

A MONOTONE SINAI THEOREM¹

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Sinai proved that a nonatomic ergodic measure-preserving system has any Bernoulli shift of no greater entropy as a factor. Given a Bernoulli shift, we show that any other Bernoulli shift that is of strictly less entropy and is stochastically dominated by the original measure can be obtained as a *monotone* factor; that is, the factor map has the property that for each point in the domain, its image under the factor map is coordinatewise smaller than or equal to the original point.

1. Introduction. Let (X, μ) be a probability space. If $T : X \rightarrow X$ is a map such that $\mu \circ T^{-1} = \mu$, then (X, μ, T) is a measure-preserving system, and if every almost-surely T -invariant set has measure zero or one, then the system is ergodic. Let S be a self-map of a measurable space Y . A measurable mapping $\phi : X \rightarrow Y$ such that $\mu \circ \phi^{-1} = \nu$ and $\phi \circ T = S \circ \phi$ on a subset of μ -full measure is a *factor map*; when a factor map exists, we say that (Y, ν, S) is a *factor* of (X, μ, T) . It is well known that in this case, $h(\nu) \leq h(\mu)$, where h is the (Kolmogorov–Sinai) entropy. For a positive integer N , let $[N] := \{0, 1, \dots, N - 1\}$. If $Y = [N]^{\mathbb{Z}}$ is the space of all bi-infinite sequences of a finite number of symbols and $\nu = p^{\mathbb{Z}}$ for some nontrivial probability measure $p = (p_i)_{i=0}^{N-1}$ on $[N]$, and S is the left-shift given by $S(y)_i = y_{i+1}$ for all $i \in \mathbb{Z}$, then we say that $B(p) := (Y, \nu, S)$ is a *Bernoulli shift* on N symbols and that ν is a *Bernoulli measure*. The entropy of the Bernoulli shift $B(p)$ is given by the positive number

$$H(p) := - \sum_{i=0}^{n-1} p_i \log p_i.$$

Sinai [42, 43] proved that if (X, μ, T) is a nonatomic invertible ergodic measure-preserving system of entropy $h > 0$, then it has any Bernoulli shift of any entropy $h' \leq h$ as a factor.

Let (E, \succeq) be a partially ordered Polish space such that the set $M := \{(x, x') \in E^2 : x \succeq x'\}$ is closed in the product topology. For two probability measures α and β on E , we say that α *stochastically dominates* β if the integrals with re-

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spect to α and β satisfy $\alpha(f) \geq \beta(f)$ for all nondecreasing bounded functions $f: E \rightarrow \mathbb{R}$. By Strassen's theorem [45], this is equivalent to the existence of a monotone coupling of α and β ; that is, a measure ρ on $E \times E$ whose projection on the first factor is α , on the second factor is β , and satisfies $\rho(M) = 1$.

In our context, the partial order is defined by $x \succeq x'$ if $x_i \geq x'_i$ for each $i \in \mathbb{Z}$. It is well known that $p^{\mathbb{Z}}$ stochastically dominates $q^{\mathbb{Z}}$ if and only if p stochastically dominates q (where the partial order on $[N]$ is the standard total order), and p stochastically dominates q if and only if $\sum_{i=0}^k p_i \leq \sum_{i=0}^k q_i$ for all $0 \leq k < N$. A factor map, mapping $[N]^{\mathbb{Z}}$ to itself, is said to be *monotone* if $\phi(x) \preceq x$ for each $x \in [N]^{\mathbb{Z}}$. In our context, by the definition of a factor, this is equivalent to the condition $\phi(x)_0 \leq x_0$ for all $x \in [N]^{\mathbb{Z}}$. Notice that if ϕ is a monotone factor from $([N]^{\mathbb{Z}}, \mu)$ to $([N]^{\mathbb{Z}}, \nu)$, then μ stochastically dominates ν .

It follows from the above that two necessary conditions for $B(q)$ to be a monotone factor of $B(p)$ are that p stochastically dominates q and $H(p) \geq H(q)$. Karen Ball and Russell Lyons [3] asked about a partial converse:

If p and q are probability measures on $[N]$ such that p stochastically dominates q and $H(p) > H(q)$, does there exist a monotone factor map from $B(p)$ to $B(q)$?

We answer this question affirmatively.

THEOREM 1. *Let $B(p)$ and $B(q)$ be Bernoulli shifts with symbols in $[N]$ (where one allows the possibility that p and q give zero mass to some symbols). If the entropy of $B(p)$ is strictly greater than that of $B(q)$ and the measure p stochastically dominates q , then $B(q)$ is a monotone factor of $B(p)$.*

Ball [3] proved Theorem 1 in the special case where q only assigns positive mass to the two symbols $\{0, 1\}$, and also in the case where the entropy of $B(p)$ is greater than logarithm of the total number of symbols with positive q -mass; in particular, this implies that if $n > k$ and $q_i = 1/k$ for all $0 \leq i < k$ and $p_i = 1/n$ for all $0 \leq i < n$, then $B(q)$ is a monotone factor of $B(p)$. Ball's proof worked by adapting and extending the methods of the Keane and Smorodinsky [23, 24] proof of the Sinai theorem for case of Bernoulli shift. We will make use of a monotone coupling of two Bernoulli shifts that was defined by Ball (see Section 3.5), having a useful product structure and independence properties.

Another idea that we make use of comes from del Junco's proof [10, 11] of the Sinai theorem. He replaces the combinatorial marriage theorem (see Section 3.3) used by Keane and Smorodinsky and by Ball with a ingenious variation of the quantile coupling (see Section 3.7) that we adapt to handle monotonicity. Our proof will also make use of a version of the marriage theorem of Keane and Smorodinsky, but in a more limited way. By combining the tools of Ball and del Junco, we are able to use the Burton–Rothstein [7, 8] method to produce a monotone factor using the Baire category theorem.

Before we discuss in more detail the idea of the proof of Theorem 1 and other related results in ergodic theory and probability in the next sections, we ask a few questions and state an extension of Theorem 1, where stochastic domination is replaced by a general relation.

QUESTION 1. Is Theorem 1 true if we allow for the possibility that the entropy of $B(p)$ is equal to the entropy of $B(q)$? For example, if $p_0 = \frac{1}{3}$, $p_1 = \frac{2}{3}$, $q_0 = \frac{2}{3}$ and $q_1 = \frac{1}{3}$, we do not know whether $B(q)$ is a monotone factor of $B(p)$. Is it possible that there is a monotone factor that is also an isomorphism?

Motivated by Theorem 1 and the fact that Sinai's theorem does not require the original space to be a Bernoulli shift, we ask if the following monotone Sinai-type theorems are true.

QUESTION 2. Let $B(q)$ be a Bernoulli shift on $[N]$, and let μ be an ergodic shift-invariant nonatomic measure on $[N]^{\mathbb{Z}}$ which stochastically dominates the product measure $q^{\mathbb{Z}}$, and assume that entropy of the system with the measure μ is no less than the entropy of $B(q)$. Sinai's theorem gives that $B(q)$ can always be obtained as a factor of the system with the measure μ , but can it be obtained as a monotone factor?

QUESTION 3. Suppose μ and ν are ergodic shift-invariant nonatomic measures on $[N]^{\mathbb{Z}}$, where μ stochastically dominates ν . Assume that the system with the measure ν can be obtained as a factor of the system with measure μ ; must there exist a monotone factor? For example, the stationary bi-infinite Markov process with transition probabilities given by $q_{00} = \frac{1}{2} = q_{01}$, $q_{10} = \frac{2}{3}$ and $q_{11} = \frac{1}{3}$ can be obtained as a factor of the Bernoulli shift $B(p)$, where $p_0 = p_1 = \frac{1}{2}$ [1], and it is also easy to see that associated Markovian measure ν is stochastically dominated by the product measure $\mu = p^{\mathbb{Z}}$.

Let $R \subset [N] \times [N]$ be a relation on $[N]$. Let p and q be probability measures on $[N]$. Motivated by Strassen's theorem and a question raised by Gurel-Gurevich and Peled [17], Section 1.3, we say that p *R-dominates* q if there exists a probability measure ρ on $[N] \times [N]$ which gives unit mass to set R , and has projections equal to p and q on the first and second copies of $[N]$, respectively. We call the measure ρ an *R-coupling*.

It is an interesting question of Gurel-Gurevich and Peled [17], Section 1.3, who ask, in the general setting of Borel spaces (B_1, ρ_1) and (B_2, ρ_2) , for what relations $R \subset B_1 \times B_2$, does the existence of a *R-coupling* imply the existence of a *deterministic R-coupling*; that is, a coupling ρ for which there exists a function $f: B_1 \rightarrow B_2$ such that $\rho\{(x, f(x)): B_1\} = 1$. We prove the following related result in the more restricted context of Bernoulli factors.

