|\int_{\mathbb{X}}\big|h(y)\big|^{1+1/\zeta}1_{\{|h|>1/a^\zeta(\varepsilon)\}}\nu(dy)\\ &\qquad+ \frac{1}{a(\varepsilon)}\int_{\mathbb{X}\times I}\big|h(y)\big|\varphi(y,s)1_{\{|h|>1/a^\zeta(\varepsilon)\}}\nu(dy)\,ds.\end{aligned}$$

Using Lemma 3.1(a) (with $a = \delta|h|/2$, $b = \varphi$, $\sigma = 1$), the third term on the right-hand side above can be bounded by

$$\begin{aligned}&\frac{2}{\delta a(\varepsilon)}\Big[|I|\int_{\mathbb{X}}e^{\delta|h(y)|/2}1_{\{|h|\ge 1/a^\zeta(\varepsilon)\}}\nu(dy) + \int_{\mathbb{X}\times I}\ell\big(\varphi(y,s)\big)\nu(dy)\,ds\Big]\\ &\quad\le 2\delta^{-1}\Big[|I|\int_{\mathbb{X}}\big|h(y)\big|^{1/\zeta}e^{\delta|h(y)|/2}1_{\{|h|\ge 1/a^\zeta(\varepsilon)\}}\nu(dy) + Ma(\varepsilon)\Big]\\ &\quad\le \big(1+|I|\big)\theta'(\varepsilon),\end{aligned} \tag{4.4}$$

where

$$\theta'(\varepsilon) \doteq 2\delta^{-1}\bigg(\int_{\mathbb{X}}\big|h(y)\big|^{1/\zeta}e^{\delta|h(y)|/2}1_{\{|h|\ge 1/a^\zeta(\varepsilon)\}}\nu(dy) + Ma(\varepsilon)\bigg).$$

The first assertion of the lemma now follows by taking

$$\theta(\varepsilon) \doteq \theta'(\varepsilon) + \int_{\mathbb{X}}\big|h(y)\big|^{1+1/\zeta}1_{\{|h|\ge 1/a^\zeta(\varepsilon)\}}\nu(dy),$$

$\vartheta(\beta) \doteq M\kappa_1(\beta)$ and by noting that by Lemma 4.2, $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and by Lemma 3.1(b), $\kappa_1(\beta) \to 0$ as $\beta \to \infty$.

Finally, by the Cauchy–Schwarz inequality and Lemma 3.2(c),

$$\begin{aligned}
&\int_{\mathbb{X}\times I} |h(y)\psi(y,s)| 1_{\{|\psi|<\beta/a(\varepsilon)\}} \nu(dy)\, ds \\
&\quad \le \left(|I| \int_{\mathbb{X}} h^2(y)\nu(dy) \int_{\mathbb{X}\times I} \psi^2 1_{\{|\psi|<\beta/a(\varepsilon)\}} \nu(dy)\, ds \right)^{1/2} \\
&\quad \le \|h\|_2 \big(M\kappa_2(\beta)\big)^{1/2} |I|^{1/2}.
\end{aligned}$$

The second assertion of the lemma now follows by taking $\rho(\beta) = \|h\|_2(M \times \kappa_2(\beta))^{1/2}$. $\square$

The following result is immediate from the previous lemma.

LEMMA 4.4. *Let $h \in L^2(\nu) \cap \mathcal{H}^\delta$ for some $\delta > 0$, and suppose $h \ge 0$. Then for any $\beta < \infty$ and $M \in \mathbb{N}$,*

$$\sup_{\psi\in\mathcal{S}_\varepsilon^M} \int_{\mathbb{X}\times[0,T]} h(y)\big|\psi(y,s)\big| 1_{\{|\psi|>\beta/a(\varepsilon)\}} \nu(dy)\, ds \to 0 \qquad \text{as } \varepsilon \to 0.$$

PROOF. The result follows by applying Lemma 4.3 for any $\zeta \in (0,1)$. $\square$

For a function $\eta : [0,T] \to \mathbb{R}^d$ and $t \in [0,T]$, let $|\eta|_{*,t} \doteq \sup_{0\le s\le t} |\eta(s)|$.

LEMMA 4.5. *Let $h \in \mathcal{H}^\delta$ for some $\delta > 0$, and suppose that $\nu(|h| > 1) < \infty$. Then for every $\gamma > 0$ and $M \in \mathbb{N}$, there exists $\tilde{c}(\gamma, M) \in (0,\infty)$ such that for all measurable maps $\tilde{h} : \mathbb{X} \to \mathbb{R}_+$ with $\tilde{h} \le |h|$, $f : [0,T] \to \mathbb{R}_+$, $\varepsilon > 0$, and $0 \le s \le t \le T$*

$$\begin{aligned}
&\sup_{\varphi\in\mathcal{S}_{+,\varepsilon}^M} \int_{\mathbb{X}\times(s,t]} f(r)\tilde{h}(y)\varphi(y,r)\nu(dy)\, dr \\
&\quad \le \tilde{c}(\gamma, M)\left[\int_{\mathbb{X}} \tilde{h}(y)\nu(dy)\right]\left[\int_s^t f(r)\, du\right] + \big(\gamma + M\delta^{-1}a^2(\varepsilon)\big)|f|_{*,T}.
\end{aligned}$$

PROOF. Let $f : [0,T] \to \mathbb{R}_+$ be a measurable map, $\varphi \in \mathcal{S}_{+,\varepsilon}^M$ and $\gamma > 0$. For $m > 0$ and $0 \le s \le t \le T$, let

$$T_1(m) \doteq \int_{\mathbb{X}\times(s,t]} f(r)\tilde{h}(y)\varphi(y,r) 1_{\{|h|\le m\}} \nu(dy)\, dr,$$

$$T_2(m) \doteq \int_{\mathbb{X}\times(s,t]} f(r)\tilde{h}(y)\varphi(y,r) 1_{\{|h|>m\}} \nu(dy)\, dr.$$

Using Lemma 3.1(a) (with $a = \delta|h|$, $b = \varphi$ and $\sigma = 1$) and the definition of $\mathcal{S}_{+,\varepsilon}^M$,

$$T_2(m) \le \frac{|f|_{*,T}}{\delta}\left(T \int_{\{|h|>m\}} e^{\delta|h(y)|}\nu(dy) + Ma^2(\varepsilon)\right).$$

For $\beta > 1$, let

$$E_1(m, \beta) \doteq \{(y, u) \in \mathbb{X} \times (s, t] : |h(y)| \leq m \text{ and } \varphi(y, u) \leq \beta\},$$
$$E_2(m, \beta) \doteq \{(y, u) \in \mathbb{X} \times (s, t] : |h(y)| \leq m \text{ and } \varphi(y, u) > \beta\}.$$

Using Lemma 3.2(b), $T_1(m)$ can be estimated as

$$\begin{aligned} T_1(m) &= \int_{E_1(m,\beta)} f(r)\tilde{h}(y)\varphi(y, r)\nu(dy)\,dr + \int_{E_2(m,\beta)} f(r)\tilde{h}(y)\varphi(y, r)\nu(dy)\,dr \\ &\leq \beta\left[\int_{\mathbb{X}} \tilde{h}(y)\nu(dy)\right]\left[\int_s^t f(u)\,du\right] + m|f|_{*,T}\kappa_1'(\beta)Ma^2(\varepsilon). \end{aligned}$$

Combining these estimates,

$$\begin{aligned} &\int_{\mathbb{X}\times(s,t]} f(r)\tilde{h}(y)\varphi(y, r)\nu(dy)\,dr \\ &\quad\leq \beta\left[\int_{\mathbb{X}} \tilde{h}(y)\nu(dy)\right]\left[\int_s^t f(u)\,du\right] \\ &\qquad + |f|_{*,T}\left(m\kappa_1'(\beta)Ma^2(\varepsilon) + \frac{T}{\delta}\int_{\{|h|>m\}} e^{\delta|h(y)|}\nu(dy) + \frac{Ma^2(\varepsilon)}{\delta}\right). \end{aligned}$$

Since $\kappa_1'(\beta) \to 0$ as $\beta \to \infty$, $h \in \mathcal{H}^\delta$ and $\nu(|h| > 1) < \infty$, we can choose m sufficiently large and then β sufficiently large such that

$$m\kappa_1'(\beta)M + \frac{T}{\delta}\int_{\{|h|>m\}} e^{\delta h(y)}\nu(dy) < \gamma.$$

Denoting this choice of β (which depends on h, γ and M) by $\tilde{c}(\gamma, M)$, we have the result. $\square$

Recalling the definition of $\mathcal{U}^M_{+,\varepsilon}$ in (2.6), we note that for every $\varphi \in \mathcal{U}^M_{+,\varepsilon}$ the integral equation

$$\begin{aligned} \bar{X}^{\varepsilon,\varphi}(t) &= x_0 + \int_0^t b(\bar{X}^{\varepsilon,\varphi}(s))\,ds \\ &\quad + \int_{\mathbb{X}\times[0,t]} \varepsilon G(\bar{X}^{\varepsilon,\varphi}(s-), y)N^{\varepsilon^{-1}\varphi}(dy, ds) \end{aligned} \tag{4.5}$$

has a unique pathwise solution. Indeed, let $\tilde{\varphi} = 1/\varphi$, and recall that $\varphi \in \mathcal{U}^M_{+,\varepsilon}$ means that $\varphi = 1$ off some compact set in y and bounded above and below away from zero on the compact set. Then it is easy to check (see Theorem III.3.24 of [43] and Lemma 2.3 of [11]) that

$$\mathcal{E}^\varepsilon_t(\tilde{\varphi}) = \exp\left\{\int_{(0,t]\times\mathbb{X}\times[0,\varepsilon^{-1}\varphi]} \log(\tilde{\varphi})\,d\bar{N} + \int_{(0,t]\times\mathbb{X}\times[0,\varepsilon^{-1}\varphi]} (-\tilde{\varphi} + 1)\,d\bar{\nu}_T\right\}$$

is an $\{\bar{\mathcal{F}}_t\}$-martingale, and consequently,

$$\mathbb{Q}_T^\varepsilon(G) = \int_G \mathcal{E}_T^\varepsilon(\tilde{\varphi})\, d\bar{\mathbb{P}} \qquad \text{for } G \in \mathcal{B}(\bar{\mathbb{M}})$$

defines a probability measure on $\bar{\mathbb{M}}$. Furthermore, $\bar{\mathbb{P}}$ and $\mathbb{Q}_T^\varepsilon$ are mutually absolutely continuous. Also it can be verified that under $\mathbb{Q}_T^\varepsilon$, $\varepsilon N^{\varepsilon^{-1}\varphi}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ under $\bar{\mathbb{P}}$. Thus it follows that $\bar{X}^{\varepsilon,\varphi} = \bar{\mathcal{G}}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi})$ is $\mathbb{Q}_T^\varepsilon$ a.s. (and hence $\bar{\mathbb{P}}$ a.s.) the unique solution of (4.5), where $\bar{\mathcal{G}}^\varepsilon$ is as in Theorem 2.5.

Define $\bar{Y}^{\varepsilon,\varphi} \equiv \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi})$, and note that this is equivalent to

$$\bar{Y}^{\varepsilon,\varphi} = \frac{1}{a(\varepsilon)}(\bar{X}^{\varepsilon,\varphi} - X^0). \tag{4.6}$$

The following estimates on $\bar{X}^{\varepsilon,\varphi}$ and $\bar{Y}^{\varepsilon,\varphi}$ will be useful for our analysis.

LEMMA 4.6. *Suppose Conditions 2.4 and 2.6 hold. For every $M \in \mathbb{N}$, there exists an $\varepsilon_0 \in (0,\infty)$ such that*

$$L(M) \doteq \sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{\varphi\in\mathcal{U}_{+,\varepsilon}^M} \bar{\mathbb{E}}|\bar{X}^{\varepsilon,\varphi}|_{*,T} < \infty.$$

PROOF. Fix $M \in \mathbb{N}$, and $\varphi \in \mathcal{U}_{+,\varepsilon}^M$. Then for some $c_1 \in (0,\infty)$ (depending only on the coefficient b) and all $t \in [0,T]$,

$$\begin{aligned}|\bar{X}^{\varepsilon,\varphi}|_{*,t} \le |x_0| &+ c_1 \int_0^t (1 + |\bar{X}^{\varepsilon,\varphi}|_{*,s})\, ds \\ &+ \varepsilon \int_{\mathbb{X}\times[0,t]} M_G(y)\bigl(1 + |\bar{X}^{\varepsilon,\varphi}(s-)|\bigr) N^{\varepsilon^{-1}\varphi}(dy, ds).\end{aligned}$$

Thus

$$\begin{aligned}\bar{\mathbb{E}}|\bar{X}^{\varepsilon,\varphi}|_{*,t} \le |x_0| + c_1 T &+ \bar{\mathbb{E}} \int_{\mathbb{X}\times[0,t]} M_G(y)\varphi(y,s)\nu(dy)\, ds \\ &+ c_1 \int_0^t \bar{\mathbb{E}}|\bar{X}^{\varepsilon,\varphi}|_{*,s}\, ds + \bar{\mathbb{E}} \int_{\mathbb{X}\times[0,t]} M_G(y)|\bar{X}^{\varepsilon,\varphi}|_{*,s}\varphi(y,s)\nu(dy)\, ds.\end{aligned}$$

By Lemma 4.5, the last term in the above inequality can be bounded by

$$(\gamma + M\delta^{-1}a^2(\varepsilon))\bar{\mathbb{E}}|\bar{X}^{\varepsilon,\varphi}|_{*,t} + \tilde{c}(\gamma, M)\|M_G\|_1 \int_0^t \bar{\mathbb{E}}|\bar{X}^{\varepsilon,\varphi}|_{*,s}\, ds,$$

where $\|M_G\|_1 \doteq \int_{\mathbb{X}} M_G(y)\nu(dy)$. Choose ε_0 and γ such that for all $\varepsilon \le \varepsilon_0$, $(\gamma + M\delta^{-1}a^2(\varepsilon)) < 1/2$. Then by Gronwall's inequality

$$\bar{\mathbb{E}}|\bar{X}^{\varepsilon,\varphi}|_{*,T} \le \mathcal{R}_T \exp\bigl((c_1 + \tilde{c}(\gamma, M)\|M_G\|_1)T\bigr),$$

where $\mathcal{R}_T = 2(|x_0| + c_1 T + \bar{\mathbb{E}} \int_{\mathbb{X}\times[0,t]} M_G(y)\varphi(y,s)\nu(dy)\,ds)$. Note that by Lemma 4.5

$$\sup_{\varphi \in \mathcal{S}^M_{+,\varepsilon}} \int_{\mathbb{X}\times I} M_G(y)\varphi(y,s)\nu(dy)\,ds < \infty.$$

The result follows. □

LEMMA 4.7. *Suppose Conditions 2.4 and 2.6 hold. For every $M \in \mathbb{N}$, there exists an $\varepsilon_1 \in (0,\infty)$ such that*

$$\{|\bar{Y}^{\varepsilon,\varphi}|_{*,T}, \varphi \in \mathcal{U}^M_{+,\varepsilon}, \varepsilon \in (0,\varepsilon_1)\}$$

is a tight collection of $\mathbb{R}_+$-valued random variables.

