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CONVERGENCE RATES FOR LOOP-ERASED RANDOM WALK AND OTHER LOEWNER CURVES

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We estimate convergence rates for curves generated by Loewner's differential equation under the basic assumption that a convergence rate for the driving terms is known. An important tool is what we call the *tip structure modulus*, a geometric measure of regularity for Loewner curves parameterized by capacity. It is analogous to Warschawski's boundary structure modulus and closely related to annuli crossings. The main application we have in mind is that of a random discrete-model curve approaching a Schramm–Loewner evolution (SLE) curve in the lattice size scaling limit. We carry out the approach in the case of loop-erased random walk (LERW) in a simply connected domain. Under mild assumptions of boundary regularity, we obtain an explicit power-law rate for the convergence of the LERW path toward the radial SLE_2 path in the supremum norm, the curves being parameterized by capacity. On the deterministic side, we show that the tip structure modulus gives a sufficient geometric condition for a Loewner curve to be Hölder continuous in the capacity parameterization, assuming its driving term is Hölder continuous. We also briefly discuss the case when the curves are *a priori* known to be Hölder continuous in the capacity parameterization and we obtain a power-law convergence rate depending only on the regularity of the curves.

1. Introduction, motivation and results.

1.1. *Introduction*. The Loewner equation is a partial differential equation that produces a Loewner chain, a family of conformal mappings from a reference domain onto a continuously decreasing sequence of simply connected domains. The evolution is controlled by a real valued function called driving term which acts as a parameter. Under smoothness assumptions on the driving term, the Loewner equation can be used to generate a growing continuous curve, by which we mean a continuous function from some interval into the reference domain. Conversely, starting from a suitable curve one can reverse the procedure to recover the driving term and so there is a correspondence between *Loewner curves* and their driving terms. Following Schramm [22], Loewner's equation has in recent years been

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successfully applied to study conformally invariant scaling limits of certain lattice models from statistical physics. By taking a scaled Brownian motion as the driving term, one obtains the one-parameter family of random fractal Schramm–Loewner evolution (SLE) curves which are essentially the only possible conformally invariant scaling limits of cluster interfaces with a certain Markovian property; see [22]. Convergence to SLE has been proved in several cases; see, for example, [23] and the references therein. The use of the Loewner equation and SLE techniques in this context has made it possible to give precise meaning to the (passage to the) scaling limit itself, but also to prove conformal invariance, and to give rigorous proofs of various predictions made by physicists. The latter is to large extent due to the fact that the SLE processes are amenable to computation via stochastic calculus.

In this paper, we will be interested in quantifying the relationship between (random) rough Loewner curves with driving terms that are close in the supremum norm. To explain our interest, let us first consider a nonrandom setting. One can view the Loewner equation as a highly nonlinear function from a space of driving terms to a suitable metric space of (parameterized) curves and it is natural to ask about continuity properties, if any. This point of view is closely related to work by Lind, Marshall and Rohde; see [16] and [14]. For example, Theorem 4.1 of [14] proves that curves driven by Hölder-$1/2$ driving terms with small semi-norm converge as curves if their driving terms converge. So the "Loewner function" is continuous when restricted to this collection of driving terms and our results can be used to show that it is Hölder continuous with an explicit exponent depending only on the semi-norm assuming it is sufficiently small. One can also ask similar questions, restricting attention to driving terms generating curves with some given regularity.

Our principal motivation, however, comes from the observation that although several discrete-model curves are known to converge (as curves up to reparameterization) to SLE curves, next to nothing appears to be known about the rates of their convergence. (See the paper [4] by Beneš, Kozdron and the author for a quantitative result of convergence of loop-erased random walk at a fixed time with respect to Hausdorff distance when the curves are viewed as compact sets.)

