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SPATIAL ASYMPTOTICS FOR THE PARABOLIC ANDERSON MODELS WITH GENERALIZED TIME–SPACE GAUSSIAN NOISE[1]

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Partially motivated by the recent papers of Conus, Joseph and Khoshnevisan [*Ann. Probab.* **41** (2013) 2225–2260] and Conus et al. [*Probab. Theory Related Fields* **156** (2013) 483–533], this work is concerned with the precise spatial asymptotic behavior for the parabolic Anderson equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + V(t,x)u(t,x), \\ u(0,x) = u_0(x), \end{cases}$$

where the homogeneous generalized Gaussian noise $V(t,x)$ is, among other forms, white or fractional white in time and space. Associated with the Cole–Hopf solution to the KPZ equation, in particular, the precise asymptotic form

$$\lim_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) = \frac{3}{4}\sqrt[3]{\frac{2t}{3}} \qquad \text{a.s.}$$

is obtained for the parabolic Anderson model $\partial_t u = \frac{1}{2}\partial_{xx}^2 u + \dot{W}u$ with the $(1+1)$-white noise $\dot{W}(t,x)$. In addition, some links between time and space asymptotics for the parabolic Anderson equation are also pursued.

1. Introduction. This work is devoted to the analysis of the spatial asymptotics for the parabolic Anderson model

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + \theta V(t,x)u(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $V(t,x)$ is a centered generalized homogeneous Gaussian field with the covariance function formally given as

$$\mathrm{Cov}\big(V(s,x), V(t,y)\big) = \gamma_0(s-t)\gamma(x-y), \qquad s,t\in\mathbb{R}^+, x,y\in\mathbb{R}^d, \tag{1.2}$$

and $\theta > 0$ is a constant playing a role as coefficient. Some remarkable progress in this direction has been made in recent papers by Conus, Joseph and Khoshnevisan [9] and Conus et al. [10] in the case when the time is white, that is, when

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$\gamma_0(\cdot) = \delta_0(\cdot)$ (Dirac function) and $\gamma(x)$ takes a variety of forms. Here we specifically mention the case when $d = 1$, $\gamma_0(u) = \delta_0(u)$ and $\gamma(x) = \delta_0(x)$ in which (1.1) is formally written as

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\partial_{xx}^2 u(t,x) + \theta \dot{W}(t,x)u(t,x), \\ u(0,x) = u_0(x) \end{cases} \tag{1.3}$$

with $V = \dot{W}$ being a space–time white noise, where $\{W(t,x); t \in \mathbb{R}^+, x \in \mathbb{R}\}$ is a time–space Brownian sheet. Under the bounded initial condition [given in (1.8) below], Conus, Joseph and Khoshnevisan prove (Theorem 1.3, [9]) in this case that

$$\begin{aligned} C^{-1} &\le \liminf_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) \\ &\le \limsup_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) \le C \qquad \text{a.s.} \end{aligned} \tag{1.4}$$

The importance of this result partially lies in the connection (see [15]) between (1.3) and the Kardar–Parisi–Zhang (KPZ) equation (see [19] and [20] for its background in the study of interface)

$$\frac{\partial h}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,x) + \frac{1}{2}\left(\frac{\partial h}{\partial x}(t,x)\right)^2 + \theta \dot{W}(t,x), \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.5}$$

through the Hopf–Cole transform

$$u(t,x) = \exp\{h(t,x)\}. \tag{1.6}$$

In particular, (1.4) leads to that $\max_{|x|\le R} h(t,x) \asymp (\log R)^{2/3}$ $(R \to \infty)$.

The objectives of this work are set up as follows:

First, we shall install the limits for the asymptotics given in (1.4) and in some other cases considered in [9] and [10]. Further, we shall identify or compute the values of these limits.

Second, we shall consider a wider class of Gaussian potentials where $V(t,x)$ can be white or colored in time. Our first theorem (Theorem 1.1) considers the case of a general $\gamma_0(\cdot)$ matching with a "nice" $\gamma(\cdot)$. In this paper, however, we are mainly interested in the cases listed in Table 1 where $V(t,x)$ is fractional (colored)/white in time and space. In the setting labeled (1) × (II), the Gaussian field $V(t,x)$ is formally given as

$$V(t,x) = c\frac{\partial^{d+1} W^H(t,x)}{\partial t\, \partial x_1 \cdots \partial x_d}, \qquad (t, x_1, \dots, x_d) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{1.7}$$

and is known as the fractional noise, where $W^H(t,x)$ is a $(d+1)$-parameter fractional Brownian sheet with the Hurst parameter $H = (H_0, H_1, \dots, H_d)$. The settings (1) × (III), (2) × (II) and (2) × (III) are also interpreted by (1.7) with

TABLE 1
Fractional/white potentials considered in this paper

Time/space	**(I)** $\gamma(x) = \lvert x\rvert^{-\alpha}$	**(II)** $\gamma(x) = \prod_{j=1}^{d} \lvert x_j\rvert^{2H_j-2}$	**(III)** $\gamma(x) = \delta_0(x)$
(1) $\gamma_0(\cdot) = \lvert\cdot\rvert^{-\alpha_0}$ $(\alpha_0 = 2 - 2H_0)$	$\alpha_0 \ge 0$, $0 < \alpha < d$, $2\alpha_0 + \alpha < 2$	$1/2 < H_0 \le 1$, $1/2 < H_j < 1$ $(1 \le j \le d)$, $2H_0 + \sum_{j=1}^{d} H_j > d + 1$	$d = 1$
(2) $\gamma_0(\cdot) = \delta_0(\cdot)$	$0 < \alpha < 2 \wedge d$	$1/2 < H_j < 1$ $(j = 1, \dots, d)$	$d = 1$

$H = (H_0, 1/2)$, $H = (1/2, H_1, \dots, H_d)$ and $H = (1/2, 1/2)$, respectively. The case of Riesz potential $\gamma(x) = |x|^{-\alpha}$ can be interpreted as a fractional noise with a radially symmetric fractional spatial component and has close ties to some classical laws in physics, such as Newton's gravity law and Coulomb's electrostatics law.

There are some major differences between regime (1) and regime (2) that lead to difference in treatment between these two regimes. In regime (1) the solutions $u(t, x)$ have a Feynman–Kac representation [see (1.9) below] and the solutions in regime (2) do not. On the other hand, the solutions in regime (1) do not possess the martingale structure that is related to the mild representation given in (1.33) below.

Similar to [9] and [10], we assume in (1.1) that $u_0(\cdot)$ is deterministic with

$$0 < \inf_{x\in\mathbb{R}^d} u_0(x) \le \sup_{x\in\mathbb{R}^d} u_0(x) < \infty. \tag{1.8}$$

The major development of this paper involves two independent random systems: one is a d-dimensional Brownian motion $B(t)$ and the other is a centered generalized homogeneous Gaussian field $V(t, x)$. Throughout the paper, by $\mathbb{E}_x$ and $\mathbb{P}_x$, we mean that, respectively, the expectation and probability law with respect to the Brownian motion with $B(0) = x$. $\mathbb{E}$ and $\mathbb{P}$ are introduced for the expectation and probability law with respect to the Gaussian field.

1.1. *Results under Feynman–Kac representation.* Solving equation (1.1) may mean different things under different definitions of stochastic integrals. The cases considered in this subsection yield the Feynman–Kac representation

$$u(t, x) = \mathbb{E}_x\left[\exp\left\{\theta \int_0^t V\big(t - s, B(s)\big)\, ds\right\} u_0\big(B(t)\big)\right], \qquad x \in \mathbb{R}^d \tag{1.9}$$

for the solution of equation (1.1). When $V(t, x)$ has sufficiently nice trajectories, (1.9) is a well-known fact. This is not a trivial matter in our context, as the Gaussian field $V(t, x)$ is not even (necessarily) point-wise defined. Mathematically speaking, a generalized centered time–space Gaussian field V can be defined as a random linear operator on a Schwartz space $\mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^d)$ of rapidly decreasing (at ∞)

and infinitely smooth functions $\varphi(t,x)$ on $\mathbb{R}^+ \times \mathbb{R}^d$ with $\lim_{t\to 0^+} \partial_t^{(n)}\varphi(t,x) = 0$ for all $n = 0, 1, \dots$ such that for each $\varphi \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^d)$, $\langle V, \varphi\rangle$ is a centered normal random variable. In the settings considered in this paper, there are (probably generalized) functions $\gamma_0(\cdot)$ on $\mathbb{R}$ and $\gamma(\cdot)$ on $\mathbb{R}^d$ such that for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^d)$

$$\begin{aligned} &\mathrm{Cov}\big(\langle V,\varphi\rangle, \langle V,\psi\rangle\big) \\ &\quad = \int_{(\mathbb{R}^+\times\mathbb{R}^d)^2} \gamma_0(s-t)\gamma(x-y)\varphi(s,x)\psi(t,y)\,ds\,dt\,dx\,dy. \end{aligned} \tag{1.10}$$

This relation is formally written as in the form given in (1.2). Given a probability density $h \in \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^d)$, write $h_\varepsilon(s,x) = \varepsilon^{-(d+1)}h(\varepsilon^{-1}s, \varepsilon^{-1}x)$. Notice that $V_\varepsilon(t,x) \equiv \langle V, h_\varepsilon(t-\cdot, x-\cdot)\rangle$ is a point-wise defined Gaussian field on $\mathbb{R}^+ \times \mathbb{R}^d$ and has a sufficient regularity if $h(t,x)$ is smooth enough. The time integral in (1.9) is defined as the $\mathcal{L}^2$-limit

$$\begin{aligned} &\int_0^t V\big(t-s, B(s)\big)\,ds \\ &\quad \overset{\text{def}}{=} \lim_{\varepsilon\to 0^+} \int_0^t V_\varepsilon\big(t-s, B(s)\big)\,ds - \mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}\otimes\mathbb{P}_x), \end{aligned} \tag{1.11}$$

provided that the right-hand side converges in $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}\otimes\mathbb{P}_x)$.

Conditioning on the Brownian motion, this integral is a centered Gaussian process (in t) with the conditional variance

$$\int_0^t\int_0^t \gamma_0(r-s)\gamma\big(B(r)-B(s)\big)\,dr\,ds \qquad (t \ge 0). \tag{1.12}$$

The exponential integrability required by the construction of Feynman–Kac representation in (1.9) can be established by the conditional Gaussian property, given the exponential integrability of the Hamiltonian in (1.12). An interested reader is referred to [17] for details.

Under the usual conditions (satisfied by the theorems in this subsection), the Feynman–Kac representation given in (1.9) is a weak solution (Theorem 4.3, [17]) to (1.1) in the sense that

$$\begin{aligned} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\,dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\,dx\,ds \\ + \int_0^t\int_{\mathbb{R}^d} u(s,x)V(s,x)\varphi(x)\,dx\,ds \end{aligned}$$

for any C^∞ and compactly supported function $\varphi(x)$ on $\mathbb{R}^d$, where the last term is a Stratonovich stochastic integral (Definition 4.1, [17]).

One such case is when $\gamma_0(\cdot)$ satisfies some local integrability and $\gamma(\cdot)$ satisfies

$$\int_{\mathbb{R}^d} \big(1+|\lambda|^\delta\big)\hat{\gamma}(\lambda)\,d\lambda < \infty \qquad \text{for some } \delta > 0, \tag{1.13}$$

where $\hat{\gamma}$ represents the Fourier transform (which is non-negative, and exists possibly in the distributional sense).

$$\hat{\gamma}(\lambda) = \int_{\mathbb{R}^d} \gamma(x) e^{i\lambda\cdot x}\, dx, \qquad \lambda \in \mathbb{R}^d. \tag{1.14}$$

By Fourier inversion

$$\gamma(x) - \gamma(y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \big(e^{-i\lambda\cdot x} - e^{-i\lambda\cdot y}\big) \hat{\gamma}(\lambda)\, d\lambda.$$

Therefore, under (1.13) $\gamma(\cdot)$ is Hölder continuous with the exponent δ given in (1.13).

THEOREM 1.1. *Assume that $\gamma(\cdot)$ satisfies (1.13) and $\gamma_0(\cdot) \geq 0$ satisfies*

$$\int_0^t \int_0^t \gamma_0(r-s)\, dr\, ds < \infty \qquad (t > 0). \tag{1.15}$$

Then for any $t > 0$ the weak solution in (1.9) obeys the asymptotic law

$$\begin{aligned} &\lim_{R\to\infty} (\log R)^{-1/2} \log \max_{|x|\leq R} u(t,x) \\ &\quad = \theta\left(2d\gamma(0) \int_0^t \int_0^t \gamma_0(r-s)\, dr\, ds\right)^{1/2} \qquad a.s. \end{aligned} \tag{1.16}$$

Theorem 1.1 here is comparable to Theorem 2.5 in [10] under a different assumption that appears to be not so comparable to (1.13).

The other cases are those labeled as (1) in Table 1 where the Gaussian potential is fractional in time. The legitimacy of the Feynman–Kac representation (1.9) is secured for (1) × (II) by Theorem 4.3, [17], for (1) × (III) by Theorem 6.2, [17] and for (1) × (I) by an obvious modification of the approach used in [17].

Let $W^{1,2}(\mathbb{R}^d)$ be the Sobolev space of all functions g on $\mathbb{R}^d$ such that $g, \nabla g \in \mathcal{L}^2(\mathbb{R}^d)$. Denote

$$\begin{aligned} \mathcal{A}_d = \bigg\{ &g(s,x);\, g(s,\cdot) \in W^{1,2}(\mathbb{R}^d),\ \int_{\mathbb{R}^d} g^2(s,x)\, dx = 1 \\ &\forall 0 \leq s \leq 1 \text{ and } \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2\, dx\, ds < \infty \bigg\}, \\ \mathcal{E}(\alpha_0, d, \gamma) = \sup_{g\in\mathcal{A}_d} \bigg\{ &\int_0^1 \int_0^1 \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\gamma(x-y)}{|r-s|^{\alpha_0}} g^2(s,x) g^2(r,y)\, dx\, dy\, dr\, ds \\ &- \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2\, dx\, ds \bigg\}. \end{aligned} \tag{1.17}$$

By Lemma 7.2, [6], $\mathcal{E}(\alpha_0, d, \gamma)$ is finite under the assumptions in any of the cases listed in Table 1 with the label (1).

Consistently with the parameter α in setting (I), we set

$$\alpha = 2d - 2\sum_{j=1}^{d} H_j \tag{1.18}$$

in the setting labeled (II). A common property shared by (I) and (II) is the spatial scaling

$$\gamma(cx) = c^{-\alpha}\gamma(x), \qquad x \in \mathbb{R}^d, c > 0, \tag{1.19}$$

which plays a major role in the formulation of the following theorem.

THEOREM 1.2. *In settings* (1) × (I) *and* (1) × (II) *listed in Table* 1, *we have that for any* $t > 0$, *the weak solution in* (1.9) *satisfies*

$$\begin{aligned} &\lim_{R\to\infty} (\log R)^{-2/(4-\alpha)} \log \max_{|x|\le R} u(t,x) \\ &\quad = \frac{4-\alpha}{4}\left(\frac{4\mathcal{E}(\alpha_0, d, \gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)} \\ &\qquad \times \theta^{4/(4-\alpha)} d^{2/(4-\alpha)} t^{(4-\alpha-2\alpha_0)/(4-\alpha)} \qquad a.s. \end{aligned} \tag{1.20}$$

Relation (1.19) remains valid for the setting of $\gamma(\cdot) = \delta_0(\cdot)$ and $d = 1$ with $\alpha = 1$. Consistently with (1.17), set

$$\begin{aligned} \mathcal{E}(\alpha_0, 1, \delta_0) = \sup_{g\in\mathcal{A}_1} \biggl\{ &\int_0^1\int_0^1\int_{-\infty}^{\infty} \frac{g^2(s,x)g^2(r,x)}{|r-s|^{\alpha_0}}\,dx\,dr\,ds \\ &- \frac{1}{2}\int_0^1\int_{-\infty}^{\infty} \bigl|\nabla_x g(s,x)\bigr|^2\,dx\,ds \biggr\}. \end{aligned}$$

THEOREM 1.3. *In setting* (1) × (III), *listed in Table* 1, *we have that for any* $t > 0$, *the weak solution in* (1.9) *satisfies*

$$\begin{aligned} &\lim_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) \\ &\quad = \frac{3}{4}\theta^{4/3} t^{(3-2\alpha_0)/3} \sqrt[3]{4\mathcal{E}(\alpha_0, 1, \delta_0)} \qquad a.s. \end{aligned} \tag{1.21}$$

For any $t > 0$, let $\mathcal{S}([0,t]\times\mathbb{R}^d)$ be the sub-class of $\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d)$ consisting of φ supported on $[0,t]$ such that $\lim_{s\to t^-} \partial_s^{(n)}\varphi(s,x) = 0$ for $n = 0, 1, \ldots$. By comparing the covariance functions one can see that

$$\bigl\{\langle V, \varphi(t-\cdot,\cdot)\rangle; \varphi \in \mathcal{S}\bigl([0,t]\times\mathbb{R}^d\bigr)\bigr\} \stackrel{d}{=} \bigl\{\langle V, \varphi\rangle; \varphi \in \mathcal{S}\bigl([0,t]\times\mathbb{R}^d\bigr)\bigr\}.$$

Therefore,

$$
\begin{aligned}
&\left\{\mathbb{E}_x \exp\left\{\theta\int_0^t V\big(s, B(s)\big)\,ds\right\}; x\in\mathbb{R}^d\right\} \\
&\quad\stackrel{d}{=}\left\{\mathbb{E}_x \exp\left\{\theta\int_0^t V\big(t-s, B(s)\big)\,ds\right\}; x\in\mathbb{R}^d\right\}.
\end{aligned}
\tag{1.22}
$$

Together, (1.9), Theorems 1.1 and 1.2 (with $u_0 \equiv 1$) lead to the following spatial asymptotics for the models of directed polymers.

COROLLARY 1.4. *Under the assumption of Theorem* 1.1,

$$
\begin{aligned}
&\lim_{R\to\infty}(\log R)^{-1/2}\log\max_{|x|\le R}\mathbb{E}_x\exp\left\{\theta\int_0^t V\big(s,B(s)\big)\,ds\right\} \\
&\quad = \theta\left(2d\gamma(0)\int_0^t\int_0^t \gamma_0(r-s)\,dr\,ds\right)^{1/2} \qquad a.s.
\end{aligned}
\tag{1.23}
$$

Under the assumption of Theorem 1.2,

$$
\begin{aligned}
&\lim_{R\to\infty}(\log R)^{-2/(4-\alpha)}\log\max_{|x|\le R}\mathbb{E}_x\exp\left\{\theta\int_0^t V\big(s,B(s)\big)\,ds\right\} \\
&\quad = \frac{4-\alpha}{4}\left(\frac{4\mathcal{E}(\alpha_0,d,\gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)} \\
&\qquad \times \theta^{4/(4-\alpha)}d^{2/(4-\alpha)}t^{(4-\alpha-2\alpha_0)/(4-\alpha)} \qquad a.s.
\end{aligned}
\tag{1.24}
$$

Under the assumption of Theorem 1.3,

$$
\begin{aligned}
&\lim_{R\to\infty}(\log R)^{-2/3}\log\max_{|x|\le R}\mathbb{E}_x\exp\left\{\theta\int_0^t V\big(s,B(s)\big)\,ds\right\} \\
&\quad = \frac{3}{4}\theta^{4/3}t^{(3-2\alpha_0)/3}\sqrt[3]{4\mathcal{E}(\alpha_0,1,\delta_0)} \qquad a.s.
\end{aligned}
\tag{1.25}
$$

We now consider the special case when $\alpha_0 = 0$ (equivalently, $H_0 = 1$) in Theorems 1.2 and 1.3. The Gaussian potential V is time-independent. Corresponding to (1) × (II), for example,

$$
V(x) = c\frac{\partial^d W^H}{\partial x_1\cdots\partial x_d}(x_1,\dots,x_d), \qquad x=(x_1,\dots,x_d)\in\mathbb{R}^d,
$$

where $W^H(x_1,\dots,x_d)$ is a spatial fractional Brownian sheet with the Hurst index $H=(H_1,\dots,H_d)$ satisfying $1/2 < H_1,\dots,H_d < 1$. As for (1) × (III) with $\alpha_0 = 0$ in Table 1, $V(x) = \dot{W}(x)$, a spatial white noise on $\mathbb{R}$.

Write

$$
\mathcal{F}_d = \left\{g\in W^{1,2}(\mathbb{R}^d); \int_{\mathbb{R}^d} g^2(x)\,dx = 1\right\}.
$$

By Lemma A.5 in the Appendix, when $\alpha_0 = 0$, $\mathcal{E}(0, d, \gamma)$ becomes

$$\mathcal{E}(d,\gamma) \equiv \sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)\,dx\,dy - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\right\}. \tag{1.26}$$

In the special case when $d = 1$ and $\gamma(\cdot) = \delta_0(\cdot)$,

$$\mathcal{E}(1,\delta_0) = \sup_{g\in\mathcal{F}_1}\left\{\int_{-\infty}^{\infty} g^4(x)\,dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2\,dx\right\} = \frac{1}{6}. \tag{1.27}$$

Indeed, the original version (page 291, [11]) of the above identity is

$$\sup_{g\in\mathcal{F}_1}\left\{2\int_{-\infty}^{\infty} g^4(x)\,dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2\,dx\right\} = \frac{2}{3}.$$

Replacing $g(x)$ by $\sqrt{2}g(2x)$ on the left-hand side, we have that

$$\begin{aligned}\sup_{g\in\mathcal{F}_1}&\left\{2\int_{-\infty}^{\infty} g^4(x)\,dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2\,dx\right\}\\ &= 4\sup_{g\in\mathcal{F}_1}\left\{\int_{-\infty}^{\infty} g^4(x)\,dx - \frac{1}{2}\int_{-\infty}^{\infty}|g'(x)|^2\,dx\right\}.\end{aligned}$$

So we have (1.27).

COROLLARY 1.5. *When $\gamma(\cdot)$ satisfies the assumptions given in Theorem 1.1,*

$$\lim_{R\to\infty}(\log R)^{-1/2}\log\max_{|x|\le R}\mathbb{E}_x\exp\left\{\theta\int_0^t V(B(s))\,ds\right\} = t\theta(2d\gamma(0))^{1/2} \quad a.s. \tag{1.28}$$

When $\gamma(\cdot)$ is given in (I) *or* (II) *with $0 < \alpha < 2\wedge d$,*

$$\lim_{R\to\infty}(\log R)^{-2/(4-\alpha)}\log\max_{|x|\le R}\mathbb{E}_x\exp\left\{\theta\int_0^t V(B(s))\,ds\right\} = \frac{4-\alpha}{4}t\left(\frac{4\mathcal{E}(d,\gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)}\theta^{4/(4-\alpha)}d^{2/(4-\alpha)} \quad a.s. \tag{1.29}$$

When $d = 1$ and $\gamma(x) = \delta_0(x)$,

$$\lim_{R\to\infty}(\log R)^{-2/3}\log\max_{|x|\le R}\mathbb{E}_x\exp\left\{\theta\int_0^t \dot{W}(B(s))\,ds\right\} = \frac{3t}{4}\theta^{4/3}\sqrt[3]{\frac{2}{3}} \quad a.s., \tag{1.30}$$

where $\dot{W}(x)$ $(-\infty < x < \infty)$ is an 1-dimensional spatial white noise.

1.2. *Results for mild solutions*. We now consider the cases labeled by (2) in Table 1, in which the Gaussian noise $V(t,x)$ is white in time. The Feynman–Kac representation (1.9) is no longer available as $\gamma(0)=\infty$. Indeed, one can easily see that the Hamiltonian in (1.12) [given as the conditional variance of the time integral (1.11) that would be conditionally Gaussian if defined] diverges in this case. In spite of this, the parabolic Anderson equation (1.1) can be solvable in a slightly different sense which is briefly described below; we refer to [12, 18] and [23] for details.