THEOREM 2. *Let $B(p)$ and $B(q)$ be Bernoulli shifts with symbols in $[N]$ (where one allows the possibility that p and q give zero mass to some symbols). Let R be any relation on $[N]$. If the entropy of $B(p)$ is strictly greater than that of $B(q)$, and the measure p R -dominates the measure q , then there exists a factor ϕ from $B(p)$ to $B(q)$ such that $(x_0, \phi(x_0)) \in R$ for all $x \in [N]^{\mathbb{Z}}$.*

We will see that the proof of Theorem 2 does not require any additional work; we will point out the necessary modifications to the proof of Theorem 1, and concentrate on the case of stochastic domination.

2. Background.

2.1. *The isomorphism problem for Bernoulli shifts.* Let (X, μ, T) and (Y, ν, S) be measure-preserving systems. A factor $\phi: X \rightarrow Y$ is an *isomorphism* if $\phi^{-1}: Y \rightarrow X$ is also a factor. A fundamental question in ergodic theory is to ask when are two systems isomorphic [19, 47]. It was an open question whether the two Bernoulli shifts given by $p = (\frac{1}{2}, \frac{1}{2})$ and $q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ were isomorphic, until Kolmogorov gave a negative answer by introducing the idea of entropy from statistical physics into ergodic theory and proving that (Kolmogorov–Sinai) entropy is an isomorphism invariant [22]. Sinai’s theorem and isomorphisms constructed for certain specific cases by Mešalkin [26], [9], page 181, and Blum and Hansen [5] suggested that entropy could be a complete isomorphism-invariant for Bernoulli shifts. Ornstein [29, 30] proved that this was true; any two Bernoulli shifts of equal entropy are isomorphic.

2.2. *Joinings and Baire category.* It is an easy application of the Baire category theorem to prove the existence of a continuous and nowhere differentiable function. Burton and Rothstein [7, 8] had the nice idea to use the Baire category theorem to give a unified treatment of three major results in ergodic theory: Sinai’s factor theorem [42], Ornstein’s isomorphism theorem [29], and Krieger’s generator theorem [25], which states any given any nonatomic invertible ergodic measure-preserving system with finite entropy less than $\log N$, the space $[N]^{\mathbb{Z}}$ can be endowed with a shift-invariant measure that makes it isomorphic to the given system.

Let (X, μ, T) and (Y, ν, S) be measure-preserving systems. A *coupling* of μ and ν is a measure (i.e., not necessarily the product measure) on the product space $X \times Y$ that has as its projections the measures μ and ν ; a coupling that is also invariant under $T \times S$ is a *joining* [13]. The set of joinings is always nonempty because of the product measure, and the set of joinings that are supported on a subset of $X \times Y$ that is a graph, is exactly the set of factors! Burton and Rothstein’s alternative to explicitly constructing factors is to prove they form a residual (large) subset in the set of joinings. Our proof of Theorem 1 will take place in this setting.

A joining ζ of μ and ν is *monotone* if

$$\zeta\{(x, y) \in [N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}} : x_0 \geq y_0\} = 1.$$

If μ stochastically dominates ν , then the space of all ergodic monotone joinings of μ and ν is nonempty. Our proof of Theorem 1 will proceed as follows. We will give more precise definitions later; here we only try to give an idea of the proof. Given a monotone joining of μ and ν , via a construction of Ball, we will perturb it to another monotone joining with large amounts of independence between blocks, then via the coupling of del Junco, we will perturb the resulting joining to obtain a monotone joining that is an ε -almost factor, one that is a factor except on a set of measure less than ε . This will be the key ingredient that will allow us to conclude in the weak-star topology (see Section 3.4) that the set of ε -almost factors is an open dense set for every ε ; intersecting over all ε and using the Baire category theorem implies that the resulting set is nonempty, and thus there exists a monotone factor. Recently, we also used the Burton–Rothstein method to prove a Krieger generator theorem for (nonhyperbolic) toral automorphisms [36].

2.3. Finitary constructions. Keane and Smorodinsky [23, 24] strengthened the results of Sinai and Ornstein by constructing factors that are *finitary*; that is, on a set of full measure the factors constructed by Keane and Smorodinsky are continuous with respect to the product topology on the space $[N]^{\mathbb{Z}}$ and thus have the property that for almost every $x \in [N]^{\mathbb{Z}}$, there is a k such that if $x_i = x'_i$ for all $|i| \leq k$, then $\phi(x')_0 = \phi(x)_0$. See also [12, 20, 37, 39–41] for background and recent developments with regards to finitary factors. Let us also note that Ball’s monotone factor is also finitary [3], but the factor we construct will *not* be. It will be interesting to see if the construction in [20] can be adapted to give monotone factors, since their construction in the case where there is a strict entropy gap, $H(p) > H(q)$, has a coding radius with exponential tails, so that the probability that k of the coordinates of x are insufficient to determine the zeroth coordinate of the image decays to zero exponentially fast as $k \rightarrow \infty$.

QUESTION 4. Is Theorem 1 true with the additional requirement that the factor be finitary?

2.4. Unilateral constructions. Sinai’s original theorem also applies in the case where the original nonatomic ergodic measure-preserving system (X, μ, T) is not invertible, in which case, any one-sided Bernoulli shift on $[N]^{\mathbb{N}}$ of no greater entropy can be obtained as a factor of (X, μ, T) . In particular, for the case of Bernoulli shifts, Sinai defined factor maps that are *unilateral* so that zeroth coordinate of the image of almost every point $x \in [N]^{\mathbb{Z}}$ depends only the future coordinates of x , given by $(x_i)_{i=0}^{\infty}$. Within the powerful framework of Ornstein theory [31], Ornstein and Weiss [32] also extended the one-sided version of the Sinai theorem to mixing Markov chains with positive transitions, but their construction is not finitary; see also the proof and extension to all mixing Markov chains given

by Propp [35]. In the case where both Bernoulli shifts give nonzero mass to at least three symbols, del Junco [10, 11] further strengthened the results of Keane and Smorodinsky by constructing unilateral finitary factors and isomorphisms. Because del Junco was interested in constructing unilateral factors, he defined what he called the star-joining to replace the more combinatorial marriage theorem that is used in the Keane and Smorodinsky proofs, but was not suitable for the unilateral case. Let us also note that the factor we construct will *not* be unilateral.

QUESTION 5. Is Theorem 1 true with the additional requirement that the factor be unilateral?

2.5. *Point processes and monotone thinning.* Ornstein theory also extends to much more general spaces. In particular, Ornstein and Weiss [33] proved that any two (homogeneous) Poisson processes on \mathbb{R}^d are isomorphic. Note that a Poisson process on \mathbb{R}^d is stochastically dominated by a Poisson process on \mathbb{R}^d of higher intensity, and given a Poisson process on \mathbb{R}^d selecting each point independently with some probability fixed probability gives a Poisson process of lower intensity; sometimes this is referred to as independent (randomized) thinning. In the case $d = 1$, Ball [4] proved that any Poisson process can be obtained as a monotone factor of a Poisson process of higher intensity; that is, as a translation-equivariant (nonrandomized) function of the higher intensity process, a set of points is removed so that the remaining set forms a Poisson process of lower intensity. Holroyd, Lyons, and Soo [21] extended this result to all dimensions d . For Poisson processes on a finite volume, Angel, Holroyd and Soo [2] proved a necessary and sufficient condition on the two intensities for the existence of a nonrandomized thinning. See also [17] for the related question of nonrandomized thickening, [16, 28] for cases where nonrandomized equivariant thinning is impossible and [27] for a case where even a monotone invariant coupling is impossible.

3. Some tools used in the proof.

3.1. *Markers.* Let ζ be a joining of the two Bernoulli measures $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$ on $[N]^{\mathbb{Z}}$. Suppose that $p_a, p_b > 0$, where $0 \leq a < b < N$. Let k_{mark} be a large positive integer that we will fix later. Given $x \in [N]^{\mathbb{Z}}$, for $n \in \mathbb{Z}$ we say that $[n, n + 2k_{\text{mark}}] \subset \mathbb{Z}$ is a *marker* if $x_{n+i} = a$ for all $0 \leq i \leq 2k_{\text{mark}} - 1$ and $x_{n+2k_{\text{mark}}} = b$; we call n the *left endpoint* and $n + 2k_{\text{mark}}$ the *right endpoint*. Note that markers have been defined so that no two markers will intersect.

3.2. *The quantile coupling.* The law of a random variable X is the measure given by $\mathbb{P}(X \in \cdot)$, and if X is real-valued, its distribution function is given by $F(x) = F_X(x) := \mathbb{P}(X \leq x)$ for all $x \in \mathbb{R}$. The generalized inverse of a distribution function is given by $F^{-1}(y) := \sup\{x : F(x) < y\}$. If two random variables X and Y have the same law, then we write $X \stackrel{d}{=} Y$.