Good control over convergence rates would allow SLE techniques to be used on mesoscopic scales, that is, scales of order ε^p with $p \in (0, 1)$ where ε is the lattice spacing. It is reasonable to believe that such results will be helpful for obtaining fine properties of corresponding discrete models; this question was raised by Schramm in connection with sharp estimation of critical exponents [23]. We may compare the present setting with a related model. So-called strong approximation results such as the KMT approximation or the Skorokhod embedding [11] yield couplings in which simple random walk and Brownian motion paths are close with high probability, with error terms expressed explicitly in terms of the lattice spacing. This gives a natural way to use techniques for Brownian motion to deduce fine properties of simple random walk that can depend on behavior on mesoscopic scales. This approach has been used by, for example, Lawler, Lawler and Puckette, and Beneš; see [12] and [3] and the references therein. It thus seems that approximation results with explicit error terms for discrete models converging to SLE could be quite useful. Presently, all known proofs of convergence to SLE goes via convergence of the driving terms in one way or another, so it seems natural to take a convergence rate for the driving terms as a starting point. We remark that the work in [4] essentially reduces the derivation of a convergence rate for the driving terms to the derivation of a convergence rate for the so-called martingale observable in rough domains. We will show that a power-law convergence rate to an SLE curve can be derived from a power-law convergence rate for the driving terms provided some additional quantitative geometric information, related to crossing events, is available for the discrete curves, along with an estimate on the growth of the derivative of the SLE map. The approach is quite general and we believe it can be applied to several models (even with nonsimple scaling limit curves) as soon as the aforementioned information is available, though we carry out the specific probabilistic estimates only in the case of loop-erased random walk.

1.2. *Overview, results and related work.* Let us briefly sketch the setup and main ideas in the (chordal) half-plane setting, though we will later work mostly in the disk. See Section 2 for precise definitions. Let $W, W_n : [0, T] \to \mathbb{R}$ be continuous functions such that

$$\sup_{t\in[0,T]} \left|W(t) - W_n(t)\right| \le \varepsilon,$$

where $\varepsilon > 0$ is small but for the moment fixed. Let $f(t,z):\mathbb{H} \to H(t)$ and $f_n(t,z):\mathbb{H} \to H_n(t)$ be the solutions to the chordal Loewner equation (Loewner chains)

$$\partial_t f(t,z) = -\partial_z f(t,z)\frac{2}{z - U(t)}, \qquad f(0,z) = z, z \in \mathbb{H}$$

with $U(t)$ replaced by $W(t)$ and $W_n(t)$, respectively. Assume that the Loewner chains are generated by the curves γ and γ_n parameterized by capacity so that for each t, $H(t)$ and $H_n(t)$ are the unbounded components of $\mathbb{H} \setminus \gamma[0,t]$ and $\mathbb{H} \setminus \gamma_n[0,t]$, respectively. (We can think of γ_n as the conformal image of a discrete-model curve on a lattice approximation of a smooth domain D, where the mesh of the lattice is n^{-1}, and the driving term of γ_n is coupled with a scaled Brownian motion W driving the chordal SLE curve γ so that the driving terms are at distance at most $\varepsilon = n^{-q}$ for some $q < 1$.) Let $y > 0$; we will later choose $y = y(\varepsilon)$. Let $t \in [0, T]$. We can write

$$\begin{aligned}
\left|\gamma(t) - \gamma_n(t)\right| \le\ & \left|\gamma(t) - f\big(t, W(t) + iy\big)\right| \\
& + \left|f\big(t, W(t) + iy\big) - f\big(t, W_n(t) + iy\big)\right| \\
& + \left|f\big(t, W_n(t) + iy\big) - f_n\big(t, W_n(t) + iy\big)\right| \\
& + \left|f_n\big(t, W_n(t) + iy\big) - \gamma_n(t)\right| \\
=:\ & A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

We wish to estimate the A_j in terms of ε. Suppose that there are $\beta < 1$ and $c < \infty$ such that

$$|f'(t, W(t) + id)| \le cd^{-\beta} \qquad \text{for all } d \le y. \tag{1}$$

If this estimate holds, then by integrating, $A_1 \le cy^{1-\beta}$. (Constants may change from line to line, and are assumed to depend only on the parameters and not on ε, y, etc.) By the distortion theorem, the same bound holds for A_2 if $y \ge \varepsilon$. The third term, A_3, represents the distance between two solutions to the Loewner equation having driving terms at supremum distance at most ε, and evaluated at the same point. In Section 2.3, we will use the reverse-time Loewner flow to estimate quantities like this. In particular, we will see that if $\operatorname{Im} z = y$, then

$$|f(t, z) - f_n(t, z)| \le c\varepsilon y^{-1}$$

with c depending only on T. Hence, $A_3 \le c\varepsilon y^{-1}$ and Cauchy's integral formula implies that