The spatial covariance functions considered here have the representation

$$\gamma(x)=\int_{\mathbb{R}^d} K(x-y)K(y)\,dy, \qquad x\in\mathbb{R}^d,$$

where $K(x)$ is symmetric and nonnegative; see (2.11) below. Assume that the spatial covariance function $\gamma(\cdot)$ satisfies $\gamma(\cdot)\ge 0$ and the Dalang condition

$$\int_{\mathbb{R}^d} \frac{\hat{\gamma}(\lambda)}{1+|\lambda|^2}\,d\lambda<\infty, \tag{1.31}$$

where $\hat{\gamma}(\cdot)$ is the Fourier transform of $\gamma(\cdot)$; see (1.14). Notice that $\hat{\gamma}(\cdot)\ge 0$ as $\gamma(\cdot)$ is nonnegative definite.

Let $W(t,x)$ be a $(d+1)$-parameter Brownian sheet, and consider the Gaussian field

$$M_t(\varphi)=\int_0^t\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\varphi(y-x)K(y)\,dy\right]W(ds\,dx), \qquad \varphi\in\mathcal{S}(\mathbb{R}^d), \tag{1.32}$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of the infinitely smooth and rapidly decaying functions on $\mathbb{R}^d$.

By the theory of Walsh (Chapter 2, [23]) and Dalang [12], this field can be extended into a martingale measure $M(t,A)=M_t(1_A)$ such that up to the distributional equivalence

$$\langle V,\varphi\rangle=\int_{\mathbb{R}^+\times\mathbb{R}^d}\varphi(s,x)M(ds\,dx), \qquad \varphi\in\mathcal{S}(\mathbb{R}^+\times\mathbb{R}^d).$$

By the Dalang–Walsh theory, (1.31) ensures the existence and uniqueness (with a.s. equivalence) of the solution to the parabolic Anderson equation in the sense that

$$u(t,x)=(p_t*u_0)+\theta\int_0^t\int_{\mathbb{R}^d} p_{t-s}(y-x)u(s,y)M(ds,dy), \tag{1.33}$$

where p_t is the density function of the d-dimensional Brownian motion $B(t)$, and the stochastic integral on the right-hand side is taken in the sense of Itô–Skorokhod. We point out that $u(t,x)\ge 0$ in regime (2), labeled as $u(t,x)\ge 0$ in Table 1. Indeed, our claim follows from the following facts: (1) the uniqueness of solution, which implies that $u(t,x)\equiv 0$ if $u_0(0)\equiv 0$; (2) the monotonicity in initial condition. In comparison to the zero solution, we conclude the solution

$u(t,x) \ge 0$ if $u_0(x) > 0$. The monotonicity in initial condition was established in [21] and [22] in the setting of (2) × (III). See (1.41) below for its generalization to whole regime (2).

An alternative but equivalent view is to interpret the product in (1.1) between $V(t,x)$ and $u(t,x)$ as the Wick product; see [18] for an over view of the Wick product. When $\gamma(\cdot)$ is bounded and continuous, the solution has a "renormalized" Feynman–Kac representation,

$$u(t,x) = e^{-(\theta^2 t/2)\gamma(0)}\mathbb{E}_x \exp\left[\exp\left\{\theta \int_0^t V\big(t-s, B(s)\big)\,ds\right\} u_0\big(B(t)\big)\right]. \tag{1.34}$$

We refer to the argument used in the proof of Theorem 7.2, [17] for a proof of (1.34). This representation is no longer valid whenever $\gamma(0) = \infty$. In the cases labeled (2) in Table 1, however, the solution $u(t,x)$ can be obtained as the L^2-limit $\lim_{\varepsilon\to 0^+} u_\varepsilon(t,x)$ of $u_\varepsilon(t,x)$, represented in (1.34), that appears as the solution of (1.1), with $V(t,x)$ being replaced by the Gaussian potential $V_\varepsilon(t,x)$ of the modified spatial covariance; see, for example, [16] for details.

By comparing (1.9) and (1.34), we observe some obvious differences between solutions in the Stratonovich sense (1.9) and in the Skorokhod sense (1.34). On the other hand, the solutions given in (1.9) and (1.34) follow the same limiting behavior as that stated in Theorem 1.1 for the case $\gamma_0(\cdot) = \delta_0(\cdot)$ in which (1.16) becomes

$$\lim_{R\to\infty} (\log R)^{-1/2} \log \max_{|x|\le R} u(t,x) = \theta\big(2dt\gamma(0)\big)^{1/2} \qquad \text{a.s.,} \tag{1.35}$$

which is the precise form of the limit law stated in Theorem 2.5, [10].

THEOREM 1.6. *In settings* (2) × (I) *and* (2) × (II) *listed in Table* 1,

$$\begin{aligned}
&\lim_{R\to\infty} (\log R)^{-2/(4-\alpha)} \log \max_{|x|\le R} u(t,x) \\
&\quad = \frac{4-\alpha}{4}\left(\frac{4t\mathcal{E}(d,\gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)} \theta^{4/(4-\alpha)} d^{2/(4-\alpha)} \qquad \textit{a.s.,}
\end{aligned} \tag{1.36}$$

where $\mathcal{E}(d,\gamma)$ *is the variation given in* (1.26).

THEOREM 1.7. *When* $d = 1$, $\gamma_0(\cdot) = \delta_0(\cdot)$ *and* $\gamma(\cdot) = \delta_0(\cdot)$ [*i.e.*, (2) × (III) *in Table* 1],

$$\lim_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) = \frac{3}{4}\theta^{4/3}\sqrt[3]{\frac{2t}{3}} \qquad \textit{a.s.} \tag{1.37}$$

In the context of Theorem 1.7, the parabolic Anderson equation (1.1) becomes (1.3), which connects the KPZ equation given in (1.5) through the Hopf–Cole transform (1.6) in some proper sense.

COROLLARY 1.8. *Under the deterministic initial condition*

$$-\infty < \inf_{x\in\mathbb{R}} h_0(x) \le \sup_{x\in\mathbb{R}} h_0(x) < \infty,$$

the Hopf–Cole solution $h(t,x)$ to the KPZ equation in (1.5) *satisfies*

$$\lim_{R\to\infty} (\log R)^{-2/3} \max_{|x|\le R} h(t,x) = \frac{3}{4}\theta^{4/3}\sqrt[3]{\frac{2t}{3}} \qquad a.s. \tag{1.38}$$

1.3. *Discussion and comment.* As expected, the spatial asymptotics given in the main theorems are mainly determined by the spatial covariance function $\gamma(\cdot)$, and more specifically, by the scaling rate α of $\gamma(\cdot)$ [see (1.19)] when it comes to the settings in Table 1. On the other hand, cases (1) and (2) (labeled in Table 1) require different approaches, as we shall see in Sections 2 and 3. The tail probability asymptotics (Theorems 5.1–5.5) that support the theorems listed above have their independent values, so we treat them as a part of major theorems of this paper and list them in Section 5.

The case of time-independence and the case of white time are two extremes: the least singular and the most singular, respectively. As we have seen, the former is associated to $\alpha_0 = 0$. Because the Fourier transform of $\gamma_0 = |u|^{-\alpha_0}$ is $\hat{\gamma}_0(\lambda) = c(\alpha_0)|\lambda|^{-(1-\alpha_0)}$ ($\lambda \in \mathbb{R}^d$) and $\hat{\delta}_0(\lambda) = 1$, the function $\gamma_0(\cdot) = \delta_0(\cdot)$ is naturally classified as the extension of $\gamma_0(\cdot) = |\cdot|^{-\alpha_0}$ ($0 \le \alpha_0 < 1$) to $\alpha_0 = 1$. A big surprise is that these two extreme settings share the same variation $\mathcal{E}(d,\gamma)$ while the cases with $0 < \alpha_0 < 1$ are formulated by the different variation $\mathcal{E}(\alpha_0, d, \gamma)$. In view of the moment representation (3.10) below, and knowing that the difference between two independent Brownian motions is a Brownian motion, one would bet on

$$\sup_{g\in\mathcal{F}_d}\left\{\int_{\mathbb{R}^d}\gamma(x)g^2(x)\,dx - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\right\}$$

rather than $\mathcal{E}(d,\gamma)$, as the variation relevant to Theorem 1.6 and Theorem 1.7. See Remark 3.8 for the discussion.

A central piece of the approach that allows us to compute the limit values is the precise high moment asymptotics

$$\log \mathbb{E}u(t,0)^m \qquad (m\to\infty)$$

given in Section 3. For the cases labeled (1) in Table 1, our treatment starts at the moment representation (3.1). The problem can be essentially reduced to the long-term asymptotics for the annealed moment, to which some results and ideas developed in the recent work [6] apply. Perhaps the hardest part of this paper is the computation of the high moment, when $V(t,x)$ is white in time [i.e., (2) in Table 1]. Unlike the cases labeled (1) in Table 1, the high moment asymptotics in (2) do not agree with the long-term asymptotics such as $\log \mathbb{E}u(t,0)^2$ ($t\to\infty$) at the constant level; see Remark 3.8 below for details. Under a proper time scaling, the problem becomes a combination of high moment and large time with the ratio

$t_m \asymp m^{2/(2-\alpha)}$. The package of methodology includes the Feynman–Kac type large deviations for time-dependent additive functionals, newly developed in [6], some ideas along the line of probability in Banach spaces and smooth approximation at the exponential scale.

Another novelty of the paper comes from the proof of the lower bounds requested by the major theorems listed above, even with the large deviation estimates given in Theorems 5.1–5.5 below. The relevant idea is clear and simple in principle: when the space points $x_1, \dots, x_n$ are sufficiently spread out, the random variables $u(t, x_1), \dots, u(t, x_n)$ are close to being independent. In practice, carrying this idea out is not easy at all, as indicated by the delicate steps taken in [9] and [10]. We adopt this idea and the estimate of localization developed in [9] and [10], and use them in our proof (given in Section 2.2) in the setting of Theorems 1.6 and 1.7. This treatment does not apply to Theorems 1.1, 1.2 and 1.3 due to its heavy dependence on the martingale structure associated to equation (1.33) that defines the mild solution. The proof (given in Section 2.1) of the lower bounds for Theorems 1.1, 1.2 and 1.3 relies on Gaussian property in a substantial way and appears to be new in methodology.

In comparing their estimates of the high moment to the literature on intermittency, Conus et al. (Remark 9.2, [10]) raise the issue of the link between the time and spatial asymptotics. In this paper, we pursue this link on two fronts: the first is the connection between the long term asymptotics for the annealed moments of $u(t, 0)$ and the high moment asymptotics for $u(t, 0)$. Indeed, the main development of our argument in Section 3 is to utilize the link between annealed intermittency and high moment asymptotics. We observe a "perfect match" when it comes to the Feynman–Kac solution given by (1.9) and a small but interesting gap when $V(t, x)$ is white in time. We refer the reader to Remark 3.8 below. Our second concern is the connection between the quenched spatial asymptotics and the quenched time asymptotics. In Section 6 we demonstrate our finding in the setting of time-independence.

We now comment on the relation between the current paper and [9] and [10]. Whenever possible, we adopt the results and ideas in [9] and [10] to our setting. The list includes the localization (Section 2.2) and estimate for modulus continuity (Lemma 4.1) in the case when $V(t, x)$ is white in time. Estimation by the martingale bound, a substantial idea in [9] and [10], does not apply to the setting labeled (1) in Table 1. The use of the moment representations (3.1) and (3.10), together with some newly developed ideas in large deviations for time–space Hamiltonians, allow us to obtain a form of high moment asymptotics sharper than those achieved in [9] and [10]. On the other hand, the dependence on the moment representations (3.1) and (3.10) makes our method unsuitable to the nonlinear stochastic heat equations labeled (SHE) in [9] and [10].

In view of the assumption (1.8) on the initial condition and the Feynman–Kac representation (1.9), we have

$$\underline{u}_0 \mathbb{E}_x \exp\left\{\int_0^t V\big(t-s, B(s)\big)\, ds\right\} \le u(t, x) \le \overline{u}_0 \mathbb{E}_x \exp\left\{\int_0^t V\big(t-s, B(s)\big)\, ds\right\}$$

in the context of Theorems 1.1, 1.2 and 1.3 where $\underline{u}_0 = \inf_{x\in\mathbb{R}^d} u_0(x)$ and $\overline{u}_0 = \sup_{x\in\mathbb{R}^d} u_0(x)$. Or,

$$\underline{u}_0\tilde{u}(t,x) \le u(t,x) \le \overline{u}_0\tilde{u}(t,x) \qquad \text{a.s.,} \tag{1.39}$$

where $\tilde{u}(t,x)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + V(t,x)u(t,x), \\ u(0,x) = 1. \end{cases} \tag{1.40}$$

Relation (1.39) remains in the setting of Theorems 1.6 and 1.7. Indeed, the monotonicity of the solution of (1.1) in the initial value u_0 was established by Mueller [21] in the special setting $\gamma(x) = \delta_0(x)$. This should be true in a more general setting. More precisely, let $\tilde{u}(t,x)$ be the solution of (1.1) in the sense of (1.33) with $u_0(x)$ being replaced by $\tilde{u}_0(x)$. We claim that

$$\begin{aligned} &\tilde{u}_0(x) \le u_0(x) \qquad (\forall x \in \mathbb{R}^d) \\ &\qquad \Longrightarrow \quad \tilde{u}(t,x) \le u(t,x) \qquad \text{a.s. } \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d. \end{aligned} \tag{1.41}$$

In fact, this becomes obvious in the case when $\gamma(x)$ is well bounded, in view of (1.34). For the cases labeled (2) in Table 1, it is well known [16] that $u(t,x)$ can be obtained as the L^2-limit

$$u(t,x) \equiv \lim_{\varepsilon\to 0^+} u_\varepsilon(t,x),$$

where $u_\varepsilon(t,x)$ is the solution of (1.1) with the modified Gaussian potential $V_\varepsilon(t,x)$ replacing $V(t,x)$ that justifies the renormalized Feynman–Kac representation (1.34). Consequently, the monotonicity in $u_0(x)$ stated in (1.41) passes to $u(t,x)$ through the limit.

Let $\underline{u}(t,x)$ and $\overline{u}(t,x)$ be the solutions of (1.1), corresponding to the initial conditions $u_0(x) = \underline{u}_0$ and $u_0(x) = \overline{u}_0$, respectively. By (1.41), $\underline{u}(t,x) \le u(t,x) \le \overline{u}(t,x)$ a.s. for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. By the linearity of (1.1), $\underline{u}(t,x) = \underline{u}_0\tilde{u}(t,x)$ and $\overline{u}(t,x) = \overline{u}_0\tilde{u}(t,x)$. This leads to (1.39).

By (1.39), it is sufficient to establish our theorems for $\tilde{u}(t,x)$. Equivalently, we replace (1.1) by (1.40) in the rest of the paper. This reduction results in the stationarity of $u(t,x)$ in x which substantially simplifies our argument.

In the following proof, we often treat Theorems 1.2 and 1.3 together; likewise, we treat Theorems 1.6 and 1.7 together. In view of (1.27), Theorems 1.3 and 1.7 can be viewed as, respectively, Theorems 1.2 and 1.6 in the special case when $d = 1$ and $\alpha = 1$. This agreement will be reinforced throughout our argument in order to have a more uniform treatment.

The rest of the paper is organized as follows: Section 2 is devoted to the lower bounds for Theorems 1.1, 1.2, 1.3, 1.6 and 1.7. Section 3 is concerned with the high moment asymptotics $\log u(t,0)^m$ as $m \to \infty$, which appears to be most critical to the main development of this work. In Section 4, the modulus continuity of $u(t,x)$ in x is established. With the high moment asymptotics and the modulus continuity, we are able to compute the exact tail asymptotics for $\log u(t,0)$ and $\log \max_{x\in D} u(t,x)$ (with bounded $D \subset \mathbb{R}^d$) in Section 5, where the upper bounds requested by Theorems 1.1, 1.2, 1.3, 1.6 and 1.7 are established as a direct consequence of these tail estimates. In Section 6, we compare the quenched spatial asymptotics to existing quenched time asymptotics in the case of time-independent Gaussian potential. Finally, we prove some auxiliary lemmas needed for this paper in the Appendix.

2. Lower bounds. The proof of the lower bound for a limit law usually appears to be the most revealing side. In this section we establish the lower bounds requested by Theorems 1.1, 1.2, 1.3, 1.6 and 1.7.

2.1. *The setting of Theorems* 1.1, 1.2 *and* 1.3. Let $m = m(R) \ge 1$ be an integer valued function satisfying

$$m \gg \begin{cases} \sqrt{\log R}, & \text{in the context of Theorem 1.1,} \\ (\log R)^{(2-\alpha)/(4-\alpha)}, & \text{in the context of Theorems 1.2 and 1.3} \end{cases} \tag{2.1}$$

as $R \to \infty$. Let $\{B_k(t)\}_{k\ge 1}$ be an i.i.d. sequence of d-dimensional Brownian motions with $B_k(0) = 0$ ($k = 1, 2, \dots$). The notation $\mathbb{E}_0$ is extended to the expectation with respect to $\{B_k(t)\}_{k\ge 1}$. Write τ_k for the exit time of $B_k(s)$,

$$\tau_k = \inf\{s \ge 0; |B_k(s)| \ge 1\}, \qquad k = 1, 2, \dots.$$

In view of (1.9), for any $x \in \mathbb{R}^d$,

$$\begin{aligned} u(t,x)^m &= \mathbb{E}_0 \exp\left\{\theta \sum_{k=1}^m \int_0^t V(t-s, x+B_k(s))\,ds\right\} \\ &\ge \mathbb{E}_0\left[\exp\left\{\theta \sum_{k=1}^m \int_0^t V(t-s, x+B_k(s))\,ds\right\}; \min_{k\le m} \tau_k \ge t\right] \\ &\ge \mathbb{E}_0\Big[\exp\{\lambda\theta\sqrt{\log R}\, S_m(t)\}; \xi_m(t,x) \ge \lambda\sqrt{\log R}\, S_m(t), \min_{k\le m}\tau_k \ge t\Big], \end{aligned}$$

where $\lambda > 0$ is a constant less than but close to $\sqrt{2d}$,

$$\xi_m(t,x) = \sum_{k=1}^m \int_0^t V(t-s, x+B_k(s))\,ds,$$

$$S_m(t) = \left(\sum_{j,k=1}^m \int_0^t \int_0^t \gamma_0(r-s)\gamma(B_j(r)-B_k(s))\,dr\,ds\right)^{1/2}.$$

Set $\mathcal{N}_R = N\mathbb{Z}^d \cap B(0, R)$, where $B(0, R) = \{x \in \mathbb{R}^d; |x| \le R\}$ and $N > 0$ is large but fixed. We have

$$\begin{aligned}
&\max_{|x|\le R} u(t,x)^m \\
&\quad \ge \max_{z\in\mathcal{N}_R} u(t,z)^m \ge \#(\mathcal{N}_R)^{-1} \sum_{z\in\mathcal{N}_R} u(t,z)^m \\
&\quad \ge \#(\mathcal{N}_R)^{-1}\mathbb{E}_0\Big[\exp\{\lambda\theta\sqrt{\log R}S_m(t)\}; \\
&\qquad\qquad \max_{z\in\mathcal{N}_R} \xi_m(t,z) \ge \lambda\sqrt{\log R}S_m(t), \min_{k\le m}\tau_k \ge t\Big] \\
&\quad = \#(\mathcal{N}_R)^{-1}\mathbb{E}_0\Big(\mathcal{Z}_m(R)1\Big\{\max_{z\in\mathcal{N}_R} \xi_m(t,z) \ge \lambda\sqrt{\log R}S_m(t)\Big\}\Big),
\end{aligned}$$

where

$$\mathcal{Z}_m(R) = \exp\{\lambda\theta\sqrt{\log R}S_m(t)\}1_{\{\min_{k\le m}\tau_k\ge t\}}.$$

The big power m is set to undo the price $\#(\mathcal{N}_R)^{-1}$ paid for pushing $\max_z$ into the expectation. Indeed,

$$\begin{aligned}
&\max_{|x|\le R} u(t,x) \\
&\quad \ge \#(\mathcal{N}_R)^{-1/m}\Big\{\mathbb{E}_0\Big(\mathcal{Z}_m(R)1\Big\{\max_{z\in\mathcal{N}_R} \xi_m(t,z) \ge \lambda\sqrt{\log R}S_m(t)\Big\}\Big)\Big\}^{1/m}.
\end{aligned} \tag{2.2}$$

Given that $\#(\mathcal{N}_R) \le CR^d$ for a universal $C > 0$, (2.1) implies the bounds

$$\#(\mathcal{N}_R)^{-1/m} = \begin{cases} \exp\{-o(\sqrt{\log R})\}, \\ \qquad \text{in the context of Theorem 1.1}, \\ \exp\{-o((\log R)^{2/(4-\alpha)})\}, \\ \qquad \text{in the context of Theorems 1.2 and 1.3}, \end{cases} \tag{2.3}$$

which are negligible in comparison to the asymptotic order we try to establish.

Write

$$\eta_R = (\mathbb{E}_0\mathcal{Z}_m(R))^{-1}\mathbb{E}_0\Big(\mathcal{Z}_m(R)1\Big\{\max_{z\in\mathcal{N}_R} \xi_m(t,z) < \lambda\sqrt{\log R}S_m(t)\Big\}\Big).$$

We have

$$\mathbb{E}_0\Big(\mathcal{Z}_m(R)1\Big\{\max_{z\in\mathcal{N}_R} \xi_m(t,z) \ge \lambda\sqrt{\log R}S_m(t)\Big\}\Big) = (\mathbb{E}_0\mathcal{Z}_m(R))(1-\eta_R). \tag{2.4}$$

An important step is to establish

$$\lim_{n\to\infty} \eta_{2^n} = 0 \qquad \text{a.s.} \tag{2.5}$$

For any $\varepsilon > 0$,

$$\mathbb{P}\{\eta_R \geq \varepsilon\} \leq \varepsilon^{-1}\mathbb{E}\eta_R$$

$$= \left(\varepsilon\mathbb{E}_0\mathcal{Z}_m(R)\right)^{-1}\mathbb{E}_0 \otimes \mathbb{E}\Big(\mathcal{Z}_m(R)1\Big\{\max_{z\in\mathcal{N}_R} \xi_m(t,z) < \lambda\sqrt{\log R}S_m(t)\Big\}\Big) \tag{2.6}$$

$$= \left(\varepsilon\mathbb{E}_0\mathcal{Z}_m(R)\right)^{-1}\mathbb{E}_0\Big(\mathcal{Z}_m(R)\mathbb{P}\Big\{\max_{z\in\mathcal{N}_R} \xi_m(t,z) < \lambda\sqrt{\log R}S_m(t)|\mathcal{B}\Big\}\Big),$$

where $\mathcal{B}$ is the σ-algebra generated by the Brownian motions $\{B_k(s)\}_{k\geq 1}$.