In probabilistic terms, a coupling of two random variables X and Y is a pair of random variables X' and Y' defined on the same probability space such that $X' \stackrel{d}{=} X$ and $Y' \stackrel{d}{=} Y$. If X and Y take values in finite sets A and B , then an element $x \in A$ is *split* by the coupling if there exist distinct $y, z \in B$ such that $\mathbb{P}(X' = x, Y' = y) > 0$ and $\mathbb{P}(X' = x, Y' = z) > 0$; given a subset $B' \subset B$, we say that $x \in A$ is *split in B'* if there exist distinct $y, z \in B'$ such that $\mathbb{P}(X' = x, Y' = y) > 0$ and $\mathbb{P}(X' = x, Y' = z) > 0$. For a probability measure α on $A \times B$, we define splitting in a similar way.

The quantile coupling is defined in the following way. Let X and Y be two real-valued random variables with distribution functions F and G . Let U be uniformly distributed on the unit interval $[0, 1]$. It is easy to verify that $X' := F^{-1}(U) \stackrel{d}{=} X$ and $Y' := G^{-1}(U) \stackrel{d}{=} Y$ and that if the law of X stochastically dominates the law of Y , then $X' \geq Y'$; see [46], Chapter 1, Section 3, for details.

REMARK 3. A very useful property of the quantile coupling is that if X and Y take values in finite sets A and B , then under the quantile coupling at most $\#B - 1$ elements of A are split.

More generally, if X is a random variable taking values in a totally ordered complete space, then the distribution function $F(x) = \mathbb{P}(X \leq x)$ and its generalized inverse are well defined, so the quantile coupling applies.

3.3. Marriage and coupling. Let A and B be finite sets. If α and α' are probability measures on $A \times B$ such that for all $x \in A$ and all $y \in B$ we have $\alpha(x, y) = 0$ implies $\alpha'(x, y) = 0$, then α' is absolutely continuous with respect to α , and we say that α' is *subordinate* to α .

We will make use of the variation of Keane and Smorodinsky's marriage theorem [23], Theorem 11, stated in the language of measures.

PROPOSITION 4 (Keane and Smorodinsky). *Let A and B be finite sets. If α is a probability measure on $A \times B$, then for all $B' \subset B$ there exists a probability measure α' such that:*

- (i) α' is subordinate to α ,
- (ii) $\alpha'(A, \cdot) = \alpha(A, \cdot)$ and $\alpha'(\cdot, B) = \alpha(\cdot, B)$ and
- (iii) α' splits at most $\#B' - 1$ elements in B' .

The proposition follows immediately from Ball's variation ([3], Lemma 6.1) of [23], Theorem 11, and [3], Lemma 3.2. For more information, see [1], Section 4 and [34], Section 6.5, in Karl Petersen's textbook; in particular, see [34], Chapter 6, Lemma 5.13 for a discussion of the relation between [23], Theorem 11 and the usual Hall marriage theorem [18].

REMARK 5. Note that in Proposition 4 that if $\alpha := \mathbb{P}(X \in \cdot, Y \in \cdot)$ is the joint distribution of random variables X and Y taking values in A and B , respectively, then by (ii) the probability measure α' given by the proposition is a coupling of X and Y . Moreover, by (i), if A and B are subsets of a poset and α is a monotone coupling of X and Y , then so is α' .

3.4. *Weak-star metric.* Let $N > 0$. For $i \geq 0$, let \mathcal{C}_i be the set of measurable $C \subset [N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}}$ that only depend on the coordinates $j \in [-i, i]$, so that $z \in C$ implies that $z' \in C$ if $z_j = z'_j$ for all $j \in [-i, i]$. We define the *weak-star* metric d^* on the space of measures on $[N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}}$ by setting

$$d^*(\zeta, \xi) := \sum_{i=0}^{\infty} 2^{-(i+1)} \sup_{C \in \mathcal{C}_i} |\zeta(C) - \xi(C)|.$$

The metric d^* generates the usual weak-star topology, and convergence in this topology is equivalent to what is sometimes referred to as *weak* convergence in probability theory [38], (7.4), [15], Chapter 11.

3.5. *Ball's joining.* For $A \subset \mathbb{Z}$ and $x \in [N]^{\mathbb{Z}}$, we let $x|_A \in [N]^A$ denote x restricted to the elements of A . Also let $(x, y)|_A = (x|_A, y|_A)$. For the measure ζ on $[N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}}$ and any $A \subset \mathbb{Z}$, we let $\zeta|_A$ denote the measure ζ restricted to $[N]^A \times [N]^A$, so that for all measurable $F \subset [N]^A \times [N]^A$, we have $\zeta|_A(F) = \zeta(F')$, where $F' := \{(x, y) : (x|_A, y|_A) \in F\}$. Sometimes we will refer to $\zeta|_A$ simply as ζ restricted to A .

Ball [3], pages 214–215, defines a joining of two Bernoulli shifts that has certain useful independence properties. Let p and q be probability measures on $[N]$, where p stochastically dominates q . Let ϱ be the quantile (monotone) coupling of p and q . Let ζ be an arbitrary ergodic monotone joining of $p^{\mathbb{Z}}$ and $q^{\mathbb{Z}}$. Then let γ be the monotone coupling of the finite product measures $p^{k_{\text{mark}}}$ and $q^{k_{\text{mark}}}$ given by $\gamma = \gamma_{\zeta} := \zeta|_{[1, k_{\text{mark}}]}$. Here let k_{mark}, a , and b be as in Section 3.1. We define a monotone coupling of $p^{\mathbb{N}}$ and $q^{\mathbb{N}}$ by alternating between γ and ϱ in the following way. If $Z = (X, Y)$ has law γ and $X = (a, \dots, a) = a^{k_{\text{mark}}}$, or if $Z = (X, Y)$ has law ϱ and $X \neq a$, then we say that a *switch* occurs. Let $\dot{\zeta}$ be given by sampling from γ independently until a switch occurs, afterwards, sample from ϱ until a switch occurs; by switching back and forth between γ and ϱ we obtain a monotone coupling $\dot{\zeta}$ of $p^{\mathbb{N}}$ and $q^{\mathbb{N}}$.

To see that $\dot{\zeta}$ is, in fact, a coupling of $p^{\mathbb{N}}$ and $q^{\mathbb{N}}$, let $k, n \geq 1$ and α be a coupling of p^k and q^k , and β be a coupling of p^n and q^n . Observe that if $W := (W_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of random variables with law α and $(R_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of random variables with law β , then for any finite deterministic ℓ the random variable given by $Z_{\ell} := (W_1, \dots, W_{\ell}, R_1, R_2, \dots)$ is a coupling of $p^{\mathbb{N}}$ and $q^{\mathbb{N}}$. Furthermore, if L is a stopping time for W , so that for all positive integers ℓ the event $\{L \leq \ell\}$ belongs to the sigma-algebra generated by $(W_i)_{i=1}^{\ell}$, then it is also true that Z_L is a coupling of $p^{\mathbb{N}}$ and $q^{\mathbb{N}}$. Since the switches also are stopping times, the result follows from repeated applications of this simple observation.

For $\dot{\zeta}$, since a marker consists of $2k_{\text{mark}}$ a 's followed by a b , we see that no matter where the marker starts relative to the switches, where the b occurs the ϱ coupling is used and a switch occurs, so that $\dot{\zeta}$ restricted to the following interval of size k_{mark} will always be obtained from the γ coupling. Recall that we defined S to be the left-shift. By first stationarizing $\dot{\zeta}$, by setting $\ddot{\zeta} := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \dot{\zeta} \circ S^i$, where the limit is taken in the weak-star topology, and then taking the natural extension (for details see, e.g., [14], Section 4.3) of $\ddot{\zeta}$ to $[N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}}$, we obtain a monotone ergodic joining of $p^{\mathbb{Z}}$ and $q^{\mathbb{Z}}$, which we denote by ζ_{alt} and refer to as the *alternating* joining. By the above observation, once we see a marker, then we can determine (using the x variable alone) which of ϱ and γ is being used for all coordinates to the right. Since with probability one, there are markers to the left of any point, we see that almost surely we can, by looking at the x variable, decide which of ϱ and γ is being used at each coordinate. In Ball's paper, the coupling γ is defined to satisfy additional properties that she needs for her argument, but are not needed here.

The joining ζ_{alt} has the following property. For a given $x \in [N]^{\mathbb{Z}}$ we define a bi-infinite sequence of *alternating* intervals $\mathbf{K}(x) = (I_i)_{i \in \mathbb{Z}}$ that partition \mathbb{Z} into intervals of length k_{mark} and 1 in the following way. Locate all the markers of x . Any $n \in \mathbb{Z}$ that belongs to the right endpoint of a marker is an interval of length 1, following a marker will always be an interval of length k_{mark} , and if x restricted to the interval of length k_{mark} is not a string of k_{mark} consecutive a 's, then the following interval will also be one of length k_{mark} , otherwise, the following intervals will all be of length 1, until a symbol that is not a occurs; the following interval will be one of length k_{mark} .

Let $\Gamma = \Gamma_{\zeta}$ be the measure $\zeta|_{[1, k_{\text{mark}}]}$ conditioned so that a switch does not occur. A random variable with law Γ takes values in $[N]^{k_{\text{mark}}} \times [N]^{k_{\text{mark}}}$. For an interval $I \subset \mathbb{Z}$ of size k , we will often make the identification $[N]^I \equiv [N]^k$.