$$\big|y|f'(t, z)| - y|f_n'(t, z)|\big| \le c\varepsilon y^{-1}.$$

From this it follows, using Koebe's estimate and (1), that if

$$\Delta_n(t, y) := \operatorname{dist}[f_n(t, W_n(t) + iy), \partial H_n(t)],$$

then

$$\Delta_n(t, y) \le cy|f_n'(t, W_n(t) + iy)| \le cy^{1-\beta} + c\varepsilon y^{-1}; \tag{2}$$

see Proposition 2.4. (Note that we have made no explicit assumption on the behavior of $|f_n'|$.) Now choose $y(\varepsilon) = \varepsilon^p$, for some $p \in (0, 1)$. Then

$$A_1 + A_2 + A_3 \le c\varepsilon^{p(1-\beta)} + c\varepsilon^{1-p}$$

and it remains to bound A_4. Clearly, $A_4 \ge \Delta_n(t, \varepsilon^p)$ but we would like an upper bound in terms of $\Delta_n(t, \varepsilon^p)$. To proceed, some additional information about the boundary behavior of f_n is necessary.

For this, we will use what we call the tip structure modulus, a geometric gauge of the regularity of a Loewner curve in the capacity parameterization, that is, for our problem, the analog of Warschawski's [26] measure with a similar name. Let $\delta > 0$ and consider $\mathcal{S}_{t,\delta}$, the set of all crosscuts of $H_n(t)$ of diameter at most δ that separate the tip, $\gamma_n(t)$, from ∞ in $H_n(t)$. Each crosscut $\mathcal{C} \in \mathcal{S}_{t,\delta}$ separates from ∞ in $H_n(t)$ a piece $\gamma_{\mathcal{C}}$ of $\gamma_n[0, t]$ obtained by tracing γ_n backward from $\gamma_n(t)$ until $\overline{\mathcal{C}}$ is first hit. (If γ_n and $\overline{\mathcal{C}}$ do not intersect, we set $\gamma_{\mathcal{C}} = \gamma$.) We then define the tip structure modulus, $\eta_{\text{tip}}(\delta)$, of $\gamma_n(t)$, $t \in [0, T]$, to be the maximum of δ and

$$\sup_{t\in[0,T]} \sup_{\mathcal{C}\in\mathcal{S}_{t,\delta}} \operatorname{diam} \gamma_{\mathcal{C}}.$$

(See Section 3 for a precise definition.) Roughly speaking, $\eta_{\text{tip}}(\delta)$ is the maximal distance the curve travels into a "bottle" with "bottleneck" opening smaller than

δ viewed from the point toward which the curve is growing. (Similar conditions have been used before; see below.) In Proposition 3.2, we show that

$$|f_n(t, W_n(t) + iy) - \gamma_n(t)| \le c_1 \eta_{\mathrm{tip}}(c_2 \Delta_n(t, y)), \tag{3}$$

where η_{tip} is the tip structure modulus for γ_n. Consequently, if we have a power-law bound on the tip structure modulus evaluated at $c\Delta_n(t, \varepsilon^p)$, that is, if

$$\eta_{\mathrm{tip}}(c\Delta_n(t, \varepsilon^p)) \le c'(\Delta_n(t, \varepsilon^p))^r$$

for some $r \in (0, 1)$, then by (2)

$$A_4 \le c\varepsilon^{p(1-\beta)r} + c\varepsilon^{(1-p)r}.$$

We stress that the estimate on η_{tip} is only required to hold on the scale of $\Delta_n(t, \varepsilon^p)$ and note that the failure of the existence of such a bound on η_{tip} implies certain crossing events for the curve. If the estimates hold uniformly in $t \in [0, T]$, then we have obtained a power-law bound in terms of ε on $\sup_{t\in[0,T]} |\gamma(t) - \gamma_n(t)|$ and we can then conclude by optimizing over exponents.

To implement these ideas in a particular setting, we need to show that the assumptions we used are satisfied uniformly in $t \in [0, T]$, with high probability in terms of ε. If a convergence rate for the driving terms (or martingale observable in rough domains) is known, then we believe it is possible to derive the remaining required information from existing results in the literature without too much effort, and we derive the needed SLE derivative estimates, from estimates in [6], in this paper. Indeed, as already mentioned, the event that the geometric condition fails implies annuli crossing events that are fairly well understood for the models known to converge to SLE.