Conditioning on the Brownian motions, $\{\xi_m(t,z); z\in\mathcal{N}_R\}$ is a mean zero, and identically distributed Gaussian family with the common (conditional) variance $S_m^2(t)$. Further, for any $z, z'\in\mathcal{N}_R$,

$$\operatorname{Cov}\left(\xi_m(t,z),\xi_m(t,z')|\mathcal{B}\right)$$
$$= \sum_{j,k=1}^m \int_0^t\int_0^t \gamma_0(r-s)\gamma\left((z-z') + \left(B_j(r)-B_k(s)\right)\right)dr\,ds.$$

We now claim that for any $0<\rho<1$, one can take $N>0$ sufficiently large so that on the event $\{\min_{k\leq m}\tau_k \geq t\}$,

$$\gamma\left((z-z') + \left(B_j(r)-B_k(s)\right)\right) \leq \rho\gamma\left(B_j(r)-B_k(s)\right), \tag{2.7}$$
$$z, z'\in\mathcal{N}_R, z\neq z', j,k = 1,\dots,m.$$

Regarding this claim, the setting associated to (II), labeled in Table 1, is the most delicate case among all due to the un-boundedness of $\gamma(\cdot)$ on each coordinate super plane, so we treat it in detail. Let the independent 1-dimensional Brownian motions $B_j^1(s),\dots,B_j^d(s)$ be the components of $B_j(s)$, and write $z=(z_1,\dots,z_d)$ for $z\in\mathcal{N}_R$. Set $\alpha_j = 2-2H_j$ $(j=1,\dots,d)$. By assumption we have that $\alpha_j>0$ $(j=1,\dots,d)$. Write

$$J(z,z') = \{1\leq i\leq d; z_i = z_i'\}, \qquad z,z'\in\mathcal{N}_R.$$

For $i\notin J(z',z)$,

$$\left|(z_i - z_i') + B_j^i(r) - B_k^i(s)\right| \geq N-2 \geq \frac{N-2}{2}\left|B_j^i(r)-B_k^i(s)\right|.$$

Consequently,

$$\gamma\left((z-z') + \left(B_j(r)-B_k(s)\right)\right)$$
$$= \prod_{i=1}^d \left|(z_i-z_i') + B_j^i(r) - B_k^i(s)\right|^{-\alpha_i}$$
$$\leq \left(\frac{2}{N-2}\right)^{\alpha'} \prod_{i=1}^d \left|B_j^i(r)-B_k^i(s)\right|^{-\alpha_i}$$
$$\leq \left(\frac{2}{N-2}\right)^{\min_{1\leq i\leq d}\alpha_i} \gamma\left(B_j(r)-B_k(s)\right)$$

for every pair $z, z' \in \mathcal{N}_d$ with $z \neq z'$, where the last step follows from

$$\alpha' \equiv \sum_{i \notin J(z,z')} \alpha_i \geq \min_{1 \leq i \leq d} \alpha_i.$$

Thus, our assertion (2.7) holds in setting (II).

The proof of (2.7) in other cases is similar, but easier, due to the fact that $\lim_{|x|\to\infty} \gamma(x) = 0$ which is automatic for the type-(I) and type-(III) $\gamma(\cdot)$ and a consequence of assumption (1.13) and the Fourier inversion

$$\gamma(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\gamma}(\lambda) e^{-i\lambda\cdot x}\, d\lambda$$

in the setting of Theorem 1.1, according to Riemann's lemma.

By (2.7),

$$\operatorname{Cov}\bigl(\xi_m(t,z), \xi_m\bigl(t,z'\bigr)|\mathcal{B}\bigr) \leq \rho S_m^2(t). \tag{2.8}$$

Recall that $\lambda < \sqrt{2d}$. Take $u, \rho > 0$ sufficiently small so

$$\frac{(1+2\rho)(\lambda+u)^2}{2} < d \quad \text{and} \quad \frac{u^2}{4\rho} > d+1.$$

Recall (Lemma 4.2, [5]) that for a mean-zero n-dimensional Gaussian vector $(\xi_1, \dots, \xi_n)$ with identically distributed components,

$$\rho \equiv \max_{i \neq j} \bigl|\operatorname{Cov}(\xi_i, \xi_j)\bigr| / \operatorname{Var}(\xi_1) \leq \frac{1}{2},$$

and for any $A, B > 0$,

$$\mathbb{P}\Bigl\{\max_{k \leq n} \xi_k \leq A\Bigr\} \leq \bigl(\mathbb{P}\bigl\{\xi_1 \leq \sqrt{1+2\rho}(A+B)\bigr\}\bigr)^n + \mathbb{P}\bigl\{U \geq B/\sqrt{2\rho \operatorname{Var}(\xi_1)}\bigr\},$$

where U is a standard normal random variable. Applying this inequality conditionally with $A = \lambda S_m(t)\sqrt{\log R}$ and $B = u S_m(t)\sqrt{\log R}$ and noticing $S_m^2(t) = \operatorname{Var}(\xi_m(t,0)|\mathcal{B})$, we have

$$\begin{aligned}
&\mathbb{P}\Bigl\{\max_{z \in \mathcal{N}_R} \xi_m(t,z) < \lambda\sqrt{\log R}\, S_m(t)|\mathcal{B}\Bigr\} \\
&\quad \leq \bigl(\mathbb{P}\bigl\{U \leq \sqrt{1+2\rho}(\lambda+u)\sqrt{\log R}\bigr\}\bigr)^{\#(\mathcal{N}_R)} + \mathbb{P}\bigl\{U \geq (u/\sqrt{2\rho})\sqrt{\log R}\bigr\} \\
&\quad \leq \bigl(1 - \exp\bigl\{-(d-v)\log R\bigr\}\bigr)^{C^{-1}R^d} + \exp\bigl\{-(d+1)\log R\bigr\} \\
&\quad = \exp\bigl\{-(1+o(1))C^{-1}R^v\bigr\} + R^{-(d+1)} \leq C R^{-(d+1)}
\end{aligned}$$

for large $R > 0$, where $v > 0$ is independent of R. Bringing this to (2.6) we have that $\mathbb{P}\{\eta_R \geq \varepsilon\} \leq C\varepsilon^{-1} R^{-(d+1)}$ for large R. In particular, (2.5) holds by the Borel–Cantelli lemma.

By the monotonicity of $\max_{|x|\le R} u(t,x)$ in R and by (2.2), (2.3) and (2.4), the limit along $R = 2^n$ established in (2.5) is sufficient for the lower bounds for Theorems 1.1, 1.2 and 1.3 if we can show that [recall that $m = m(R) \to \infty$ as $R \to \infty$]

$$\begin{aligned}\liminf_{\lambda\to(\sqrt{2d})^-}\liminf_{R\to\infty} m^{-1}(\log R)^{-1/2}\log \mathbb{E}_0 \mathcal{Z}_m(R)\\ \ge \theta\left(2d\gamma(0)\int_0^t\int_0^t \gamma_0(r-s)\,dr\,ds\right)^{1/2}\end{aligned} \tag{2.9}$$

in the context of Theorem 1.1 and that

$$\begin{aligned}\liminf_{\lambda\to(\sqrt{2d})^-}\liminf_{R\to\infty} m^{-1}(\log R)^{-2/(4-\alpha)}\log \mathbb{E}_0 \mathcal{Z}_m(R)\\ \ge \frac{4-\alpha}{4}\left(\frac{4\mathcal{E}(\alpha_0,d,\gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)}\\ \times \theta^{4/(4-\alpha)} d^{2/(4-\alpha)} t^{(4-\alpha-2\alpha_0)/(4-\alpha)}\end{aligned} \tag{2.10}$$

in the context of Theorems 1.2 and 1.3. By now, the problem of the almost sure limits has been reduced to pursuing the deterministic limits. Unlike setting (2), the discussion here does not rely on, but contributes to the development in later sections.

We now prove (2.9). By the continuity of $\gamma(x)$ at $x = 0$, given $\varepsilon > 0$ one can take $0 < \delta < 1$ sufficiently small so that $\gamma(x) \ge \gamma(0) - \varepsilon$ as long as $|x| \le 2\delta$. Set

$$\tau_k(\delta) = \inf\{s \ge 0;\ |B_k(s)| \ge \delta\}.$$

By the definition of $\mathcal{Z}_m(R)$,

$$\begin{aligned}\mathcal{Z}_m(R) \ge \exp\left\{\lambda\theta m\sqrt{\log R}\left((\gamma(0)-\varepsilon)\int_0^t\int_0^t \gamma_0(r-s)\,dr\,ds\right)^{1/2}\right\}\\ \times \mathbb{P}_0\left\{\min_{k\le m}\tau_k(\delta) \ge t\right\}.\end{aligned}$$

Therefore, (2.9) follows from the bound given by the following relation:

$$\mathbb{P}_0\left\{\min_{k\le m}\tau_k(\delta) \ge t\right\} = \left(\mathbb{P}_0\left\{\max_{s\le t}|B(s)| \le \delta\right\}\right)^m.$$

It remains to prove (2.10). First, by the Brownian scaling

$$\begin{aligned}\mathbb{E}_0\mathcal{Z}_m(R) = \mathbb{E}_0\Bigg[\exp\left\{\theta\lambda\sqrt{\log R}\left(\sum_{j,k=1}^m \int_0^t\int_0^t \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\right)^{1/2}\right\};\\ \min_{k\le m}\tau_k \ge t\Bigg]\end{aligned}$$

$$= \mathbb{E}_0\left[\exp\left\{\theta\lambda t_R^{\alpha_0/2}\left(\sum_{j,k=1}^m \int_0^{t_R}\int_0^{t_R} \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\right)^{1/2}\right\}; \min_{k\le m}\tilde{\tau}_k \ge t_R\right],$$

where

$$t_R = t^{(4-\alpha-2\alpha_0)/(4-\alpha)}(\log R)^{2/(4-\alpha)}$$

and

$$\tilde{\tau}_k = \inf\{s\ge 0; |B_k(s)| \ge t^{-\alpha_0/(4-\alpha)}(\log R)^{1/(4-\alpha)}\}.$$

Notice the representations

$$|u|^{-\alpha_0} = C_0\int_{\mathbb{R}} |v|^{-(1+\alpha_0)/2}|v-u|^{-(1+\alpha_0)/2}\,dv \quad \text{and} \tag{2.11}$$
$$\gamma(x) = \int_{\mathbb{R}^d} K(y)K(y-x)\,dy,$$

where $C_0 > 0$ is a constant independent of u and the function $K(x)\ge 0$ is a positive constant multiple of

$$|x|^{-(d+\alpha)/2}, \qquad \prod_{i=1}^d |x_i|^{-(1+\alpha_i)/2} \quad \text{and} \quad \delta_0(x),$$

in connection to, respectively, the space covariances $\gamma(\cdot)$ of type-(I), type-(II) and type-(III) (labeled in Table 1). This leads to

$$t_R^{\alpha_0/2}\left(\sum_{j,k=1}^m \int_0^{t_R}\int_0^{t_R} \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\right)^{1/2}$$
$$= \left(\sum_{j,k=1}^m \int_0^{t_R}\int_0^{t_R} \frac{\gamma(B_j(r)-B_k(s))}{|t_R^{-1}(r-s)|^{\alpha_0}}\,dr\,ds\right)^{1/2}$$
$$= \left(C_0\int_{\mathbb{R}\times\mathbb{R}^d}\left[\sum_{j=1}^m\int_0^{t_R}|u-t_R^{-1}s|^{-(1+\alpha_0)/2}K(x-B_j(s))\,ds\right]^2 du\,dx\right)^{1/2}.$$

Let $f(u,x)$ be a bounded, continuous and locally supported function on $\mathbb{R}\times\mathbb{R}^d$ with $\|f\|_2 = 1$. By the Cauchy–Schwarz inequality, the right-hand side of the above equation is no less than

$$\sqrt{C_0}\int_{\mathbb{R}\times\mathbb{R}^d} f(u,x)\left[\sum_{j=1}^m\int_0^{t_R}|u-t_R^{-1}s|^{-(1+\alpha_0)/2}K(x-B_j(s))\,ds\right]du\,dx$$
$$= \sqrt{C_0}\sum_{j=1}^m\int_0^{t_R}\bar{f}\left(\frac{s}{t_R}, B_j(s)\right)ds,$$

where

$$\bar{f}(s,x)=\int_{\mathbb{R}\times\mathbb{R}^d} f(u,y)|u-s|^{-(1+\alpha_0)/2}K(y-x)\,du\,dy.$$

Summarizing our computation and by independence we have

$$\mathbb{E}_0\mathcal{Z}_m(R)\ge\left(\mathbb{E}_0\left[\exp\left\{\theta\lambda\sqrt{C_0}\int_0^{t_R}\bar{f}\left(\frac{s}{t_R},B(s)\right)ds\right\};\tilde{\tau}\ge t_R\right]\right)^m. \tag{2.12}$$

According to Proposition 3.1 and (3.18) in [6], for a bounded open domain $D\subset\mathbb{R}^d$ containing 0, and for a bounded function $h(s,x)$ defined on $[0,1]\times\mathbb{R}^d$ that is continuous in x and equicontinuous (over $x\in\mathbb{R}^d$) in s,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_0\left[\exp\left\{\int_0^t h\left(\frac{s}{t},B(s)\right)ds\right\};\tau_D\ge t\right]\\&\quad=\sup_{g\in\mathcal{A}_d(D)}\left\{\int_0^1\int_D h(s,x)g^2(s,x)\,dx\,ds-\frac{1}{2}\int_0^1\int_D\left|\nabla_x g(x)\right|^2dx\,ds\right\},\end{aligned}\tag{2.13}$$

where $\tau_D=\inf\{s\ge 0;\,B(s)\notin D\}$ and $\mathcal{A}_d(D)$ is the subspace of $\mathcal{A}_d$ consisting of the functions $g(s,x)$ vanishing for $x\notin D$. Applying this to (2.12) we can get

$$\begin{aligned}&\liminf_{R\to\infty}\frac{1}{mt_R}\log\mathbb{E}_0\mathcal{Z}_m(R)\\&\quad\ge\sup_{g\in\mathcal{A}_d}\left\{\theta\lambda\sqrt{C_0}\int_0^1\int_{\mathbb{R}^d}\bar{f}(s,x)g^2(s,x)\,dx\,ds\right.\\&\qquad\left.-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2dx\,ds\right\}.\end{aligned}\tag{2.14}$$

By Fubini's theorem

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^d}\bar{f}(s,x)g^2(s,x)\,dx\,ds\\&\quad=\int_{\mathbb{R}\times\mathbb{R}^d}f(u,y)\left[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2}K(y-x)g^2(s,x)\,dx\,ds\right]du\,dy.\end{aligned}$$

We now take the supremum over f on the right-hand side of (2.14). Notice that the supremums over g and over f are commutative and that for any dense subset set $\mathcal{S}$ of the unit sphere of $\mathcal{L}^2(\mathbb{R}\times\mathbb{R}^d)$, by the Hahn–Banach theorem,

$$\sup_{f\in\mathcal{S}}\int_{\mathbb{R}\times\mathbb{R}^d}f(u,x)h(u,x)\,dx=\left(\int_{\mathbb{R}\times\mathbb{R}^d}\left|h(u,x)\right|^2dx\,du\right)^{1/2},\qquad h\in\mathcal{L}^2(\mathbb{R}\times\mathbb{R}^d).$$

Therefore, the right-hand side of (2.14) becomes

$$\sup_{g\in\mathcal{A}_d}\biggl\{\theta\lambda\biggl(C_0\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^1\int_{\mathbb{R}^d}|u-s|^{-(1+\alpha_0)/2} \times K(y-x)g^2(s,x)\,dx\,ds\biggr]^2\,du\,dy\biggr)^{1/2} - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(x)\bigr|^2\,dx\,ds\biggr\}$$

$$= \sup_{g\in\mathcal{A}_d}\biggl\{\theta\lambda\biggl(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(r,x)g^2(s,y)\,dx\,dy\biggr)^{1/2} - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(x)\bigr|^2\,dx\,ds\biggr\},$$

where the equality follows from Fubini's theorem and the relations in (2.11). By rescaling the space variable (Lemma 4.1, [6]) this variation is further equal to

$$(\theta\lambda)^{4/(4-\alpha)}\sup_{g\in\mathcal{A}_d}\biggl\{\biggl(\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\alpha_0}}g^2(r,x)g^2(s,y)\,dx\,dy\biggr)^{1/2} - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(x)\bigr|^2\,dx\,ds\biggr\}$$

$$= (\theta\lambda)^{4/(4-\alpha)}\frac{4-\alpha}{4}2^{-\alpha/(4-\alpha)}\biggl(\frac{2\mathcal{E}(\alpha_0,d,\gamma)}{2-\alpha}\biggr)^{(2-\alpha)/(4-\alpha)},$$

where the equality follows from Lemma 7.2, [6].

Summarizing our computation since (2.14),

$$\liminf_{R\to\infty}\frac{1}{mt_R}\log\mathbb{E}_0\mathcal{Z}_m(R) \geq (\theta\lambda)^{4/(4-\alpha)}\frac{4-\alpha}{4}2^{-\alpha/(4-\alpha)}\biggl(\frac{2\mathcal{E}(\alpha_0,d,\gamma)}{2-\alpha}\biggr)^{(2-\alpha)/(4-\alpha)}. \tag{2.15}$$

This clearly leads to (2.10).

2.2. *The setting of Theorems* 1.6 *and* 1.7. Our approach is based on the method of localization developed in [9] and [10]. The construction is specifically designed for the scheme (2) × (I) in [10] and for (2) × (III) in [9]. This method also works for (2) × (II) with minor modification. In the following we carry out this approach.

Given $\beta > 0$ set

$$l_\beta(x) = \prod_{j=1}^d\biggl(1-\frac{|x_j|}{\beta}\biggr)^+ \quad\text{and}\quad K_\beta(x) = K(x)l_\beta(x),$$

where $K(x)$ is given in (2.11). Let $M_\beta(t, A)$ be the martingale measure constructed through (1.32) with $K(x)$ being replaced by $K_\beta(x)$. For each $n \geq 1$, define $U^{(\beta,n)}(t,x)$ as the nth Picard iteration given in the following integral equation: $U^{(\beta,0)} = 1$ and

$$U^{(\beta,n+1)}(t,x) = 1 + \theta \int_0^t \int_{[x-\beta\sqrt{t},x+\beta\sqrt{t}]^d} p_{t-s}(y-x) U^{(\beta,n)}(s,y) M_\beta(ds\,dy)$$

$$n = 1, 2, \ldots.$$

In [9] and [10], the process $U_\beta(t,x) \equiv U^{(\beta,[\log\beta]+1)}(t,x)$ is used to proximate $u(t,x)$, where $\beta = \beta(R)$ increases in R with a suitable speed. By [10], Lemma 9.8, for any $\{x^{(k)}\} \subset \mathbb{R}^d$ with $|x^{(j)} - x^{(k)}| \geq 2\beta([\log\beta]+1)(1+\sqrt{t})$, $\{U_\beta(t, x^{(k)})\}$ is an i.i.d. sequence. By [10], Lemma 9.7, for every $\eta \in (0, 1 \wedge \alpha)$ there are finite and positive constants $l_i = l_i(d, \alpha, \eta)$ $(i = 1, 2)$ such that uniformly for $\beta > 0$ and $m \geq 2$

$$\mathbb{E}\big|u(t,0) - U_\beta(t,0)\big|^m \leq \left(\frac{l_2 m}{\beta^\eta}\right)^{m/2} \exp\{l_1 m^{(4-\alpha)/(2-\alpha)}\}. \tag{2.16}$$

Let η be fixed, and set $\beta = \exp\{M(\log R)^{2/(4-\alpha)}\}$ where $M > 0$ is large but fixed (will be specified later). Let $N = \beta([\log\beta]+1)(1+\sqrt{t})$ and $\mathcal{N}_R = N\mathbb{Z}^d \cap B(0, R-N)$. By the fact that $\alpha < 2$, $\#(\mathcal{N}_R) = CR^{d+o(1)}$ $(R \to \infty)$ where the constant $C > 0$ does not depend on R.

Given $\varepsilon > 0$

$$\mathbb{P}\Big\{\log \max_{z \in \mathcal{N}_R} \big|u(t,z) - U_\beta(t,z)\big| \geq \varepsilon(\log R)^{2/(4-\alpha)}\Big\}$$
$$\leq \#(\mathcal{N}_R)\mathbb{P}\big\{\log\big|u(t,0) - U_\beta(t,0)\big| \geq \varepsilon(\log R)^{2/(4-\alpha)}\big\}.$$

By Chebyshev's inequality and the moment bound given in (2.16),

$$\mathbb{P}\big\{\log\big|u(t,0) - U_\beta(t,0)\big| \geq \varepsilon(\log R)^{2/(4-\alpha)}\big\}$$
$$\leq \exp\{-\varepsilon \log R\}\mathbb{E}\big|u(t,0) - U_\beta(t,0)\big|^{(\log R)^{(2-\alpha)/(4-\alpha)}}$$
$$\leq \exp\{-\varepsilon \log R\} \exp\big\{-\tfrac{1}{2}\big(\eta M - l_1 - o(1)\big)\log R\big\}$$

when R is large. Make M sufficiently large, and we have

$$\mathbb{P}\big\{\log\big|u(t,0) - U_\beta(t,0)\big| \geq \varepsilon(\log R)^{2/(4-\alpha)}\big\} \leq R^{-(d+2)} \tag{2.17}$$

for large R. Consequently,

$$\mathbb{P}\Big\{\log \max_{z \in \mathcal{N}_R} \big|u(t,z) - U_\beta(t,z)\big| \geq \varepsilon(\log R)^{2/(4-\alpha)}\Big\} \leq R^{-2} \tag{2.18}$$

for large R. By the Borel–Cantelli lemma

$$\limsup_{n\to\infty} (\log 2^n)^{-2/(4-\alpha)} \log \max_{z \in \mathcal{N}_{2^n}} \big|u(t,z) - U_{\beta(2^n)}(t,z)\big| = 0 \qquad \text{a.s.} \tag{2.19}$$

On the other hand

$$\max_{z\in\mathcal{N}_R}\big|U_\beta(t,z)\big| \le \max_{z\in\mathcal{N}_R} u(t,z) + \max_{z\in\mathcal{N}_R}\big|u(t,z)-U_\beta(t,z)\big|.$$

Consequently,

$$\begin{aligned}&\log \max_{z\in\mathcal{N}_R}\big|U_\beta(t,z)\big| \\ &\quad\le \log 2 + \max\Big\{\log \max_{z\in\mathcal{N}_R} u(t,z), \log \max_{z\in\mathcal{N}_R}\big|u(t,z)-U_\beta(t,z)\big|\Big\}.\end{aligned} \tag{2.20}$$

For any $\lambda > 0$ satisfying

$$\lambda + \varepsilon < \frac{4-\alpha}{4}\bigg(\frac{4t\mathcal{E}(d,\gamma)}{2-\alpha}\bigg)^{(2-\alpha)/(4-\alpha)}\theta^{4/(4-\alpha)}d^{2/(4-\alpha)},$$

by independence

$$\begin{aligned}&\mathbb{P}\Big\{\log \max_{z\in\mathcal{N}_R}\big|U_\beta(t,z)\big| \le \lambda(\log R)^{2/(4-\alpha)}\Big\} \\ &\qquad = \big(1-\mathbb{P}\big\{\log\big|U_\beta(t,0)\big| > \lambda(\log R)^{2/(4-\alpha)}\big\}\big)^{\#(\mathcal{N}_R)}.\end{aligned}$$

By (2.17)

$$\begin{aligned}&\mathbb{P}\big\{\log\big|U_\beta(t,0)\big| > \lambda(\log R)^{2/(4-\alpha)}\big\} \\ &\qquad \ge \mathbb{P}\big\{\log u(t,0) > (\lambda+\varepsilon)(\log R)^{2/(4-\alpha)}\big\} - R^{-(d+2)}.\end{aligned}$$

By the large deviation result given in (5.7), Theorem 5.4 below,

$$\mathbb{P}\big\{\log u(t,0) > (\lambda+\varepsilon)(\log R)^{2/(4-\alpha)}\big\} \ge \exp\{-(d-\delta)\log R\}$$

for sufficiently large R. Thus we have established the bound

$$\mathbb{P}\Big\{\log \max_{z\in\mathcal{N}_R}\big|U_\beta(t,z)\big| \le \lambda(\log R)^{2/(4-\alpha)}\Big\} \le \exp\{-R^{v}\}$$

for some $v > 0$. By the Borel–Cantelli lemma,

$$\liminf_{n\to\infty}\big(\log 2^n\big)^{-2/(4-\alpha)} \log \max_{z\in\mathcal{N}_{2^n}}\big|U_{\beta(2^n)}(t,z)\big| \ge \lambda \qquad \text{a.s.} \tag{2.21}$$

Combining (2.19), (2.20) and (2.21),

$$\liminf_{n\to\infty}\big(\log 2^n\big)^{-2/(4-\alpha)} \log \max_{z\in\mathcal{N}_{2^n}} u(t,z) \ge \lambda \qquad \text{a.s.}$$

By the fact that

$$\max_{z\in\mathcal{N}_R} u(t,z) \le \max_{|x|\le R} u(t,x)$$

and by the monotonicity of $\max_{|x|\le R} u(t,x)$ in R,

$$\liminf_{R\to\infty}(\log R)^{-2/(4-\alpha)} \log \max_{|x|\le R} u(t,x) \ge \lambda \qquad \text{a.s.}$$

This leads to the lower bound for (1.36) as λ can be made arbitrarily close to the limit value appearing in the right-hand side of (1.36).