PROPOSITION 6 (Ball). *Let ζ be an ergodic monotone joining of two Bernoulli measures μ and ν . The alternating joining ζ_{alt} is another ergodic monotone joining of μ and ν . If $Z = (X, Y)$ has law ζ_{alt} , then conditional on the alternating intervals $\mathbf{K}(X) = (I_i)_{i \in \mathbb{Z}}$, the random variable Z has the following properties:*

- *The random variables $(Z|_{I_i})_{i \in \mathbb{Z}}$ are independent.*
- *On each alternating interval I of size 1 not immediately to the left of an interval of size k_{mark} , the law of $Z|_I = (a, Y|_I)$ is ϱ conditioned on the event a switch does not occur, otherwise the law of $Z|_I$ is ϱ conditioned on the event that a switch occurs.*
- *On each alternating interval I of size k_{mark} that is not immediately left of an interval of size 1, the law of $Z|_I$ is Γ (a switch does not occur); otherwise it is γ conditioned so that a switch does occur.*

PROOF. The result follows from the definition of ζ_{alt} . \square

Let ζ be a monotone joining of Bernoulli measures. Let $k_{\text{mark}} > 0$, and ζ_{alt} be the associated alternating joining. Let (x, y) be in the support of ζ_{alt} . For each $n \in \mathbb{Z}$, we say n is *frozen* if n belongs an alternating interval in $\mathbf{K}(x)$ of size 1 or an alternating interval of size k_{mark} where a switch occurs. Similarly, we say that any alternating interval of size 1 is *frozen* and any alternating interval where a switch occurs is *frozen*. We say that any coordinate or alternating interval that is not frozen is *free*.

LEMMA 7. *Let ζ be a monotone joining of two Bernoulli measures $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$. Given k_{mark} , let ζ_{alt} be the associated alternating joining. For k_{mark} sufficiently large the probability that an integer $n \in \mathbb{Z}$ belongs to a frozen interval can be made arbitrary small.*

PROOF. Note that if the origin is in an alternating interval of size k_{mark} , then the probability that this interval is a switch is exactly $p_a^{k_{\text{mark}}}$, which goes to zero as $k_{\text{mark}} \rightarrow \infty$. A simple calculation will show that the probability that origin is in an alternating interval of size one can be made arbitrarily small. Let F_i be the event that $i \in \mathbb{Z}$ is an alternating interval of size 1. Note that $\mathbb{P}(F_0) = \mathbb{P}(F_1)$. We have that

$$\begin{aligned} \mathbb{P}(F_1) &= \mathbb{P}(F_1|F_0)\mathbb{P}(F_0) + \mathbb{P}(F_1|F_0^c)\mathbb{P}(F_0^c) \\ &= \mathbb{P}(F_1|F_0)\mathbb{P}(F_1) + \mathbb{P}(F_1|F_0^c)\mathbb{P}(F_0^c) \\ &\leq (1 - p_a)\mathbb{P}(F_1) + p_a^{k_{\text{mark}}}\mathbb{P}(F_0^c); \end{aligned}$$

thus $\mathbb{P}(F_1) \leq p_a^{k_{\text{mark}}-1}$. \square

LEMMA 8. *Let ζ be a monotone joining of two Bernoulli measures $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$. For any $\varepsilon > 0$, there exists k_{mark} sufficiently large so that $d^*(\zeta, \zeta_{\text{alt}}) < \varepsilon$.*

PROOF. It suffices to show that for any integer $n > 0$ and $\varepsilon > 0$, there exists a k_{mark} sufficiently large such that $|\zeta(C) - \zeta_{\text{alt}}(C)| < \varepsilon$ for all $C \in \mathcal{C}_n$.

Let Z and Z' be random variables with laws ζ and ζ_{alt} , respectively. Take $k_{\text{mark}} > 2n + 1$. Let G be the event (measurable with respect to Z') such that the interval $[-n, n]$ is contained in an alternating interval of size k_{mark} , and G^c denote the complement. We have

$$\mathbb{P}(Z' \in C) = \mathbb{P}(Z' \in C|G)\mathbb{P}(G) + \mathbb{P}(Z' \in C|G^c)\mathbb{P}(G^c),$$

for all $C \in \mathcal{C}_n$. By Proposition 6, $\mathbb{P}(Z' \in C|G) = \mathbb{P}(Z \in C)$. By Lemma 7, we can also choose k_{mark} so that $\mathbb{P}(Z' \in G) > 1 - \varepsilon/2$. \square

3.6. *The Shannon–McMillan–Breiman theorem.* The Shannon–McMillan–Breiman theorem [6] states that for an ergodic invariant measure μ on $[N]^{\mathbb{Z}}$ with entropy $h(\mu)$, for μ -almost every $x \in [N]^{\mathbb{Z}}$, we have

$$-\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(C_n(x)) = h(\mu),$$

where $C_n(x) := \{x' \in [N]^{\mathbb{Z}} : x|_{[0,n)} = x'|_{[0,n)}\}$.

Let ζ be an ergodic monotone joining of Bernoulli measures. Recall that Γ was $\zeta|_{[1,k_{\text{mark}}]}$ conditioned so that a switch does not occur. Consider the identification $[N]^{k_{\text{mark}}} \equiv [N^{k_{\text{mark}}}]$, and the measure ζ_{fill} on $[N^{k_{\text{mark}}}]^{\mathbb{Z}} \times [N^{k_{\text{mark}}}]^{\mathbb{Z}}$ given by $\Gamma_{\zeta}^{\mathbb{Z}}$. Let μ_{fill} and ν_{fill} be the respective projections of ζ_{fill} . Note that μ_{fill} and ν_{fill} are Bernoulli measures.

LEMMA 9. *Let ζ be a monotone joining of two Bernoulli measures $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$. Suppose that $H(p) > H(q)$. For k_{mark} sufficiently large, we have $h(\mu_{\text{fill}}) > h(\nu_{\text{fill}})$.*

The proof of Lemma 9 follows from the following lemma, the proof of which will involve some entropy calculations. If X is a discrete random variable taking values in a countable set $E = (e_i)_{i=1}^{\infty}$, with probability distribution r , we set $H(X) = H(r)$. Similarly, if $\mathcal{R} = (R_i)_{i=1}^{\infty}$ is a partition of a probability space, where $r_i = \mathbb{P}(R_i)$, then we also set $H(\mathcal{R}) = H(r)$. We also let $X_{\text{part}} = (\{X = e_i\})_{i=1}^{\infty}$, so that $H(X_{\text{part}}) = H(X)$. For $t \in [0, 1]$, let $\Phi(t) = -t \log t - (1-t) \log(1-t)$, the entropy of a two-element partition with elements of size t and $1-t$.

LEMMA 10. *Let $Z = (X, Y)$ be a jointly distributed pair of random variables, each taking values in a finite set E . Let $e^* \in E$. Let \tilde{X} be the variable X conditioned on $\{X \neq e^*\}$ and \tilde{Y} be the variable Y conditioned on $\{X \neq e^*\}$. Let $u := \mathbb{P}(X = e^*)$. Then:*

- $H(\tilde{X}) \geq H(X) - \Phi(u)$.
- $H(\tilde{Y}) \leq H(Y) + \Phi(u) + u \log(\#E)$.

PROOF. We may assume by relabeling that $E = \{1, 2, \dots, M\}$ and that $e^* = M$. Let $r_i = \mathbb{P}(X = i)$. Then for the first inequality, we have

$$\begin{aligned} H(\tilde{X}) &= -\sum_{i=1}^{M-1} \frac{r_i}{1-r_M} \log\left(\frac{r_i}{1-r_M}\right) \\ &= -\frac{1}{1-r_M} \sum_{i=1}^{M-1} r_i \log r_i + \log(1-r_M) \\ &= \frac{1}{1-r_M} (H(X) + r_M \log r_M + (1-r_M) \log(1-r_M)) \\ &\geq H(X) - \Phi(u). \end{aligned}$$

For the second inequality, let \tilde{Y}' be an independent copy of \tilde{Y} , and define

$$W = \begin{cases} Y, & \text{if } X \neq e^*, \\ \tilde{Y}', & \text{otherwise.} \end{cases}$$

Clearly W has the same distribution as \tilde{Y} . Let \mathcal{Q} be the partition of the probability space into the two sets $\{X = e^*\}$ and $\{X \neq e^*\}$.

We then have

$$\begin{aligned} H(\tilde{Y}) &= H(W) \leq H(W_{\text{part}} \vee Y_{\text{part}} \vee \mathcal{Q}) \\ &= H(W_{\text{part}} | Y_{\text{part}} \vee \mathcal{Q}) + H(Y_{\text{part}} \vee \mathcal{Q}) \\ &\leq H(W_{\text{part}} | Y_{\text{part}} \vee \mathcal{Q}) + H(Y) + H(\mathcal{Q}). \end{aligned}$$

By definition, $H(\mathcal{Q}) = \Phi(u)$.