The organization of the paper is as follows. In Section 2.3, we discuss some preliminaries and prove the quantitative comparison estimates for solutions to the Loewner equation. These estimates might be of some independent interest; see, for example, [8]. We also consider a natural case when the curves are *a priori* known to be Hölder continuous in the capacity parameterization and derive a power-law convergence rate depending only on the regularity of the curves. See Corollaries 2.6 and 2.7.

In Section 3, we define the tip structure modulus and prove the estimates implying (3). Then in Theorem 3.5, we show that if a Loewner curve γ has the property that there is $M < \infty$ such that $\eta_{\mathrm{tip}}(\delta) \le M\delta$, $\delta < \delta_0$, and the driving term is Hölder continuous, then γ is also Hölder continuous in the capacity parameterization with exponent depending only on M and the exponent for the driving term. A linear bound on the structure modulus is a natural analog of the John condition for simply connected domains; see, for example, Chapter 5 of [20]. Theorem 3.5 can thus be viewed as the analog for Loewner curves of the well-known fact that a John domain is also a Hölder domain [20].

In Section 4, we apply the above ideas to obtain a power-law estimate on the convergence rate to radial SLE_2 for the loop-erased random walk (LERW) path.

Here is an informal version of the result; see Theorem 4.3 for a precise statement. Let D_n be a $n^{-1}\mathbb{Z}^2$ grid-domain approximation of a fixed simply connected Jordan domain D containing 0 and with $\mathcal{C}^{1+\alpha}$ boundary and inner radius from 0 equal to 1. (The proof works for the larger class of quasidisks [20], but we then get a slower convergence rate which depends on the constant in the Ahlfors three-point condition for D.) Let γ_n be the time-reversal of LERW in D_n from 0 to ∂D_n and let $\tilde{\gamma}_n$ be its image in $\mathbb{D}$ under the conformal map $\psi_n : D_n \to \mathbb{D}$ with the usual normalization. Let $\tilde{\gamma}$ be the radial SLE_2 path in $\mathbb{D}$ started uniformly on $\partial\mathbb{D}$. Our main result can now be given as follows.

THEOREM. *For each n sufficiently large, there is a coupling of $\tilde{\gamma}_n$ with $\tilde{\gamma}$ such that*

$$\mathbb{P}\Big\{ \sup_{t\in[0,\sigma]} \big|\tilde{\gamma}_n(t) - \tilde{\gamma}(t)\big| > \varepsilon_n^{1/41} \Big\} < \varepsilon_n^{1/41},$$

where both curves are parameterized by capacity, $\varepsilon_n = n^{-1/24}$ is the convergence rate of the driving terms from [4], *and σ is a stopping time. The same estimate holds for the preimages of the curves in D_n.*

[The stopping time $\sigma = \sigma(\varepsilon, T)$, which is needed for technical reasons, can be taken as the minimum of some fixed $T < \infty$ and the first time such that the forward SLE_2 flow of $-\tilde{\gamma}(0)$ is smaller than some given $\varepsilon > 0$. We have $\lim_{\varepsilon\to 0} \sigma(\varepsilon, T) = T$ almost surely, see Appendix A.] This quantifies the convergence result [13], Theorem 3.9, of Lawler, Schramm and Werner. As indicated, the proof considers the couplings of [4] in which if $s < 1/24$, then with probability at least $1 - n^{-s}$ the estimate $\sup_{t\in[0,T]} |W_n(t) - W(t)| < n^{-s}$ holds. Here, W_n is of the LERW in D_n and W is a Brownian motion with speed 2 on $\partial\mathbb{D}$. Using the Brownian motion as driving term in the Loewner equation, we have a coupling of the LERW image and SLE_2 for each n, with their driving terms close. To prove Theorem 4.3, we then show that the above reasoning can be carried out on an event with large probability in terms of n. Some work is required to establish the needed geometric condition for the LERW path; see Proposition 4.5.

In Appendix A, we derive an estimate on the probability (in terms of y) that a bound of the type (1) holds for *radial* SLE from a corresponding estimate for chordal SLE from [6].

Finally, in Appendix B we discuss a convergence rate result for a sequence of grid-domain approximations of a quasidisk which allows us to directly "transfer" the required geometric condition to $\mathbb{D}$.