According to the agreement made at the end of Section 1, the lower bound requested by (1.37) can be viewed as the special case of the lower bound for (1.36) under the identification $\alpha = 1$ and $d = 1$.

3. High moment asymptotics. Associated to the main theorems are the tail behaviors of $\log u(t,0)$, which are relevant to the high moment asymptotics for $\mathbb{E}u(t,0)^m$ as $m \to \infty$, in light of the Gärtner–Ellis theorem. The objective of this section is to find the exact high moment asymptotics required by our main theorems.

3.1. *The setting of Theorems* 1.1, 1.2 *and* 1.3. Recall our extra assumption $u_0(x) \equiv 1$. We begin with the moment representations (Corollary 4.5 and Remark 4.6, [6])

$$\mathbb{E}u(t,x)^m = \mathbb{E}_0 \exp\left\{\frac{1}{2}\theta^2 \sum_{j,k=1}^m \int_0^t\int_0^t \gamma_0(r-s)\gamma\big(B_j(r)-B_k(s)\big)\,dr\,ds\right\} \tag{3.1}$$

for each integer $m \geq 1$, where $\{B_k(s)\}_{k\geq 1}$ is an i.i.d. sequence of Brownian motions.

PROPOSITION 3.1. *Under the assumption of Theorem* 1.1,

$$\lim_{m\to\infty} m^{-2}\log \mathbb{E}u(t,0)^m = \frac{1}{2}\theta^2\gamma(0)\int_0^t\int_0^t \gamma_0(r-s)\,dr\,ds. \tag{3.2}$$

PROOF. We first notice that $\gamma(x)$ reaches its maximum at $x=0$. Indeed, for a infinitely smooth and rapidly decreasing (at ∞) function $\varphi_0(\cdot) \geq 0$ on $\mathbb{R}^+$, $\varepsilon > 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\operatorname{Cov}\big(\langle V, \varphi_0 p_\varepsilon\rangle, \langle V, \varphi_0 p_\varepsilon(\cdot - x)\rangle\big)\\
&\quad= \left(\int_{\mathbb{R}^+\times\mathbb{R}^+} \gamma_0(r-s)\varphi_0(r)\varphi_0(s)\,dr\,ds\right)\\
&\qquad\times \int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(y-z)p_\varepsilon(y)p_\varepsilon(z-x)\,dy\,dz.
\end{aligned}$$

Here we recall our notation $p_s(x)$ for d-dimensional Brownian density.

On the other hand, by homogeneity

$$\begin{aligned}
&\operatorname{Cov}\big(\langle V, \varphi_0 p_\varepsilon\rangle, \langle V, \varphi_0 p_\varepsilon(\cdot - x)\rangle\big)\\
&\quad\leq \operatorname{Var}\big(\langle V, \varphi_0 p_\varepsilon\rangle\big)\\
&\quad= \left(\int_{\mathbb{R}^+\times\mathbb{R}^+} \gamma_0(r-s)\varphi_0(r)\varphi_0(s)\,dr\,ds\right)\int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(y-z)p_\varepsilon(y)p_\varepsilon(z)\,dy\,dz.
\end{aligned}$$

Consequently

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(y-z)p_\varepsilon(y)p_\varepsilon(z-x)\,dy\,dz \\ \le \int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(y-z)p_\varepsilon(y)p_\varepsilon(z)\,dy\,dz.$$

Letting $\varepsilon\to 0^+$, by continuity of $\gamma(\cdot)$ we have $\gamma(x)\le\gamma(0)$.

Therefore, the requested upper bound follows from (3.1).

As for the lower bound, we essentially follow the strategy used in the previous section: by continuity, for any $\varepsilon>0$ there is $\delta>0$ such that $\gamma(x)\ge\gamma(0)-\varepsilon$ as long as $|x|\le 2\delta$. Thus

$$\begin{aligned}&\mathbb{E}_0\exp\left\{\frac{1}{2}\sum_{j,k=1}^m\int_0^t\int_0^t\gamma_0(r-s)\gamma\big(B_j(r)-B_k(s)\big)\,dr\,ds\right\}\\ &\quad\ge\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\sum_{j,k=1}^m\int_0^t\int_0^t\gamma_0(r-s)\gamma\big(B_j(r)-B_k(s)\big)\,dr\,ds\right\};\min_{k\le m}\tau_k(\delta)\ge t\right]\\ &\quad\ge\exp\left\{\frac{m^2}{2}(\gamma(0)-\varepsilon)\int_0^t\int_0^t\gamma_0(r-s)\,dr\,ds\right\}\mathbb{P}_0\Big\{\min_{k\le m}\tau_k(\delta)\ge t\Big\},\end{aligned}$$

where $\tau_k(\delta)$ is the time for $B_k(s)$ to exit from the δ-ball. Therefore, the requested lower bound follows from the facts that $\varepsilon>0$ can be arbitrarily small and that the probability

$$\mathbb{P}_0\Big\{\min_{k\le m}\tau_k(\delta)\ge t\Big\}=\Big(\mathbb{P}\Big\{\max_{s\le t}\big|B(s)\big|\le\delta\Big\}\Big)^m$$

decays at a speed no faster than exponential rate. □

PROPOSITION 3.2. *Under the assumptions of Theorem 1.2*

$$\lim_{m\to\infty}m^{-(4-\alpha)/(2-\alpha)}\log\mathbb{E}u(t,0)^m \\ =\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)}t^{(4-\alpha-2\alpha_0)/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma). \tag{3.3}$$

Under the assumptions of Theorem 1.3

$$\lim_{m\to\infty}m^{-3/2}\log\mathbb{E}u(t,0)^m=\frac{\theta^4}{4}t^{3-2\alpha_0}\mathcal{E}(\alpha_0,1,\delta_0). \tag{3.4}$$

PROOF. We need only to prove (3.3) as (3.4) can be viewed as a special case under the identification $d=\alpha=1$. Recall that $\gamma_0(u)=|u|^{-\alpha_0}$ in this setting. For any $1 \le j, k \le m$, by (2.11)

$$\begin{aligned}
&\int_0^t\int_0^t \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\\
&\quad = C_0\int_{\mathbb{R}\times\mathbb{R}^d}\biggl[\int_0^t |u-s|^{-(1+\alpha_0)/2}K\bigl(x-B_j(s)\bigr)\,ds\biggr]\\
&\qquad\times\biggl[\int_0^t |u-s|^{-(1+\alpha_0)/2}K\bigl(x-B_k(s)\bigr)\,ds\biggr]\,du\,dx.
\end{aligned} \tag{3.5}$$

Applying the Cauchy–Schwarz inequality

$$\begin{aligned}
&\int_0^t\int_0^t \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\\
&\quad\le \biggl(\int_0^t\int_0^t \frac{\gamma(B_j(r)-B_j(s))}{|r-s|^{\alpha_0}}\,dr\,ds\biggr)^{1/2}\biggl(\int_0^t\int_0^t \frac{\gamma(B_k(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\biggr)^{1/2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{j,k=1}^m \int_0^t\int_0^t \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\\
&\quad\le \biggl\{\sum_{k=1}^m\biggl(\int_0^t\int_0^t \frac{\gamma(B_k(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds\biggr)^{1/2}\biggr\}^2\\
&\quad\le m\sum_{k=1}^m \int_0^t\int_0^t \frac{\gamma(B_k(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds.
\end{aligned}$$

Write

$$X_m(t) = \sum_{j,k=1}^m \int_0^t\int_0^t \frac{\gamma(B_j(r)-B_k(s))}{|r-s|^{\alpha_0}}\,dr\,ds.$$

By independence, for any $\beta > 0$

$$\begin{aligned}
\mathbb{E}_0\exp\{\beta X_m(t)\} &\le \biggl(\mathbb{E}_0\exp\biggl\{m\beta\int_0^t\int_0^t \frac{\gamma(B(r)-B(s))}{|r-s|^{\alpha_0}}\,dr\,ds\biggr\}\biggr)^m\\
&= \biggl(\mathbb{E}_0\exp\biggl\{\beta\int_0^{t_m}\int_0^{t_m} \frac{\gamma(B(r)-B(s))}{|r-s|^{\alpha_0}}\,dr\,ds\biggr\}\biggr)^m,
\end{aligned}$$

where $t_m = tm^{2/(4-\alpha-2\alpha_0)}$, and the equality follows from the Brownian scaling. Recall (Theorem 1.1, [6]) that

$$\begin{aligned}
&\lim_{m\to\infty} t_m^{-(4-\alpha-2\alpha_0)/(2-\alpha)}\log\mathbb{E}_0\exp\biggl\{\beta\int_0^{t_m}\int_0^{t_m} \frac{\gamma(B(r)-B(s))}{|r-s|^{\alpha_0}}\,dr\,ds\biggr\}\\
&\quad = \beta^{2/(2-\alpha)}\mathcal{E}(\alpha_0, d, \gamma).
\end{aligned}$$

We conclude that

$$\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\{\beta X_m(t)\} \le \beta^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)} \mathcal{E}(\alpha_0, d, \gamma) \tag{3.6}$$

for any $\beta > 0$.

On the other hand, let $\tilde{t}_m = tm^{2/(2-\alpha)}$. An obvious modification of the argument for (2.15) [with $(\log R)^{2/(2-\alpha)}$ being replaced by $\tilde{t}_m$] shows that for any $\lambda > 0$,

$$\liminf_{m\to\infty} \frac{1}{m\tilde{t}_m} \log \mathbb{E}_0 \exp\{\lambda \tilde{t}_m^{\alpha_0/2} X_m(\tilde{t}_m)^{1/2}\} \ge \lambda^{4/(4-\alpha)} \frac{4-\alpha}{4} 2^{-\alpha/(4-\alpha)} \left(\frac{2\mathcal{E}(\alpha_0, d, \gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)}.$$

Let $\lambda = \beta t^{-\alpha_0/2}$. By Brownian scaling, the above limiting bound can be rewritten as

$$\liminf_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\{\beta m^{(4-\alpha)/(2(2-\alpha))} X_m(t)^{1/2}\} \ge t^{(4-\alpha-2\alpha_0)/(4-\alpha)} \beta^{4/(4-\alpha)} \frac{4-\alpha}{4} 2^{-\alpha/(4-\alpha)} \left(\frac{2\mathcal{E}(\alpha_0, d, \gamma)}{2-\alpha}\right)^{(2-\alpha)/(4-\alpha)} \quad (\beta > 0). \tag{3.7}$$

By the first half of Lemma A.2 in the Appendix, (3.6) and (3.7),

$$\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\{\beta X_m(t)\} = \beta^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)} \mathcal{E}(\alpha_0, d, \gamma).$$

Let $\beta = \theta^2/2$. Proposition 3.2 follows from (3.1). □

3.2. *The setting of Theorems* 1.6 *and* 1.7. The goal here is to establish the following:

PROPOSITION 3.3. *Under the assumptions given in Theorem* 1.6,

$$\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E} u(t,0)^m = t\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} \mathcal{E}(d, \gamma). \tag{3.8}$$

Under the assumptions given in Theorem 1.7,

$$\lim_{m\to\infty} m^{-3} \log \mathbb{E} u(t,0)^m = t\frac{\theta^4}{24}. \tag{3.9}$$

In view of (1.27), we need only to prove (3.8), as (3.9) can be viewed as a special case under a proper identification. Our starting point is the following moment representation (see Theorem 5.3 in [16] and Theorem 3.1 in [8]):

$$\mathbb{E} u(t,0)^m = \mathbb{E}_0 \exp\left\{\theta^2 \sum_{1\le j<k\le m} \int_0^t \gamma\big(B_j(s) - B_k(s)\big)\,ds\right\}. \tag{3.10}$$

The approach here is much more delicate due to the absence of the diagonal terms in the (j,k)-summation in (3.10) and the fact that the missing diagonal terms blow up. The proof consists of several steps.

Let $t_m = tm^{2/(2-\alpha)}$ and $\varepsilon > 0$ be small but fixed. Set

$$\gamma_\varepsilon(x) = \int_{\mathbb{R}^d} p_{2\varepsilon}(x-y)\gamma(y)\,dy, \qquad x \in \mathbb{R}^d.$$

Here we recall that $p_t(x)$ represents the density function of a d-dimensional Brownian motion $B(t)$ starting at 0. Let the kernel $K(x)$ be defined in (2.11). Clearly,

$$\gamma_\varepsilon(x) = \int_{\mathbb{R}^d} K_\varepsilon(y)K_\varepsilon(y-x)\,dy, \qquad x \in \mathbb{R}^d, \tag{3.11}$$

where

$$K_\varepsilon(x) = \int_{\mathbb{R}^d} p_\varepsilon(x-y)K(y)\,dy, \qquad x \in \mathbb{R}^d. \tag{3.12}$$

Our first step is to prove the following:

LEMMA 3.4. *For any $\beta > 0$*

$$\begin{aligned}\lim_{m\to\infty} & m^{-(4-\alpha)/(2-\alpha)} \\ & \times \log \mathbb{E}_0 \exp\left\{\beta\left(t_m \int_0^{t_m}\int_{\mathbb{R}^d}\left[\sum_{j=1}^m K_\varepsilon(x - B_j(s))\right]^2 dx\,ds\right)^{1/2}\right\} \\ & = tM_\varepsilon(\beta),\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}M_\varepsilon(\beta) = \sup_{g\in\mathcal{A}_d}\Bigg\{&\beta\left(\int_0^1\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} K_\varepsilon(y-x)g^2(s,y)\,dy\right]^2 dx\,ds\right)^{1/2} \\ &- \frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,dx\,dy\Bigg\}.\end{aligned}$$

PROOF. Indeed,

$$\begin{aligned}&\left(t_m \int_0^{t_m}\int_{\mathbb{R}^d}\left[\sum_{j=1}^m K_\varepsilon(x - B_j(s))\right]^2 dx\,ds\right)^{1/2} \\ &\quad = t_m\left(\int_0^1\int_{\mathbb{R}^d}\left[\sum_{j=1}^m K_\varepsilon(x - B_j(t_m s))\right]^2 dx\,ds\right)^{1/2} \\ &\quad \ge t_m \int_0^1\int_{\mathbb{R}^d} f(s,x)\left[\sum_{j=1}^m K_\varepsilon(x - B_j(t_m s))\right] dx\,ds \\ &\quad = \sum_{j=1}^m \int_0^{t_m} \bar{f}\left(\frac{s}{t_m}, B_j(s)\right) ds,\end{aligned}$$

where $f(s,x)\ge 0$ is a compactly supported and continuous function on $[0,1]\times\mathbb{R}^d$ with

$$\int_0^1\int_{\mathbb{R}^d} f^2(s,x)\,dx\,ds=1,$$
$$\bar f(s,x)=\int_{\mathbb{R}^d} f(s,y)K_\varepsilon(y-x)\,dy,\qquad x\in\mathbb{R}^d,$$

and the second step follows from the Cauchy–Schwarz inequality. By independence,

$$\mathbb{E}_0\exp\biggl\{\beta\biggl(t_m\int_0^{t_m}\int_{\mathbb{R}^d}\biggl[\sum_{j=1}^m K_\varepsilon\bigl(x-B_j(s)\bigr)\biggr]^2dx\,ds\biggr)^{1/2}\biggr\}$$
$$\ge\biggl(\mathbb{E}_0\exp\biggl\{\beta\int_0^{t_m}\bar f\biggl(\frac{s}{t_m},B(s)\biggr)ds\biggr\}\biggr)^m.$$

Applying Proposition 3.1, [6] or (2.13) to the right-hand side,

$$\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)}\log\mathbb{E}_0\exp\biggl\{\beta\biggl(t_m\int_0^{t_m}\int_{\mathbb{R}^d}\biggl[\sum_{j=1}^m K_\varepsilon\bigl(x-B_j(s)\bigr)\biggr]^2dx\,ds\biggr)^{1/2}\biggr\}$$
$$\ge t\sup_{g\in\mathcal{A}_d}\biggl\{\beta\int_0^1\int_{\mathbb{R}^d}\bar f(s,x)g^2(s,x)\,dx\,ds-\frac12\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(s,x)\bigr|^2dx\,ds\biggr\}$$
$$=t\sup_{g\in\mathcal{A}_d}\biggl\{\beta\int_0^1\int_{\mathbb{R}^d}f(s,y)\biggl[\int_{\mathbb{R}^d}K_\varepsilon(y-x)g^2(s,x)\,dx\biggr]dy\,ds$$
$$-\frac12\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(s,x)\bigr|^2dx\,ds\biggr\}.$$

Taking supremum over f on the right-hand side leads to the lower bound requested by (3.13).

The proof of the upper bound is harder. First, we perform the following smooth truncation: let $l:\mathbb{R}^+\longrightarrow[0,1]$ be a smooth function satisfying the following properties: $l(u)=1$ for $u\in[0,1]$, $l(u)=0$ for $u\ge 3$ and $-1\le l'(u)\le 0$ for all $u>0$. Let $M>0$ be a large number, and write

$$Q(x)=K_\varepsilon(x)l\bigl(M^{-1}|x|\bigr).$$

One can easily see that $Q(x)$ is supported on $B(0,3M)=\{x\in\mathbb{R}^d;\,|x|\le 3M\}$ and that

$$\int_{\mathbb{R}^d}\bigl[K_\varepsilon(x)-Q(x)\bigr]^2dx\longrightarrow 0\qquad(M\to\infty).$$

By the triangle inequality,

$$\begin{aligned}
&\left(t_m \int_0^{t_m}\int_{\mathbb{R}^d}\left[\sum_{j=1}^m \big(K_\varepsilon\big(x-B_j(s)\big)-Q\big(x-B_j(s)\big)\big)\right]^2 dx\,ds\right)^{1/2}\\
&\quad\le t_m^{1/2}\sum_{j=1}^m\left(\int_0^{t_m}\int_{\mathbb{R}^d}\big[K_\varepsilon\big(x-B_j(s)\big)-Q\big(x-B_j(s)\big)\big]^2 dx\,ds\right)^{1/2}\\
&\quad= mt_m\left(\int_{\mathbb{R}^d}\big[K_\varepsilon(x)-Q(x)\big]^2 dx\right)^{1/2}\\
&\quad= tm^{(4-\alpha)/(2-\alpha)}\left(\int_{\mathbb{R}^d}\big[K_\varepsilon(x)-Q(x)\big]^2 dx\right)^{1/2}.
\end{aligned}$$

This estimate shows that it suffices to establish the upper bound with $K_\varepsilon(x)$ being replaced by $Q(x)$ for an arbitrarily large M.

Let $M>0$ be fixed. For $N>3M$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left[\sum_{j=1}^m Q\big(x-B_j(s)\big)\right]^2 dx\\
&\qquad=\sum_{z\in\mathbb{Z}^d}\int_{[-N,N]^d}\left[\sum_{j=1}^m Q\big(2zN+x-B_j(s)\big)\right]^2 dx\\
&\qquad\le\int_{[-N,N]^d}\left[\sum_{j=1}^m Q_N\big(x-B_j(s)\big)\right]^2 dx\\
&\qquad=m^2\int_{[-N,N]^d}\eta_m^2(s,x)\,dx,
\end{aligned}$$

where

$$Q_N(x)=\sum_{z\in\mathbb{Z}^d}Q(2zN+x)\quad\text{and}$$
$$\eta_m(s,x)=\frac{1}{m}\sum_{j=1}^m Q_N\big(x-B_j(s)\big). \tag{3.14}$$

Notice in the z-summation that defines $Q_N(\cdot)$, there is at most one nonzero term for any $x\in\mathbb{R}^d$ by the assumption that $N>3M$. Consequently, $Q_N(x)$ is a continuous periodic extension (with the period $2N$) of $Q(x)$.

Further, by integration substitution

$$\int_0^{t_m}\int_{[-N,N]^d}\eta_m^2(s,x)\,dx\,ds=t_m\int_0^1\int_{[-N,N]^d}\eta_m^2(t_m s,x)\,dx\,ds.$$

To establish the upper bound requested by (3.13), therefore, all we need is to show that for any $M > 0$,

$$\limsup_{N\to\infty}\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \times \log \mathbb{E}_0 \exp\left\{\beta m t_m \left(\int_0^1\int_{\mathbb{R}^d} \eta_m^2(t_m s, x)\,dx\,ds\right)^{1/2}\right\} \le t M_\varepsilon(\beta). \tag{3.15}$$

We let $N > 3M$ be fixed for a while and concentrate on the m-lim sup. Unfortunately, $\eta_m(t_m(\cdot),\cdot)$ is not exponentially tight when embedded into $\mathcal{L}^2([0,1]\times[-N,N]^d)$. In the following we prove that with overwhelming probability, for any $\delta > 0$ there is a $C > 0$ such that the range of the $\mathcal{L}^2([0,1]\times[-N,N]^d)$-valued random variable $\eta_m(t_m(\cdot),\cdot)$ is covered by at most $\exp(Ct_m)$ δ-balls in $\mathcal{L}^2([0,1]\times[-N,N]^d)$.

Let $\upsilon > 0$ be a small number, and define $[s]_\upsilon = \upsilon[\upsilon^{-1}s]$. By Jensen inequality,

$$\begin{aligned}&\int_0^{t_m}\int_{[-N,N]^d}\big[\eta_m(s,x) - \eta_m([s]_\upsilon, x)\big]^2\,dx\,ds\\ &\quad\le \frac{1}{m}\sum_{j=1}^m \int_0^{t_m}\int_{[-N,N]^d}\big[Q_N\big(x - B_j(s)\big) - Q_N\big(x - B_j([s]_\upsilon)\big)\big]^2\,dx\,ds.\end{aligned}$$

By independence,

$$\begin{aligned}&\mathbb{E}_0\exp\left\{\beta m\int_0^{t_m}\int_{[-N,N]^d}\big[\eta_m(s,x) - \eta_m([s]_\upsilon, x)\big]^2\,dx\,ds\right\}\\ &\quad\le \left(\mathbb{E}_0\exp\left\{\beta\int_0^{t_m}\int_{[-N,N]^d}\big[Q_N\big(x - B(s)\big) - Q_N\big(x - B([s]_\upsilon)\big)\big]^2\,dx\,ds\right\}\right)^m.\end{aligned}$$

Notice that

$$\begin{aligned}&\int_0^{t_m}\int_{[-N,N]^d}\big[Q_N\big(x - B(s)\big) - Q_N\big(x - B([s]_\upsilon)\big)\big]^2\,dx\,ds\\ &\quad\le \sum_k \int_{(k-1)\upsilon}^{k\upsilon}\int_{[-N,N]^d}\big[Q_N\big(x - B(s)\big) - Q_N\big(x - B((k-1)\upsilon)\big)\big]^2\,dx\,ds\\ &\quad= \sum_k \int_{(k-1)\upsilon}^{k\upsilon}\int_{[-N,N]^d}\big[Q_N(x) - Q_N\big(x + B(s) - B((k-1)\upsilon)\big)\big]^2\,dx\,ds,\end{aligned}$$

where the summation over k runs from $k = 1$ until $k = [\upsilon^{-1}t_m] + 1$, and the second step follows from the periodicity of the function $Q_N(\cdot)$. By increment-independence of the Brownian motion,

$$\begin{aligned}&\mathbb{E}_0\exp\left\{\beta m\int_0^{t_m}\int_{[-N,N]^d}\big[\eta_m(s,x) - \eta_m([s]_\upsilon, x)\big]^2\,dx\,ds\right\}\\ &\quad\le \left(\mathbb{E}_0\exp\left\{\beta\int_0^{\upsilon}\int_{[-N,N]^d}\big[Q_N(x) - Q_N\big(x + B(s)\big)\big]^2\,dx\,ds\right\}\right)^{m([\upsilon^{-1}t_m]+1)}.\end{aligned}$$

By the continuity of $Q_N(\cdot)$ one can easily see that

$$\mathbb{E}_0 \exp\left\{\beta \int_0^{\upsilon} \int_{[-N,N]^d} \left[Q_N(x) - Q_N\big(x + B(s)\big)\right]^2 dx\, ds\right\} = \exp\{o(\upsilon)\} \qquad (\upsilon \to 0^+).$$

Thus we have proved that for any $\beta > 0$,

$$\lim_{\upsilon \to 0^+} \limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \times \log \mathbb{E}_0 \exp\left\{\beta m \int_0^{t_m} \int_{[-N,N]^d} \left[\eta_m(s,x) - \eta_m\big([s]_\upsilon, x\big)\right]^2 dx\, ds\right\} = 0.$$

Write

$$\Omega(\upsilon, \delta, m) = \left\{\int_0^1 \int_{[-N,N]^d} \left[\eta_m(t_m s, x) - \eta_m\big([t_m s]_\upsilon, x\big)\right]^2 dx\, ds \le \frac{\delta^2}{4}\right\}.$$

By variable substitution and Chebyshev's inequality, for any $L > 0$ one can take υ sufficiently small so that

$$\mathbb{P}_0\big(\Omega(\upsilon,\delta,m)^c\big) \le \exp\{-L m^{(4-\alpha)/(2-\alpha)}\}$$

for large m. Define

$$\tau_j(H) = \inf\{s \ge 0; |B_j(s)| \ge H m^{(6-\alpha)/(2(2-\alpha))}\} \quad \text{and} \quad \tau_*(H) = \min_{j\le m} \tau_j(H),$$

where $H > 0$ is a large but fixed constant. By Gaussian tail,

$$\begin{aligned}\mathbb{P}_0\{\tau_*(H) < t_m\} &\le m \mathbb{P}_0\Big\{\max_{s\le t_m} |B(s)| \ge H m^{(6-\alpha)/(2(2-\alpha))}\Big\} \\ &\le m \exp\{-C H^2 m^{(4-\alpha)/(2-\alpha)}\},\end{aligned}$$

where $C > 0$ is a universal constant.