If $X \neq e^*$ (an event with probability $1 - u$), then knowing in which element of $Y_{\text{part}} \vee \mathcal{Q}$ a point lies, determines W and hence in which element of W_{part} it lies. Otherwise, on a set of measure u , we simply know that W takes values in E . Hence $H(W_{\text{part}} | Y_{\text{part}} \vee \mathcal{Q})$, which is the expected amount of additional information gained by knowing W_{part} when $Y_{\text{part}} \vee \mathcal{Q}$ is already known is at most $u \log(\#E)$. \square

PROOF OF LEMMA 9. Let ζ be a joining of two Bernoulli measures μ and ν on $[N]^{\mathbb{Z}}$, and let (X, Y) have law $\zeta|_{[1, k_{\text{mark}}]} = \gamma$. Note that $H(X) = k_{\text{mark}} H(p)$ and $H(Y) = k_{\text{mark}} H(q)$. Let (\tilde{X}, \tilde{Y}) have law Γ ; that is, γ conditioned on the event that X is not a string of k_{mark} consecutive a 's; note that $\mathbb{P}(X = a^{k_{\text{mark}}}) = p_a^{k_{\text{mark}}}$. Thus with Lemma 10 we have

$$\begin{aligned} (1) \quad h(\mu_{\text{fill}}) - h(\nu_{\text{fill}}) &= H(\tilde{X}) - H(\tilde{Y}) \\ &\geq (H(X) - H(Y)) - (2\Phi(p_a^{k_{\text{mark}}}) - p_a^{k_{\text{mark}}} \log N^{k_{\text{mark}}}) \\ &= k_{\text{mark}}(H(p) - H(q)) - (2\Phi(p_a^{k_{\text{mark}}}) - k_{\text{mark}} p_a^{k_{\text{mark}}} \log N). \end{aligned}$$

Since we assume that $H(p) > H(q)$, the first term on the right-hand side of (1) grows linearly as a function of k_{mark} , whereas the second term decreases to zero exponentially as a function k_{mark} . \square

Note that in our proof of Lemma 9, we made use of the strictness of the inequality $H(p) > H(q)$.

3.7. The star-coupling. Let (X_1, Y_1) and (X_2, Y_2) be finite valued random variables taking values in (E_1, F_1) and (E_2, F_2) , where E_1 and F_2 are totally ordered via $<_1$ and $<_2$. Following del Junco [10, 11], we define the *star-coupling* of (X_1, Y_1) and (X_2, Y_2) in the following way. Set $s_{f_1}(e_1) := \mathbb{P}(X_1 \leq_1 e_1 | Y_1 = f_1)$

and $t_{e_2}(f_2) := \mathbb{P}(Y_2 \leq f_2 | X_2 = e_2)$. Let V_2, V_1 and U be independent random variables uniformly distributed in $[0, 1]$. Set

$$X'_2 := F_{X_2}^{-1}(V_2) \quad \text{and} \quad Y'_1 := F_{Y_1}^{-1}(V_1),$$

so that X'_2 and Y'_1 are independently sampled copies of X_2 and Y_1 . For all $e_2 \in E_2$ and $f_1 \in F_1$, if $X'_2 = e_2$ and $Y'_1 = f_1$, then we define Y'_2 and X'_1 via the following conditional quantile coupling:

$$Y'_2 := t_{e_2}^{-1}(U) \quad \text{and} \quad X'_1 := s_{f_1}^{-1}(U).$$

Clearly $(X'_1, Y'_1) \stackrel{d}{=} (X_1, Y_1)$ and $(X'_2, Y'_2) \stackrel{d}{=} (X_2, Y_2)$.

REMARK 11. In the star-coupling of (X_1, Y_1) and (X_2, Y_2) , X'_2 is independent of (X'_1, Y'_1) and Y'_1 is independent of (X'_2, Y'_2) .

REMARK 12. It follows from Remark 3 that the star-coupling of the random variables (X_1, Y_1) and (X_2, Y_2) taking values on (E_1, F_1) and (E_2, F_2) , respectively, has the property that for a fixed $e_2 \in E_2$ and $f_1 \in F_1$, the number of $e_1 \in E_1$ such that there are distinct $f_2, h_2 \in F_2$ with both (e_1, f_1, e_2, f_2) and (e_1, f_1, e_2, h_2) receiving positive mass under the star-coupling (X'_1, Y'_1, X'_2, Y'_2) is at most $\#F_2 - 1$.

Remark 12, Proposition 4, and the Shannon–McMillan–Breiman theorem [6] lead to the following useful modification of a proposition of del Junco [11], Proposition 4.8.

Let $Z_i := (X_i, Y_i)$ be finite valued random variables where each of the X_i 's and Y_i 's take values on ordered spaces E_i and F_i . We define the *iterative* star-coupling of Z_1, \dots, Z_n to be a random variable W_n taking values on $(E_1 \times \dots \times E_n) \times (F_1 \times \dots \times F_n)$ in the following way for the case $n = 3$; the definition for general n will follow inductively. Let $Z'_1 := (X'_1, Y'_1)$ and $Z'_2 := (X'_2, Y'_2)$ be the star-coupling of Z_1 and Z_2 . Set $W_2 := ((X'_1, X'_2), (Y'_1, Y'_2))$. Note that (X'_1, X'_2) takes values in the space $E_1 \times E_2$, which we endow with the lexicographic ordering. Now let the star-coupling of W_2 and Z_3 be given by $W'_2 := ((X''_1, X''_2), (Y''_1, Y''_2))$ and $Z'_3 := (X'_3, Y'_3)$. Set $W_3 := ((X''_1, X''_2, X'_3), (Y''_1, Y''_2, Y'_3))$.

REMARK 13. Note that in general, even if the star-coupling of (X_1, X_2) and (Y_1, Y_2) is defined, the star-coupling of (X_2, Y_2) and (X_1, Y_1) may not be defined, since the required spaces may not be ordered, and even if they are, there is a lack of commutativity. Note the iterative star-coupling is defined in a certain order, so that the iterative star-coupling of Z_1, Z_2, Z_3 is given by the star-coupling of the star-coupling of (Z_1, Z_2) and Z_3 . It is possible to define the star-coupling so that it is associative [11], Lemma 4.3; this observation is important for del Junco's construction of isomorphisms, but will not be important for us.

PROPOSITION 14 (del Junco). *Let $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$ be Bernoulli measures on $[N]^{\mathbb{Z}}$, where $H(p) > H(q)$ and $p \geq q$. Let ζ be an ergodic monotone joining of μ and ν . Given a sufficiently large integer $k_{\text{mark}} \in \mathbb{Z}^+$ so that the conclusion of Lemma 9 holds, and $\eta > 0$, there exists a $n_{\text{initial}} \in \mathbb{Z}^+$, and random variable \bar{Z}_0 taking values on $[N]^{n_{\text{initial}}k_{\text{mark}}} \times [N]^{n_{\text{initial}}k_{\text{mark}}}$ with law β_{sub} that is subordinate to the measure $\Gamma^{n_{\text{initial}}}$ and has the same marginals, such that for all $n > 0$, the following holds.*

Define $(Z_i)_{i=1}^n$ to be independent random variables with law Γ . Define the following product space:

$$\mathbf{I}_j := [N]^{n_{\text{initial}}k_{\text{mark}}} \times \prod_{i=1}^j [N]^{k_{\text{mark}}} \equiv [N_{\text{mark}}^k]^{n_{\text{initial}}+j}.$$

Let $\mathbf{W}_n = (\mathbf{X}_n, \mathbf{Y}_n)$ be a random variable given by the iterative star-coupling of $\bar{Z}_0, Z_1, \dots, Z_n$. There exists a deterministic function $\Psi : \mathbf{I}_n \rightarrow \mathbf{I}_n$ such that $\mathbb{P}(\mathbf{Y}_n = \Psi(\mathbf{X}_n)) > 1 - \eta$.

The key feature of this proposition is that n can be taken arbitrarily large, independently of k_{mark} and η . When appealing to Proposition 14, we will refer to the set $[N]^{n_{\text{initial}}k_{\text{mark}}} \times [N]^{n_{\text{initial}}k_{\text{mark}}}$ as the *initial block*.