Besides classical articles by Ahlfors, Warschawski, Becker, Pommerenke and others, which develop (Euclidean) geometric conditions for regularity estimates on Riemann maps (see, e.g., [2, 17, 18, 25, 26] and the references therein), there are close connections between the results and methods of this paper and more recent work. Let us highlight some. We mentioned the work by Lind, Marshall and

Rohde [14] and by Marshall and Rohde [16]; see also Wong's paper [27]. The paper by Aizenman and Burchard [1] characterizes tightness for probability measures on a space of (discrete model) curves modulo reparameterization in terms of estimates on probabilities of annuli crossing events. The event that the geometric condition fails is contained in a union of crossing events of this type and this is what allows for estimation of probabilities. Kemppainen and Smirnov consider related questions and use similar conditions in [9] and a quantity somewhat similar to the tip structure modulus has been used by Lind and Rohde in [15].

2. Preliminaries and the deterministic Loewner equation.

2.1. *Preliminaries.* We start by setting some notation. We will write $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for the unit disk in the complex plane. This is the basic *reference domain*, although we will occasionally also consider the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $D \ni 0$ be a simply connected domain. By the Riemann mapping theorem, there exists a unique conformal map $\psi : D \to \mathbb{D}$ with $\psi(0) = 0$ and $\psi'(0) > 0$. If we do not state otherwise, we will always assume that uniformizing conformal maps like ψ are normalized in this way.

A *crosscut* $\mathcal{C}$ of a simply connected domain D is an open Jordan arc in D such that $\overline{\mathcal{C}} = \mathcal{C} \cup \{\zeta, \eta\}$ with $\zeta, \eta \in \partial D$. A crosscut partitions D into exactly two disjoint components; see Chapter 2 of [20].

A (parameterized) *curve* γ is a continuous function $\gamma(t) : I \to \mathbb{C}$ defined on some interval I which we will usually assume to be $[0, T]$ for some fixed $T > 0$. Given two curves γ_1, γ_2 defined on the same interval, we measure their distance by the supremum norm

$$\sup_{t \in [0,T]} \big|\gamma_1(t) - \gamma_2(t)\big|.$$

Let $\gamma : [0, T] \to \overline{\mathbb{D}}$ be a curve with $\gamma(0) \in \partial\mathbb{D}$, $0 \notin \gamma[0, T]$, and for $t \in [0, T]$, let D_t be the connected component of 0 of $\mathbb{D} \setminus \gamma[0, t]$. We say that γ is *parameterized by capacity* if the normalized conformal maps $g_t : D_t \to \mathbb{D}$ satisfy $g_t'(0) = e^t$ for $t \in [0, T]$. (Clearly, not all curves in $\overline{\mathbb{D}}$ can be parameterized in this way.) A reparameterization of a curve γ is a new curve $\tilde{\gamma}$ obtained by $\tilde{\gamma}(t) = \gamma \circ \alpha(t)$, where $\alpha(t) : [0, \widetilde{T}] \to [0, T]$ is a strictly increasing and continuous function. We will often, when no confusion is possible, treat a curve and its reparameterizations as the same. A ($\mathbb{D}$-) *Loewner curve* is a curve γ in $\overline{\mathbb{D}}$ as above, parameterized by capacity, for which the following continuity condition holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $s, t \in [0, T]$ with $0 < t - s < \delta$ there is a crosscut $\mathcal{C}$ with $\operatorname{diam} \mathcal{C} < \varepsilon$ that separates $K_t \setminus K_s$ from 0 in D_t, where $K_t = \overline{\mathbb{D} \setminus D_t}$. Intuitively, a $\mathbb{D}$-Loewner curve γ is a continuous curve such that: the conformal radius from 0 of the complement of the curve is strictly and continuously decreasing, it has no transversal self-crossings, and the tip $\gamma(t)$ is always "visible" from 0. For example, if γ is piecewise smooth with no double points and is contained in $\mathbb{D}$

for $t \in (0, T]$, then it is a Loewner curve. By Theorem 1 of [19], the $\mathbb{D}$-Loewner curves are exactly the curves that can be described using the radial Loewner equation driven by a continuous driving term, as discussed in the next section. We will also consider (chordal) Loewner curves in $\mathbb{H}$ which are defined in a similar manner; we refer to Chapter 4 of [10] for more information. We just note that in this case it is convenient to parameterize γ by the so-called half-plane capacity, that is, so that the conformal maps $g_t : H_t \to \mathbb{H}$, where H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$, satisfy $g_t(z) = z + 2t/z + o(1/|z|)$ at ∞. (In this case, the normalization is at a boundary point, and the tip of the curve is to be "visible" from this point at all times.)