By the fact that $\eta_m(t_m s, x)$ is bounded by a deterministic constant C_N independent of m, for sufficiently small υ and sufficiently large $H > 0$,

$$\begin{aligned}&\mathbb{E}_0 \exp\left\{\beta m t_m \left(\int_0^1 \int_{\mathbb{R}^d} \eta_m^2(t_m s, x)\, dx\, ds\right)^{1/2}\right\} \\ &\quad = \left[\mathbb{E}_0 \exp\left\{\beta m t_m \left(\int_0^1 \int_{\mathbb{R}^d} \eta_m^2(t_m s, x)\, dx\, ds\right)^{1/2}\right\} 1_{\Omega(\upsilon,\delta,m)}; \tau_* \ge t_m\right] \\ &\qquad + \exp\{-(L - \beta C_N) m^{(4-\alpha)/(2-\alpha)}\} \\ &\qquad + m \exp\{-(H^2 C - \beta C_N) m^{(4-\alpha)/(2-\alpha)}\}.\end{aligned} \tag{3.16}$$

The second and the third terms on the right-hand side are negligible for sufficiently large L and H.

We view $\eta_m(s,\cdot)$ $(s\ge 0)$ as a process (in s) taking values in $\mathcal{L}^2([-N,N]^d)$. Notice that the function $Q(x)$ is bounded and Lipschitz continuous. These properties are inherited by $Q_N(x)$ as a continuous periodic extension of $Q(x)$. Consequently, $Q_N(\cdot-B_j(s))$ is bounded and Lipschitz continuous on $[-N,N]^d$ uniformly in $s\ge 0$ and $j\ge 1$ with a deterministic bound and a deterministic Lipschitz constant. Hence there is a deterministic and convex compact set $\mathcal{K}\subset\mathcal{L}^2([-N,N]^d)$ such that $Q_N(\cdot-B_j(s))\in\mathcal{K}$ a.s. for every $s\ge 0$ and $j=1,2,\dots$. As a convex linear combination of $Q_N(\cdot-B_j(s))$ $(j=1,\dots,m)$, $\eta_m(s,\cdot)\in\mathcal{K}$ a.s. for any $s\ge 0$ and $m=1,2,\dots$. Let $g_1,\dots,g_l\in\mathcal{K}$ be a $(2^{-1}\delta)$-net of $\mathcal{K}$. On the set $\Omega(\upsilon,\delta,m)$ the functions of the form

$$g(s,x)=g_{i_k}(x)\qquad \text{as } s\in\left[\frac{(k-1)\upsilon}{t_m},\frac{k\upsilon}{t_m}\right), k=1,2,\dots,[\upsilon^{-1}t_m]+1$$

make a δ-net (denoted as $\mathcal{N}_m^\delta$) of the range of the $\mathcal{L}^2([0,1]\times[-N,N]^d)$-valued random variable $\eta_m(t_m(\cdot),\cdot)$. Indeed, for any $k\ge 1$ there is $g_{i_k}(x)=g_{i_k}(\omega,x)$ out of $\{g_1,\dots,g_l\}$ such that

$$\int_{[-N,N]^d}\left|\eta_m\left(\frac{(k-1)\upsilon}{t_m},x\right)-g_{i_k}(x)\right|^2dx<\frac{\delta^2}{4}.$$

Here the notation $g_{i_k}(\omega,x)$ indicates the randomness of picking g_{i_k}.

Consequently,

$$\begin{aligned}&\int_0^1\int_{[-N,N]^d}\big|\eta_m([t_m s]_\upsilon,x)-g(s,x)\big|^2dx\,ds\\&\qquad\le\frac{\upsilon}{t_m}\sum_k\int_{[-N,N]^d}\left|\eta_m\left(\frac{(k-1)\upsilon}{t_m},x\right)-g_{i_k}(x)\right|^2dx<\frac{\delta^2}{4}.\end{aligned}$$

So our assertion follows from the restriction by the set $\Omega(\upsilon,\delta,m)$.

In addition, we can see that $\#(\mathcal{N}_m^\delta)\le l^{[\upsilon^{-1}t_m]+1}$. Further, by our construction of $g\in\mathcal{N}_m^\delta$,

$$\begin{aligned}&\int_0^1\int_{[-N,N]^d}\big|g(s,x)\big|^2dx\,ds\\&\qquad\le\frac{\upsilon}{t_m}\sum_k\int_{[-N,N]^d}\big|g_{i_k}(x)\big|^2dx\\&\qquad\le 2\sup_{h\in\mathcal{K}}\int_{[-N,N]^d}\big|h(x)\big|^2dx<\infty,\qquad g\in\mathcal{N}_m^\delta,\end{aligned}\tag{3.17}$$

and similarly, for any $0<u<1$,

$$\int_0^u\int_{[-N,N]^d}\big|g(s,x)\big|^2dx\,ds\le u\sup_{h\in\mathcal{K}}\int_{[-N,N]^d}\big|h(x)\big|^2dx,\qquad g\in\mathcal{N}_m^\delta.\tag{3.18}$$

We emphasize the fact that the bounds in (3.17) and (3.18) do not depends on δ.

By the Hahn–Banach theorem, for each $g \in \mathcal{N}_m^\delta$ there is $f \in \mathcal{L}^2([0,1] \times [-N,N]^d)$ such that

$$\int_0^1 \int_{[-N,N]^d} |f(s,x)|^2 \, dx \, ds = 1,$$

$$\int_0^1 \int_{[-N,N]^d} f(s,x) g(s,x) \, dx \, ds = \left(\int_0^1 \int_{[-N,N]^d} |g(s,x)|^2 \, dx \, ds \right)^{1/2}.$$

In view of the uniform bound (3.17) on $g \in \mathcal{N}_m^\delta$, for any given $\sigma > 0$ one can take $\delta > 0$ sufficiently small so that

$$\int_0^1 \int_{[-N,N]^d} f(s,x) h(s,x) \, dx \, ds$$
$$> (1-\sigma) \left(\int_0^1 \int_{[-N,N]^d} |h(s,x)|^2 \, dx \, ds \right)^{1/2}$$

for every $h \in B(g,\delta)$ and $g \in \mathcal{N}_m^\delta$. By bound (3.18), we may make $u > 0$ sufficiently small (but independent of f) so that $|f(s,x)| \le 1$ for $0 \le s \le u$ and $x \in [-N,N]^d$, due to the fact that one can change the definition of $f(s,x)$ on $[0,u] \times [-N,N]^d$ without drastically changing the value of the integral on $[0,1] \times [-N,N]^d$. Finally, we may make each $f(s,x)$ continuous and bounded on $[0,1] \times [-N,N]^d$ by (3.17) and the fact that these kinds of functions are dense in $\mathcal{L}^2([0,1] \times [-N,N]^d)$. Denote the collection of such f by $(\mathcal{N}_m^\delta)^*$. Our way of using the Hahn–Banach theorem defines a surjective map from $\mathcal{N}_m^\delta$ to $(\mathcal{N}_m^\delta)^*$. Consequently, $\#((\mathcal{N}_m^\delta)^*) \le \#(\mathcal{N}_m^\delta) = l^{[\upsilon^{-1} t_m]+1} \le \exp(Ct_m)$, where the constant $C > 0$ is independent of m (but dependent on l and υ).

On the set $\Omega(\upsilon,\delta,m)$, in particular,

$$\left(\int_0^1 \int_{[-N,N]^d} \eta_m^2(t_m s, x) \, dx \, ds \right)^{1/2}$$
$$\le (1-\sigma)^{-1} \max_{f \in (\mathcal{N}_m^\delta)^*} \int_0^1 \int_{[-N,N]^d} f(s,x) \eta_m(t_m s, x) \, dx \, ds.$$

Therefore,

$$\mathbb{E}_0 \left[\exp\left\{ \beta m t_m \left(\int_0^1 \int_{[-N,N]^d} \eta_m^2(t_m s, x) \, dx \, ds \right)^{1/2} \right\} 1_{\Omega(\upsilon,\delta,m)}; \tau_*(H) \ge t_m \right]$$
$$\le \exp(Ct_m) \max_{f \in (\mathcal{N}_m^\delta)^*} \mathbb{E}_0 \left[\exp\left\{ \frac{\beta m t_m}{1-\sigma} \int_0^1 \int_{[-N,N]^d} f(s,x) \eta_m(t_m s, x) \, dx \, ds \right\}; \right.$$
$$\left. \tau_*(H) \ge t_m \right].$$

Notice that

$$\begin{aligned}(3.19)\qquad &\int_0^1\int_{[-N,N]^d} f(s,x)\eta_m(t_m s,x)\,dx\,ds\\ &= \frac{1}{m}\sum_{j=1}^m \int_0^1\biggl[\int_{[-N,N]^d} f(s,x)Q_N\bigl(x-B_j(t_m s)\bigr)\,dx\biggr]\,ds\\ &= \frac{1}{mt_m}\sum_{j=1}^m \int_0^{t_m} \tilde f\Bigl(\frac{s}{t_m}, B_j(s)\Bigr)\,ds,\end{aligned}$$

where

$$\tilde f(s,x) = \int_{[-N,N]^d} f(s,y)Q_N(y-x)\,dy.$$

Summarizing our argument since (3.16), we conclude that

$$\begin{aligned}(3.20)\qquad &\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)}\log\mathbb{E}_0\exp\biggl\{\beta m t_m\biggl(\int_0^1\int_{\mathbb{R}^d}\eta_m^2(t_m s,x)\,dx\,ds\biggr)^{1/2}\biggr\}\\ &\le t\limsup_{m\to\infty}\frac{1}{t_m}\log\max_{f\in(\mathcal{N}_m^\delta)^*}\mathbb{E}_0\biggl[\exp\biggl\{\frac{\beta}{1-\sigma}\int_0^{t_m}\tilde f\Bigl(\frac{s}{t_m},B(s)\Bigr)\,ds\biggr\};\\ &\qquad\qquad \tau(H)\ge t_m\biggr].\end{aligned}$$

Here we recall our notation

$$\tau(H) = \inf\bigl\{s\ge 0; \bigl|B(s)\bigr|\ge Hm^{(6-\alpha)/(2(2-\alpha))}\bigr\}.$$

Let $f\in(\mathcal{N}_m^\delta)^*$. For large m

$$\begin{aligned}&\mathbb{E}_0\biggl[\exp\biggl\{\frac{\beta}{1-\sigma}\int_0^{t_m}\tilde f\Bigl(\frac{s}{t_m},B(s)\Bigr)\,ds\biggr\};\tau(H)\ge t_m\biggr]\\ &\quad\le \exp\biggl\{\frac{\beta}{1-\sigma}\biggr\}\mathbb{E}_0\biggl[\exp\biggl\{\frac{\beta}{1-\sigma}\int_1^{t_m}\tilde f\Bigl(\frac{s}{t_m},B(s)\Bigr)\,ds\biggr\};\tau(H)\ge t_m\biggr]\\ &\quad= \exp\biggl\{\frac{\beta}{1-\sigma}\biggr\}\\ &\qquad\times\int_{B(0,Hm^{(6-\alpha)/(2(2-\alpha))})} p_1(x)\mathbb{E}_x\biggl[\exp\biggl\{\frac{\beta}{1-\sigma}\\ &\qquad\qquad\times\int_0^{t_m-1}\tilde f\Bigl(\frac{s+1}{t_m},B(s)\Bigr)\,ds\biggr\};\\ &\qquad\qquad\qquad\tau(H)\ge t_m\biggr]\,dx,\end{aligned}$$

where $p_1(x)$ is the density function of $B(1)$ and the second step follows from Markov's property. Here and elsewhere, we adopt the notation $B(0,r)$ for the d-dimensional ball with the center 0 and the radius $r>0$.

By the bound $p_1(x)\le(2\pi)^{-d/2}$, the right-hand side is bounded by a constant multiple of

$$\begin{aligned}&\int_{B(0,Hm^{(6-\alpha)/(2(2-\alpha))})}\mathbb{E}_x\biggl[\exp\biggl\{\frac{\beta}{1-\sigma}\int_0^{t_m-1}\tilde f\biggl(\frac{s+1}{t_m},B(s)\biggr)ds\biggr\};\tau(H)\ge t_m\biggr]\\&\quad\le\bigl|B\bigl(0,Hm^{(6-\alpha)/(2(2-\alpha))}\bigr)\bigr|\\&\qquad\times\exp\biggl\{\int_0^{t_m-1}\sup_{g\in\mathcal{F}_d}\biggl(\frac{\beta}{1-\sigma}\int_{\mathbb{R}^d}\tilde f\biggl(\frac{s+1}{t_m},x\biggr)g^2(x)\,dx\\&\qquad\qquad-\frac12\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\biggr)ds\biggr\},\end{aligned}$$

where the inequality follows from (A.4) in Lemma A.1 in the Appendix. By variable substitution,

$$\begin{aligned}&\int_0^{t_m-1}\sup_{g\in\mathcal{F}_d}\biggl(\frac{\beta}{1-\sigma}\int_{\mathbb{R}^d}\tilde f\biggl(\frac{s+1}{t_m},x\biggr)g^2(x)\,dx-\frac12\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\biggr)ds\\&\quad=t_m\int_{t_m^{-1}}^1\sup_{g\in\mathcal{F}_d}\biggl(\frac{\beta}{1-\sigma}\int_{\mathbb{R}^d}\tilde f(s,x)g^2(x)\,dx\\&\qquad\qquad-\frac12\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\biggr)ds\\&\quad\le t_m\int_0^1\sup_{g\in\mathcal{F}_d}\biggl(\frac{\beta}{1-\sigma}\int_{\mathbb{R}^d}\tilde f(s,x)g^2(x)\,dx-\frac12\int_{\mathbb{R}^d}|\nabla g(x)|^2\,dx\biggr)ds\\&\quad=t_m\sup_{g\in\mathcal{A}_d}\biggl(\frac{\beta}{1-\sigma}\int_0^1\int_{\mathbb{R}^d}\tilde f(s,x)g^2(s,x)\,dx\,ds\\&\qquad\qquad-\frac12\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,dx\,ds\biggr).\end{aligned}$$

Further,

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^d}\tilde f(s,x)g^2(s,x)\,dx\,ds\\&\quad=\int_0^1\int_{[-N,N]^d}f(s,x)\biggl[\int_{\mathbb{R}^d}Q_N(y-x)g^2(s,y)\,dy\biggr]dx\\&\quad\le\biggl(\int_0^1\int_{[-N,N]^d}\biggl[\int_{\mathbb{R}^d}Q_N(y-x)g^2(s,y)\,dy\biggr]^2dx\biggr)^{1/2}.\end{aligned}$$

Summarizing our estimate,

$$\max_{f\in(\mathcal{N}_m^\delta)^*}\mathbb{E}_0\biggl[\exp\biggl\{\frac{\beta}{1-\sigma}\int_0^{t_m}\tilde{f}\biggl(\frac{s}{t_m},B(s)\biggr)\,ds\biggr\};\tau(H)\ge t_m\biggr]$$
$$\le Cm^{(6-\alpha)d/(2(2-\alpha))}\exp\biggl\{M_{\varepsilon,N}\biggl(\frac{\beta}{1-\sigma}\biggr)t_m\biggr\}.$$

Here we introduce the notation

$$M_{\varepsilon,N}(\beta)=\sup_{g\in\mathcal{A}_d}\biggl\{\biggl(\int_0^1\int_{[-N,N]^d}\biggl[\int_{\mathbb{R}^d}Q_N(y-x)g^2(s,y)\,dy\biggr]^2dx\biggr)^{1/2}$$
$$-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,dx\,ds\biggr\}.$$

By (3.20), therefore,

$$\limsup_{m\to\infty}m^{-(4-\alpha)/(2-\alpha)}\log\mathbb{E}_0\exp\biggl\{\beta m t_m\biggl(\int_0^1\int_{\mathbb{R}^d}\eta_m^2(t_m s,x)\,dx\,ds\biggr)^{1/2}\biggr\}$$
$$\le tM_{\varepsilon,N}\biggl(\frac{\beta}{1-\sigma}\biggr). \tag{3.21}$$

By Lemma A.3 in the Appendix,

$$\limsup_{N\to\infty}M_{\varepsilon,N}\biggl(\frac{\beta}{1-\sigma}\biggr)\le M_\varepsilon\biggl(\frac{\beta}{1-\sigma}\biggr).$$

Finally, the requested (3.15) follows from the obvious fact that the right-hand side of the above inequality tends to $M_\varepsilon(\beta)$ as $\sigma\to 0^+$. □

By (3.11),

$$\int_0^{t_m}\int_{\mathbb{R}^d}\biggl[\sum_{j=1}^m K_\varepsilon(x-B_j(s))\biggr]^2dx\,ds$$
$$=mt_m\gamma_\varepsilon(0)+2\sum_{1\le j<k\le m}\int_0^{t_m}\gamma_\varepsilon(B_j(s)-B_k(s))\,ds.$$

The first term on the right-hand side is deterministic and negligible. Thus Lemma 3.4 (with β replaced by $\beta/\sqrt{2}$) can be restated as

$$\lim_{m\to\infty}m^{-(4-\alpha)/(2-\alpha)}$$
$$\times\log\mathbb{E}_0\exp\biggl\{\beta\biggl(t_m\sum_{1\le j<k\le m}\int_0^{t_m}\gamma_\varepsilon(B_j(s)-B_k(s))\,ds\biggr)^{1/2}\biggr\} \tag{3.22}$$
$$=tM_\varepsilon\biggl(\frac{\beta}{\sqrt{2}}\biggr).$$

The next step is to squash ε to zero.

LEMMA 3.5. *For any integer $n \ge 1$,*

$$\mathbb{E}_0\left[\sum_{1\le j<k\le m}\int_0^{t_m}\gamma_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds\right]^n \le \mathbb{E}_0\left[\sum_{1\le j<k\le m}\int_0^{t_m}\gamma\big(B_j(s)-B_k(s)\big)\,ds\right]^n.$$

PROOF. By Fourier transform

$$\sum_{1\le j<k\le m}\int_0^{t_m}\gamma_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds = (2\pi)^{-d} \times\int_{\mathbb{R}^d}\exp\{-\varepsilon|\lambda|^2\}\hat{\gamma}(\lambda)\left[\int_0^{t_m}\sum_{1\le j<k\le m}\exp\{-i\lambda\cdot\big(B_j(s)-B_k(s)\big)\}\,ds\right]d\lambda,$$

where $\hat{\gamma}(\lambda)$ is the Fourier transform of $\gamma(x)$; see (1.14). Here we shall use the fact that $\hat{\gamma}(\lambda)>0$ in our setting. Hence

$$\begin{aligned}\mathbb{E}_0&\left[\sum_{1\le j<k\le m}\int_0^{t_m}\gamma_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds\right]^n\\ &=(2\pi)^{-nd}\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_n\exp\left\{-\varepsilon\sum_{l=1}^n|\lambda_l|^2\right\}\left(\prod_{l=1}^n\hat{\gamma}(\lambda_l)\right)\\ &\qquad\times\int_{[0,t_m]^n}\mathbb{E}_0\prod_{l=1}^n\sum_{1\le j<k\le m}\exp\{-i\lambda_l\cdot\big(B_j(s)-B_k(s)\big)\}\,ds_1\cdots ds_n.\end{aligned}$$

Notice that

$$\mathbb{E}_0\prod_{l=1}^n\sum_{1\le j<k\le m}\exp\{-i\lambda_l\cdot\big(B_j(s)-B_k(s)\big)\}>0.$$

The right-hand side is less than or equal to

$$\begin{aligned}&(2\pi)^{-nd}\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_n\left(\prod_{l=1}^n\hat{\gamma}(\lambda_l)\right)\\ &\quad\times\int_{[0,t_m]^n}\mathbb{E}_0\prod_{l=1}^n\sum_{1\le j<k\le m}\exp\{-i\lambda_l\cdot\big(B_j(s)-B_k(s)\big)\}\,ds_1\cdots ds_n\\ &\qquad=\mathbb{E}_0\left[\sum_{1\le j<k\le m}\int_0^{t_m}\gamma\big(B_j(s)-B_k(s)\big)\,ds\right]^n.\end{aligned}$$

$\square$

By using [4], Lemma 1.2.6, page 13, twice with $p = 2$, Lemma 3.5 and (3.22) lead to

$$\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\left\{\beta\left(t_m \sum_{1\le j<k\le m} \int_0^{t_m} \gamma\big(B_j(s) - B_k(s)\big)\,ds\right)^{1/2}\right\} \ge t M_\varepsilon\left(\frac{\beta}{\sqrt{2}}\right)$$

for every $\varepsilon > 0$. Notice that

$$\liminf_{\varepsilon\to 0^+} M_\varepsilon\left(\frac{\beta}{\sqrt{2}}\right) \ge M\left(\frac{\beta}{\sqrt{2}}\right) = \left(\frac{\beta}{\sqrt{2}}\right)^{4/(4-\alpha)} M(1),$$

where

$$M(\beta) = \sup_{g\in\mathcal{A}_d}\left\{\beta\left(\int_0^1\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} K(y-x) g^2(s,y)\,dy\right]^2 dx\right)^{1/2} - \frac{1}{2}\int_0^1\int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2\,dx\,dy\right\}$$

and the second step comes from the fact that $M(\beta) = \beta^{4/(4-\alpha)} M(1)$ resulted from replacing $g(s,x)$ by $\beta^{d/(4-\alpha)} g(s, \beta^{2/(4-\alpha)} x)$ in the variation $M(\beta)$.