PROOF OF PROPOSITION 14. The proof is an adaptation of [11], Proposition 4.7. We will place conditions on n_{initial} later. Set

$$\mathbf{L}_j := n_{\text{initial}} + j \quad \text{for } 0 \leq j \leq n.$$

Recall that we assumed that k_{mark} was chosen to ensure that $h_{\text{gap}} := h(\mu_{\text{fill}}) - h(\nu_{\text{fill}}) > 0$. Let $\varepsilon \in (0, h_{\text{gap}}/2)$ and

$$(2) \quad \delta := h_{\text{gap}} - 2\varepsilon > 0.$$

Let $\mathbf{x} \in \mathbf{I}_j$ be given by $\mathbf{x} = (x_0, \dots, x_j)$. We say that \mathbf{x} is μ_{fill} -good if

$$(3) \quad \mu_{\text{fill}}|_{[1, \mathbf{L}_j]}(\mathbf{x}) < e^{-(h(\mu_{\text{fill}}) - \varepsilon)\mathbf{L}_j},$$

and is μ_{fill} -completely good if for all $0 \leq i \leq j$, we have $(x_0, \dots, x_i) \in \mathbf{I}_i$ is good. Similarly, for $\mathbf{y} = (y_0, \dots, y_j) \in \mathbf{I}_j$, we say that \mathbf{y} is ν_{fill} -good if

$$(4) \quad \nu_{\text{fill}}|_{[1, \mathbf{L}_j]}(\mathbf{y}) > e^{-(h(\nu_{\text{fill}}) + \varepsilon)\mathbf{L}_j},$$

and is ν_{fill} -completely good if for all $0 \leq i \leq j$, we have $(y_0, \dots, y_i) \in \mathbf{I}_i$ is good.

Let $\mathbf{I}_0(\nu_{\text{fill}})^{\text{good}}$ denote the set of ν_{fill} -good elements of \mathbf{I}_0 . Note that

$$\#\mathbf{I}_0(\nu_{\text{fill}})^{\text{good}} \leq e^{(h(\nu_{\text{fill}}) + \varepsilon)\mathbf{L}_0}.$$

By Proposition 4, there exists a random variable \bar{Z}_0 taking values on $[N]^{n_{\text{initial}}k_{\text{mark}}} \times [N]^{n_{\text{initial}}k_{\text{mark}}}$ with law β_{sub} , that is:

- subordinate to $\Gamma^{n_{\text{initial}}}$,
- has the same marginals as $\Gamma^{n_{\text{initial}}}$ and
- where at most $e^{(h(\nu_{\text{fill}})+\varepsilon)\mathbf{L}_0} - 1$ elements of \mathbf{I}_0 are split in $\mathbf{I}_0(\nu_{\text{fill}})^{\text{good}}$.

Let

$$\mathbf{J}_j := \mathbf{I}_0(\nu_{\text{fill}})^{\text{good}} \times \prod_{i=1}^j [N]^{k_{\text{mark}}}.$$

For $j \geq 0$, let $\mathbf{W}_j = (\mathbf{X}_j, \mathbf{Y}_j)$ be a random variable given by the iterative star-coupling of $\bar{Z}_0, Z_1, \dots, Z_j$, where we set $\mathbf{W}_0 := \bar{Z}_0$; thus \mathbf{X}_j and \mathbf{Y}_j take values in \mathbf{I}_j . We say that $\mathbf{x} \in \mathbf{I}_j$ is *desirable* if the following properties are satisfied:

- (a) The element \mathbf{x} is μ_{fill} -completely good.
- (b) The element \mathbf{x} is not split in \mathbf{J}_j by $\mathbf{W}_j = (\mathbf{X}_j, \mathbf{Y}_j)$.
- (c) Furthermore, there exists a unique ν_{fill} -completely good $\mathbf{y} \in \mathbf{J}_j$ for which (\mathbf{x}, \mathbf{y}) receives positive mass under \mathbf{W}_j .

For desirable $\mathbf{x} \in \mathbf{I}_j$, set $\Psi_j(\mathbf{x}) = \mathbf{y}$, where \mathbf{y} is determined by condition (c); otherwise if \mathbf{x} is not desirable simply set $\Psi_j(\mathbf{x}) = \mathbf{y}'$ for the fixed $\mathbf{y}' \in \mathbf{I}_j$ that is just a block of 0's. Note that

$$\mathbb{P}(\mathbf{Y}_j = \Psi_j(\mathbf{X}_j)) \geq \mathbb{P}(\mathbf{X}_j \text{ is desirable}).$$

Using Remark 12, we will use the inductive argument in the proof of [11], Lemma 4.6, to show that for all $j \geq 0$,

$$(5) \quad \begin{aligned} & \mathbb{P}(\mathbf{X}_j \text{ is not desirable}) \\ & \leq \mathbb{P}(\mathbf{X}_j \text{ is not c.g.}) + \mathbb{P}(\mathbf{Y}_j \text{ is not c.g.}) + e^{-\delta\mathbf{L}_0} + N^{k_{\text{mark}}} \sum_{i=0}^{j-1} e^{-\delta\mathbf{L}_i}, \end{aligned}$$

where ‘‘c.g.’’ is short for completely good.

The case $j = 0$ is easy, since being good implies being completely good, and under \bar{Z}_0 at most $e^{(h(\nu_{\text{fill}})+\varepsilon)\mathbf{L}_0} - 1$ elements of \mathbf{I}_0 are split in \mathbf{J}_0 ; thus by (3), the μ_{fill} -measure of all the μ_{fill} -good elements that are split by \mathbf{J}_0 is at most $e^{-(h(\mu_{\text{fill}})-\varepsilon)\mathbf{L}_0} \times e^{(h(\nu_{\text{fill}})+\varepsilon)\mathbf{L}_0} \leq e^{-\delta\mathbf{L}_0}$, by (2).

Assume (5) for the case $j - 1 \geq 0$. We show that (5) holds for the case j . Let E be the event that \mathbf{X}_{j-1} is desirable but \mathbf{X}_j is not desirable. Clearly,

$$(6) \quad \mathbb{P}(\mathbf{X}_j \text{ is not desirable}) \leq \mathbb{P}(\mathbf{X}_{j-1} \text{ is not desirable}) + \mathbb{P}(E).$$

Note that on the event E , the random variables \mathbf{X}_{j-1} and \mathbf{Y}_{j-1} are completely good. Observe that the event E is contained in the following three events:

- (I) $E_1 :=$ The random variable \mathbf{X}_j is not good, but \mathbf{X}_{j-1} is completely good.
- (II) $E_2 :=$ The random variable \mathbf{X}_j is completely good, but is split in \mathbf{J}_j under the iterative star-coupling \mathbf{W}_j , even though \mathbf{X}_{j-1} is desirable.

(III) $E_3 :=$ The random variable \mathbf{Y}_j is not good, but \mathbf{Y}_{j-1} is completely good.

Clearly,

$$(7) \quad \mathbb{P}(E_1) + \mathbb{P}(\mathbf{X}_{j-1} \text{ is not c.g.}) = \mathbb{P}(\mathbf{X}_j \text{ is not c.g.}).$$

Similarly,

$$(8) \quad \mathbb{P}(E_3) + \mathbb{P}(\mathbf{Y}_{j-1} \text{ is not c.g.}) = \mathbb{P}(\mathbf{Y}_j \text{ is not c.g.}).$$

Let us focus on the event E_2 . Let $\mathbf{X}_j = (\mathbf{X}_{j-1}, X)$, so that X takes values in $[N]^{k_{\text{mark}}}$. We show that for any $x \in [N]^{k_{\text{mark}}}$ and any completely good $\mathbf{y} \in \mathbf{J}_{j-1}$ that

$$(9) \quad \mathbb{P}(E_2 | X = x, \mathbf{Y}_{j-1} = \mathbf{y}) \leq N^{k_{\text{mark}}} e^{-\delta \mathbf{L}_{j-1}},$$

so that $\mathbb{P}(E_2) \leq N^{k_{\text{mark}}} e^{-\delta \mathbf{L}_{j-1}}$ and it follows that (5) holds by (6), (7), (8) and the inductive hypothesis.

Note that if \mathbf{x} and \mathbf{y} are good, then

$$(10) \quad \begin{aligned} \mathbb{P}(\mathbf{X}_{j-1} = \mathbf{x} | X = x, \mathbf{Y}_{j-1} = \mathbf{y}) &= \frac{\mathbb{P}(\mathbf{X}_{j-1} = \mathbf{x}, \mathbf{Y}_{j-1} = \mathbf{y}, X = x)}{\mathbb{P}(\mathbf{Y}_{j-1} = \mathbf{y}, X = x)} \\ &= \frac{\mathbb{P}(\mathbf{X}_{j-1} = \mathbf{x}, \mathbf{Y}_{j-1} = \mathbf{y})}{\mathbb{P}(\mathbf{Y}_{j-1} = \mathbf{y})} \\ &\leq \frac{\mathbb{P}(\mathbf{X}_{j-1} = \mathbf{x})}{\mathbb{P}(\mathbf{Y}_{j-1} = \mathbf{y})} \\ (11) \quad &\leq e^{-\delta \mathbf{L}_{j-1}}, \end{aligned}$$

where (10) follows from Remark 11 (with $X = X'_2$ and $\mathbf{Y}_{j-1} = \mathbf{Y}'_1$) and (11) follows from (3), (4) and (2). Also note that if \mathbf{x} is desirable, then (\mathbf{x}, x) is split under \mathbf{W}_j if and only if for the unique \mathbf{y} for which (\mathbf{x}, \mathbf{y}) receives positive mass under \mathbf{W}_{j-1} there exist distinct $y, y' \in [N]^{k_{\text{mark}}}$ for which both $((\mathbf{x}, x), (\mathbf{y}, y))$ and $((\mathbf{x}, x), (\mathbf{y}, y'))$ receive positive mass under \mathbf{W}_j . By Remark 12, for a fixed $x \in [N]^{k_{\text{mark}}}$ and $\mathbf{y} \in \mathbf{J}_{j-1}$, the set of all \mathbf{x} such that there exists distinct $y, y' \in [N]^{k_{\text{mark}}}$ for which both $((\mathbf{x}, x), (\mathbf{y}, y))$ and $((\mathbf{x}, x), (\mathbf{y}, y'))$ receive positive mass under \mathbf{W}_j has at most $N^{k_{\text{mark}}} - 1$ elements; thus summing over all such \mathbf{x} yields (9).