PROOF. Fix $\varphi \in \mathcal{U}^M_{+,\varepsilon}$, and let $\psi = (\varphi - 1)/a(\varepsilon)$. Then

$$\begin{aligned}
&\bar{X}^{\varepsilon,\varphi}(t) - X^0(t) \\
&\quad = \int_0^t \bigl(b(\bar{X}^{\varepsilon,\varphi}(s)) - b(X^0(s))\bigr)\,ds \\
&\qquad + \int_{\mathbb{X}\times[0,t]} \varepsilon G(\bar{X}^{\varepsilon,\varphi}(s-), y)\tilde{N}^{\varepsilon^{-1}\varphi}(dy,ds) \\
&\qquad + \int_{\mathbb{X}\times[0,t]} \bigl(G(\bar{X}^{\varepsilon,\varphi}(s), y) - G(X^0(s), y)\bigr)\varphi(y,s)\nu(dy)\,ds \\
&\qquad + \int_{\mathbb{X}\times[0,t]} G(X^0(s), y)\bigl(\varphi(y,s) - 1\bigr)\nu(dy)\,ds.
\end{aligned}$$

Write

$$\bar{Y}^{\varepsilon,\varphi} = A^{\varepsilon,\varphi} + M^{\varepsilon,\varphi} + B^{\varepsilon,\varphi} + \mathcal{E}_1^{\varepsilon,\varphi} + C^{\varepsilon,\varphi}, \tag{4.7}$$

where

$$\begin{aligned}
M^{\varepsilon,\varphi}(t) &= \frac{\varepsilon}{a(\varepsilon)} \int_{\mathbb{X}\times[0,t]} G(\bar{X}^{\varepsilon,\varphi}(s-), y)\tilde{N}^{\varepsilon^{-1}\varphi}(dy,ds), \\
A^{\varepsilon,\varphi}(t) &= \frac{1}{a(\varepsilon)} \int_0^t \bigl(b(\bar{X}^{\varepsilon,\varphi}(s)) - b(X^0(s))\bigr)\,ds, \\
B^{\varepsilon,\varphi}(t) &= \frac{1}{a(\varepsilon)} \int_{\mathbb{X}\times[0,t]} \bigl(G(\bar{X}^{\varepsilon,\varphi}(s), y) - G(X^0(s), y)\bigr)\nu(dy)\,ds, \qquad (4.8) \\
\mathcal{E}_1^{\varepsilon,\varphi}(t) &= \int_{\mathbb{X}\times[0,t]} \bigl(G(\bar{X}^{\varepsilon,\varphi}(s), y) - G(X^0(s), y)\bigr)\psi(y,s)\nu(dy)\,ds, \\
C^{\varepsilon,\varphi}(t) &= \int_{\mathbb{X}\times[0,t]} G(X^0(s), y)\psi(y,s)\nu(dy)\,ds.
\end{aligned}$$

Note that for $\beta > 1$ and $m \in \mathbb{N}$,

$$\begin{aligned}
\bar{\mathbb{E}}|M^{\varepsilon,\varphi}|_{*,T} &\leq \frac{\varepsilon}{a(\varepsilon)}\bar{\mathbb{E}}\left|\int_{\mathbb{X}\times[0,\cdot]} G(\bar{X}^{\varepsilon,\varphi}(s-),y)1_{\{\varphi\leq\beta\}}\tilde{N}^{\varepsilon^{-1}\varphi}(dy,ds)\right|_{*,T} \\
&\quad + \frac{\varepsilon}{a(\varepsilon)}\bar{\mathbb{E}}\left|\int_{\mathbb{X}\times[0,\cdot]} G(\bar{X}^{\varepsilon,\varphi}(s-),y)1_{\{\varphi>\beta,M_G\leq m\}}\tilde{N}^{\varepsilon^{-1}\varphi}(dy,ds)\right|_{*,T} \\
&\quad + \frac{\varepsilon}{a(\varepsilon)}\bar{\mathbb{E}}\left|\int_{\mathbb{X}\times[0,\cdot]} G(\bar{X}^{\varepsilon,\varphi}(s-),y)1_{\{\varphi>\beta,M_G>m\}}\tilde{N}^{\varepsilon^{-1}\varphi}(dy,ds)\right|_{*,T} \\
&= T_1 + T_2 + T_3.
\end{aligned}$$

Let ε_0 and $L(M)$ be as in Lemma 4.6, and assume henceforth that $\varepsilon \in (0, \varepsilon_0)$. Then the Lenglart–Lepingle–Pratelli inequality gives, for some $c < \infty$,

$$\begin{aligned}
T_1 &\leq \frac{c\varepsilon^{1/2}}{a(\varepsilon)}\bar{\mathbb{E}}\left((1+|\bar{X}^{\varepsilon,\varphi}|_{*,T})\left(\int_{\mathbb{X}\times[0,T]} M_G^2(y)1_{\{\varphi\leq\beta\}}\varphi(y,s)\nu(dy)\,ds\right)^{1/2}\right) \\
&\leq \frac{(\varepsilon\beta T)^{1/2}}{a(\varepsilon)}\|M_G\|_2(1+L(M)).
\end{aligned}$$

Also, using Lemma 3.2(b),

$$T_2 \leq 2a(\varepsilon)\kappa_1'(\beta)Mm(1+L(M)).$$

Next, whenever $\varphi > \beta > 1$,

$$\begin{aligned}
\ell\left(\frac{\varphi}{a(\varepsilon)}\right) &\leq \frac{\ell(\varphi)}{a(\varepsilon)} - \frac{\varphi}{a(\varepsilon)}\log a(\varepsilon) + 1 \\
&\leq \frac{\ell(\varphi)}{a(\varepsilon)} - \kappa_1'(\beta)\frac{\ell(\varphi)}{a(\varepsilon)}\log a(\varepsilon) + 1.
\end{aligned}$$

From this and Lemma 3.1(a) [with $\sigma = 1$, $a = \delta M_G$, $b = \varphi/a(\varepsilon)$], it follows that

$$\begin{aligned}
T_3 &\leq \frac{2(L(M)+1)}{\delta} \\
&\quad \times \left(T\int_{\mathbb{X}}(e^{\delta M_G(y)}+1)1_{\{M_G>m\}}\nu(dy) + M(a(\varepsilon) - \kappa_1'(\beta)a(\varepsilon)\log a(\varepsilon))\right).
\end{aligned}$$

Combining these estimates, for some $c_1 < \infty$ (depending only on T and M),

$$\begin{aligned}
\bar{\mathbb{E}}|M^{\varepsilon,\varphi}|_{*,T} &\leq c_1\left(\frac{(\varepsilon\beta T)^{1/2}}{a(\varepsilon)} + 2a(\varepsilon)\kappa_1'(\beta)Mm\right. \\
&\qquad \left. + \int_{\mathbb{X}}(e^{\delta M_G(y)}+1)1_{\{M_G>m\}}\nu(dy) + a(\varepsilon)(1-\log a(\varepsilon))\right).
\end{aligned}$$

Sending $\varepsilon \to 0$ and then $m \to \infty$, we have that

$$\limsup_{\varepsilon\to 0}\sup_{\varphi\in\mathcal{U}_{+,\varepsilon}^M}\bar{\mathbb{E}}|M^{\varepsilon,\varphi}|_{*,T} = 0. \tag{4.9}$$

Next by the Lipschitz condition on G [Condition 2.4(b)] and Condition 2.6(a), there is a $\gamma_2 \in (0, \infty)$ such that for all $t \in [0, T]$, $\varphi \in \mathcal{U}^M_{+,\varepsilon}$ and ε sufficiently small

$$\begin{aligned}
|\mathcal{E}_1^{\varepsilon,\varphi}|_{*,t} &\le a(\varepsilon) \int_{\mathbb{X}\times[0,t]} L_G(y) |\bar{Y}^{\varepsilon,\varphi}(s)| |\psi(y,s)| \nu(dy)\, ds \\
&\le a(\varepsilon) |\bar{Y}^{\varepsilon,\varphi}|_{*,t} \int_{\mathbb{X}\times[0,t]} L_G(y) |\psi(y,s)| \nu(dy)\, ds \qquad (4.10)\\
&\le \gamma_2 a(\varepsilon) |\bar{Y}^{\varepsilon,\varphi}|_{*,t},
\end{aligned}$$

where the last inequality follows from Lemma 4.3.

Again using the Lipschitz condition on G we have, for all $t \in [0, T]$,

$$|B^{\varepsilon,\varphi}|_{*,t} \le \|L_G\|_1 \int_0^t |\bar{Y}^{\varepsilon,\varphi}|_{*,s}\, ds.$$

Similarly, the Lipschitz condition on b gives

$$|A^{\varepsilon,\varphi}|_{*,t} \le L_b \int_0^t |\bar{Y}^{\varepsilon,\varphi}|_{*,s}\, ds.$$

Using Condition 2.6(a) and Lemma 4.3 again we have that for some $\gamma_3 \in (0, \infty)$, all $\varphi \in \mathcal{U}^M_{+,\varepsilon}$ and all $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
|C^{\varepsilon,\varphi}|_{*,T} &\le \int_{\mathbb{X}\times[0,T]} |G(X^0(s), y)\psi(y,s)| \nu(dy)\, ds \\
&\le (1 + |X^0|_{*,T}) \int_{\mathbb{X}\times[0,T]} M_G(y) |\psi(y,s)| \nu(dy)\, ds \\
&\le \gamma_3.
\end{aligned}$$

Collecting these estimates we have, for some $\gamma_4 \in (0, \infty)$, $\varepsilon_1 > 0$ and all $\varphi \in \mathcal{U}^M_{+,\varepsilon}$, $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon_1)$,

$$|\bar{Y}^{\varepsilon,\varphi}|_{*,t} \le \gamma_4 \left(1 + \int_0^t |\bar{Y}^{\varepsilon,\varphi}|_{*,s}\, ds\right) + r_\varepsilon,$$

where as shown in (4.9) $\{r_\varepsilon, \varepsilon \in (0, \varepsilon_1)\}$ is a tight family of $\mathbb{R}_+$-valued random variables. The result now follows from Gronwall's inequality. □

LEMMA 4.8. *Let $\{\psi^\varepsilon\}_{\varepsilon>0}$ be such that for some $M < \infty$, $\psi^\varepsilon \in \mathcal{S}^M_\varepsilon$ for all $\varepsilon > 0$. Let $f : \mathbb{X} \times [0, T] \to \mathbb{R}^d$ be such that*

$$|f(y, s)| \le h(y), \qquad y \in \mathbb{X}, s \in [0, T]$$

for some h in $L^2(\nu) \cap \mathcal{H}^\delta$, $\delta > 0$. Suppose for some $\beta \in (0, 1]$ that $\psi^\varepsilon 1_{\{|\psi^\varepsilon| \le \beta/a(\varepsilon)\}}$ converges in $B_2((M\kappa_2(1))^{1/2})$ to ψ. Then

$$\int_{\mathbb{X}\times[0,t]} f\psi^\varepsilon\, d\nu_T \to \int_{\mathbb{X}\times[0,t]} f\psi\, d\nu_T \qquad \textit{for all } t \in [0, T].$$

PROOF. From Lemma 4.4 we have that

$$\int_{\mathbb{X}\times[0,T]} |f\psi^\varepsilon| 1_{\{|\psi^\varepsilon|>\beta/a(\varepsilon)\}}\, d\nu_T \to 0 \qquad \text{as } \varepsilon \to 0.$$

Also, since $f1_{[0,t]} \in L^2(\nu_T)$ for all $t \in [0,T]$ and $\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}} \to \psi$ in $B_2((M\kappa_2(1))^{1/2})$, we have

$$\int_{\mathbb{X}\times[0,t]} f\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}\, d\nu_T \to \int_{\mathbb{X}\times[0,t]} f\psi\, d\nu_T.$$

The result follows by combining the last two displays. $\square$

4.1. *Proof of Theorem* 2.7. The following is the key result needed for the proof of the theorem. It gives tightness of the joint distribution of controls and controlled processes, and indicates how limits of these two quantities are related.

LEMMA 4.9. *Suppose Conditions* 2.4 *and* 2.6 *hold. Let* $\{\varphi^\varepsilon\}_{\varepsilon>0}$ *be such that for some* $M<\infty$, $\varphi^\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ *for every* $\varepsilon>0$. *Let* $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$ *and* $\beta \in (0,1]$. *Suppose that* $\bar{Y}^{\varepsilon,\varphi^\varepsilon} = \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon})$, *and recall that* $\bar{Y}^{\varepsilon,\varphi^\varepsilon} = (\bar{X}^{\varepsilon,\varphi^\varepsilon} - X^0)/a(\varepsilon)$, *where* $\bar{X}^{\varepsilon,\varphi^\varepsilon} = \bar{\mathcal{G}}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon})$. *Then for some* $\varepsilon_2>0$, $\{(\bar{Y}^{\varepsilon,\varphi^\varepsilon}, \psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}), \varepsilon \in (0,\varepsilon_2)\}$ *is tight in* $D([0,T]:\mathbb{R}^d) \times B_2((M\kappa_2(1))^{1/2})$, *and any limit point* $(\bar{Y},\psi)$ *satisfies* (2.14) *with* η *replaced by* $\bar{Y}$, *w.p.*1.