Hence we reach the lower bound

$$\liminf_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \times \log \mathbb{E}_0 \exp\left\{\beta\left(t_m \sum_{1\le j<k\le m} \int_0^{t_m} \gamma\big(B_j(s) - B_k(s)\big)\,ds\right)^{1/2}\right\} \ge t\left(\frac{\beta}{\sqrt{2}}\right)^{4/(4-\alpha)} M(1). \tag{3.23}$$

Write $\zeta_\varepsilon(x) = \gamma(x) - \gamma_\varepsilon(x)$. To have the correspondent upper bound, we prove the following:

LEMMA 3.6. *For every $\beta > 0$,*

$$\lim_{\varepsilon\to 0^+} \limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\left\{\frac{\beta}{m} \sum_{1\le j<k\le m} \int_0^{t_m} \zeta_\varepsilon\big(B_j(s) - B_k(s)\big)\,ds\right\} = 0.$$

PROOF. By Jensen's inequality

$$\mathbb{E}_0 \exp\left\{\frac{\beta}{m} \sum_{1\le j<k\le m} \int_0^{t_m} \zeta_\varepsilon\big(B_j(s) - B_k(s)\big)\,ds\right\} \ge \exp\left\{\frac{\beta}{m} \mathbb{E}_0 \sum_{1\le j<k\le m} \int_0^{t_m} \zeta_\varepsilon\big(B_j(s) - B_k(s)\big)\,ds\right\} \ge 1,$$

where the second inequality follows from Lemma 3.5 with $n=1$. Thus we only need to prove the upper bound estimate.

Write

$$\sum_{1\le j<k\le m}\int_0^{t_m}\zeta_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds=\frac{1}{2}\sum_{j=1}^m\sum_{k:\,k\neq j}\int_0^{t_m}\zeta_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds.$$

By Hölder's inequality

$$\begin{aligned}\mathbb{E}_0\exp&\bigg\{\frac{\beta}{m}\sum_{1\le j<k\le m}\int_0^{t_m}\zeta_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds\bigg\}\\&\le\prod_{j=1}^m\bigg(\mathbb{E}_0\exp\bigg\{\frac{\beta}{2}\sum_{k:\,k\neq j}\int_0^{t_m}\zeta_\varepsilon\big(B_j(s)-B_k(s)\big)\,ds\bigg\}\bigg)^{1/m}\\&=\mathbb{E}_0\exp\bigg\{\frac{\beta}{2}\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_1(s)-B_k(s)\big)\,ds\bigg\}.\end{aligned}$$

We now make use of Fourier transform again. Notice that $\hat{\zeta}_\varepsilon(\lambda)=(1-e^{-\varepsilon|\lambda|^2})\hat{\gamma}(\lambda)\ge 0$. By Fourier inversion

$$\begin{aligned}&\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_1(s)-B_k(s)\big)\,ds\\&\qquad=(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{\zeta}_\varepsilon(\lambda)\bigg[\int_0^{t_m}\sum_{k=2}^m\exp\big\{-i\lambda\cdot\big(B_1(s)-B_k(s)\big)\big\}\,ds\bigg]\,d\lambda.\end{aligned}$$

For any integer $n\ge 1$, by the independence between $B_1(s)$ and $\{B_2(s),\dots,B_m(s)\}$,

$$\begin{aligned}\mathbb{E}_0&\bigg[\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_1(s)-B_k(s)\big)\,ds\bigg]^n\\&=(2\pi)^{-nd}\int_{(\mathbb{R}^d)^n}d\lambda_1\cdots d\lambda_m\bigg(\prod_{l=1}^n\hat{\zeta}_\varepsilon(\lambda_l)\bigg)\\&\quad\times\int_{[0,t_m]^n}\mathbb{E}_0\exp\bigg\{-i\sum_{l=1}^n\lambda_l\cdot B_1(s_l)\bigg\}\\&\qquad\times\mathbb{E}_0\bigg(\prod_{l=1}^n\sum_{k=2}^m\exp\{i\lambda_l\cdot B_k(s_l)\}\bigg)\,ds_1\cdots ds_n.\end{aligned}$$

By the fact that

$$0<\mathbb{E}_0\exp\bigg\{-i\sum_{l=1}^n\lambda_l\cdot B_1(s_l)\bigg\}\le 1\quad\text{and}\quad\mathbb{E}_0\bigg(\prod_{l=1}^n\sum_{k=2}^m\exp\{i\lambda_l\cdot B_k(s_l)\}\bigg)>0,$$

the right-hand side is less than or equal to

$$(2\pi)^{-nd}\int_{(\mathbb{R}^d)^n} d\lambda_1\cdots d\lambda_m\left(\prod_{l=1}^n \hat{\zeta}_\varepsilon(\lambda_l)\right)$$
$$\times\int_{[0,t_m]^n}\mathbb{E}_0\left(\prod_{l=1}^n\sum_{k=2}^m \exp\{i\lambda_l\cdot B_k(s_l)\}\right)ds_1\cdots ds_n$$
$$=\mathbb{E}_0\left[\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_k(s)\big)\,ds\right]^n.$$

Therefore, for any $n=1,2,\ldots$

$$\mathbb{E}_0\left[\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_1(s)-B_k(s)\big)\,ds\right]^n\le\mathbb{E}_0\left[\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_k(s)\big)\,ds\right]^n.$$

By Taylor expansion we conclude that

$$\mathbb{E}_0\exp\left\{\frac{\beta}{2}\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_1(s)-B_k(s)\big)\,ds\right\}$$
$$\le\mathbb{E}_0\exp\left\{\frac{\beta}{2}\sum_{k=2}^m\int_0^{t_m}\zeta_\varepsilon\big(B_k(s)\big)\,ds\right\}=\left(\mathbb{E}_0\exp\left\{\frac{\beta}{2}\int_0^{t_m}\zeta_\varepsilon\big(B(s)\big)\,ds\right\}\right)^{m-1}.$$

Summarizing our argument, we have reduced the problem to the proof of

$$\limsup_{\varepsilon\to 0^+}\lim_{m\to\infty}\frac{1}{t_m}\log\mathbb{E}_0\exp\left\{\beta\int_0^{t_m}\zeta_\varepsilon\big(B(s)\big)\,ds\right\}\le 0 \tag{3.24}$$

for any $\beta>0$.

For the sake of simplicity we consider the case when t_m goes to infinity along the integer times. Notice that $\hat{\zeta}_\varepsilon(\lambda)>0$ for all $\lambda\in\mathbb{R}^d$. Using the same argument as that used in the proof of Lemma 3.5, one can show that for any $x\in\mathbb{R}^d$,

$$\mathbb{E}_x\left[\int_0^1\zeta_\varepsilon\big(B(s)\big)\,ds\right]^n\le\mathbb{E}_0\left[\int_0^1\zeta_\varepsilon\big(B(s)\big)\,ds\right]^n,\qquad n=1,2,\ldots.$$

By Taylor expansion

$$\mathbb{E}_x\exp\left\{\beta\int_0^1\zeta_\varepsilon\big(B(s)\big)\,ds\right\}\le\mathbb{E}_0\exp\left\{\beta\int_0^1\zeta_\varepsilon\big(B(s)\big)\,ds\right\},\qquad x\in\mathbb{R}^d.$$

By Markov's property,

$$\mathbb{E}_0\exp\left\{\beta\int_0^{t_m}\zeta_\varepsilon\big(B(s)\big)\,ds\right\}$$
$$=\mathbb{E}_0\left[\exp\left\{\beta\int_0^{t_m-1}\zeta_\varepsilon\big(B(s)\big)\,ds\right\}\mathbb{E}_{B(t_m-1)}\exp\left\{\beta\int_0^1\zeta_\varepsilon\big(B(s)\big)\,ds\right\}\right]$$
$$\le\mathbb{E}_0\exp\left\{\beta\int_0^{t_m-1}\zeta_\varepsilon\big(B(s)\big)\,ds\right\}\mathbb{E}_0\exp\left\{\beta\int_0^1\zeta_\varepsilon\big(B(s)\big)\,ds\right\}.$$

Continuing this procedure we have

$$\mathbb{E}_0 \exp\left\{\beta \int_0^{t_m} \zeta_\varepsilon\big(B(s)\big)\, ds\right\} \le \left(\mathbb{E}_0 \exp\left\{\beta \int_0^1 \zeta_\varepsilon\big(B(s)\big)\, ds\right\}\right)^{t_m}.$$

Finally, the requested (3.24) follows from the obvious fact that

$$\lim_{\varepsilon\to 0^+} \mathbb{E}_0 \exp\left\{\beta \int_0^1 \zeta_\varepsilon\big(B(s)\big)\, ds\right\} = 1. \qquad \square$$

Write

$$Z_m = \sum_{1\le j<k\le m} \int_0^{t_m} \zeta_\varepsilon\big(B_j(s) - B_k(s)\big)\, ds$$

and $Z_m^+ = \max\{0, Z_m\}$. Given $\delta > 0$ and $\beta > 0$

$$\begin{aligned}\mathbb{E}_0 \exp\left\{\beta\sqrt{t_m Z_m^+}\right\} &\le \exp\{\beta m t_m \delta\} + \mathbb{E}_0 \exp[\{\beta\sqrt{t_m Z_m}\}; Z_m \ge \delta^2 m^2 t_m]\\ &\le \exp\{\beta m t_m \delta\} + \mathbb{E}_0 \exp\left\{\frac{\beta}{\delta m} Z_m\right\}.\end{aligned}$$

By Lemma 3.6,

$$\limsup_{\varepsilon\to 0^+} \limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\left\{\beta\sqrt{t_m Z_m^+}\right\} \le \beta t \delta.$$

Since $\delta > 0$ can be arbitrarily small,

$$\lim_{\varepsilon\to 0^+} \limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\left\{\beta\sqrt{t_m Z_m^+}\right\} = 0. \tag{3.25}$$

We now return to the variation $M_\varepsilon(\beta)$ introduced at the beginning of this subsection. By Jensen's inequality for any $g \in \mathcal{A}_d$,

$$\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} K_\varepsilon(y-x) g^2(s,y)\, dy\right]^2 dx \le \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} K(y-x) g^2(s,y)\, dy\right]^2 dx.$$

Consequently, $M_\varepsilon(\beta) \le M(\beta)$ for any $\beta > 0$.

In view of (3.22) and (3.25), a standard argument of exponential approximation via Hölder inequality leads to

$$\begin{aligned}&\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)}\\ &\quad\times \log \mathbb{E}_0 \exp\left\{\beta\left(t_m \sum_{1\le j<k\le m} \int_0^{t_m} \gamma\big(B_j(s) - B_k(s)\big)\, ds\right)^{1/2}\right\}\\ &\quad\le t M\left(\frac{\beta}{\sqrt{2}}\right) = t\left(\frac{\beta}{\sqrt{2}}\right)^{4/(4-\alpha)} M(1).\end{aligned} \tag{3.26}$$

Combining (3.23) and (3.26)

$$\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\biggl\{\beta\biggl(t_m \sum_{1\le j<k\le m} \int_0^{t_m} \gamma\bigl(B_j(s)-B_k(s)\bigr)\,ds\biggr)^{1/2}\biggr\}$$
$$= t\biggl(\frac{\beta}{\sqrt{2}}\biggr)^{4/(4-\alpha)} M(1).$$

Replacing β by $t^{-1/2}\beta$ leads to

$$\begin{aligned}\lim_{m\to\infty} & m^{-(4-\alpha)/(2-\alpha)} \\ & \times \log \mathbb{E}_0 \exp\biggl\{\beta m^{(4-\alpha)/(2(2-\alpha))} \\ & \qquad \times \biggl(\frac{1}{m}\sum_{1\le j<k\le m} \int_0^{t_m} \gamma\bigl(B_j(s)-B_k(s)\bigr)\,ds\biggr)^{1/2}\biggr\} \\ & = t^{(2-\alpha)/(4-\alpha)}\biggl(\frac{\beta}{\sqrt{2}}\biggr)^{4/(4-\alpha)} M(1).\end{aligned} \tag{3.27}$$

In addition, notice that for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $\gamma_\varepsilon(x) \le C_\varepsilon$ for all $x \in \mathbb{R}^d$. Thus

$$\frac{1}{m}\sum_{1\le j<k\le m} \int_0^{t_m} \gamma_\varepsilon\bigl(B_j(s)-B_k(s)\bigr)\,ds \le C_\varepsilon m t_m = C_\varepsilon t m^{(4-\alpha)/(2-\alpha)}.$$

Together with Lemma 3.6, this implies that for every $\beta > 0$

$$\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\biggl\{\frac{\beta}{m}\sum_{1\le j<k\le m} \int_0^{t_m} \gamma\bigl(B_j(s)-B_k(s)\bigr)\,ds\biggr\} < \infty.$$

Using the second half of Lemma A.2 in the Appendix with

$$b_m = m^{(4-\alpha)/(2-\alpha)}, \qquad p = \frac{2}{2-\alpha}, \qquad C_0 = t\frac{2-\alpha}{4}\biggl(\frac{4M(1)}{4-\alpha}\biggr)^{(4-\alpha)/(2-\alpha)}$$

and

$$X_m = \frac{1}{m}\sum_{1\le j<k\le m} \int_0^{t_m} \gamma\bigl(B_j(s)-B_k(s)\bigr)\,ds,$$

we obtain

$$\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E}_0 \exp\biggl\{\frac{\beta}{m}\sum_{1\le j<k\le m} \int_0^{t_m} \gamma\bigl(B_j(s)-B_k(s)\bigr)\,ds\biggr\}$$
$$= t\frac{2-\alpha}{4}\biggl(\frac{4M(1)}{4-\alpha}\biggr)^{(4-\alpha)/(2-\alpha)} \beta^{2/(2-\alpha)} = t\mathcal{E}(d,\gamma)\biggl(\frac{\beta}{2}\biggr)^{2/(2-\alpha)},$$

where the last equality follows from Lemma A.4 in the Appendix. By the identity in the law

$$\frac{1}{m}\sum_{1\le j<k\le m}\int_0^{t_m}\gamma\big(B_j(s)-B_k(s)\big)\,ds \overset{d}{=} \sum_{1\le j<k\le m}\int_0^{t}\gamma\big(B_j(s)-B_k(s)\big)\,ds,$$

we have

$$\begin{aligned}&\lim_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)}\log\mathbb{E}_0\exp\Big\{\beta\sum_{1\le j<k\le m}\int_0^t\gamma\big(B_j(s)-B_k(s)\big)\,ds\Big\}\\&\quad= t\mathcal{E}(d,\gamma)\Big(\frac{\beta}{2}\Big)^{2/(2-\alpha)}.\end{aligned}\tag{3.28}$$

Proposition 3.3 follows from representations (3.10) and (3.28) (with $\beta=\theta^2$).

REMARK 3.7. Bertini and Cancrini (Theorem 2.6, [1]) claimed a precise formula for $\mathbb{E}u(t,0)^m$ in the setting of Theorem 1.7. Unfortunately, their result is false due to incorrectly using the Skorokhod lemma. On the other hand, (3.9) shows that the relation in Bertini–Cancrini's formulation is asymptotically sound.

REMARK 3.8. By Theorem 6.1 in [6], under the assumptions of Theorem 1.2,

$$\begin{aligned}&\lim_{t\to\infty} t^{-(4-\alpha-2\alpha_0)/(2-\alpha)}\log\mathbb{E}u(t,0)^m\\&\quad= m^{(4-\alpha)/(2-\alpha)}\Big(\frac{\theta^2}{2}\Big)^{2/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma)\end{aligned}\tag{3.29}$$

for every integer $m\ge 1$. Comparing this to (3.3), we find the m-limit and the t-limit are completely consistent. The same can be claimed in the context of Theorem 1.3. The situation is slightly different when it comes to the cases labeled (2) in Table 1 where $V(t,x)$ is white in time. Take the setting of Theorem 1.7, for example. Let $m=2$ in (3.10),

$$\begin{aligned}\mathbb{E}u(t,0)^2&=\mathbb{E}_0\exp\Big\{\theta^2\int_0^t\delta_0\big(B_1(s)-B_2(s)\big)\,ds\Big\}\\&=\mathbb{E}_0\exp\Big\{\frac{\theta^2}{\sqrt{2}}\int_0^t\delta_0\big(B(s)\big)\,ds\Big\}=\mathbb{E}_0\exp\Big\{\frac{\theta^2}{\sqrt{2}}\big|B(t)\big|\Big\},\end{aligned}\tag{3.30}$$

where the last equation follows the well-known identity in law between the Brownian local time and the reflected Brownian motion. Thus

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}u(t,0)^2=\frac{\theta^4}{4}=2^3\frac{\theta^4}{32}.\tag{3.31}$$

In comparison to (3.9) [keep in mind that $m=2$ in (3.31)], we witness a small but interesting gene mutation occurring during the course $m\to\infty$.

4. Modulus continuity. The main goal of this section is to measure the degree of the continuity of $u(t,x)$ in the space variable x by estimating the difference $u(t,x)-u(t,y)$. In the settings of Theorems 1.6 and 1.7 we use the bound

$$\begin{aligned}&\big(\mathbb{E}\big|u(t,x)-u(t,y)\big|^{2m}\big)^{1/(2m)}\\ &\quad\le C|x-y|+\left(8m\int_0^t\big(\mathbb{E}u(s,0)^{2m}\big)^{1/m}\mathcal{I}_s\,ds\right)^{1/2}\\ &\quad\le C|x-y|+\big(\mathbb{E}u(t,0)^{2m}\big)^{1/(2m)}\left(8m\int_0^t\mathcal{I}_{t-s}\,ds\right)^{1/2}\end{aligned}$$

established by Conus et al. ((9.49), [10]), where

$$\mathcal{I}_s=\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(z_1-z_2)h_s(z_1)h_s(z_2)\,dz_1\,dz_2,$$

$$h_s(z)=\big|p_s(z-x)-p_s(z-y)\big|,\qquad z\in\mathbb{R}^d$$

and $C>0$ represents, here and else where in this section, a constant independent of m and x,y that takes possibly different values when appearing in different places.

By (9.51) in [10],

$$\mathcal{I}_s\le C(t-s)^{-\alpha/2}\cdot\left(\frac{|x-y|}{(t-s)^{1/2}}\wedge 1\right).$$

Thus, for $|x-y|\le\sqrt{t}$,

$$\int_0^t\mathcal{I}_{t-s}\,ds\le C\left\{\int_0^{|x-y|^2}s^{-\alpha/2}\,ds+|x-y|\int_{|x-y|^2}^t s^{-(\alpha+1)/2}\,ds\right\}\le C|x-y|.$$

This estimate gives the bound

$$\mathbb{E}\big|u(t,x)-u(t,y)\big|^{2m}\le C^m m!|x-y|^m\mathbb{E}u(t,0)^{2m}$$

or

$$\mathbb{E}\big|u(t,x)-u(t,y)\big|^{m}\le C^m(m!)^{1/2}|x-y|^{m/2}\mathbb{E}u(t,0)^{m}. \tag{4.1}$$

By the classic theory on chaining method (see, e.g., Lemma 9, [7]), (4.1) leads to:

LEMMA 4.1. *In the settings of Theorems 1.6 and 1.7, $u(t,x)$ yields a continuous modification. Moreover, for any $0<\delta<1$ and bounded domain $D\subset\mathbb{R}^d$ there is a $C_\delta(D)>0$ such that for $m\delta>2d$,*

$$\mathbb{E}\sup_{\substack{x\neq y\\ x,y\in D}}\left|\frac{u(t,x)-u(t,y)}{|x-y|^{\delta/2}}\right|^m\le C_\delta(D)(m!)^{1/2}\mathbb{E}u(t,0)^m. \tag{4.2}$$

We now consider the setting of Theorems 1.1, 1.2 and 1.3 where the solution $u(t,x)$ yields the Feynman–Kac representation (1.9) [with $u_0(x)\equiv 1$ according to our agreement]. For any $p>1$, write

$$u_p(t,x)=\mathbb{E}_x\exp\left\{p\theta\int_0^t V\big(t-s,B(s)\big)\,ds\right\}.$$

LEMMA 4.2. *In the setting of Theorems 1.1, 1.2 and 1.3, $u(t,x)$ yields a continuous modification. Moreover, for any $p>1$ such that $q\equiv p(p-1)^{-1}$ is an even number and for any bounded domain $D\subset\mathbb{R}^d$, there is a $C_p(D)>0$ and $\delta>0$ such that for $m\delta>2d$:*

(1) *in the setting of Theorem 1.1,*

$$\mathbb{E}\sup_{\substack{x\neq y\\ x,y\in D}}\left|\frac{u(t,x)-u(t,y)}{|x-y|^{\delta/2}}\right|^m\le (m!)^{1/2}C_p(D)^m\{\mathbb{E}u_p(t,0)^m\}^{1/p};\tag{4.3}$$

(2) *in the settings of Theorems 1.2 and 1.3,*

$$\mathbb{E}\sup_{\substack{x\neq y\\ x,y\in D}}\left|\frac{u(t,x)-u(t,y)}{|x-y|^{\delta/2}}\right|^m\le m!C_p(D)^m\{\mathbb{E}u_p(t,0)^m\}^{1/p}.\tag{4.4}$$

PROOF. The main part of the proof is to establish a bound similar to (4.1). By the mean-value theorem,

$$\left|e^\xi-e^\eta\right|\le|\xi-\eta|\max\{e^\xi,e^\eta\}.$$

By the Feynman–Kac representation (1.9) and Hölder's inequality, for any $y\in\mathbb{R}^d$

$$\begin{aligned}
&\big|u(t,0)-u(t,y)\big|\\
&\quad\le\mathbb{E}_0\left|\exp\left\{\theta\int_0^t V\big(t-s,B(s)\big)\,ds\right\}-\exp\left\{\theta\int_0^t V\big(t-s,y+B(s)\big)\,ds\right\}\right|\\
&\quad\le\theta\mathbb{E}_0\Bigg(\left|\int_0^t V\big(t-s,B(s)\big)\,ds-\int_0^t V\big(t-s,y+B(s)\big)\,ds\right|\\
&\qquad\qquad\times\max\left\{\exp\left\{\theta\int_0^t V\big(t-s,B(s)\big)\,ds\right\},\right.\\
&\qquad\qquad\qquad\left.\exp\left\{\theta\int_0^t V\big(t-s,y+B(s)\big)\,ds\right\}\right\}\Bigg)\\
&\quad\le 2\theta\left(\mathbb{E}_0\left|\int_0^t V\big(t-s,B(s)\big)\,ds-\int_0^t V\big(s,y+B(s)\big)\,ds\right|^q\right)^{1/q}\\
&\qquad\times\{u_p(t,0)+u_p(t,y)\}^{1/p}.
\end{aligned}$$

By Hölder's inequality again,

$$\begin{aligned}&\mathbb{E}\big|u(t,0)-u(t,y)\big|^m\\&\quad\le(2\theta)^m\biggl\{\mathbb{E}\biggl(\mathbb{E}_0\biggl|\int_0^t V\bigl(t-s,B(s)\bigr)\,ds-\int_0^t V\bigl(t-s,y+B(s)\bigr)\,ds\biggr|^q\biggr)^m\biggr\}^{1/q}\\&\qquad\times\bigl\{\mathbb{E}\bigl(u_p(t,0)+u_p(t,y)\bigr)^m\bigr\}^{1/p}.\end{aligned}$$

Notice that

$$\begin{aligned}&\mathbb{E}\biggl(\mathbb{E}_0\biggl|\int_0^t V\bigl(t-s,B(s)\bigr)\,ds-\int_0^t V\bigl(t-s,y+B(s)\bigr)\,ds\biggr|^q\biggr)^m\\&\quad\le\mathbb{E}\otimes\mathbb{E}_0\biggl|\int_0^t V\bigl(t-s,B(s)\bigr)\,ds-\int_0^t V\bigl(t-s,y+B(s)\bigr)\,ds\biggr|^{qm}.\end{aligned}$$

By the triangle inequality and by the stationarity of $u_p(t,x)$ in x,

$$\bigl\{\mathbb{E}\bigl(u_p(t,0)+u_p(t,y)\bigr)^m\bigr\}^{1/p}\le 2^{(p+1)/pm}\bigl(\mathbb{E}u_p(t,0)^m\bigr)^{1/p}.$$

Set

$$S_t(y)=\biggl\{\int_0^t\int_0^t\gamma_0(r-s)\bigl(\gamma\bigl(B(r)-B(s)\bigr)-\gamma\bigl(y+B(r)-B(s)\bigr)\bigr)\,dr\,ds\biggr\}^{1/2}.$$

Notice the fact that the difference

$$\int_0^t V\bigl(t-s,B(s)\bigr)\,ds-\int_0^t V\bigl(t-s,y+B(s)\bigr)\,ds$$

is a Gaussian conditioning on the Brownian motion with conditional variance $2S_t(y)^2$. By the (conditional) Gaussian property,

$$\begin{aligned}&\mathbb{E}_0\otimes\mathbb{E}\biggl[\int_0^t V(t-s,B_s)\,ds-\int_0^t V(t-s,y+B_s)\,ds\biggr]^{qm}\\&\quad=(qm-1)!!(\sqrt{2})^{qm}\mathbb{E}_0 S_t(y)^{qm}.\end{aligned}$$

So we have

$$\begin{aligned}&\mathbb{E}\big|u(t,0)-u(t,y)\big|^m\\&\quad\le 2^{(p+1)/pm}(\sqrt{2}\theta)^m\bigl((qm-1)!!\mathbb{E}_0 S_t(y)^{qm}\bigr)^{1/q}\bigl(\mathbb{E}u_p(t,0)^m\bigr)^{1/p}.\end{aligned}\tag{4.5}$$

Let $\hat{\gamma}(\lambda)$ be the Fourier transform [see (1.14)] of $\gamma(x)$. By Fourier inversion

$$\begin{aligned}S_t(y)^2=(2\pi)^{-d}&\int_{\mathbb{R}^d}\hat{\gamma}(\lambda)\bigl[1-e^{-i\lambda\cdot y}\bigr]\\&\times\biggl[\int_0^t\int_0^t\gamma_0(r-s)\exp\bigl\{i\lambda\cdot\bigl(B(s)-B(r)\bigr)\bigr\}\,dr\,ds\biggr]\,d\lambda.\end{aligned}$$

In the setting of Theorem 1.1

$$S_t^2 \le C_\delta |y|^\delta \int_{\mathbb{R}^d} |\lambda|^\delta |\hat{\gamma}(\lambda)| \left| \int_0^t \int_0^t \gamma_0(r-s) \exp\{i\lambda \cdot (B(s) - B(r))\} \, dr \, ds \right| d\lambda$$
$$\le C_\delta |y|^\delta \left(\int_{\mathbb{R}^d} |\lambda|^\delta |\hat{\gamma}(\lambda)| \, d\lambda \right) \int_0^t \int_0^t \gamma_0(r-s) \, dr \, ds,$$

where $\delta > 0$ is chosen according to the assumption (1.13). By (4.5), by Stirling's formula and by the stationary of $u(t,x)$ in x, we reach the bound

$$\mathbb{E}|u(t,0) - u(t,y)|^m \le C(D)^m (m!)^{1/2} |y|^{m\delta/2} \big(\mathbb{E} u_p(t,0)^m\big)^{1/p},$$

which leads to (4.3) with possibly smaller δ and larger $C(D)$.