The Shannon–McMillan–Breiman theorem implies that n_{initial} can be chosen so that all four terms in (5) can be made smaller than $\eta/4$. This is done in the following way.

Set

$$S_\mu(k, K) := \{\mathbf{x} \in [N^{k_{\text{mark}}}]^K : \mu_{\text{fill}}|_{[1, \ell]}(\mathbf{x}) < e^{-(h(\mu_{\text{fill}}) - \varepsilon)\ell} \text{ for all } k \leq \ell \leq K\}$$

and

$$S_\nu(k, K) := \{\mathbf{y} \in [N^{k_{\text{mark}}}]^K : \nu_{\text{fill}}|_{[1, \ell]}(\mathbf{y}) > e^{-(h(\nu_{\text{fill}}) + \varepsilon)\ell} \text{ for all } k \leq \ell \leq K\}.$$

By the Shannon–McMillan–Breiman theorem choose n_{initial} so that for all $K > n_{\text{initial}}$, we have

$$(12) \quad \mu_{\text{fill}}|_{[1, K]}(S_\mu(n_{\text{initial}}, K)) > 1 - \eta/4 \quad \text{and}$$

$$(13) \quad \nu_{\text{fill}}|_{[1, K]}(S_\nu(n_{\text{initial}}, K)) > 1 - \eta/4;$$

we can also require that

$$(14) \quad N^{k_{\text{mark}}} \sum_{i=n_{\text{initial}}}^{\infty} e^{-\delta i} < \eta/4.$$

Conditions (12) and (13) give that $\mathbb{P}(\mathbf{X}_j \text{ is not c.g.}) < \eta/4$ and $\mathbb{P}(\mathbf{Y}_j \text{ is not c.g.}) < \eta/4$, and (14) ensures that $e^{-\delta L_0} \leq \eta/4$, and $N^{k_{\text{mark}}} \sum_{i=0}^{j-1} e^{-\delta L_j} < \eta/4$; thus all four terms on the right-hand side of inequality (5) are less than $\eta/4$. \square

REMARK 15. In proof of Proposition 14, recall that we appealed to Remark 12 which is the reason for the term $N^{k_{\text{mark}}}$ in (5). Since our proof of Theorem 14 relies on Proposition 14, we do not know if the analogue of Theorem 1 is true if q gives positive mass to a countable number of symbols. Note the Sinai and Ornstein theorems include the case where the entropy is possibly infinite and there are a countable number of symbols. See, for example, [14], Section 4.5, for a recent treatment.

4. Proof of Theorem 1.

4.1. *The alternating star-joining.* Let ζ be a monotone joining of the two Bernoulli measures μ and ν , and let $k_{\text{mark}} > 0$, and ζ_{alt} be its associated alternating joining. Assume that we have already applied Proposition 14 to obtain an n_{initial} and a probability measure β_{sub} on $[N]^{n_{\text{initial}}k_{\text{mark}}} \times [N]^{n_{\text{initial}}k_{\text{mark}}}$ that is subordinate to $\Gamma^{n_{\text{initial}}}$ and has the same marginals. By re-sampling on (most of the) free intervals of ζ_{alt} by using the star-coupling, we will produce another monotone joining $\zeta_{\text{alt}*}$ of μ and ν . We define $\zeta_{\text{alt}*}$ in the following way.

Let r_{mark} be a large integer to be chosen later. A *super marker* is the maximal union of at least r_{mark} consecutive markers, and we call the set of integers between and not including two super markers a *large block*.

Let $Z = (X, Y)$ have law ζ_{alt} . Call any large block with at least n_{initial} free intervals an *action block*; we re-sample only on the action blocks. We define a new random variable $Z' = (X', Y')$ taking values on $[N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}}$ by first declaring that on every frozen interval or free interval I not belonging to an action block that $Z'|_I = Z|_I$. Next, for a fixed action block, let Z_0 (taking values in $[N]^{k_{\text{mark}}n_{\text{initial}}} \times [N]^{k_{\text{mark}}n_{\text{initial}}}$) be Z restricted to the first n_{initial} free intervals. Let $(I_i)_{i=1}^n$ be the remaining free intervals, and let $Z_i = (X_i, Y_i)$ (taking values in $[N]_{\text{mark}}^k \times [N]_{\text{mark}}^k$) be Z restricted to I_i . By Proposition 3.5, conditional on the

alternating intervals $\mathbf{K}(X)$, we have that $(Z_i)_{i=1}^n$ is an i.i.d. sequence of random variables with law Γ_ζ , and the law of $Z_0 = (X_0, Y_0)$ is given by the law of $\Gamma_\zeta^{n_{\text{initial}}}$, and is also independent of $(Z_i)_{i=1}^n$. Let $\bar{Z}_0 = (\bar{X}_0, \bar{Y}_0)$ have law given by the measure β_{sub} . Take the iterative star-coupling of the random variables $\bar{Z}_0, Z_1, \dots, Z_n$ to obtain a random variable

$$\mathbf{W} = ((\bar{X}'_0, X'_1, \dots, X'_n), (\bar{Y}'_0, Y'_1, \dots, Y'_n)),$$

taking values on $[N]^{k_{\text{mark}} n_{\text{initial}}} [N]^{k_{\text{mark}} n} \times [N]^{k_{\text{mark}} n_{\text{initial}}} [N]^{k_{\text{mark}} n}$, which we call the *star-filler* for an action block.

By Remark 11, independence of the $(Z_i)_{i=0}^n$ and the fact that β_{sub} is a coupling of X_0 and Y_0 , we have

$$(15) \quad \begin{aligned} (X_0, X_1, \dots, X_n) &\stackrel{d}{=} (\bar{X}'_0, X'_1, \dots, X'_n) \quad \text{and} \\ (Y_0, Y_1, \dots, Y_n) &\stackrel{d}{=} (\bar{Y}'_0, Y'_1, \dots, Y'_n). \end{aligned}$$

For each $k > 0$, let \succeq_k be the partial order on $[N]^k$ defined by $x \succeq_k x'$ if and only if $x_i \geq x'_i$ for all $1 \leq i \leq k$. Since ζ is a monotone joining, ζ_{alt} is a monotone joining, and we have that $X_i \succeq_{k_{\text{mark}}} Y_i$ for all $1 \leq i \leq n$, and we also have that $\bar{X}_0 \succeq_{k_{\text{mark}} n_{\text{initial}}} \bar{Y}_0$ since β_{sub} is subordinate to $\Gamma_\zeta^{n_{\text{initial}}}$. By the definition of the iterative star-coupling we also have that

$$(16) \quad \begin{aligned} (\bar{X}_0, \bar{Y}_0) &\stackrel{d}{=} (\bar{X}'_0, \bar{Y}'_0) \quad \text{and} \\ (X_i, Y_i) &\stackrel{d}{=} (X'_i, Y'_i) \quad \text{for all } 1 \leq i \leq n; \end{aligned}$$

in particular, this implies that

$$(17) \quad \bar{X}'_0 \succeq_{k_{\text{mark}} n_{\text{initial}}} \bar{Y}'_0 \quad \text{and} \quad X'_i \succeq_{k_{\text{mark}}} Y'_i.$$

On each action block, by using the star-filler, we re-sample all of its free intervals, using independent randomization on each action block. Call $\zeta_{\text{alt}*}$ the law of the resulting random variable Z' , the *alternating star-joining*.

LEMMA 16. *Let ζ be an ergodic monotone joining of two Bernoulli measures μ and ν . The alternating star-joining $\zeta_{\text{alt}*}$ is also an ergodic monotone joining of μ and ν . In addition, for any integer $n_{\text{rel}} \geq n_{\text{initial}}$, let R be the set of all elements of $[N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}}$ for which the origin is contained in an action block that contains less than n_{rel} number of alternating intervals of size k_{mark} . Then for any $\varepsilon > 0$ for all sufficiently large k_{mark} we have*

$$(18) \quad d^*(\zeta, \zeta_{\text{alt}*}) < \varepsilon + 2\zeta_{\text{alt}*}(R) + n_{\text{initial}}/n_{\text{rel}},$$

where the inequality holds independently of the choice of n_{initial} and the probability measure β_{sub} that is subordinate to $\Gamma^{n_{\text{initial}}}$ and has the same marginals.