PROOF. We will use the notation from the proof of Lemma 4.7. Assume without loss of generality that $\varepsilon \le \varepsilon_1$. From (4.9) we have that $\bar{\mathbb{E}}|M^{\varepsilon,\varphi^\varepsilon}|_{*,T} \to 0$ as $\varepsilon \to 0$. Also, since from Lemma 4.7

$$\{|\bar{Y}^{\varepsilon,\varphi}|_{*,T}, \varphi \in \mathcal{U}^M_{+,\varepsilon}, \varepsilon \in (0,\varepsilon_1)\}$$

is a tight family, (4.10) implies that $|\mathcal{E}_1^{\varepsilon,\varphi^\varepsilon}|_{*,T} \to 0$ in probability. Next, noting that $\bar{X}^{\varepsilon,\varphi^\varepsilon}(t) = X^0(t) + a(\varepsilon)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t)$, we have by Taylor's theorem that

$$G(\bar{X}^{\varepsilon,\varphi^\varepsilon}(s),y) - G(X^0(s),y) = a(\varepsilon)D_xG(X^0(s),y)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t) + R^{\varepsilon,\varphi^\varepsilon}(s,y),$$

where

$$|R^{\varepsilon,\varphi^\varepsilon}(s,y)| \le L_{DG}(y)a^2(\varepsilon)|\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)|^2.$$

Hence

$$B^{\varepsilon,\varphi^\varepsilon}(t) = \int_{\mathbb{X}\times[0,t]} D_xG(X^0(s),y)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\nu(dy)\,ds + \mathcal{E}_2^{\varepsilon,\varphi^\varepsilon}(t),$$

where with $K_{DG} = \int_{\mathbb{X}} L_{DG}(y)\nu(dy)$,

$$|\mathcal{E}_2^{\varepsilon,\varphi^\varepsilon}|_{*,T} \le K_{DG}a(\varepsilon)\int_0^T |\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)|^2\, ds.$$

Thus using Lemma 4.7 again, $|\mathcal{E}_2^{\varepsilon,\varphi^\varepsilon}|_{*,T} \to 0$ in probability. Similarly,

$$A^{\varepsilon,\varphi^\varepsilon}(t) = \int_0^t Db(X^0(s))\bar{Y}^{\varepsilon,\varphi}(s)\,ds + \mathcal{E}_3^{\varepsilon,\varphi^\varepsilon}(t),$$

where $|\mathcal{E}_3^{\varepsilon,\varphi^\varepsilon}|_{*,T} \to 0$ in probability.

Putting these estimates together we have from (4.7) that

$$\begin{aligned}\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t) = \mathcal{E}^{\varepsilon,\varphi^\varepsilon}(t) &+ \int_0^t Db(X^0(s))\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} D_x G(X^0(s), y)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\nu(dy)\,ds \\ &+ \int_{\mathbb{X}\times[0,t]} G(X^0(s), y)\psi^\varepsilon(y,s)\nu(dy)\,ds,\end{aligned} \tag{4.11}$$

where $\mathcal{E}^{\varepsilon,\varphi^\varepsilon} = M^{\varepsilon,\varphi^\varepsilon} + \mathcal{E}_1^{\varepsilon,\varphi^\varepsilon} + \mathcal{E}_2^{\varepsilon,\varphi^\varepsilon} + \mathcal{E}_3^{\varepsilon,\varphi^\varepsilon} \Rightarrow 0$.

We now prove the tightness of

$$\tilde{B}^{\varepsilon,\varphi^\varepsilon}(\cdot) = \int_{\mathbb{X}\times[0,\cdot]} D_x G(X^0(s), y)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\nu(dy)\,ds,$$

$$C^{\varepsilon,\varphi^\varepsilon}(\cdot) = \int_{\mathbb{X}\times[0,\cdot]} G(X^0(s), y)\psi^\varepsilon(y,s)\nu(dy)\,ds$$

and

$$\tilde{A}^{\varepsilon,\varphi^\varepsilon}(\cdot) = \int_0^\cdot Db(X^0(s))\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\,ds.$$

Recall that $m_T = \sup_{s\in[0,T]}|X^0(s)|$. Applying Lemma 4.3 with $h = M_G$, for $\gamma \in (0,1)$

$$\begin{aligned}\left|C^{\varepsilon,\varphi^\varepsilon}(t+\gamma) - C^{\varepsilon,\varphi^\varepsilon}(t)\right| &= \int_{\mathbb{X}\times[t,t+\gamma]} \left|G(X^0(s), y)\right|\left|\psi^\varepsilon(y,s)\right|\nu(dy)\,ds \\ &\le (1+m_T)\int_{\mathbb{X}\times[t,t+\gamma]} M_G(y)\left|\psi^\varepsilon(y,s)\right|\nu(dy)\,ds \\ &\le (1+m_T)\left(\rho(\beta)\gamma^{1/2} + \vartheta(\beta) + 2\theta(\varepsilon)\right).\end{aligned} \tag{4.12}$$

Since $\vartheta(\beta) \to 0$ as $\beta \to \infty$ and $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$, tightness of $\{C^{\varepsilon,\varphi^\varepsilon}\}_{\varepsilon>0}$ in $C([0,T]:\mathbb{R}^d)$ is now immediate.

Next we argue the tightness of $\tilde{B}^{\varepsilon,\varphi^\varepsilon}$. For $0 \le t \le t+\gamma \le T$

$$\begin{aligned}\left|\tilde{B}^{\varepsilon,\varphi^\varepsilon}(t+\gamma) - \tilde{B}^{\varepsilon,\varphi^\varepsilon}(t)\right| &= \int_{\mathbb{X}\times[t,t+\gamma]} \left|D_x G(X^0(s), y)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\right|\nu(dy)\,ds \\ &\le \left(\sup_{|x|\le m_T}\int_{\mathbb{X}} \left|D_x G(x,y)\right|_{\mathrm{op}}\nu(dy)\right)\int_{[t,t+\gamma]}\left|\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\right|ds \\ &\le K_1\gamma\left|\bar{Y}^{\varepsilon,\varphi^\varepsilon}\right|_{*,T},\end{aligned}$$

where $K_1 = \sup_{|x|\le m_T} \int_{\mathbb{X}} |D_x G(x,y)|_{\mathrm{op}} \nu(dy)$, which is finite from Condition 2.6(b). Tightness of $\{\tilde{B}^{\varepsilon,\varphi^\varepsilon}\}_{\varepsilon>0}$ in $C([0,T]:\mathbb{R}^d)$ now follows as a consequence of Lemma 4.7. Similarly it can be seen that $\tilde{A}^{\varepsilon,\varphi^\varepsilon}$ is tight in $C([0,T]:\mathbb{R}^d)$, and consequently, $\bar{Y}^{\varepsilon,\varphi^\varepsilon}$ is tight in $D([0,T]:\mathbb{R}^d)$. Also, from Lemma 3.2(c), $\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}$ takes values in $B_2((M\kappa_2(1))^{1/2})$ for all $\varepsilon > 0$ and by the compactness of the latter space, the tightness of $\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}$ is immediate. This completes the proof of the first part of the lemma. Suppose now that $(\bar{Y}^{\varepsilon,\varphi^\varepsilon}, \psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}})$ along a subsequence converges in distribution to $(\bar{Y}, \psi)$. From Lemma 4.8 and the tightness of $C^{\varepsilon,\varphi^\varepsilon}$ established above,

$$\left(\bar{Y}^{\varepsilon,\varphi^\varepsilon}, \int_{\mathbb{X}\times[0,\cdot]} G\big(X^0(s), y\big)\psi^\varepsilon(y,s)\nu(dy)\,ds\right)$$

converges in distribution, in $D([0,T]:\mathbb{R}^{2d})$, to $(\bar{Y}, \int_{\mathbb{X}\times[0,\cdot]} G(X^0(s), y)\psi(y,s) \times \nu(dy)\,ds)$. The result now follows by using this in (4.11) and recalling that $\mathcal{E}^{\varepsilon,\varphi^\varepsilon} \Rightarrow 0$. $\square$

We now complete the proof of Theorem 2.7.

PROOF OF THEOREM 2.7. It suffices to show that Condition 2.2 holds with $\mathcal{G}^\varepsilon$ and $\mathcal{G}_0$ defined as in the beginning of the section. Part (a) of the condition was verified in Lemma 4.1. Consider now part (b). Fix $M \in (0,\infty)$ and $\beta \in (0,1]$. Let $\{\varphi^\varepsilon\}_{\varepsilon>0}$ be such that for every $\varepsilon > 0$, $\varphi^\varepsilon \in \mathcal{U}^M_{+,\varepsilon}$ and $\psi^\varepsilon 1_{\{|\psi^\varepsilon|<\beta/a(\varepsilon)\}} \Rightarrow \psi$ in $B_2((M\kappa_2(1))^{1/2})$, where $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$. To complete the proof we need to show that

$$\mathcal{G}^\varepsilon\big(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}\big) \Rightarrow \mathcal{G}_0(\psi).$$

Recall that $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi^\varepsilon}) = \bar{Y}^{\varepsilon,\varphi^\varepsilon}$. From Lemma 4.9 $\{(\bar{Y}^{\varepsilon,\varphi^\varepsilon}, \psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}})\}$ is tight in $D([0,T]:\mathbb{R}^d) \times B_2((M\kappa_2(1))^{1/2})$, and every limit point of $\bar{Y}^{\varepsilon,\varphi^\varepsilon}$ must equal $\mathcal{G}_0(\psi)$. The result follows. $\square$

4.2. *Proof of Theorem* 2.8. Fix $\eta \in C([0,T]:\mathbb{R}^d)$ and $\delta > 0$. Let $u \in L^2([0,T]:\mathbb{R}^d)$ be such that

$$\frac{1}{2}\int_0^T \big|u(s)\big|^2\,ds \le I(\eta) + \delta$$

and (η, u) satisfy (2.15). Define $\psi : \mathbb{X} \times [0,T] \to \mathbb{R}$ by

$$\psi(y,s) \doteq \sum_{i=1}^d u_i(s)e_i(y,s), \qquad (y,s) \in \mathbb{X} \times [0,T]. \tag{4.13}$$

From the orthonormality of $e_i(\cdot,s)$ it follows that

$$\frac{1}{2}\int_{\mathbb{X}\times[0,T]} |\psi|^2\,d\nu_T = \frac{1}{2}\int_0^T \big|u(s)\big|^2\,ds. \tag{4.14}$$

Also,

$$
\begin{aligned}
[A(s)u(s)]_i &= \sum_{j=1}^{d} \langle G_i(X^0(s), \cdot), e_j(\cdot, s)\rangle_{L^2(\nu)} u_j(s) \\
&= \left\langle G_i(X^0(s), \cdot), \sum_{j=1}^{d} e_j(\cdot, s) u_j(s) \right\rangle_{L^2(\nu)} \\
&= \langle G_i(X^0(s), \cdot), \psi(\cdot, s)\rangle_{L^2(\nu)},
\end{aligned}
$$

so that $A(s)u(s) = \int \psi(y, s) G(X^0(s), y)\nu(dy)\, ds$. Consequently, η satisfies (2.14) with ψ as in (4.13). Combining this with (4.14) we have $\bar{I}(\eta) \le I(\eta) + \delta$. Since $\delta > 0$ is arbitrary, we have $\bar{I}(\eta) \le I(\eta)$.

Conversely, suppose $\psi \in L^2(\nu_T)$ is such that

$$
\frac{1}{2} \int_{\mathbb{X} \times [0,T]} |\psi|^2 \, d\nu_T \le \bar{I}(\eta) + \delta
$$

and (2.14) holds. For $i = 1, \dots, d$ define $u_i : [0, T] \to \mathbb{R}$ by

$$
u_i(s) = \langle \psi(\cdot, s), e_i(\cdot, s)\rangle_{L^2(\nu)}.
$$

Note that with $u = (u_1, \dots, u_d)$,

$$
\begin{aligned}
\frac{1}{2} \int_0^T |u(s)|^2 \, ds &= \frac{1}{2} \int_0^T \sum_{j=1}^{d} \langle \psi(\cdot, s), e_j(\cdot, s)\rangle_{L^2(\nu)}^2 \\
&\le \frac{1}{2} \int_0^T \int_{\mathbb{X}} \psi^2(y, s)\nu(dy)\, ds \qquad (4.15) \\
&\le \bar{I}(\eta) + \delta.
\end{aligned}
$$

For $s \in [0, T]$, let $\{e_j(\cdot, s)\}_{j=d+1}^{\infty}$ be defined in such a manner that $\{e_j(\cdot, s)\}_{j=1}^{\infty}$ is a complete orthonormal system in $L^2(\nu)$. Then for every $s \in [0, T]$,

$$
\begin{aligned}
[A(s)u(s)]_i &= \sum_{j=1}^{d} \langle G_i(X^0(s), \cdot), e_j(\cdot, s)\rangle_{L^2(\nu)} \langle \psi(\cdot, s), e_j(\cdot, s)\rangle_{L^2(\nu)} \\
&= \sum_{j=1}^{\infty} \langle G_i(X^0(s), \cdot), e_j(\cdot, s)\rangle_{L^2(\nu)} \langle \psi(\cdot, s), e_j(\cdot, s)\rangle_{L^2(\nu)} \\
&= \langle G_i(X^0(s), \cdot), \psi(\cdot, s)\rangle_{L^2(\nu)},
\end{aligned}
$$

where the second equality follows by observing that $G_i(X^0(s), \cdot)$ is in the linear span of $\{e_j(\cdot, s)\}_{j=1}^{d}$ for every $i = 1, \dots, d$. So $A(s)u(s) = \int \psi(y, s)G(X^0(s), y)\nu(dy)\, ds$, and therefore (η, u) satisfy (2.15). Combining this with (4.15), we get $I(\eta) \le \bar{I}(\eta) + \delta$. Since $\delta > 0$ is arbitrary, $I(\eta) \le \bar{I}(\eta)$ which completes the proof.