We now come to the setting of Theorems 1.2 and 1.3. Notice that $\hat{\gamma}(\lambda)$ is equal to a positive constant multiple of $|\lambda|^{-(d-\alpha)}$, $\prod_{i=1}^d |\lambda_i|^{-(1-\alpha_i)}$ [in the notation of $\lambda = (\lambda_1, \ldots, \lambda_d)$] and 1 in connection to (1) × (I), (1) × (II) and (1) × (III) (labeled in Table 1), respectively. By the first relation in (2.11) and the representation of $S_t(y)$ given above,

$$S_t^2(y) = C \int_{\mathbb{R}\times\mathbb{R}^d} [1 - e^{-i\lambda\cdot y}] \hat{\gamma}(\lambda) \left| \int_0^t |u-s|^{-(1+\alpha_0)/2} \exp\{i\lambda \cdot B(s)\} \, ds \right|^2 du \, d\lambda$$
$$\le C_\delta |y|^\delta \int_{\mathbb{R}\times\mathbb{R}^d} \Gamma_\delta(\lambda) \left| \int_0^t |u-s|^{-(1+\alpha_0)/2} \exp\{i\lambda \cdot B(s)\} \, ds \right|^2 du \, d\lambda,$$

where $\Gamma_\delta(\lambda) = |\lambda|^\delta \hat{\gamma}(\lambda)$ and $\delta > 0$ is a small number. We claim that for sufficiently small δ the process

$$Z_T = \left(\int_{\mathbb{R}\times\mathbb{R}^d} \Gamma_\delta(\lambda) \left| \int_0^T |u-s|^{-(1+\alpha_0)/2} \exp\{i\lambda \cdot B(s)\} \, ds \right|^2 du \, d\lambda \right)^{1/2}, \quad T \ge 0$$

takes finite values almost surely. For the sake of simplicity we show this by controlling $\mathbb{E} Z_T^2$ through a "usual" computation without justification, which is easy to be installed.

By the first relation in (2.11) Z_T^2 is a constant multiple of

$$\int_{\mathbb{R}^d} \Gamma_\delta(\lambda) \left[\int_0^T \int_0^T |r-s|^{-\alpha_0} \exp\{i\lambda \cdot (B(r) - B(s))\} \, dr \, ds \right] d\lambda$$

whose expectation is equal to

$$\int_{\mathbb{R}^d} \Gamma_\delta(\lambda) \left[\int_0^T \int_0^T |r-s|^{-\alpha_0} \mathbb{E}_0 \exp\{i\lambda \cdot (B(r) - B(s))\} \, dr \, ds \right] d\lambda$$
$$= \int_{\mathbb{R}^d} \Gamma_\delta(\lambda) \left[\int_0^T \int_0^T |r-s|^{-\alpha_0} \exp\left\{ -\frac{|r-s|}{2} |\lambda|^2 \right\} dr \, ds \right] d\lambda$$

$$
\begin{aligned}
&= \int_0^T \int_0^T |r-s|^{-d/2} |r-s|^{-\alpha_0} \\
&\qquad \times \left[\int_{\mathbb{R}^d} \Gamma_\delta\left(\frac{\lambda}{\sqrt{|r-s|}} \right) \exp\left\{ -\frac{1}{2}|\lambda|^2 \right\} d\lambda \right] dr\, ds \\
&= \left(\int_{\mathbb{R}^d} \Gamma_\delta(\lambda) \exp\left\{ -\frac{1}{2}|\lambda|^2 \right\} d\lambda \right) \int_0^T \int_0^T |r-s|^{-\alpha_0-(1/2)(\alpha+\delta)}\, dr\, ds,
\end{aligned}
$$

where the second equality follows from the Fubini theorem and integration substitution, and the third equality follows from the fact that $\Gamma_\delta(C\lambda) = C^{-(d-\alpha-\delta)}\Gamma_\delta(\lambda)$ for $C > 0$ and $\lambda \in \mathbb{R}^d$. It is easy to see that the λ-integral on the right-hand side is finite. We mention the fact that $\alpha_0 + \frac{1}{2}\alpha < 1$ under our assumptions. Consequently, one can make the time-integral finite by making $\delta > 0$ sufficiently small so $\alpha_0 + \frac{1}{2}(\alpha + \delta) < 1$.

Clearly, Z_T is a continuous and nonnegative process in this case. By the triangle inequality, for any $S, T > 0$ $Z_{S+T} \le Z_S + Z'_T$ where

$$
\begin{aligned}
Z'_T &= \left(\int_{\mathbb{R}\times\mathbb{R}^d} \Gamma_\delta(\lambda) \left| \int_S^{S+T} |u-s|^{-(1+\alpha_0)/2} \exp\{ i\lambda \cdot B(s) \}\, ds \right|^2 du\, d\lambda \right)^{1/2} \\
&= \left(\int_{\mathbb{R}\times\mathbb{R}^d} \Gamma_\delta(\lambda) \left| \int_0^T |u-s|^{-(1+\alpha_0)/2} \right.\right. \\
&\qquad \left.\left. \times \exp\{ i\lambda \cdot (B(T+s) - B(T)) \}\, ds \right|^2 du\, d\lambda \right)^{1/2}.
\end{aligned}
$$

Consequently, Z'_T is independent of $\{B(s); 0 \le s \le T\}$ and $Z'_T \overset{d}{=} Z_T$. By [4], Theorem 1.3.5, page 21, Z_T is exponential integrable, and the limit

$$
L \equiv \lim_{T\to\infty} \frac{1}{T} \log \mathbb{E}_0 \exp\{Z_T\}
$$

exists.

By Brownian scaling, on the other hand,

$$
Z_T = \left(\frac{T}{t} \right)^{(4-\alpha-\delta-2\alpha_0)/4} Z_t.
$$

Applying a suitable variable substitution, we conclude

$$
\begin{aligned}
&\lim_{a\to\infty} a^{-4/(\alpha+\delta+2\alpha_0)} \log \mathbb{E}_0 \exp\{ \beta a^{(4-\alpha-\delta-2\alpha_0)/(\alpha+\delta+2\alpha_0)} Z_t \} \\
&\quad = Lt\beta^{4/(4-\alpha-\delta-2\alpha_0)}
\end{aligned}
$$

for any $\beta > 0$. By Gärtner–Ellis theorem (Theorem 1.2.4, page 11, [4])

$$
\lim_{a\to\infty} a^{-4/(\alpha+\delta+2\alpha_0)} \log \mathbb{P}_0\{Z_t \ge a\} = -C
$$

for some $C > 0$. Consequently,

$$\mathbb{E}_0 Z_t^{qm} = qm \int_0^\infty a^{qm-1} \mathbb{P}_0\{Z_t \ge a\}\, da \le C^m (m!)^{q(\alpha+\delta+2\alpha_0)/4}, \qquad m = 1, 2, \ldots .$$

In view of (4.5), by the stationary of $u(t, x)$ in x, we obtain the bound

$$\mathbb{E}_0 \big|u(t, x) - u(t, y)\big|^m \le C^m |x - y|^{m\delta/2} (m!)^{(2+\alpha+\delta+2\alpha_0)/4} \big(\mathbb{E} u_p(t, 0)^m\big)^{1/p}, \qquad m = 1, 2, \ldots$$

uniformly for all $x, y \in \mathbb{R}^d$. Notice that $\frac{2+\alpha+\delta+2\alpha_0}{4} < 1$. This bound leads to (4.4). $\square$

5. Tail probability and proof of the upper bounds. A central piece of our approach relies on the precise large deviations for $u(t, x)$. These kinds of results certainly have their independent values. We list them as part of the major theorems. Recall our assumption that $u_0(x) = 1$.

THEOREM 5.1. *Under the assumption of Theorem 1.1,*

$$\lim_{a\to\infty} a^{-2} \log \mathbb{P}\{\log u(t, 0) \ge \lambda a\} = -\frac{\lambda^2}{2\theta^2}\left(\gamma(0) \int_0^t \int_0^t \gamma_0(r - s)\, dr\, ds\right)^{-1}, \tag{5.1}$$

$$\lim_{a\to\infty} a^{-2} \log \mathbb{P}\Big\{\log \max_{x\in D} u(t, x) \ge \lambda a\Big\} = -\frac{\lambda^2}{2\theta^2}\left(\gamma(0) \int_0^t \int_0^t \gamma_0(r - s)\, dr\, ds\right)^{-1} \tag{5.2}$$

for any $t > 0$, $\lambda > 0$ and bounded domain $D \subset \mathbb{R}^d$.

THEOREM 5.2. *Under the assumption of Theorem 1.2,*

$$\lim_{a\to\infty} a^{-(4-\alpha)/2} \log \mathbb{P}\{\log u(t, 0) \ge \lambda a\} = -\frac{4}{\theta^2}\left(\frac{2-\alpha}{\mathcal{E}(\alpha_0, d, \gamma)}\right)^{(2-\alpha)/2} \left(\frac{\lambda}{4-\alpha}\right)^{(4-\alpha)/2} t^{-(4-\alpha-2\alpha_0)/2}, \tag{5.3}$$

$$\lim_{a\to\infty} a^{-(4-\alpha)/2} \log \mathbb{P}\Big\{\log \max_{x\in D} u(t, x) \ge \lambda a\Big\} = -\frac{4}{\theta^2}\left(\frac{2-\alpha}{\mathcal{E}(\alpha_0, d, \gamma)}\right)^{(2-\alpha)/2} \left(\frac{\lambda}{4-\alpha}\right)^{(4-\alpha)/2} t^{-(4-\alpha-2\alpha_0)/2} \tag{5.4}$$

for any $t > 0$, $\lambda > 0$ and bounded domain $D \subset \mathbb{R}^d$, where $\mathcal{E}(\alpha_0, d, \gamma)$ is the variation given in (1.17).

THEOREM 5.3. *Under the assumption of Theorem 1.3,*

$$\lim_{a\to\infty} a^{-3/2}\log\mathbb{P}\{\log u(t,0)\ge\lambda a\} = -\frac{4}{\theta^2}\sqrt{\frac{1}{\mathcal{E}(\alpha_0,1,\delta_0)}}\left(\frac{\lambda}{3}\right)^{3/2} t^{-(3-2\alpha_0)/2}, \tag{5.5}$$

$$\lim_{a\to\infty} a^{-3/2}\log\mathbb{P}\Big\{\log\max_{x\in D} u(t,x)\ge\lambda a\Big\} = -\frac{4}{\theta^2}\sqrt{\frac{1}{\mathcal{E}(\alpha_0,1,\delta_0)}}\left(\frac{\lambda}{3}\right)^{3/2} t^{-(3-2\alpha_0)/2} \tag{5.6}$$

for any $t>0$, $\lambda>0$ *and bounded domain* $D\subset\mathbb{R}^d$.

THEOREM 5.4. *Under the assumption of Theorem 1.6,*

$$\lim_{a\to\infty} a^{-(4-\alpha)/2}\log\mathbb{P}\{\log u(t,0)\ge\lambda a\} = -\frac{4}{\theta^2}\left(\frac{2-\alpha}{t\mathcal{E}(d,\gamma)}\right)^{(2-\alpha)/2}\left(\frac{\lambda}{4-\alpha}\right)^{(4-\alpha)/2}, \tag{5.7}$$

$$\lim_{a\to\infty} a^{-(4-\alpha)/2}\log\mathbb{P}\Big\{\log\max_{x\in D} u(t,x)\ge\lambda a\Big\} = -\frac{4}{\theta^2}\left(\frac{2-\alpha}{t\mathcal{E}(d,\gamma)}\right)^{(2-\alpha)/2}\left(\frac{\lambda}{4-\alpha}\right)^{(4-\alpha)/2} \tag{5.8}$$

for any $t>0$, $\lambda>0$ *and bounded domain* $D\subset\mathbb{R}^d$, *where* $\mathcal{E}(d,\gamma)$ *is the variation given in* (1.26).

THEOREM 5.5. *When* $\gamma_0(\cdot)=\delta_0(\cdot)$, $\gamma(\cdot)=\delta_0(\cdot)$ *and* $\alpha=d=1$,

$$\lim_{a\to\infty} a^{-3/2}\log\mathbb{P}\{\log u(t,0)\ge\lambda a\} = -\frac{4}{\theta^2}\left(\frac{6}{t}\right)^{1/2}\left(\frac{\lambda}{3}\right)^{3/2}, \tag{5.9}$$

$$\lim_{a\to\infty} a^{-3/2}\log\mathbb{P}\Big\{\log\max_{x\in D} u(t,x)\ge\lambda a\Big\} = -\frac{4}{\theta^2}\left(\frac{6}{t}\right)^{1/2}\left(\frac{\lambda}{3}\right)^{3/2}. \tag{5.10}$$

Due to similarity we only prove Theorem 5.2. By Hölder's inequality, for any $b>1$,

$$\big(\mathbb{E}u(t,0)^{[b]}\big)^{1/[b]}\le\big(\mathbb{E}u(t,0)^{b}\big)^{1/b}\le\big(\mathbb{E}u(t,0)^{[b]+1}\big)^{1/([b]+1)}.$$

Thus the limit in (3.3) (Proposition 3.2) can be extended to noninteger m. So (3.3) can be re-written as

$$\lim_{a\to\infty} a^{-(4-\alpha)/2}\log\mathbb{E}\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\} = \beta^{(4-\alpha)/(2-\alpha)}\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma)$$

for every $\beta > 0$.

We now face a problem in using the Gärtner–Ellis theorem: the exponential moment asymptotics is established only for $\beta > 0$, and the random variable $\log u(t,0)$ takes negative values with positive probability. To resolve this problem, we notice that $\mathbb{P}\{u(t,0) \geq 1\} > 0$ and

$$\begin{aligned}
&\mathbb{E}\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\} \\
&\quad = \mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}; u(t,0) < 1\big] \\
&\qquad + \mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}; u(t,0) \geq 1\big] \\
&\quad \leq 1 + \mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}; u(t,0) \geq 1\big].
\end{aligned}$$

We have that for any $\beta > 0$

$$\begin{aligned}
&\liminf_{a\to\infty} a^{-(4-\alpha)/2}\log \mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}|u(t,0) \geq 1\big] \\
&\quad \geq \beta^{(4-\alpha)/(2-\alpha)}\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma).
\end{aligned}$$

On the other hand, by the bound

$$\begin{aligned}
&\mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}|u(t,0) \geq 1\big] \\
&\quad \leq \big(\mathbb{P}\{u(t,0) \geq 1\}\big)^{-1}\mathbb{E}\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\},
\end{aligned}$$

we have that for any $\beta > 0$

$$\begin{aligned}
&\limsup_{a\to\infty} a^{-(4-\alpha)/2}\log \mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}|u(t,0) \geq 1\big] \\
&\quad \leq \beta^{(4-\alpha)/(2-\alpha)}\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma).
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{a\to\infty} a^{-(4-\alpha)/2}\log \mathbb{E}\big[\exp\{\beta a^{(2-\alpha)/2}\log u(t,0)\}|u(t,0) \geq 1\big] \\
&\quad = \beta^{(4-\alpha)/(2-\alpha)}\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma).
\end{aligned}$$

By the Gärtner–Ellis theorem for nonnegative random variables (Theorem 1.2.4, page 11, [4]),

$$\begin{aligned}
&\lim_{a\to\infty} a^{-(4-\alpha)/2}\log \mathbb{P}\{\log u(t,0) \geq \lambda a|u(t,0) \geq 1\} \\
&\quad = -\sup_{\beta>0}\left\{\beta\lambda - \beta^{(4-\alpha)/(2-\alpha)}\left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)}\mathcal{E}(\alpha_0,d,\gamma)\right\} \\
&\quad = -\frac{4}{\theta^2}\left(\frac{2-\alpha}{\mathcal{E}(\alpha_0,d,\gamma)}\right)^{(2-\alpha)/2}\left(\frac{\lambda}{4-\alpha}\right)^{(4-\alpha)/2} t^{-(4-\alpha-2\alpha_0)/2}.
\end{aligned}$$

Therefore, (5.3) follows from the fact that

$$\mathbb{P}\{\log u(t,0) \geq \lambda a\} = \mathbb{P}\{u(t,0) \geq 1\} \cdot \mathbb{P}\{\log u(t,0) \geq \lambda a | u(t,0) \geq 1\}.$$

It remains to prove (5.4). By (5.3) and the stationary of $u(t,x)$ in x, we only need to prove the upper bound. Without loss of generality, we may assume that $0 \in D$. Notice that

$$\sup_{x\in D} |u(t,x) - u(t,0)|^m \leq \operatorname{diam}(D)^{m\delta/2} \sup_{x\in D} \left| \frac{u(t,x) - u(t,0)}{|x|^{\delta/2}} \right|^m,$$

where $\delta > 0$ is determined by (4.4) in Lemma 4.2. By (4.4)

$$\begin{aligned} &\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E} \sup_{x\in D} |u(t,x) - u(t,0)|^m \\ &\quad \leq \frac{1}{p} \limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E} u_p(t,0)^m \end{aligned}$$

for any $p > 1$ with $q \equiv p(p-1)^{-1}$ being an even number. Here we point out that $u_p(t,x)$ is the solution of the parabolic Anderson equation (1.1) satisfying the assumption given in Theorem 1.2 with θ being replaced by θp. By Proposition 3.2, the lim sup on the right-hand side is equal to

$$\left(\frac{(p\theta)^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)} \mathcal{E}(\alpha_0, d, \gamma).$$

Since $p > 1$ can be made arbitrarily close to 1, we conclude that

$$\begin{aligned} &\limsup_{m\to\infty} m^{-(4-\alpha)/(2-\alpha)} \log \mathbb{E} \sup_{x\in D} |u(t,x) - u(t,0)|^m \\ &\quad \leq \left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)} \mathcal{E}(\alpha_0, d, \gamma). \end{aligned}$$

Using Chebyshev's inequality instead of the Gärtner–Ellis theorem,

$$\begin{aligned} &\limsup_{a\to\infty} a^{-(4-\alpha)/2} \log \mathbb{P}\left\{\log \sup_{x\in D} |u(t,x) - u(t,0)| \geq \lambda a\right\} \\ &\quad \leq -\sup_{\beta>0} \left\{\beta\lambda - \beta^{(4-\alpha)/(2-\alpha)} \left(\frac{\theta^2}{2}\right)^{2/(2-\alpha)} t^{(4-\alpha-2\alpha_0)/(2-\alpha)} \mathcal{E}(\alpha_0, d, \gamma)\right\} \qquad (5.11) \\ &\quad = -\frac{4}{\theta^2} \left(\frac{2-\alpha}{\mathcal{E}(\alpha_0, d, \gamma)}\right)^{(2-\alpha)/2} \left(\frac{\lambda}{4-\alpha}\right)^{(4-\alpha)/2} t^{-(4-\alpha-2\alpha_0)/2}. \end{aligned}$$

By the triangle inequality,

$$\sup_{x\in D} u(t,x) \leq u(t,0) + \sup_{x\in D} |u(t,x) - u(t,0)|.$$

Hence,

$$\begin{aligned}\log \sup_{x\in D} u(t,x) &\le \log\Big(u(t,0) + \sup_{x\in D}\big|u(t,x)-u(t,0)\big|\Big)\\ &\le \log 2 + \max\Big\{\log u(t,0), \log \sup_{x\in D}\big|u(t,x)-u(t,0)\big|\Big\}.\end{aligned}$$

For any $0<\lambda'<\lambda$, therefore,

$$\begin{aligned}&\mathbb{P}\Big\{\log \max_{x\in D} u(t,x) \ge \lambda a\Big\}\\ &\quad\le \mathbb{P}\{\log u(t,0)\ge \lambda' a\} + \mathbb{P}\Big\{\log \max_{x\in D}\big|u(t,x)-u(t,0)\big| \ge \lambda' a\Big\}\end{aligned}$$

for large a. Thus

$$\begin{aligned}&\limsup_{a\to\infty} a^{-(4-\alpha)/2}\log \mathbb{P}\Big\{\log\max_{x\in D} u(t,x)\ge \lambda a\Big\}\\ &\quad\le \max\Big\{\limsup_{a\to\infty} a^{-(4-\alpha)/2}\log\mathbb{P}\{\log u(t,0)\ge\lambda' a\},\\ &\qquad\qquad \limsup_{a\to\infty} a^{-(4-\alpha)/2}\log\mathbb{P}\Big\{\log\max_{x\in D}\big|u(t,x)-u(t,0)\big|\ge\lambda' a\Big\}\Big\}\\ &\quad\le -\frac{4}{\theta^2}\bigg(\frac{2-\alpha}{\mathcal{E}(\alpha_0,d,\gamma)}\bigg)^{(2-\alpha)/2}\bigg(\frac{\lambda'}{4-\alpha}\bigg)^{(4-\alpha)/2} t^{-(4-\alpha-2\alpha_0)/2},\end{aligned}$$

where the last step follows from (5.7) and (5.11). Since λ' can be arbitrarily close to λ, we have finally established the upper bound requested by (5.4).