PROOF. It follows from Proposition 6, the definition of a star-filler, (15), and (17) that ζ_{alt^*} is an ergodic monotone joining of μ and ν .

The proof of inequality (18) is similar to that of Lemma 8, except for the following modification. Note that $\zeta_{\text{alt}^*}(R) = \zeta_{\text{alt}}(R)$ and by (16) that restricted to every alternating interval of size k_{mark} that is not part of an initial block, ζ_{alt^*} and ζ_{alt} are equal; the probability that an alternating interval of size k_{mark} is part of an initial block is bounded by $n_{\text{initial}}/n_{\text{rel}}$, when there are more than n_{rel} alternating intervals of size k_{mark} . \square

We will show using Proposition 14 that with a proper choice of parameters that ζ_{alt^*} will be a suitable almost factor and weak-star close to ζ .

4.2. Baire category and the choice of parameters.

LEMMA 17 (The Baire space). *Let $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$ be two Bernoulli measures on $[N]^{\mathbb{Z}}$, where $p \succeq q$. The space $\mathcal{M} = \mathcal{M}(\mu, \nu)$ of all monotone ergodic joinings of μ and ν is a Baire space.*

PROOF. It is well known that space of all joinings of μ and ν is nonempty (since it contains the product measure), compact and convex; furthermore its extreme points are the ergodic joinings which form a (relatively) G_δ subset in the space of all joinings of μ and ν [14], page 122, [13], Proposition 1.5.

Note that the subset of monotone joinings of μ and ν is closed and nonempty; the ergodic monotone joining $\varrho^{\mathbb{Z}}$ is a witness to the latter fact, where ϱ is the (monotone) quantile coupling of p and q . Hence \mathcal{M} is a nonempty G_δ subset.

A G_δ subset of a complete metric space is a Polish space by a theorem of Alexandrov [44], Theorem 2.2.1; and the Baire category theorem tells us that every Polish space is a Baire space [44], Theorem 2.5.5. \square

Let \mathcal{F} denote the product sigma-algebra for $[N]^{\mathbb{Z}}$. Let

$$\mathcal{P} := \{P_i : 0 \leq i \leq N - 1\}$$

denote the partition of $[N]^{\mathbb{Z}}$ according the zeroth coordinate so that $P_i := \{x \in [N]^{\mathbb{Z}} : x_0 = i\}$. Let ζ be a joining of the Bernoulli measures μ and ν , and let $\varepsilon > 0$. If for every set $P \in \sigma(\mathcal{P})$ there exists a $P' \in \mathcal{F}$ such that

$$\zeta((P' \times [N]^{\mathbb{Z}}) \Delta ([N]^{\mathbb{Z}} \times P)) < \varepsilon,$$

then we say that ζ is an ε -almost factor.

LEMMA 18. *Let ζ be a joining of two Bernoulli measures μ and ν . If ζ is an ε -almost factor for all $\varepsilon > 0$, then there exists a factor ϕ such that*

$$\zeta(F \times G) = \mu(F \cap \phi^{-1}(G)) \quad \text{for all } (F, G) \in \mathcal{F} \times \mathcal{F}.$$

PROOF. See [13], Theorem 2.8. \square

Thus if ζ is an ε -almost factor for all $\varepsilon > 0$, then we say that ζ is a *factor*.

PROPOSITION 19. *Let $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$ be two Bernoulli measures on $[N]^{\mathbb{Z}}$, where $H(p) > H(q)$ and $p \geq q$. For each $n > 0$, let \mathcal{E}^n be the set of elements ξ of \mathcal{M} that are $1/n$ -almost factors. For each $n > 0$, the following hold:*

- (A) *The set \mathcal{E}^n is a relatively open subset of \mathcal{M} .*
- (B) *The set \mathcal{E}^n is a dense subset of \mathcal{M} .*

Proposition 19(A) is a standard argument [14], pages 123–124, that we give for completeness. The proof of Proposition 19(B) will require the use of the alternating star-joining.

PROOF OF THEOREM 1. By Proposition 19, the set defined by

$$\mathcal{E} := \bigcap_{n \geq 1} \mathcal{E}^n$$

is an intersection of relatively open dense subsets of \mathcal{M} . By Lemma 17, \mathcal{E} is a nonempty subset of \mathcal{M} , and by Lemma 18, its elements are factors. \square

PROOF OF PROPOSITION 19(A). Let $\zeta \in \mathcal{E}^n$. Recall that S is the left-shift. Since the sigma-field \mathcal{F} is generated by $\bigvee_{i \in \mathbb{Z}} S^{-i} \mathcal{P}$, and there are only a finite number of elements in \mathcal{P} , there exists $m \in \mathbb{N}$ so that for all $P \in \mathcal{P}$, there is a corresponding $P' \in \bigvee_{|i| < m} S^{-i} \mathcal{P}$ for which

$$(19) \quad \zeta((P' \times [N]^{\mathbb{Z}}) \Delta ([N]^{\mathbb{Z}} \times P)) < 1/n.$$

Note that (19) persists for all sufficiently small perturbations of ζ since each P and corresponding P' are clopen sets. \square

PROOF OF PROPOSITION 19(B). Let $\zeta \in \mathcal{M}$, and $\varepsilon > 0$ and $n > 0$. We show that with a proper choice of parameters that for the alternating star-joining, we have $\zeta_{\text{alt}^*} \in \mathcal{E}^n$ and $d^*(\zeta, \zeta_{\text{alt}^*}) < \varepsilon$. Note that by Lemma 16, ζ_{alt^*} is a monotone joining of μ and ν . The following is a list of the parameters, chosen in order:

- (i) By reducing ε if necessary, we may assume $\varepsilon < 1/n$.
- (ii) Set $\varepsilon' = \varepsilon/10$.
- (iii) Using Lemmas 7, 9 and 16, choose k_{mark} large enough so that the probability that the origin is in a frozen interval is less than ε' , $h(\mu_{\text{fill}}) > h(\nu_{\text{fill}})$, and so for any choice of n_{initial} and probability measure β_{sub} that is subordinate to $\Gamma^{n_{\text{initial}}}$ and has the same marginals, we have

$$d^*(\zeta, \zeta_{\text{alt}^*}) < \varepsilon' + 2\zeta_{\text{alt}^*}(R) + n_{\text{initial}}/n_{\text{rel}},$$

where $n_{\text{rel}} \geq n_{\text{initial}}$ and the set R was defined in Lemma 16.

(iv) Appealing to Proposition 14, with k_{mark} as chosen above, and $\eta = \varepsilon'$, we get an n_{initial} and a β_{sub} which realizes the conclusion of the proposition and in particular is subordinate to $\Gamma_{\zeta}^{n_{\text{initial}}}$ and has the same marginals.

(v) Choose an integer $n_{\text{rel}} > n_{\text{initial}}$ so that $n_{\text{initial}}/n_{\text{rel}} < \varepsilon'$.

(vi) Choose r_{mark} so that $\zeta_{\text{alt}*}(R) < \varepsilon'$ and so that the probability that origin is not in an action block is less ε' .

By (i) it remains to argue that $\zeta_{\text{alt}*}$ is an ε -factor. It suffices to define a deterministic function $\psi : [N]^{\mathbb{Z}} \rightarrow [N]$ so that

$$\zeta_{\text{alt}*}((x, y) : (x, y_0) = (x, \psi(x))) > 1 - \varepsilon.$$

By the definition of $\zeta_{\text{alt}*}$ and Proposition 14 and (iv), it follows that ψ can be easily defined from the deterministic function Ψ of the proposition, provided that the origin is in a free interval on an action block. On the other hand, by (iii) with probability less than ε' the origin belongs to a frozen interval, and by (vi) with probability greater than $1 - \varepsilon'$ the origin belongs to an action block where the star-filling is applied to the free intervals. \square

Finally, we discuss the proof of Theorem 2. Let p and q be probability measures on $[N]$, and let R be a relation on $[N]$. Call a joining ζ of $\mu = p^{\mathbb{Z}}$ and $\nu = q^{\mathbb{Z}}$ an R -joining if

$$\zeta \{ (x, y) \in [N]^{\mathbb{Z}} \times [N]^{\mathbb{Z}} : (x_0, y_0) \in R \} = 1.$$

The proof of Theorem 2 is the same as the proof of Theorem 1, except we work with R -couplings and R -joinings instead of their monotone counterparts.

PROOF OF THEOREM 2. We check the crucial details. By assumption there exists a probability measure ρ that is an R -coupling of p and q . Thus the set of R -joinings is nonempty. Given an R -joining of μ and ν , the alternating joining is defined as before, except instead of using the quantile coupling on individual coordinates, we use the one given to us by assumption, ρ ; clearly the resulting alternating joining is still an R -joining. The alternating star-joining is defined as before. To check that it is a R -joining, we use the same facts that were used to check monotonicity. The main point is that the measure given Proposition 4 is subordinate to the original one, and del Junco's star-coupling is a coupling. It follows from (16) and the observation that if α is an R -coupling, then any measure subordinate to α must also be an R -coupling. \square

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