5. Proofs for the infinite-dimensional problem (Theorem 2.14). From Theorem 2.12 there is a measurable map $\mathcal{G}^\varepsilon:\mathbb{M}\to D([0,T]:\Phi_{-q})$ such that $Y^\varepsilon = \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}})$. Also, in Theorem 5.1 below we will show that for every $\psi\in L^2(\nu_T)$, there is a unique solution of (2.27). We denote this unique solution as $\mathcal{G}_0(\psi)$. In order to prove the theorem, it suffices to show that Condition 2.2 holds with $\mathcal{G}^\varepsilon$ and $\mathcal{G}_0$ defined as above.

Recall that Conditions 2.11 and 2.13 involve numbers $p<q<q_1$. We start by proving the unique solvability of (2.27).

THEOREM 5.1. *Suppose Conditions 2.11 and 2.13 hold. Then for every $\psi\in L^2(\nu_T)$, there exists a unique $\eta_\psi\in C([0,T],\Phi_{-q_1})$ that solves (2.27). Furthermore, for every $M\in(0,\infty)$, $\sup_{\psi\in B_2(M)}\sup_{0\le t\le T}\|\eta_\psi(t)\|_{-q}<\infty$.*

PROOF. Fix $M\in(0,\infty)$, and for $\psi\in B_2(M)$, let $\tilde\eta_\psi(\cdot)=\int_{\mathbb{X}\times[0,\cdot]}G(X^0(s), y)\psi(y,s)\nu(dy)\,ds$. By an application of the Cauchy–Schwarz inequality, Condition 2.11(b) and the definition of m_T in (2.22), we see that for every such ψ and $0\le s\le t\le T$,

$$\begin{aligned}\|\tilde\eta_\psi(t)-\tilde\eta_\psi(s)\|_{-p} &= \left\|\int_{\mathbb{X}\times[s,t]}G(X^0(r),y)\psi(y,r)\nu(dy)\,dr\right\|_{-p}\\ &\le (t-s)^{1/2}M(1+m_T)\left(\int_{\mathbb{X}}M_G^2(y)\nu(dy)\right)^{1/2}\\ &= (t-s)^{1/2}m_T^1,\end{aligned}\tag{5.1}$$

where

$$m_T^1 \doteq M(1+m_T)\left(\int_{\mathbb{X}}M_G^2(y)\nu(dy)\right)^{1/2}.\tag{5.2}$$

This shows that $\tilde\eta_\psi$ is in $C([0,T]:\Phi_{-p})$. Henceforth we suppress ψ from the notation unless needed. With A_v as in Condition 2.13, define $\tilde b:[0,T]\times\Phi_{-q}\to\Phi_{-q_1}$ by

$$\begin{aligned}\tilde b(s,v) \doteq A_{X^0(s)}(v+\tilde\eta(s)) + \int_{\mathbb{X}}D_x(G(X^0(s),y)[\cdot])[v+\tilde\eta(s)]\nu(dy),\\ (s,v)\in[0,T]\times\Phi_{-q}.\end{aligned}\tag{5.3}$$

The right-hand side in (5.3) indeed defines an element in Φ_{-q_1} as is seen from the definition of A_v and the estimate in (2.26). Note that η solves (2.27) if and only if $\bar\eta=\eta-\tilde\eta$ solves the equation

$$\bar\eta(t)=\int_0^t \tilde b(s,\bar\eta(s))\,ds.\tag{5.4}$$

We now argue that (5.4) has a unique solution; namely, there is a unique $\bar{\eta} \in C([0,T]:\Phi_{-q_1})$ such that for all $\phi \in \Phi$,

$$\bar{\eta}(t)[\phi] = \int_0^t \tilde{b}(s, \bar{\eta}(s))[\phi]\,ds, \qquad t \in [0,T].$$

To do this, in view of Theorem 3.7 in [8], it suffices to check that for some $K < \infty$, $\tilde{b}$ satisfies the following properties:

(a) for all $t \in [0,T]$ and $u \in \Phi_{-q}$, $\tilde{b}(t,u) \in \Phi_{-q_1}$, and the map $u \mapsto \tilde{b}(t,u)$ is continuous;

(b) for all $t \in [0,T]$ and $\phi \in \Phi$, $2\tilde{b}(t,\phi)[\theta_q \phi] \le K(1 + \|\phi\|_{-q}^2)$;

(c) for all $t \in [0,T]$ and $u \in \Phi_{-q}$, $\|\tilde{b}(t,u)\|_{-q_1}^2 \le K(1 + \|u\|_{-q}^2)$;

(d) for all $t \in [0,T]$ and $u_1, u_2 \in \Phi_{-q}$,

$$2\langle \tilde{b}(t,u_1) - \tilde{b}(t,u_2), u_1 - u_2 \rangle_{-q_1} \le K\|u_1 - u_2\|_{-q_1}^2. \tag{5.5}$$

Consider first part (a). For $s \in [0,T]$ and $\eta \in \Phi_{-q}$, define $\tilde{A}_{X^0(s)}(\eta) : \Phi_q \to \mathbb{R}$ by

$$\tilde{A}_{X^0(s)}(\eta)[\phi] = \int_{\mathbb{X}} D_x\big(G(X^0(s), y)[\phi]\big)[\eta]\nu(dy).$$

Note that $\tilde{b}(s,v) = A_{X^0(s)}(v + \tilde{\eta}(s)) + \tilde{A}_{X^0(s)}(v + \tilde{\eta}(s))$. Let $K_1 = \max\{\sqrt{T} m_T^1, 1\}$, with m_T^1 defined in (5.2). Then using Condition 2.13(c) and (5.1) we have, for each fixed $s \in [0,T]$, that

$$\begin{aligned} &\big|\tilde{A}_{X^0(s)}(v + \tilde{\eta}(s))[\phi]\big| \le K_1 M_{DG}^*(1 + \|v\|_{-q_1})\|\phi\|_{q_1}, \\ &\qquad\qquad \phi \in \Phi_{q_1}, v \in \Phi_{-q}. \end{aligned} \tag{5.6}$$

Consequently, for each fixed s, $v \mapsto \tilde{A}_{X^0(s)}(v + \tilde{\eta}(s))$ is a map from Φ_{-q} to Φ_{-q_1}. Also, from Condition 2.13(b) for each fixed s, $v \mapsto A_{X^0(s)}(v + \tilde{\eta}(s))$ is a map from Φ_{-q} to Φ_{-q_1}. By the same condition, the map $v \mapsto A_{X^0(s)}(v + \tilde{\eta}(s))$ is continuous for each s. Also, from Condition 2.13(c) we have for each fixed $s \in [0,T]$ and $v, v' \in \Phi_{-q}$,

$$\begin{aligned} &\big|\tilde{A}_{X^0(s)}(v + \tilde{\eta}(s))[\phi] - \tilde{A}_{X^0(s)}(v' + \tilde{\eta}(s))[\phi]\big| \le M_{DG}^*\|v - v'\|_{-q_1}\|\phi\|_{q_1}, \\ &\qquad\qquad \phi \in \Phi_{q_1}. \end{aligned} \tag{5.7}$$

Consequently, the map $v \mapsto \tilde{A}_{X^0(s)}(v + \tilde{\eta}(s))$ is continuous as well. This proves (a).

For (b) note that again using Condition 2.13(c) and (5.1), for $\phi \in \Phi$,

$$\begin{aligned} \tilde{A}_{X^0(s)}(\phi + \tilde{\eta}(s))[\theta_q \phi] &\le \int_{\mathbb{X}} M_{DG}(X^0(s), y)\|\theta_q \phi\|_q \|\phi + \tilde{\eta}(s)\|_{-q}\nu(dy) \\ &\le M_{DG}^*(m_T^1\sqrt{T} + \|\phi\|_{-q})\|\phi\|_{-q}. \end{aligned} \tag{5.8}$$

Also, from (2.25)

$$2A_{X^0(s)}(\phi + \tilde{\eta}(s))[\theta_q \phi] \le C_A(\|\phi\|_{-q} + \sqrt{T} m_T^1)\|\phi\|_{-q}.$$

Combining this estimate with (5.9), we have (b).

Consider now part (c). Note that, from (5.6) we have

$$\|\tilde{A}_{X^0(s)}(v + \tilde{\eta}(s))\|_{-q_1} \le K_1 M_{DG}^*(1 + \|v\|_{-q_1}), \qquad v \in \Phi_{-q}.$$

Also, from (2.23) and (5.1),

$$\|A_{X^0(s)}(v + \tilde{\eta}(s))\|_{-q_1} \le M_A(1 + \sqrt{T} m_1^T + \|v\|_{-q}), \qquad v \in \Phi_{-q}.$$

Combining the last two estimates we have

$$\|\tilde{b}(t, v)\|_{-q_1} \le (K_1 M_{DG}^* + M_A)(1 + \sqrt{T} m_1^T + \|v\|_{-q}), \qquad v \in \Phi_{-q}, \tag{5.9}$$

which verifies part (c).

Finally, for (d) note that from (2.24), for all $u_1, u_2 \in \Phi_{-q}$ and $s \in [0, T]$,

$$\langle u_1 - u_2, A_{X^0(s)}(u_1 + \tilde{\eta}(s)) - A_{X^0(s)}(u_2 + \tilde{\eta}(s))\rangle_{-q_1} \le L_A \|u_1 - u_2\|_{-q_1}^2. \tag{5.10}$$

Also, from (5.7), for all $u_1, u_2 \in \Phi_{-q}$ and $s \in [0, T]$,

$$\begin{aligned}&\langle u_1 - u_2, \tilde{A}_{X^0(s)}(u_1 + \tilde{\eta}(s)) - \tilde{A}_{X^0(s)}(u_2 + \tilde{\eta}(s))\rangle_{-q_1}\\ &\qquad \le M_{DG}^* \|u_1 - u_2\|_{-q_1}^2.\end{aligned} \tag{5.11}$$

Part (d) now follows by combining these two displays.

As noted earlier, we now have from Theorem 3.7 in [8] that (5.4) and therefore (2.28) have a unique solution in $C([0, T], \Phi_{-q_1})$. Also, from the same theorem it follows that

$$\sup_{\psi \in B_2(M)} \sup_{0 \le t \le T} \|\bar{\eta}(t)\|_{-q} < \infty.$$

The second part of the theorem is now immediate by noting that

$$\sup_{\psi \in B_2(M)} \sup_{0 \le t \le T} \|\eta_\psi(t)\|_{-q} \le \sqrt{T} m_1^T + \sup_{\psi \in B_2(M)} \sup_{0 \le t \le T} \|\bar{\eta}(t)\|_{-q}. \qquad \square$$

The following lemma verifies part (a) of Condition 2.2.

LEMMA 5.2. *Suppose that Conditions 2.11 and 2.13 hold. Fix $M \in (0, \infty)$ and $g^\varepsilon, g \in B_2(M)$ such that $g^\varepsilon \to g$. Let $\mathcal{G}_0$ be the mapping that was shown to be well defined in Theorem 5.1. Then $\mathcal{G}_0(g^\varepsilon) \to \mathcal{G}_0(g)$.*

PROOF. From (5.1) and the compact embedding of Φ_{-p} into Φ_{-q}, we see that the collection

$$\left\{\tilde{\eta}_\varepsilon(\cdot) = \int_{\mathbb{X} \times [0,\cdot]} G(X^0(r), y) g^\varepsilon(y, r) \nu(dy)\, dr\right\}_{\varepsilon > 0}$$

is precompact in $C([0,T]:\Phi_{-q})$. Combining this with the convergence $g^\varepsilon \to g$ and the fact that $(s,y) \mapsto G(X^0(s),y)[\phi]$ is in $L^2(\nu_T)$ for every $\phi \in \Phi$, we see that

$$\tilde{\eta}_\varepsilon \to \tilde{\eta} \qquad \text{as } \varepsilon \to 0 \text{ in } C\big([0,T]:\Phi_{-q}\big), \tag{5.12}$$

where $\tilde{\eta} = \int_{\mathbb{X}\times[0,\cdot]} G(X^0(r),y)g(y,r)\nu(dy)\,dr$. Next, let η_ε denote the unique solution of (2.27) with ψ replaced by g^ε and, as in the proof of Theorem 5.1, define $\bar{\eta}_\varepsilon = \eta_\varepsilon - \tilde{\eta}_\varepsilon$. From Theorem 5.1

$$M_{\bar{\eta}} = \sup_{\varepsilon>0}\sup_{0\le t\le T} \|\bar{\eta}_\varepsilon(t)\|_{-q} < \infty. \tag{5.13}$$

Also, for every fixed $\phi \in \Phi$, $\bar{\eta}_\varepsilon$ solves

$$\bar{\eta}_\varepsilon(t)[\phi] = \int_0^t \tilde{b}_\varepsilon\big(s,\bar{\eta}_\varepsilon(s)\big)[\phi]\,ds, \tag{5.14}$$

where $\tilde{b}_\varepsilon$ is defined by the right-hand side of (5.3) by replacing $\tilde{\eta}$ with $\tilde{\eta}_\varepsilon$.