Having Theorems 5.1–5.5 installed, we are ready to prove the upper bounds in Theorems 1.1, 1.2, 1.3, 1.6 and 1.7. Again, due to similarity we only prove the upper bound requested by Theorem 1.6. That is,

$$\begin{aligned}&\limsup_{R\to\infty}(\log R)^{-2/(4-\alpha)}\log\max_{|x|\le R} u(t,x)\\ &\quad\le \frac{4-\alpha}{4}\bigg(\frac{4t\mathcal{E}(d,\gamma)}{2-\alpha}\bigg)^{(2-\alpha)/(4-\alpha)}\theta^{4/(4-\alpha)}d^{2/(4-\alpha)}\qquad \text{a.s.}\end{aligned}\tag{5.12}$$

To this end, we set $\mathcal{N}_R=\mathbb{Z}^d\cap B(0,R)$ and write $Q=[-1,1]^d$. Notice that

$$\max_{|x|\le R} u(t,x)\le \max_{z\in\mathcal{N}_R}\max_{x\in z+Q} u(t,x).$$

For any $\lambda>0$ satisfying

$$\lambda > \frac{4-\alpha}{4}\bigg(\frac{4t\mathcal{E}(d,\gamma)}{2-\alpha}\bigg)^{(2-\alpha)/(4-\alpha)}\theta^{4/(4-\alpha)}d^{2/(4-\alpha)},$$

by stationarity of $u(t,x)$ in x

$$\begin{aligned}&\mathbb{P}\Big\{\log\max_{|x|\le R} u(t,x)\ge\lambda(\log R)^{2/(4-\alpha)}\Big\}\\ &\quad\le \#(\mathcal{N}_R)\mathbb{P}\Big\{\log\max_{x\in Q} u(t,x)\ge\lambda(\log R)^{2/(4-\alpha)}\Big\}.\end{aligned}$$

By (5.8) in Theorem 5.4 there is a $\delta > 0$ such that

$$\mathbb{P}\Big\{\log \max_{x\in Q} u(t,x) \geq \lambda(\log R)^{2/(4-\alpha)}\Big\} \leq \exp\{-(d+\delta)\log R\}$$

as R is sufficiently large. Consequently,

$$\mathbb{P}\Big\{\log \max_{|x|\leq R} u(t,x) \geq \lambda(\log R)^{2/(4-\alpha)}\Big\} \leq CR^{-\delta}$$

with the constant $C > 0$ independent of R. With this bound

$$\sum_{n=1}^{\infty}\mathbb{P}\Big\{\log \max_{|x|\leq 2^n} u(t,x) \geq \lambda\big(\log 2^n\big)^{2/(4-\alpha)}\Big\} < \infty.$$

By the Borel–Cantelli lemma

$$\limsup_{n\to\infty}\big(\log 2^n\big)^{-2/(4-\alpha)} \log \max_{|x|\leq 2^n} u(t,x) \leq \lambda \qquad \text{a.s.}$$

The lim sup can be extended from the sequence 2^n to R due to the monotonicity of the quantity $\log \max_{|x|\leq R} u(t,x)$ in R. Finally, (5.12) follows from the fact that λ can be arbitrarily close to the limit value appearing on the right-hand side of (5.12).

6. Link to the long-term asymptotics: The case of time independence. A classic quenched law (Theorem 5.1, [2]) by Carmona and Molchanov stated that for a homogeneous and time-independent Gaussian potential $V(x)$ whose covariance function $\gamma(x)$ satisfies the conditions comparable to the ones assumed in Theorem 1.1,

$$\lim_{t\to\infty}\frac{1}{t\sqrt{\log t}}\log \mathbb{E}_0 \exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\} = \theta\sqrt{2d\gamma(0)} \qquad \text{a.s.} \tag{6.1}$$

In his recent work, Chen [5] considers the case of the time independent Gaussian field $V(x)$ with the covariance function $\gamma(\cdot)$ in the forms given in Table 1. More specifically, under the assumption $0 < \alpha < 2\wedge d$, and for the $\gamma(\cdot)$ of types (I) and (II) (labeled in Table 1) (Corollary 1.2 and Theorem 1.3, [5]),

$$\begin{aligned}&\lim_{t\to\infty} t^{-1}(\log t)^{-2/(4-\alpha)}\log \mathbb{E}_0\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\}\\ &\quad= \frac{4-\alpha}{4}\bigg(\frac{4\mathcal{E}(d,\gamma)}{2-\alpha}\bigg)^{(2-\alpha)/(4-\alpha)}\theta^{4/(4-\alpha)}d^{2/(4-\alpha)} \qquad \text{a.s.}\end{aligned} \tag{6.2}$$

When $d = 1$ and $\gamma(\cdot) = \delta_0(\cdot)$ (Theorem 1.4, [5]),

$$\lim_{t\to\infty} t^{-1}(\log t)^{-2/3}\log \mathbb{E}_0\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\} = \frac{3}{4}\sqrt[3]{\frac{2}{3}} \qquad \text{a.s.} \tag{6.3}$$

We mention that the right-hand side of (6.2) was initially given in terms of the best constant $\kappa(\gamma,d)$ of the Soblev-type inequality

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)f^2(x)f^2(y)\,dx\,dy\le C\|f\|_2^{4-\alpha}\|\nabla f\|_2^{\alpha}\qquad f\in W^{1,2}(\mathbb{R}^d)$$

and can be switched into the current form, thanks to the identity

$$\mathcal{E}(d,\gamma)=\frac{2-\alpha}{2}\alpha^{\alpha/(2-\alpha)}\kappa(\gamma,d)^{2/(2-\alpha)}$$

which can be derived in the same way as (7.3) in [6].

The striking resemblance of the pairs (1.28) versus (6.1), (1.29) versus (6.2) and (1.30) versus (6.3) suggests a possible link between the time asymptotics and the spatial asymptotics. In this section we explore this problem by providing an alternative treatment to the long-term asymptotics. For similarity, we only consider (6.2).

For the sake of simplicity we assume that t goes ∞ along the integer points. Given $R>0$, define $\tau(R)=\inf\{s\ge 0; |B(s)|\ge R\}$. For any function $R(t)\uparrow\infty$ $(t\to\infty)$, by Markov's property

$$\begin{aligned}&\mathbb{E}_0\Big[\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\};\tau\big(R(t)\big)\ge t\Big]\\&\qquad\le\Big(\max_{|x|\le R(t)}\mathbb{E}_x\exp\Big\{\theta\int_0^1 V\big(B(s)\big)\,ds\Big\}\Big)^t.\end{aligned}\tag{6.4}$$

Applying (1.29) we have

$$\begin{aligned}&\limsup_{t\to\infty}t^{-1}\big(\log R(t)\big)^{-2/(4-\alpha)}\log\mathbb{E}_0\Big[\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\};\tau\big(R(t)\big)\ge t\Big]\\&\qquad\le\frac{4-\alpha}{4}\Big(\frac{4\mathcal{E}(d,\gamma)}{2-\alpha}\Big)^{(2-\alpha)/(4-\alpha)}\theta^{4/(4-\alpha)}d^{2/(4-\alpha)}\qquad\text{a.s.}\end{aligned}\tag{6.5}$$

Further, let $R_k(t)=t(\log t)^{k+1}$ $(k=0,1,2,\ldots)$.

$$\begin{aligned}&\mathbb{E}_0\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\}\\&\quad=\mathbb{E}_0\Big[\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\};\tau\big(R_0(t)\big)\ge t\Big]\\&\qquad+\sum_{k=1}^\infty\mathbb{E}_0\Big[\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\};\tau\big(R_{k-1}(t)\big)<t\le\tau\big(R_k(t)\big)\Big]\\&\quad\le\mathbb{E}_0\Big[\exp\Big\{\theta\int_0^t V\big(B(s)\big)\,ds\Big\};\tau\big(R_0(t)\big)\ge t\Big]\\&\qquad+\sum_{k=1}^\infty\Big(\mathbb{E}_0\Big[\exp\Big\{2\theta\int_0^t V\big(B(s)\big)\,ds\Big\};\tau\big(R_k(t)\big)\ge t\Big]\Big)^{1/2}\\&\qquad\quad\times\big(\mathbb{P}_0\{\tau\big(R_{k-1}(t)\big)<t\}\big)^{1/2}.\end{aligned}$$

By Gaussian tail

$$\mathbb{P}_0\{\tau(R_{k-1}(t)) < t\} \le \exp\left\{-C\frac{R_{k-1}(t)^2}{t}\right\} = \exp\{-Ct(\log t)^{2k}\}, \quad k = 1, 2, \ldots.$$

Together with (6.5), this shows that the infinite series on the right-hand side of the decomposition is negligible. Applying (6.5) to the first term [with $R(t) = R_0(t)$] on the right-hand side of the decomposition leads to the upper bound requested by (6.2).

Relation (6.4) is reversible with some nonsubstantial but technically involved modification, so (1.29) also applies to the lower bound for (6.2). We skip this part of the argument.

REMARK 6.1. An asymptotic bound similar to (6.5) can be extended to the setting of time-dependence with some obvious modification. However, it is unlikely to be sharp in the settings given in Table 1 (with $\alpha_0 > 0$, of course). Compared with the case of time independence, much less is known about the quenched long-term asymptotics in the setting of time-dependence.

APPENDIX

A.1. Feynman–Kac bounds. For any open domain $D \in \mathbb{R}^d$, define $\mathcal{F}_d(D)$ as the class of the functions g supported in D such that $\|g\|_2 = 1$ and $\|\nabla g\|_2 < \infty$. Write

$$\tau_D = \inf\{s \ge 0; B(s) \notin D\}. \tag{A.1}$$

For a function f defined on D, set

$$\lambda_D(f) = \sup_{g \in \mathcal{F}_d(D)} \left\{\int_D f(x) g^2(x)\, dx - \frac{1}{2}\int_D |\nabla g(x)|^2\, dx\right\}. \tag{A.2}$$

LEMMA A.1. *Let $t > 0$, and let the function $f(s, x)$ be continuous and bounded on $[0, t] \times \mathrm{cl}(D)$. Then for any $t > 0$,*

$$\begin{aligned}\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t f(t-s, B(s))\, ds\right\}; \tau_D \ge t\right] dx \\ \le |D| \exp\left\{\int_0^t \lambda_D(f(s,\cdot))\, ds\right\},\end{aligned} \tag{A.3}$$

$$\begin{aligned}\int_D \mathbb{E}_x\left[\exp\left\{\int_0^t f(s, B(s))\, ds\right\}; \tau_D \ge t\right] dx \\ \le |D| \exp\left\{\int_0^t \lambda_D(f(s,\cdot))\, ds\right\}.\end{aligned} \tag{A.4}$$

PROOF. By the Feynman–Kac formula (e.g., Theorem 2.3, page 133, [14] with $g(s,x)=0$), the function

$$u(s,x)=\mathbb{E}_x\bigg[\exp\bigg\{\int_0^s f\big(s-u, B(u)\big)\,du\bigg\}; \tau_D\ge s\bigg], \qquad x\in D$$

solves the initial-boundary problem

$$\begin{cases} \partial_s u(s,x)=\frac{1}{2}\Delta u(s,x)+f(s,x)u(s,x), & (s,x)\in(0,t]\times D,\\ u(s,x)=0, & x\in\partial D,\\ u(0,x)=1, & x\in D. \end{cases}$$

Hence

$$\begin{aligned} \frac{d}{ds}\int_D u^2(s,x)\,dx &= 2\int_D u(s,x)\partial_s u(s,x)\,dx\\ &= 2\bigg\{\int_D f(s,x)u^2(s,x)\,dx-\frac{1}{2}\int_D\big|\nabla_x u(s,x)\big|^2\,dx\bigg\}\\ &\le 2\lambda_D\big(f(s,\cdot)\big)\int_D u^2(s,x)\,dx. \end{aligned}$$

Notice that the function

$$U(s)=\int_D u^2(s,x)\,dx$$

has the initial value $U(0)=|D|$. Thus by Gronwall's inequality

$$\int_D u^2(t,x)\,dx\le |D|\exp\bigg\{2\int_0^t\lambda_D\big(f(s,\cdot)\big)\,ds\bigg\}.$$

Therefore, (A.3) follows from the Cauchy–Schwarz inequality:

$$\int_D u(t,x)\,dx\le\sqrt{|D|}\bigg\{\int_D u^2(t,x)\,dx\bigg\}^{1/2}.$$

Replacing $f(s,x)$ by $f_t(s,x)=f(t-s,x)$ in (A.3) leads to (A.4). □

A.2. A lemma on the large deviations. Let $\{X_m\}$ be a sequence of nonnegative random variables and b_m be a sequence of positive numbers such that $b_m\to\infty$ as $m\to\infty$.

LEMMA A.2. *Assume that there is $p>1$ and $C_0>0$ such that for any $\beta>0$,*

$$\limsup_{m\to\infty}\frac{1}{b_m}\log\mathbb{E}\exp\{\beta X_m\}\le C_0\beta^p, \tag{A.5}$$

$$\liminf_{m\to\infty}\frac{1}{b_m}\log\mathbb{E}\exp\{\beta b_m^{1/2}X_m^{1/2}\}\ge\frac{p+1}{p}(pC_0)^{1/(p+1)}\bigg(\frac{\beta}{2}\bigg)^{(2p)/(p+1)}. \tag{A.6}$$

Then we have

$$\lim_{m\to\infty} \frac{1}{b_m} \log \mathbb{E} \exp\{\beta X_m\} = C_0 \beta^p \qquad \forall \beta > 0. \tag{A.7}$$

The same claim holds if we weaken the first assumption (A.5) *to*

$$\limsup_{m\to\infty} \frac{1}{b_m} \log \mathbb{E} \exp\{\beta X_m\} < \infty \qquad \forall \beta > 0 \tag{A.8}$$

and strengthen the second assumption (A.6) *into*

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{b_m} \log \mathbb{E} \exp\{\beta b_m^{1/2} X_m^{1/2}\} \\ &\quad = \frac{p+1}{p} (pC_0)^{1/(p+1)} \left(\frac{\beta}{2}\right)^{(2p)/(p+1)} \qquad \forall \beta > 0. \end{aligned} \tag{A.9}$$

PROOF. Due to similarity, we only prove the first claim. By (A.5) and by a standard way of using Chebyshev's inequality, for any $\lambda > 0$,

$$\begin{aligned} \limsup_{m\to\infty} \frac{1}{b_m} \log \mathbb{P}\{X_m \ge \lambda b_m\} &\le -\sup_{\beta>0} \{\lambda\beta - C_0 \beta^p\} \\ &= -\frac{p-1}{p} (C_0 p)^{-1/(p-1)} \lambda^{p/(p-1)}, \end{aligned}$$

and for any $\beta > 0$,

$$\limsup_{m\to\infty} \frac{1}{b_m} \log \mathbb{E} \exp\{\beta b_m^{1/2} X_m^{1/2}\} < \infty.$$

By Varadhan's integral lemma (Lemma 4.3.6, [13]),

$$\begin{aligned} &\limsup_{m\to\infty} \frac{1}{b_m} \log \mathbb{E} \exp\{\beta b_m^{1/2} X_m^{1/2}\} \\ &\quad \le \sup_{\lambda>0} \left\{\beta\lambda^{1/2} - \frac{p-1}{p} (C_0 p)^{-1/(p-1)} \lambda^{p/(p-1)}\right\} \\ &\quad = \frac{p+1}{p} (pC_0)^{1/(p+1)} \left(\frac{\beta}{2}\right)^{(2p)/(p+1)}. \end{aligned}$$

Together with (A.6) and the Gärtner–Ellis theorem (Theorem 1.2.4, page 11, [4]), we have

$$\begin{aligned} &\lim_{m\to\infty} \frac{1}{b_m} \log \mathbb{P}\{X_m \ge \lambda b_m\} \\ &\quad = -\sup_{\beta>0} \left\{\beta\sqrt{\lambda} - \frac{p+1}{p} (pC_0)^{1/(p+1)} \left(\frac{\beta}{2}\right)^{(2p)/(p+1)}\right\} \\ &\quad = -\frac{p-1}{p} (C_0 p)^{-1/(p-1)} \lambda^{p/(p-1)}. \end{aligned}$$

Finally, by Varadhan's integral lemma (Lemma 4.3.6, [13])

$$\lim_{m\to\infty}\frac{1}{b_m}\log\mathbb{E}\exp\{\beta X_m\}=\sup_{\lambda>0}\left\{\beta\lambda-\frac{p-1}{p}(C_0p)^{-1/(p-1)}\lambda^{p/(p-1)}\right\}$$
$$=C_0\beta^p. \qquad \square$$

A.3. Variations. Recall that for any $\varepsilon>0$ and $\beta>0$,

$$M_\varepsilon(\beta)=\sup_{g\in\mathcal{A}_d}\left\{\beta\left(\int_0^1\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}K_\varepsilon(y-x)g^2(s,y)\,dy\right]^2dx\,ds\right)^{1/2}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,dx\,dy\right\},$$

$$M_{\varepsilon,N}(\beta)=\sup_{g\in\mathcal{A}_d}\left\{\beta\left(\int_0^1\int_{[-N,N]^d}\left[\int_{\mathbb{R}^d}Q_N(y-x)g^2(s,y)\,dy\right]^2dx\right)^{1/2}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,dx\,ds\right\},$$

where $K_\varepsilon(x)$ and $Q_N(x)$ are defined in (3.12) and (3.14), respectively.

LEMMA A.3. *In the settings marked* (2) *in Table* 1, *for any* $\varepsilon>0$ *and* $\beta>0$,

$$\limsup_{N\to\infty}M_{\varepsilon,N}(\beta)\le M_\varepsilon(\beta).$$

PROOF. Notice that for any $0\le s\le 1$,

$$\int_{\mathbb{R}^d}Q_N(y-x)g^2(s,y)\,dy=\int_{\mathbb{R}^d}Q(y-x)\tilde{g}^2(s,y)\,dy,$$

where

$$\tilde{g}(x)=\sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d}g^2(2\mathbf{k}N+x)},\qquad x\in\mathbb{R}^d.$$

Here we recall

$$Q(x)=K_\varepsilon(x)l\big(M^{-1}|x|\big),$$

where l: $\mathbb{R}^+\longrightarrow[0,1]$ is a smooth function satisfying the following properties: $l(u)=1$ for $u\in[0,1]$, $l(u)=0$ for $u\ge 3$ and $-1\le l'(u)\le 0$ for all $u>0$. By the fact that $Q(\cdot)$ is supported on $[-3M,3M]^d$,

$$M_{\varepsilon,N}(\beta)=\sup_{g\in\mathcal{A}_d}\left\{\left(\int_0^1\int_{[-N,N]^d}\left[\int_{[-\tilde{N},\tilde{N}]^d}Q(y-x)\tilde{g}^2(s,y)\,dy\right]^2dx\right)^{1/2}-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}|\nabla_x g(s,x)|^2\,dx\,ds\right\},$$

where $\widetilde{N} = N + 3M$. We omit the rest of the proof as it follows from the constructive argument used in [3], Lemma A.1, with some minor modification. $\square$

We also use the following notation:

$$M(\beta) = \sup_{g \in \mathcal{A}_d} \left\{ \beta \left(\int_0^1 \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} K(y-x) g^2(s,y)\, dy \right]^2 dx\, ds \right)^{1/2} - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2\, dx\, dy \right\},$$

$$\mathcal{E}(d,\gamma) \equiv \sup_{g \in \mathcal{F}_d} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y)\, dx\, dy - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2\, dx \right\}.$$

LEMMA A.4. *Under the assumptions of Theorems 1.6 and 1.7,*

$$\mathcal{E}(d,\gamma) = \frac{2-\alpha}{2} 2^{\alpha/(2-\alpha)} \left(\frac{4M(1)}{4-\alpha} \right)^{(4-\alpha)/(2-\alpha)}.$$

PROOF. By (2.11), $M(1)$ can be rewritten as

$$M(1) = \sup_{g \in \mathcal{A}_d} \left\{ \left(\int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(s,x) g^2(s,y)\, dx\, dy \right)^{1/2} - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2\, dx\, dy \right\}.$$

Define

$$\mathcal{E}'(d,\gamma) = \sup_{g \in \mathcal{A}_d} \left\{ \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} \gamma(x-y) g^2(s,x) g^2(s,y)\, dx\, dy\, ds - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla_x g(s,x)|^2\, dx\, ds \right\}.$$

Replacing $\gamma_0(s) = |s|^{-\alpha_0}$ by $\gamma_0(s) = \delta_0(s)$ in (7.4), [6], we have

$$\mathcal{E}'(d,\gamma) = \frac{2-\alpha}{2} 2^{\alpha/(2-\alpha)} \left(\frac{4M(1)}{4-\alpha} \right)^{(4-\alpha)/(2-\alpha)}.$$

Therefore, it remains to show that

$$\mathcal{E}'(d,\gamma) = \mathcal{E}(d,\gamma).$$

Indeed, taking $g(s,\cdot)=g(\cdot)\in\mathcal{F}_d$ leads to $\mathcal{E}'(d,\gamma)\ge\mathcal{E}(d,\gamma)$. On the other hand, by the relation $\mathcal{A}_d=\{g(\cdot,\cdot);\, g(s,\cdot)\in\mathcal{F}_d\ \forall 0\le s\le 1\}$,

$$\begin{aligned}\mathcal{E}'(d,\gamma)&\le\int_0^1\sup_{g\in\mathcal{A}_d}\biggl\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(s,x)g^2(s,y)\,dx\,dy\\&\qquad\qquad-\frac{1}{2}\int_{\mathbb{R}^d}\bigl|\nabla_x g(s,x)\bigr|^2\,dx\,dy\biggr\}\,ds\\&=\mathcal{E}(d,\gamma).\end{aligned}$$

□

In connection to the variation $\mathcal{E}(\alpha_0,d,\gamma)$ given in (1.17), write

$$\begin{aligned}\mathcal{E}(0,d,\gamma)=\sup_{g\in\mathcal{A}_d}\biggl\{&\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(s,x)g^2(r,y)\,dx\,dy\,dr\,ds\\&-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(s,x)\bigr|^2\,dx\,ds\biggr\}.\end{aligned}$$

LEMMA A.5. *Under the assumptions of Corollary* 1.5, $\mathcal{E}(0,d,\gamma)=\mathcal{E}(d,\gamma)$.

PROOF. The direction of $\ge$ is obvious. We now consider opposite direction. Let $g\in\mathcal{A}_d$, and write

$$\tilde{g}(x)=\biggl(\int_0^1 g^2(s,x)\,ds\biggr)^{1/2}.$$

Then $\tilde{g}\in\mathcal{F}_d$ and

$$\begin{aligned}&\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(s,x)g^2(r,y)\,dx\,dy\,dr\,ds\\&\qquad=\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)\tilde{g}^2(x)\tilde{g}^2(y)\,dx\,dy\end{aligned}$$

and

$$\nabla\tilde{g}(x)=\biggl(\int_0^1 g^2(s,x)\,ds\biggr)^{-1/2}\int_0^1 g(s,x)\nabla_x g(s,x)\,ds.$$

Hence

$$\begin{aligned}\bigl|\nabla\tilde{g}(x)\bigr|&\le\biggl(\int_0^1 g^2(s,x)\,ds\biggr)^{-1/2}\int_0^1\bigl|g(s,x)\bigr|\cdot\bigl|\nabla_x g(s,x)\bigr|\,ds\\&\le\biggl(\int_0^1\bigl|\nabla_x g(s,x)\bigr|^2\,ds\biggr)^{1/2}.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^d}\bigl|\nabla\tilde{g}(x)\bigr|^2\,dx\le\int_0^1\int_{\mathbb{R}^d}\bigl|\nabla_x g(s,x)\bigr|^2\,dx\,ds.$$

Summarizing our estimate,

$$\begin{aligned}
&\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(s,x)g^2(r,y)\,dx\,dy\,dr\,ds\\
&\quad-\frac{1}{2}\int_0^1\int_{\mathbb{R}^d}\left|\nabla_x g(s,x)\right|^2 dx\,ds\\
&\qquad\le\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)\tilde{g}^2(x)\tilde{g}^2(y)\,dx\,dy-\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla\tilde{g}(x)\right|^2 dx\le\mathcal{E}(d,\gamma).
\end{aligned}$$

Taking supremum over $g\in\mathcal{A}_d$ on the left-hand side completes the proof. □

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