Next, let $\bar{\eta} \in C([0,T]:\Phi_{-q_1})$ be the unique solution of

$$\bar{\eta}(t)[\phi] = \int_0^t \tilde{b}\big(s,\bar{\eta}(s)\big)[\phi]\,ds, \qquad \phi \in \Phi,$$

where $\tilde{b}$ is as in (5.3). Let $\hat{A}_v = A_v + \tilde{A}_v$ and $a_\varepsilon(s) = \hat{A}_{X^0(s)}(\bar{\eta}_\varepsilon(s) + \tilde{\eta}_\varepsilon(s)) - \hat{A}_{X^0(s)}(\bar{\eta}(s) + \tilde{\eta}(s))$. Using the same bounds as those used in (5.9), (5.10) and (5.11), there is $K < \infty$ such that

$$\begin{aligned}
&\|\bar{\eta}_\varepsilon(t) - \bar{\eta}(t)\|_{-q_1}^2 \\
&\quad = 2\int_0^t \big\langle \tilde{b}_\varepsilon\big(s,\bar{\eta}_\varepsilon(s)\big) - \tilde{b}\big(s,\bar{\eta}(s)\big), \bar{\eta}_\varepsilon(s) - \bar{\eta}(s)\big\rangle_{-q_1}\,ds \\
&\quad = 2\int_0^t \big\langle a_\varepsilon(s), \bar{\eta}_\varepsilon(s) - \bar{\eta}(s)\big\rangle_{-q_1}\,ds \\
&\quad = 2\int_0^t \big\langle a_\varepsilon(s), \big(\bar{\eta}_\varepsilon(s) + \tilde{\eta}_\varepsilon(s)\big) - \big(\bar{\eta}(s) + \tilde{\eta}(s)\big)\big\rangle_{-q_1}\,ds \\
&\qquad + 2\int_0^t \big\langle a_\varepsilon(s), \tilde{\eta}(s) - \tilde{\eta}_\varepsilon(s)\big\rangle_{-q_1}\,ds \\
&\quad \le K\int_0^t \|(\bar{\eta}_\varepsilon(s) + \tilde{\eta}_\varepsilon(s) - \big(\bar{\eta}(s) + \tilde{\eta}(s)\big)\|_{-q_1}^2\,ds \\
&\qquad + K_2\int_0^t \|\tilde{\eta}(s) - \tilde{\eta}_\varepsilon(s)\|_{-q_1}\,ds,
\end{aligned}$$

where $K_2 = 2(K_1 M_{DG}^* + M_A)(1 + \sqrt{T}m_T^1 + M_{\bar{\eta}})$ and $M_{\bar{\eta}}$ is from (5.13). Thus

$$\begin{aligned}
\|\bar{\eta}_\varepsilon(t) - \bar{\eta}(t)\|_{-q_1}^2 &\le 2K\int_0^t \|\bar{\eta}_\varepsilon(s) - \bar{\eta}(s)\|_{-q_1}^2\,ds \\
&\qquad + K_3\int_0^t \big(\|\tilde{\eta}_\varepsilon(s) - \tilde{\eta}(s)\|_{-q_1}^2 + \|\tilde{\eta}_\varepsilon(s) - \tilde{\eta}(s)\|_{-q_1}\big)\,ds,
\end{aligned}$$

where $K_3 = K_2 + 2K$. The result now follows by combining this with (5.12) and $q < q_1$, and using Gronwall's inequality. $\square$

We now consider part (b) of Condition 2.2. For $\varphi \in \mathcal{U}^M_{+,\varepsilon}$ let $\bar{X}^{\varepsilon,\varphi} = \bar{\mathcal{G}}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi})$. As in Section 4 it follows by an application of Girsanov's theorem that $\bar{X}^{\varepsilon,\varphi}$ is the unique solution of the integral equation

$$\begin{aligned}\bar{X}^{\varepsilon,\varphi}_t[\phi] &= x_0[\phi] + \int_0^t b(\bar{X}^{\varepsilon,\varphi}_s)[\phi]\,ds \\ &\quad + \varepsilon \int_{\mathbb{X}\times[0,t]} G(\bar{X}^{\varepsilon,\varphi}_{s-}, y)[\phi] N^{\varepsilon^{-1}\varphi}(dy, ds), \qquad \phi \in \Phi.\end{aligned} \tag{5.15}$$

Define $\bar{Y}^{\varepsilon,\varphi}$ as in (4.6). Then $\bar{Y}^{\varepsilon,\varphi} = \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi})$.

The following moment bounds on $\bar{X}^{\varepsilon,\varphi}$ and $\bar{Y}^{\varepsilon,\varphi}$ will be the key. The proof of part (a) is given in Proposition 3.13 of [8]. However, equation (3.33) in [8] contains an error, in view of which we give a corrected proof below. The idea is to first approximate $\bar{X}^{\varepsilon,\varphi}$ by a sequence of finite-dimensional processes $\{\bar{X}^{\varepsilon,d,\varphi}\}_{d\in\mathbb{N}}$ and obtain an analogous equation for the d-dimensional process for every value of d. The desired estimate follows by first obtaining an estimate for the finite-dimensional processes that is uniform in d and then sending $d \to \infty$.

LEMMA 5.3. *Suppose Conditions* 2.11 *and* 2.13(d) *hold. Fix $M < \infty$. Then there exists an $\varepsilon_0 \in (0, 1)$ such that*:

(a) $\sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{\varphi\in\mathcal{U}^M_{+,\varepsilon}} \bar{\mathbb{E}}[\sup_{0\le s\le T} \|\bar{X}^{\varepsilon,\varphi}(s)\|^2_{-p}] < \infty$,

(b) $\sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{\varphi\in\mathcal{U}^M_{+,\varepsilon}} \bar{\mathbb{E}}[\sup_{0\le s\le T} \|\bar{Y}^{\varepsilon,\varphi}(s)\|^2_{-q}] < \infty$.

PROOF. We first prove part (a). We follow the steps in the proof of Theorem 6.2.2 of [44]; see also the proof of Theorem 3.7 in [8]. Recall that $\{\phi_j\}_{j\in\mathbb{N}}$ is a CONS in Φ_0 and a COS in Φ_n, $n \in \mathbb{N}$. For $d \in \mathbb{N}$, let $\pi_d : \Phi_{-p} \to \mathbb{R}^d$ be defined by

$$\pi_d(u) \doteq (u[\phi^p_1], \dots, u[\phi^p_d])', \qquad u \in \Phi_{-p}.$$

Let $x^d_0 = \pi_d(x_0)$. Define $\beta^d : \mathbb{R}^d \to \mathbb{R}^d$ and $g^d : \mathbb{R}^d \times \mathbb{X} \to \mathbb{R}^d$ by

$$\beta^d(x)_k \doteq b\left(\sum_{j=1}^d x_j \phi^{-p}_j\right)[\phi^p_k],$$

$$g^d(x, y)_k \doteq G\left(\sum_{j=1}^d x_j \phi^{-p}_j, y\right)[\phi^p_k], \qquad k = 1, \dots, d,$$

where $\eta[\phi]$ was defined in (2.16). Next define $\gamma^d : \Phi' \to \Phi'$ by

$$\gamma^d u \doteq \sum_{k=1}^{d} u[\phi_k^p]\phi_k^{-p},$$

and define $b^d : \Phi' \to \Phi'$ and $G^d : \Phi' \times \mathbb{X} \to \Phi'$ by

$$b^d(u) \doteq \gamma^d b(\gamma^d u), \qquad G^d(u, y) \doteq \gamma^d G(\gamma^d u, y).$$

It is easy to check that for each $d \in \mathbb{N}$, b^d and G^d satisfy Condition 2.11 [with $(M_{b^d}, M_{G^d}, C_{b^d}, L_{b^d}, L_{G^d})$ equal to $(M_b, M_G, C_b, L_b, L_G)$ for all d]; see the proof of Theorem 6.2.2 in [44]. Also, from Lemma 6.2.2 in [44] and an argument based on Girsanov's theorem (as in Section 4), it follows that the following integral equation has a unique solution in $D([0,T] : \mathbb{R}^d)$ for all $\varphi \in \mathcal{U}^M_{+,\varepsilon}$:

$$\bar{x}^{\varepsilon,d,\varphi}(s) = x_0^d + \int_0^t \beta^d(\bar{x}^{\varepsilon,d,\varphi}(s))\,ds + \int_{\mathbb{X}\times[0,t]} \varepsilon g^d(\bar{x}^{\varepsilon,d,\varphi}(s-), y) N^{\varepsilon^{-1}\varphi}(dy, ds).$$

Let

$$\bar{X}^{\varepsilon,d,\varphi}(t) \doteq \sum_{k=1}^{d} \bar{x}_k^{\varepsilon,d,\varphi}(t)\phi_k^{-p}, \qquad t \in [0, T].$$

Then, with $X_0^d = \sum_{k=1}^d (x_0^d)_k \phi_k^{-p}$, for all $t \in [0, T]$

$$\bar{X}^{\varepsilon,d,\varphi}(t) = X_0^d + \int_0^t b^d(\bar{X}^{\varepsilon,d,\varphi}(s))\,ds + \int_{\mathbb{X}\times[0,t]} \varepsilon G^d(\bar{X}^{\varepsilon,d,\varphi}(s-), y) N^{\varepsilon^{-1}\varphi}(dy, ds).$$

We next prove that there exists $\varepsilon_0 > 0$ such that

$$\sup_{d\in\mathbb{N}} \sup_{\varepsilon\in(0,\varepsilon_0)} \bar{\mathbb{E}}\Big[\sup_{0\le t\le T} \|\bar{X}^{\varepsilon,d,\varphi}(s)\|_{-p}^2\Big] < \infty.$$

The proof is similar to Lemma 6.2.2 in [44] (see also the proof of [8], Proposition 3.13), and therefore we just outline the main steps. By Itô's lemma

$$\begin{aligned}
&\|\bar{X}^{\varepsilon,d,\varphi}(t)\|_{-p}^2 \\
&\quad = \|X^0(t)\|_{-p}^2 + 2\int_0^t b^d(\bar{X}^{\varepsilon,d,\varphi}(s))[\theta_p \bar{X}^{\varepsilon,d,\varphi}(s)]\,ds \\
&\qquad + 2\int_{\mathbb{X}\times[0,t]} \langle \bar{X}^{\varepsilon,d,\varphi}(s), G^d(\bar{X}^{\varepsilon,d,\varphi}(s), y)\rangle_{-p} \varphi\, d\nu_T \\
&\qquad + \int_{\mathbb{X}\times[0,t]} \varepsilon \|G^d(\bar{X}^{\varepsilon,d,\varphi}(s), y)\|_{-p}^2 \varphi\, d\nu_T \\
&\qquad + \int_{\mathbb{X}\times[0,t]} 2\big[\langle \bar{X}^{\varepsilon,d,\varphi}(s-), \varepsilon G^d(\bar{X}^{\varepsilon,d,\varphi}(s-), y)\rangle_{-p} \\
&\qquad\qquad + \|\varepsilon G^d(\bar{X}^{\varepsilon,d,\varphi}(s-), y)\|_{-p}^2\big]\, d\tilde{N}^{\varepsilon^{-1}\varphi}.
\end{aligned} \tag{5.16}$$

Recalling that Condition 2.11(c) holds with $b = b^d$ (with the same constant C_b for all d), we have

$$2\int_0^t \langle \bar{X}^{\varepsilon,d,\varphi}(s), b^d(\bar{X}^{\varepsilon,d,\varphi}(s))\rangle_{-p}\, ds \le C_b \int_0^t (1 + \|\bar{X}^{\varepsilon,d,\varphi}(s)\|_{-p}^2)\, ds.$$

Now exactly as in [8], Proposition 3.13, it follows that there exists $L_1 \in (0,\infty)$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\varphi \in \mathcal{U}_{+,\varepsilon}^M$,

$$\sup_{0\le s\le T} \|\bar{X}^{\varepsilon,d,\varphi}(s)\|_{-p}^2 \le L_1\Big(1 + \sup_{0\le s\le T} |M^d(s)|\Big), \tag{5.17}$$

where $M^d(t)$ is the last term on the right-hand side of (5.16); see (3.35) in [8]. Once more, exactly as in [8] [see (3.36) and (3.37) therein] one has that there is a $L_2 \in (0,\infty)$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\varphi \in \mathcal{U}_{+,\varepsilon}^M$,

$$\bar{\mathbb{E}} \sup_{0\le s\le T} |M^d(s)| \le \varepsilon L_2\Big(1 + \bar{\mathbb{E}} \sup_{0\le t\le T} \|\bar{X}^{\varepsilon,d,\varphi}(t)\|_{-p}^2\Big) + \frac{1}{8}\bar{\mathbb{E}} \sup_{0\le t\le T} \|\bar{X}^{\varepsilon,d,\varphi}(t)\|_{-p}^2.$$

Using the last estimate in (5.17) we now have that, for some $\varepsilon_0 > 0$,

$$\sup_{d\in\mathbb{N}} \sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{\varphi\in\mathcal{U}_{+,\varepsilon}^M} \bar{\mathbb{E}} \sup_{0\le t\le T} \|\bar{X}^{\varepsilon,d,\varphi}(t)\|_{-p}^2 < \infty. \tag{5.18}$$

Also an application of Girsanov's theorem and Theorem 6.1.2 of [44] shows that $\bar{X}^{\varepsilon,d,\varphi}$ converges in distribution, in $D([0,T]:\Phi_{-q})$ to the solution of (5.15). The estimate in part (a) of the lemma now follows from (5.18) and an application of Fatou's lemma.

We now prove part (b) of the lemma. By Itô's formula,

$$\begin{aligned}
&\|\bar{X}^{\varepsilon,\varphi}(t) - X^0(t)\|_{-q}^2 \\
&\quad = 2\int_0^t \langle \bar{X}^{\varepsilon,\varphi}(s) - X^0(s), b(\bar{X}^{\varepsilon,\varphi}(s)) - b(X^0(s))\rangle_{-q}\, ds \\
&\qquad + 2\int_{\mathbb{X}\times[0,t]} \langle \bar{X}^{\varepsilon,\varphi}(s) - X^0(s), G(\bar{X}^{\varepsilon,\varphi}(s), y) - G(X^0(s), y)\rangle_{-q} \nu(dy)\, ds \\
&\qquad + 2\int_{\mathbb{X}\times[0,t]} \langle \bar{X}^{\varepsilon,\varphi}(s) - X^0(s), G(\bar{X}^{\varepsilon,\varphi}(s), y)\rangle_{-q} (\varphi - 1)\nu(dy)\, ds \\
&\qquad + \int_{\mathbb{X}\times[0,t]} \varepsilon\|G(\bar{X}^{\varepsilon,\varphi}(s), y)\|_{-q}^2 \varphi\nu(dy)\, ds \\
&\qquad + \int_{\mathbb{X}\times[0,t]} (2\langle \bar{X}^{\varepsilon,\varphi}(s-) - X^0(s-), \varepsilon G(\bar{X}^{\varepsilon,\varphi}(s-), y)\rangle_{-q} \\
&\qquad\qquad + \|\varepsilon G(\bar{X}^{\varepsilon,\varphi}(s-), y)\|_{-q}^2) \tilde{N}^{\varepsilon^{-1}\varphi}(dy, ds) \\
&\quad = a^2(\varepsilon)(A^{\varepsilon,\varphi} + B^{\varepsilon,\varphi} + C^{\varepsilon,\varphi} + \mathcal{E}_1^{\varepsilon,\varphi} + M_1^{\varepsilon,\varphi} + M_2^{\varepsilon,\varphi}).
\end{aligned}$$

By Condition 2.11(d), for all $t \in [0,T]$

$$\sup_{0\le r\le t} A^{\varepsilon,\varphi}(r) \le 2L_b \int_0^t \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\, ds.$$

Also by Condition 2.11(e),

$$\sup_{0\le r\le t} \left|B^{\varepsilon,\varphi}(r)\right| \le \|L_G\|_1 \int_0^t \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\, ds.$$

Next, note that with $\psi = (\varphi - 1)/a(\varepsilon)$,

$$\begin{aligned}
&\sup_{0\le r\le t} \left|C^{\varepsilon,\varphi}(r)\right| \\
&\quad \le 2\int_{\mathbb{X}\times[0,t]} \left|\langle \bar{Y}^{\varepsilon,\varphi}(s), G(\bar{X}^{\varepsilon,\varphi}(s),y)\rangle_{-q}\right| |\psi|\, d\nu_T \\
&\quad \le 2\int_{\mathbb{X}\times[0,t]} \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q} \|G(\bar{X}^{\varepsilon,\varphi}(s),y)\|_{-q} |\psi|\, d\nu_T \\
&\quad \le 2\int_{\mathbb{X}\times[0,t]} \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q} \big[\|G(\bar{X}^{0}(s),y)\|_{-q} \\
&\qquad\qquad + L_G(y)a(\varepsilon)\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}\big] |\psi|\, d\nu_T \\
&\quad \le 2\int_{\mathbb{X}\times[0,t]} \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q} R_G(y)|\psi| \big(1 + a(\varepsilon)\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}\big)\, d\nu_T,
\end{aligned}$$

where for $y \in \mathbb{X}$, $R_G(y) \doteq M_G(y)(1+m_T) + L_G(y)$. Thus

$$\begin{aligned}
\sup_{r\le t}\left|C^{\varepsilon,\varphi}(r)\right| &\le 2a(\varepsilon)\sup_{r\le t}\|\bar{Y}^{\varepsilon,\varphi}(r)\|_{-q}^2 \int_{\mathbb{X}\times[0,t]} R_G(y)|\psi|\, d\nu_T \\
&\quad + 2\int_{\mathbb{X}\times[0,t]} \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q} R_G(y)|\psi|\, d\nu_T \qquad (5.19)\\
&= T_1 + T_2.
\end{aligned}$$

Consider now T_2. Note that $R_G \in L^2(\nu)\cap\mathcal{H}^\delta$. We can therefore apply Lemma 4.3 with h replaced by R_G. For any $\beta < \infty$,

$$\begin{aligned}
T_2 &= 2\int_{\mathbb{X}\times[0,t]} \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q} R_G(y)|\psi| [1_{\{|\psi|\le\beta/a(\varepsilon)\}} + 1_{\{|\psi|>\beta/a(\varepsilon)\}}]\, d\nu_T \\
&\le 2\sup_{r\le t}\|\bar{Y}^{\varepsilon,\varphi}(r)\|_{-q}\big(\vartheta(\beta) + (1+T)\theta(\varepsilon)\big) \\
&\quad + \int_{\mathbb{X}\times[0,t]} [\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2 R_G^2(y) + |\psi|^2 1_{\{|\psi|\le\beta/a(\varepsilon)\}}]\, d\nu_T \qquad (5.20)\\
&\le 2\sup_{r\le t}\|\bar{Y}^{\varepsilon,\varphi}(r)\|_{-q}\big(\vartheta(\beta) + (1+T)\theta(\varepsilon)\big) \\
&\quad + L_1\int_{[0,t]} \|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\, ds + M\kappa_2(\beta),
\end{aligned}$$

where $L_1 \doteq \int_{\mathbb{X}} R_G^2(y)\nu(dy)$ and in the last inequality we have used Lemma 3.2(c). Once again from Lemma 4.3

$$L_2 \doteq \sup_{\varepsilon\in(0,1)} \sup_{\psi\in\mathcal{S}_\varepsilon^M} 2\int_{\mathbb{X}\times[0,T]} R_G(y)|\psi|\,d\nu_T < \infty.$$

Using $a \le 1+a^2$ and the last two estimates in (5.19), we have that

$$\begin{aligned}\sup_{r\le t}\left|C^{\varepsilon,\varphi}(r)\right| &\le L(\beta) + \sup_{r\le t}\|\bar{Y}^{\varepsilon,\varphi}(r)\|_{-q}^2\big(L_2 a(\varepsilon)+\tilde{\vartheta}(\beta)\big) \\ &\quad + L_1\int_{[0,t]}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\,ds,\end{aligned}\tag{5.21}$$

where $\tilde{\vartheta}(\beta) \doteq 2(\vartheta(\beta)+(1+T)\bar{\theta})$, $\bar{\theta} \doteq \sup_{\varepsilon\in(0,\varepsilon_0)}\theta(\varepsilon)$, and $L(\beta) \doteq M\kappa_2(\beta) + \tilde{\vartheta}(\beta)$. Without loss of generality we assume that ε_0 is small enough that $(1+T)\bar{\theta} < 1/4$. Next note that

$$\begin{aligned}\sup_{r\le t}\mathcal{E}_1^{\varepsilon,\varphi}(r) &\le \frac{\varepsilon}{a^2(\varepsilon)}\int_{\mathbb{X}\times[0,t]}(1+\|\bar{X}^{\varepsilon,\varphi}(s)\|_{-p})^2 M_G^2(y)\varphi(y,s)\,d\nu_T \\ &\le \frac{\varepsilon}{a^2(\varepsilon)}\Big(1+\sup_{s\le T}\|\bar{X}^{\varepsilon,\varphi}(s)\|_{-p}\Big)^2\int_{\mathbb{X}\times[0,t]} M_G^2(y)\varphi(y,s)\,d\nu_T.\end{aligned}$$

Since $M_G^2 \in L^1(\nu)\cap\mathcal{H}^\delta$, we have from Lemma 4.5 that

$$L_3 = \sup_{\varepsilon\in(0,1)}\sup_{\varphi\in\mathcal{S}_{+,\varepsilon}^M}\int_{\mathbb{X}\times[0,t]} M_G^2\varphi\,d\nu_T < \infty,$$

and consequently, for all $\varepsilon\in(0,\varepsilon_0)$,

$$\bar{\mathbb{E}}\Big[\sup_{r\le t}\mathcal{E}_1^{\varepsilon,\varphi}(r)\Big] \le L_4\frac{\varepsilon}{a^2(\varepsilon)},$$

where $L_4 \doteq L_3\sup_{\varepsilon\in(0,\varepsilon_0)}\sup_{\varphi\in\mathcal{U}_{+,\varepsilon}^M}(1+\bar{\mathbb{E}}\sup_{0\le s\le T}\|\bar{X}^{\varepsilon,\varphi}(s)\|_{-p}^2) < \infty$ by part (a) of the lemma. Next, an application of Lenglart–Lepingle–Pratelli inequality gives that for some $L_5\in(0,\infty)$,

$$\begin{aligned}&\bar{\mathbb{E}}\Big[\sup_{0\le s\le T} M_1^{\varepsilon,\varphi}(s)\Big] \\ &\quad\le \frac{L_5}{a^2(\varepsilon)}\bar{\mathbb{E}}\Big[\int_{\mathbb{X}\times[0,T]}\langle\bar{X}^{\varepsilon,\varphi}(s)-X^0(s),\varepsilon G(\bar{X}^{\varepsilon,\varphi}(s),y)\rangle_{-q}^2\varepsilon^{-1}\varphi\nu(dy)\,ds\Big]^{1/2} \\ &\quad\le \frac{L_5\sqrt{\varepsilon}}{a^2(\varepsilon)}\bar{\mathbb{E}}\Big[\int_{\mathbb{X}\times[0,t]}\|\bar{X}^{\varepsilon,\varphi}(s)-X^0(s)\|_{-q}^2\|G(\bar{X}^{\varepsilon,\varphi}(s-),y)\|_{-q}^2\varphi\nu(dy)\,ds\Big]^{1/2} \\ &\quad\le \frac{L_5\sqrt{\varepsilon}}{a(\varepsilon)}\bar{\mathbb{E}}\Big[\sup_{s\le t}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}\Big(\int_{\mathbb{X}\times[0,t]}\|G(\bar{X}^{\varepsilon,\varphi}(s),y)\|_{-q}^2\varphi\nu(dy)\,ds\Big)^{1/2}\Big]\end{aligned}$$

$$
\begin{aligned}
&\le \frac{L_5\sqrt{\varepsilon}}{2a(\varepsilon)}\Big[\bar{\mathbb{E}}\sup_{s\le t}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2 \\
&\qquad\qquad + \bar{\mathbb{E}}\Big(1+\sup_{s\le t}\|\bar{X}^{\varepsilon,\varphi}(s)\|_{-p}^2\Big)\int_{\mathbb{X}\times[0,t]} M_G^2\varphi\nu(dy)\,ds\Big] \\
&\le \frac{L_5\sqrt{\varepsilon}}{2a(\varepsilon)}\bar{\mathbb{E}}\sup_{s\le t}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2 + \frac{L_5L_4\sqrt{\varepsilon}}{2a(\varepsilon)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{0\le s\le t} M_2^{\varepsilon,\varphi}(s)\Big] &\le \frac{1}{a^2(\varepsilon)}\bar{\mathbb{E}}\int_{\mathbb{X}\times[0,T]}\|\varepsilon G(\bar{X}^{\varepsilon,\varphi}(s-),y)\|_{-q}^2 N^{\varepsilon^{-1}\varphi}(dy,ds) \\
&\quad + \frac{1}{a^2(\varepsilon)}\bar{\mathbb{E}}\int_{\mathbb{X}\times[0,T]}\varepsilon\|G(\bar{X}^{\varepsilon,\varphi}(s),y)\|_{-q}^2\varphi\,d\nu_T \\
&\le \frac{2\varepsilon}{a^2(\varepsilon)}\bar{\mathbb{E}}\int_{\mathbb{X}\times[0,T]}\|G(\bar{X}^{\varepsilon,\varphi}(s),y)\|_{-q}^2\varphi\,d\nu_T \\
&\le \frac{2\varepsilon L_4}{a^2(\varepsilon)}.
\end{aligned}
$$

Let $\varepsilon_1 \in (0,\varepsilon_0)$ be such that for all $\varepsilon \in (0,\varepsilon_1)$, $\max\{\varepsilon, a(\varepsilon), \frac{\varepsilon}{a^2(\varepsilon)}\} < 1$. Collecting terms together, we now have for all $\varepsilon \in (0,\varepsilon_1)$,

$$
\begin{aligned}
\bar{\mathbb{E}}\Big[\sup_{s\le t}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\Big] &\le K_1\int_0^t \bar{\mathbb{E}}\Big[\sup_{r\le s}\|\bar{Y}^{\varepsilon,\varphi}(r)\|_{-q}^2\Big]ds + \Big(L(\beta)+2L_4+\frac{L_5L_4}{2}\Big) \\
&\quad + \Big[L_2a(\varepsilon)+L_5\frac{\sqrt{\varepsilon}}{2a(\varepsilon)}+\tilde{\vartheta}(\beta)\Big]\bar{\mathbb{E}}\Big[\sup_{s\le t}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\Big],
\end{aligned}
$$

where $K_1 \doteq 2L_b + \|L_G\|_1 + L_1$. Since $\tilde{\vartheta}(\beta) \to 0$ as $\beta \to \infty$, we can find $\beta_0 < \infty$ and $\varepsilon_2 \in (0,\varepsilon_1)$ such that for all $\varepsilon \in (0,\varepsilon_2)$, $L_2a(\varepsilon) + L_5\frac{\varepsilon}{2a(\varepsilon)} + \tilde{\vartheta}(\beta_0) \le 1/2$. Using this in the above inequality, for all $\varepsilon \in (0,\varepsilon_2)$,

$$
\bar{\mathbb{E}}\Big[\sup_{s\le t}\|\bar{Y}^{\varepsilon,\varphi}(s)\|_{-q}^2\Big] \le K_2 + 2K_1\int_0^t \bar{\mathbb{E}}\Big[\sup_{r\le s}\|\bar{Y}^{\varepsilon,\varphi}(r)\|_{-q}^2\Big]ds,
$$

where $K_2 \doteq 2(L(\beta_0) + 2L_4 + \frac{L_5L_4}{4})$. The result now follows from Gronwall's inequality. □

The following result will be used in verifying part (b) of Condition 2.2. Recall the integer $q_1 > q$ introduced in Condition 2.13.

LEMMA 5.4. *Suppose Conditions 2.11 and 2.13 hold. Let $\varepsilon_0 > 0$ be as in Lemma 5.3, and let $\{\varphi^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ be such that for some $M < \infty$, $\varphi^\varepsilon \in \mathcal{U}_{+,\varepsilon}^M$ for every $\varepsilon \in (0,\varepsilon_0)$. Let $\psi^\varepsilon = (\varphi^\varepsilon - 1)/a(\varepsilon)$, and fix $\beta \in (0,1]$. Then*

$\{(\bar{Y}^{\varepsilon,\varphi^\varepsilon}, \psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}})\}_{\varepsilon\in(0,\varepsilon_0)}$ *is tight in* $D([0,T]:\Phi_{-q_1})\times B_2((M\kappa_2(1))^{1/2})$, *and any limit point* (η,ψ) *solves* (2.27).

PROOF. In order to prove the tightness of $\{\bar{Y}^{\varepsilon,\varphi^\varepsilon}\}_{\varepsilon\in(0,\varepsilon_0)}$, we will apply Theorem 2.5.2 of [44], according to which it suffices to verify that:

(a) $\{\sup_{0\le t\le T}\|\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t)\|_{-q}\}_{\varepsilon\in(0,\varepsilon_0)}$ is a tight family of $\mathbb{R}_+$-valued random variables;

(b) for every $\phi\in\Phi$, $\{\bar{Y}^{\varepsilon,\varphi^\varepsilon}[\phi]\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in $D([0,T]:\mathbb{R})$.

Note that (a) is immediate from Lemma 5.3(b). Consider now (b).

As in the proof for the finite-dimensional case (see the proof of Lemma 4.7), we write $\bar{Y}^{\varepsilon,\varphi^\varepsilon} = M^{\varepsilon,\varphi^\varepsilon} + A^{\varepsilon,\varphi^\varepsilon} + B^{\varepsilon,\varphi^\varepsilon} + \mathcal{E}_1^{\varepsilon,\varphi^\varepsilon} + C^{\varepsilon,\varphi^\varepsilon}$, where the processes on the right-hand side are as defined in (4.8). Fix $\phi\in\Phi$. Using Condition 2.11 parts (b) and (e), it follows as in the proof of Lemma 4.7 that

$$\begin{aligned}&\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}\big|M^{\varepsilon,\varphi^\varepsilon}(s)[\phi]\big|\Big]\to 0,\\ &\bar{\mathbb{E}}\Big[\sup_{0\le s\le T}\big|\mathcal{E}_1^{\varepsilon,\varphi^\varepsilon}(s)[\phi]\big|\Big]\to 0 \qquad \text{as } \varepsilon\to 0.\end{aligned}\tag{5.22}$$

Next, by Taylor's theorem and Condition 2.13(c),

$$\begin{aligned}&G(\bar{X}^{\varepsilon,\varphi^\varepsilon}(s),y)[\phi]-G(X^0(s),y)[\phi]\\ &\quad = a(\varepsilon)D_x\big(G(X^0(s),y)[\phi]\big)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)+R^{\varepsilon,\varphi^\varepsilon,\phi}(y,s),\end{aligned}$$

where

$$\big|R^{\varepsilon,\varphi^\varepsilon,\phi}(y,t)\big|\le L_{DG}(\phi,y)a^2(\varepsilon)\big\|\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t)\big\|_{-q}^2.\tag{5.23}$$

Hence

$$B^{\varepsilon,\varphi^\varepsilon}(t)[\phi]=\int_{\mathbb{X}\times[0,t]}D_x\big(G(X^0(s),y)[\phi]\big)\bar{Y}^{\varepsilon,\varphi}(s)\nu(dy)\,ds+\mathcal{E}_2^{\varepsilon,\varphi^\varepsilon,\phi}(t),\tag{5.24}$$

where, from (5.23), Lemma 5.3(b) and Condition 2.13(c),

$$\begin{aligned}&\bar{\mathbb{E}}\Big[\sup_{0\le t\le T}\big|\mathcal{E}_2^{\varepsilon,\varphi^\varepsilon,\phi}(t)\big|\Big]\\ &\quad\le Ta(\varepsilon)\big\|L_{DG}(\phi,\cdot)\big\|_1\bar{\mathbb{E}}\sup_{0\le t\le T}\big\|\bar{Y}^{\varepsilon,\varphi^\varepsilon,\phi}(t)\big\|_{-q}^2\to 0 \qquad \text{as } \varepsilon\to 0.\end{aligned}\tag{5.25}$$

Similarly, using Condition 2.13(a)

$$A^{\varepsilon,\varphi^\varepsilon}(t)[\phi]=\int_0^t D\big(b(X^0(s))[\phi]\big)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\,ds+\mathcal{E}_3^{\varepsilon,\varphi^\varepsilon,\phi}(t),$$

where

$$\bar{\mathbb{E}}\Big[\sup_{0\le t\le T}\big|\mathcal{E}_3^{\varepsilon,\varphi^\varepsilon}(t)\big|\Big]\to 0 \qquad \text{as } \varepsilon\to 0.\tag{5.26}$$

Combining (5.22)–(5.26), we have

$$
\begin{aligned}
\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t)[\phi] &= \mathcal{E}^{\varepsilon,\varphi^\varepsilon,\phi}(t) + \int_0^t D\big(b\big(X^0(s)\big)[\phi]\big)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t)\,ds \\
&\quad + \int_{\mathbb{X}\times[0,t]} D_x\big(G\big(X^0(s),y\big)[\phi]\big)\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\,d\nu_T \\
&\quad + \int_{\mathbb{X}\times[0,t]} G\big(X^0(s),y\big)[\phi]\psi^\varepsilon\,d\nu_T \\
&\equiv \mathcal{E}^{\varepsilon,\varphi^\varepsilon,\phi}(t) + A_1^{\varepsilon,\varphi^\varepsilon,\phi}(t) + B_1^{\varepsilon,\varphi^\varepsilon,\phi}(t) + C_1^{\varepsilon,\varphi^\varepsilon,\phi}(t),
\end{aligned}
\tag{5.27}
$$

where $\bar{\mathbb{E}}[\sup_{0\le t\le T}|\mathcal{E}^{\varepsilon,\varphi^\varepsilon,\phi}(t)|]\to 0$.

Next, from Condition 2.11(b) we have, applying Lemma 4.3 with $h = M_G$ as in the proof of (4.12), that for all $\gamma > 0$, $t\in[0,T-\gamma]$, $\varepsilon > 0$,

$$
\big|C_1^{\varepsilon,\varphi^\varepsilon}(t+\gamma)[\phi] - C_1^{\varepsilon,\varphi^\varepsilon}(t)[\phi]\big| \le (1+m_T)\|\phi\|_p\big(\rho(\beta)\gamma^{1/2} + \vartheta(\beta) + 2\theta(\varepsilon)\big),
$$

where ρ, ϑ and θ are as in Lemma 4.3 and m_T is as in (2.22). Tightness of $C_1^{\varepsilon,\varphi^\varepsilon}(\cdot)[\phi]$ in $C([0,T]:\mathbb{R})$ is now immediate.

For the tightness of $B_1^{\varepsilon,\varphi^\varepsilon}(t)[\phi]$ note that from Condition 2.13(c), for all $\gamma > 0$, $t\in[0,T-\gamma]$, $\varepsilon > 0$,

$$
\begin{aligned}
&\big|B_1^{\varepsilon,\varphi^\varepsilon}(t+\gamma)[\phi] - B_1^{\varepsilon,\varphi^\varepsilon}(t)[\phi]\big| \\
&\quad\le \int_{\mathbb{X}\times[t,t+\gamma]} \big\|D_x\big(G\big(X^0(s),y\big)[\phi]\big)\big\|_{\mathrm{op},-q_1}\big\|\bar{Y}^{\varepsilon,\varphi^\varepsilon}(s)\big\|_{-q}\,d\nu_T \\
&\quad\le \|\phi\|_{q_1} M^*_{DG}\gamma \sup_{0\le t\le T}\big\|\bar{Y}^{\varepsilon,\varphi^\varepsilon}(t)\big\|_{-q}.
\end{aligned}
$$

Tightness of $B_1^{\varepsilon,\varphi^\varepsilon}(t)[\phi]$ in $C([0,T]:\mathbb{R})$ now follows from Lemma 5.3(b). A similar estimate using (2.23) shows that $A_1^{\varepsilon,\varphi^\varepsilon}(t)[\phi]$ is tight in $C([0,T]:\mathbb{R})$ as well. Combining these tightness properties we have from (5.27) that $\{\bar{Y}^{\varepsilon,\varphi^\varepsilon}(\cdot)[\phi]\}_{\varepsilon>0}$ is tight in $D([0,T]:\mathbb{R})$ for all $\phi\in\Phi$, which proves part (b) of the tightness criterion stated at the beginning of the proof. Thus $\{\bar{Y}^{\varepsilon,\varphi^\varepsilon}(\cdot)\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in $D([0,T]:\Phi_{-q_1})$. Tightness of $\{\psi^\varepsilon 1_{\{|\psi^\varepsilon|\le\beta/a(\varepsilon)\}}\}_{\varepsilon\in(0,\varepsilon_0)}$ holds for the same reason as in the proof of Lemma 4.9, that is, because they take values in a compact set.

Suppose now that $(\bar{Y}^{\varepsilon,\varphi^\varepsilon}(\cdot), \psi^\varepsilon 1_{\{|\psi^\varepsilon|<\beta/a(\varepsilon)\}})$ converges along a subsequence in distribution to (η,ψ). To prove the result it suffices to show that for all $\phi\in\Phi$, (2.28) is satisfied. From Lemma 4.8, Condition 2.11(b) and the tightness of $C_1^{\varepsilon,\varphi^\varepsilon}(t)[\phi]$ shown above, it follows that

$$
\left(\bar{Y}^{\varepsilon,\varphi^\varepsilon}, \int_{\mathbb{X}\times[0,\cdot]} G\big(X^0(s),y\big)[\phi]\psi^\varepsilon\,d\nu_T\right) \to \left(\eta, \int_{\mathbb{X}\times[0,\cdot]} G\big(X^0(s),y\big)[\phi]\psi\,d\nu_T\right)
$$

in $D([0,T]:\Phi_{-q_1}\times\mathbb{R})$. The result now follows by using this convergence in (5.27). $\square$

6. Example. The following equation was introduced in [44] to model the spread of Poissonian point source chemical agents in the d-dimensional bounded domain $[0, l]^d$. The time instants, sites and the magnitudes of chemical injections into the domain are modeled using a Poisson random measure on $\mathbb{X} \times [0, T]$, where $\mathbb{X} = [0, l]^d \times \mathbb{R}_+$, with a finite intensity measure. Formally, the model can be written as follows. Denote by $\tau_i^\varepsilon(\omega)$, $i \in \mathbb{N}$, the jump times of the Poisson process with rate $\varepsilon^{-1}\nu(\mathbb{X})$, where ν is a finite measure on $\mathbb{X}$, and let (κ_i, A_i) be an i.i.d. sequence of $\mathbb{X}$-valued random variables with common distribution $\nu_0(dy) = \nu(dy)/\nu(\mathbb{X})$. Let $\zeta > 0$ be a small fixed parameter, and let $c_\zeta = \int_{\mathbb{R}^d} 1_{B_\zeta(0)}(x)\,dx$, where for $y \in \mathbb{R}^d$, $B_\zeta(y) = \{x \in \mathbb{R}^d : |y - x| \le \zeta\}$. Then the model can be described by the following equation:

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= D\Delta u(t,x) - V \cdot \nabla u(t,x) - \alpha u(t,x) \\ &\quad + \sum_{i=1}^{\infty} A_i(\omega) c_\zeta^{-1} 1_{B_\zeta(\kappa_i)}(x) 1_{\{t=\tau_i(\omega)\}},\end{aligned} \tag{6.1}$$

where for a smooth function f on $\mathbb{R}^d$, $\Delta f = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}$ and $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d})'$, $\alpha \in (0, \infty)$, $D > 0$, $V \in \mathbb{R}^d$ and $\varepsilon > 0$ is a scaling parameter. The last term on the right-hand side of (6.1) says that at the time instant τ_i, A_i amount of contaminant is introduced which is distributed uniformly over a ball of radius ζ in $\mathbb{R}^d$ centered at κ_i, where for simplicity we assume that κ_i a.s. takes values in the ζ-interior of $[0, l]^d$; see Condition 6.1.

The equation is considered with a Neumann boundary condition on the boundary of the box. A precise formulation of equation (6.1) is given in terms of an SPDE driven by a Poisson random measure of the form in (2.17). We now introduce a convenient CHNS to describe the solution space. Let $\rho_0(x) = e^{-2\sum_{i=1}^d c_i x_i}$, $x = (x_1, \dots, x_d)'$, where $c_i = \frac{V_i}{2D}$, $i = 1, \dots, d$. Let $\mathbb{H} = L^2([0, l]^d, \rho_0(x)\,dx)$. It can be checked that the operator $A = D\Delta - V \cdot \nabla$ with Neumann boundary condition on $[0, l]^d$ has eigenvalues and eigenfunctions

$$\{-\lambda_{\mathbf{j}}, \phi_{\mathbf{j}}\}_{\mathbf{j}=(j_1,\dots,j_d)\in\mathbb{N}_0^d},$$

where $\lambda_{\mathbf{j}} = \sum_{k=1}^d \lambda_{j_k}^{(k)}$, $\phi_{\mathbf{j}} = \prod_{k=1}^d \phi_{j_k}^{(k)}$, $\mathbf{j} = (j_1, \dots, j_d)$,

$$\begin{aligned}\phi_0^{(i)}(x) &= \sqrt{\frac{2c_i}{1 - e^{-2c_i l}}}, \qquad \phi_j^{(i)}(x) = \sqrt{\frac{2}{l}} e^{c_i x} \sin\left(\frac{j\pi}{l}x + \alpha_j^i\right), \\ \alpha_j^i &= \tan^{-1}\left(-\frac{j\pi}{l c_i}\right)\end{aligned}$$

and

$$\lambda_0^{(i)} = 0, \qquad \lambda_j^{(i)} = D\left(c_i^2 + \left(\frac{j\pi}{l}\right)^2\right).$$

Note that $\{\phi_{\mathbf{j}}\}$ forms a complete orthonormal set in $\mathbb{H}$. For $h \in \mathbb{H}$ and $n \in \mathbb{Z}$, define

$$\|h\|_n^2 = \sum_{\mathbf{j}\in\mathbb{N}_0^d} \langle h, \phi_{\mathbf{j}}\rangle_{\mathbb{H}}^2 (1+\lambda_{\mathbf{j}})^{2n},$$

and let

$$\Phi \doteq \{\phi \in \mathbb{H} : \|\phi\|_n < \infty, \forall n \in \mathbb{Z}\}.$$

Let Φ_n be the completion of $\mathbb{H}$ with respect to the norm $\|\cdot\|_n$. In particular $\Phi_0 = \mathbb{H}$. Then the sequence $\{\Phi_n\}$ has all the properties stated in Section 2.4 for $\Phi = \bigcap_{n\in\mathbb{Z}} \Phi_n$ to be a CHNS. Also, for each $n \in \mathbb{Z}$, $\{\|\phi_{\mathbf{j}}\|_n^{-1}\phi_{\mathbf{j}}\}$ is a complete orthonormal system in Φ_n.

We will make the following assumption on ν:

$$\text{(6.2)}\quad \text{For some } \delta > 0, \qquad \int_{\mathbb{X}} e^{\delta a^2}\nu(dy) < \infty, \qquad y = (x,a) \in [0,l]^d \times \mathbb{R}_+.$$

Here ν is a joint distribution on the possible locations and amounts of pollutants. We now describe the precise formulation of equation (6.1). In fact we will consider a somewhat more general equation that permits the magnitude of chemical injection to depend on the concentration profile and also allows for nonlinear dependence on the field. Consider the equation

$$X^\varepsilon(t) = x_0 + \int_0^t b(X^\varepsilon(s))\,ds + \varepsilon \int_{\mathbb{X}\times[0,t]} G(X^\varepsilon(s-), y) N^{\varepsilon^{-1}}(dy, ds), \qquad t \in [0,T],$$

where $N^{\varepsilon^{-1}}$ is as in Section 2.1. The function $b : \Phi' \to \Phi'$ is defined as follows: for $v \in \Phi'$ and $\phi \in \Phi$, $b(v)[\phi] \doteq b_1(v)[\phi] + b_0(v)[\phi]$, where $b_1(v)[\phi] \doteq v[A\phi] - \alpha v[\phi]$ and $b_0 : \Phi' \to \Phi'$ is defined by

$$b_0(v)[\phi] \doteq \sum_{i=1}^{\ell} K_i(v[\eta_1], \dots, v[\eta_m])\zeta_i[\phi], \qquad v \in \Phi', \phi \in \Phi,$$

where $K_i : \mathbb{R}^m \to \mathbb{R}$ and $\{\eta_j\}_{j=1}^m$, $\{\zeta_i\}_{i=1}^\ell$ are given elements in Φ. Also, $G : \Phi' \times \mathbb{X} \to \Phi'$ is as follows. For $v \in \Phi'$, $y = (x,a) \in \mathbb{X}$ and $\phi \in \Phi$,

$$G(v,y)[\phi] \doteq a G_1(v) c_\zeta^{-1} \int_{B_\zeta(x)\cap[0,l]^d} \phi(z)\rho_0(z)\,dz,$$

where $G_1 : \Phi' \to \mathbb{R}$ is given by

$$G_1(v) \doteq K_0(v[\eta_1], \dots, v[\eta_m]), \qquad v \in \Phi',$$

and $K_0 : \mathbb{R}^m \to \mathbb{R}$. Equation (6.1) corresponds to cases $b_0 = 0$ and $G_1 = 1$. We will make the following assumption on $\{K_i\}_{i=0}^p$:

CONDITION 6.1. (a) For some $L_K \in (0,\infty)$,

$$\max_{i=0,\dots,\ell} \left|K_i(x) - K_i(x')\right| \le L_K|x - x'| \qquad \text{for all } x, x' \in \mathbb{R}^m.$$

(b) For each $i = 0,\dots,\ell$, K_i is differentiable, and for some $L_{DK} \in (0,\infty)$

$$\max_{i=0,\dots,\ell} \left|\nabla K_i(x) - \nabla K_i(x')\right| \le L_{DK}|x - x'| \qquad \text{for all } x, x' \in \mathbb{R}^m.$$

(c) $\nu_0\{(x,a) : B_\zeta(x) \subset [0,l]^d\} = 1$.

Suppose that $x_0 \in \Phi_{-p}$. We next verify that the functions b and G satisfy Conditions 2.11 and 2.13. Choose $q = p + r$ and $q_1 = p + 2r$ where $r > 0$ is such that $\sum_{\mathbf{j}\in\mathbb{N}_0^d} \lambda_{\mathbf{j}}^2 (1 + \lambda_{\mathbf{j}})^{-2r} < \infty$. Then the embeddings $\Phi_{-p} \subset \Phi_{-q}$ and $\Phi_{-q} \subset \Phi_{-q_1}$ are Hilbert–Schmidt.

We first verify that b satisfies the required conditions. Clearly, b is a continuous function from Φ_{-p} to Φ_{-q}. Also, for $v \in \Phi_{-p}$,

$$\|b_1(v)\|_{-q}^2 = \sum_{\mathbf{j}\in\mathbb{N}_0^d} \left(v[A\phi_{\mathbf{j}}^q] - \alpha v[\phi_{\mathbf{j}}^q]\right)^2 = \sum_{\mathbf{j}\in\mathbb{N}_0^d} (\lambda_{\mathbf{j}} + \alpha)^2 \left(v[\phi_{\mathbf{j}}^q]\right)^2 \le c_\lambda \|v\|_{-p}^2,$$

where $c_\lambda = \sup_{\mathbf{j}\in\mathbb{N}_0^d}\{(\lambda_{\mathbf{j}} + \alpha)^2 (1 + \lambda_{\mathbf{j}})^{-2r}\}$, and the last inequality follows by noting that for $n \in \mathbb{Z}$, $\|\phi_{\mathbf{j}}\|_n = (1 + \lambda_{\mathbf{j}})^n$.

Also, using Condition 6.1(a) it is easily verified that for some $C_1 \in (0,\infty)$,

$$\|b_0(v)\|_{-p}^2 \le C_1\left(1 + \|v\|_{-p}\right)^2 \qquad \text{for all } v \in \Phi_{-p}. \tag{6.3}$$

Combining the above two estimates we see that b satisfies Condition 2.11(b). Next, using the observation that $\alpha \ge 0$ and $\lambda_{\mathbf{j}} \ge 0$ for all $\mathbf{j}$, we see that $2b_1(\phi)[\theta_p\phi] \le 0$ for all $\phi \in \Phi$. Also, using (6.3) it is immediate that

$$2b_0(\phi)[\theta_p\phi] \le C_1\|\phi\|_{-p}\left(1 + \|\phi\|_{-p}\right) \qquad \text{for all } \phi \in \Phi.$$

This shows that b satisfies Condition 2.11(c).

Once again using the nonnegativity of $-A$ and α, we see that

$$\langle u - u', b_1(u) - b_1(u')\rangle_{-q} \le 0 \qquad \text{for all } u, u' \in \Phi_{-p}.$$

Also, by the Lipschitz property of K_i [Condition 6.1(a)], we see that

$$\|b_0(u) - b_0(u')\|_{-q} \le C_2\|u - u'\|_{-q} \qquad \text{for all } u, u' \in \Phi_{-p}.$$

Combining the two inequalities shows that b satisfies Condition 2.11(d).

Next we verify that b satisfies Condition 2.13. Note that for $\phi \in \Phi$, the map $\Phi_{-q} \ni v \mapsto b_1(v)[\phi] \in \mathbb{R}$ is Fréchet differentiable and

$$D\left(b_1(v)[\phi]\right)[\eta] = \eta[A\phi] - \alpha\eta[\phi] \qquad \text{for all } \eta \in \Phi_{-q}.$$

Thus Condition 2.13(a) holds trivially for b_1. Also, from the differentiability of K_i, it follows that $b_0(v)[\phi]$ is Fréchet differentiable, and for $\eta \in \Phi_{-q}$,

$$D\big(b_0(v)[\phi]\big)[\eta] = \sum_{i=1}^{\ell}\sum_{j=1}^{m} \frac{\partial}{\partial x_j} K_i\big(v[\eta_1], \dots, v[\eta_m]\big)\eta[\eta_j]\zeta_i[\phi].$$

Using the Lipschitz property of ∇K_i [Condition 6.1(b)], it is now easy to check that b_0 and consequently, b satisfy Condition 2.13(a) as well.

Next, for $v \in \Phi_{-p}$ and $\eta \in \Phi_{-q}$,

$$\begin{aligned}\sum_{\mathbf{j}\in\mathbb{N}_0^d} \big|D\big(b_1(v)[\phi_{\mathbf{j}}^{q_1}]\big)[\eta]\big|^2 &= \sum_{\mathbf{j}\in\mathbb{N}_0^d} \big|\eta\big[(A-\alpha)\phi_{\mathbf{j}}^{q_1}\big]\big|^2 \\ &= \sum_{\mathbf{j}\in\mathbb{N}_0^d} (\alpha+\lambda_{\mathbf{j}})^2(1+\lambda_{\mathbf{j}})^{-2r}\big|\eta\big[\phi_{\mathbf{j}}^{q_1}\big]\big|^2 \le c_\lambda \|\eta\|_{-q}^2.\end{aligned} \tag{6.4}$$

Also using the linear growth of ∇K_i [which follows from Condition 6.1(b)], there is a $C_3 \in (0,\infty)$ such that

$$\begin{aligned}&\sup_{\{v\in\Phi_{-p}\,:\,\|v\|_{-p}\le m_T\}} \sum_{\mathbf{j}\in\mathbb{N}_0^d} \big|D\big(b_0(v)[\phi_{\mathbf{j}}^{q_1}]\big)[\eta]\big|^2 \\ &\qquad \le C_3\|\eta\|_{-q}^2 \sum_{i=1}^{\ell}\sum_{\mathbf{j}\in\mathbb{N}_0^d} \big|\zeta_i[\phi_{\mathbf{j}}^{q_1}]\big|^2.\end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5) we get that for $\eta \in \Phi_{-q}$ and $\phi \in \Phi_{q_1}$,

$$\begin{aligned}\sup_{\{v\in\Phi_{-p}\,:\,\|v\|_{-p}\le m_T\}} \big|A_v(\eta)[\phi]\big|^2 &\le \Big(\sum_{\mathbf{j}\in\mathbb{N}_0^d} \big|\langle\phi,\phi_{\mathbf{j}}^{q_1}\rangle_{q_1} A_v(\eta)\big[\phi_{\mathbf{j}}^{q_1}\big]\big|\Big)^2 \\ &\le 2(c_\lambda + C_4)\|\phi\|_{q_1}^2\|\eta\|_{-q}^2,\end{aligned}$$

where $C_4 = C_3\sum_{i=1}^{\ell}\sum_{\mathbf{j}\in\mathbb{N}_0^d}|\zeta_i[\phi_{\mathbf{j}}^{q_1}]|^2$. This shows that $\eta \mapsto A_v(\eta)$ is continuous and that b satisfies (2.23).

Using the nonnegativity of $-A$ and α, it follows that for all $\eta \in \Phi_{-q}$, $A_v^1(\eta) \doteq D(b_1(v)[\cdot])[\eta] \in \Phi_{-q_1}$ satisfies

$$\langle\eta, A_v^1(\eta)\rangle_{-q_1} \le 0.$$

Also, from the linear growth of DK_i, $A_v^0(\eta) \doteq D(b_0(v)[\cdot])[\eta] \in \Phi_{-q_1}$ satisfies, for some $C_5 \in (0,\infty)$,

$$\sup_{\{v\in\Phi_{-p}\,:\,\|v\|_{-p}\le m_T\}} \langle\eta, A_v^0(\eta)\rangle_{-q_1} \le C_5\|\eta\|_{-q_1}^2 \qquad \text{for all } \eta \in \Phi_{-q}.$$

Combining the last two estimates, for all $\eta_1, \eta_2 \in \Phi_{-q}$,

$$\begin{aligned}\sup_{\{v\in\Phi_{-p}:\,\|v\|_{-p}\le m_T\}} &\langle \eta_1-\eta_2, A_v(\eta_1)-A_v(\eta_2)\rangle_{-q_1} \\ &\le \sup_{\{v\in\Phi_{-p}:\,\|v\|_{-p}\le m_T\}} \langle \eta_1-\eta_2, A_v^0(\eta_1-\eta_2)\rangle_{-q_1} \\ &\le C_5\|\eta_1-\eta_2\|_{-q_1}^2.\end{aligned}$$

This verifies (2.24).

Next noting for all $u \in \Phi_{-p}$ and $\phi \in \Phi$ that $u[(A-\alpha)(\theta_q\phi)] \le C_6\|u\|_{-p}\|\phi\|_{-q}$ and $\phi[(A-\alpha)(\theta_q\phi)] \le 0$, we see that for all $u \in \Phi_{-p}$,

$$\sup_{\{v\in\Phi_{-p}:\,\|v\|_{-p}\le m_T\}} 2A_v^1(\phi+u)[\theta_q\phi] \le 2C_6\|u\|_{-p}\|\phi\|_{-q}.$$

Also, using the linear growth of ∇K_i, we see that for some $C_7 \in (0,\infty)$,

$$\sup_{\{v\in\Phi_{-p}:\,\|v\|_{-p}\le m_T\}} 2A_v^0(\phi+u)[\theta_q\phi] \le C_7\big(\|\phi\|_{-q}+\|u\|_{-p}\big)\|\phi\|_{-q} \qquad \text{for all } u\in\Phi_{-p}, \phi\in\Phi.$$

From the last two inequalities we have (2.25).

Conditions for G are verified in a similar fashion. In particular, note that for $\phi \in \Phi_0$ and $x \in \mathbb{R}^d$ such that $B_\zeta(x) \subset [0,l]^d$,

$$ac_\zeta^{-1}\int_{B_\zeta(x)} |\phi(z)|\rho_0(z)\,dz \le aC_8\|\phi\|_0,$$

where $C_8 = c_\zeta^{-1/2}\sup_{z\in[0,l]^d}\rho_0(z)$. From this it is immediate that for some $C_9 < \infty$, G satisfies Condition 2.11 with $M_G(y) = L_G(y) = aC_9$, $y = (x,a) \in \mathbb{X}$. Note that in view of (6.2) M_G, L_G satisfy part (d) of Condition 2.13. Remaining parts of this condition are verified similarly, and we omit the details.

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