We will often write "constants" depending on parameters as $c = c(a, b)$, etc. It is then to be understood that c depends only on these parameters.

2.2. *Loewner equations.* We will be interested in two versions of Loewner's differential equation. We define radial and chordal Loewner vector fields by

$$\Phi_{\mathbb{D}}(z, \zeta) = -z\frac{\zeta + z}{\zeta - z}, \qquad \Phi_{\mathbb{H}}(z, \xi) = -\frac{2}{z - \xi}.$$

The radial and chordal Loewner equations are then given by

$$\partial_t f(t, z) = \partial_z f(t, z)\Phi_X\big(z, W(t)\big), \qquad f_0(z) = z, z \in X, \tag{4}$$

$X = \mathbb{D}$ and $X = \mathbb{H}$, respectively. (We will sometimes refer to these equations the $\mathbb{D}$- and $\mathbb{H}$-Loewner PDEs and their solutions as $\mathbb{D}$- and $\mathbb{H}$-Loewner chains, etc.) Here, $W : [0, \infty) \to \partial X$ is a (continuous) function called the driving term. In the radial case, we will sometimes write the driving term as $W(t) = e^{i\xi(t)}$ for a real valued function ξ which, when no confusion is possible, for brevity is also referred to as the driving term.

Let us discuss a few properties in the radial setting. (Similar results hold for the chordal version.) For each $t_0 \ge 0$, the solution $f(t_0, \cdot) : \mathbb{D} \to D_{t_0}$ is a conformal map onto a simply connected domain $D_{t_0} \subset \mathbb{D}$. The family $(f(t, z))_{t \ge 0}$ of conformal mappings is called a *Loewner chain*. A *Loewner pair* (f, W) consists of a function $f(t, z)$ and a (continuous) function $W(t)$, $t \ge 0$, such that f is the solution to the Loewner equation with W as driving term. Under some rather mild regularity assumptions on W [e.g., that W is Hölder-$(1/2 + \varepsilon)$ for some $\varepsilon > 0$], there exists a curve $\gamma(t)$ such that D_t is the component of the origin of $\mathbb{D} \setminus \gamma[0, t]$, and in this case we say that the Loewner chain is generated by the Loewner curve γ. Conversely, given a Loewner curve, one can associate via the Loewner equation a unique driving term such that the Loewner chain (f_t) in the Loewner pair (f, W) is generated by γ. In fact, the driving term is the preimage in $\partial\mathbb{D}$ of the tip of the growing curve. In terms of the inverse relationship, we have

$$\gamma(t) = \lim_{d \to 0+} f\big(t, (1 - d)W(t)\big). \tag{5}$$

A sufficient condition for (f, W) to be generated by a curve γ is that the limit (5) exists for all $t \geq 0$ and that $t \mapsto \gamma(t)$ is continuous; see Theorem 4.1 of [21]. The parameterization of γ given by (5) is the capacity parameterization.

We will use the notation $f_t(z) = f(t, z)$, $f' = \partial_z f$ and $\dot{f} = \partial_t f$.

LEMMA 2.1. *There exists a constant* $c_0 < \infty$ *such that the following holds. Let* $X \in \{\mathbb{D}, \mathbb{H}\}$. *Suppose that* f_t *satisfies the* X*-Loewner PDE and that* $\operatorname{dist}(z, \partial X) = d$. *Then for* $s \geq 0$

$$e^{-c_0 s/d^2}\left|f_t'(z)\right| \leq \left|f_{t+s}'(z)\right| \leq e^{c_0 s/d^2}\left|f_t'(z)\right| \tag{6}$$

and

$$\left|f_{t+s}(z) - f_t(z)\right| \leq c_0\, d\left|f_t'(z)\right|\left(e^{c_0 s/d^2} - 1\right). \tag{7}$$

PROOF. See Lemma 3.5 of [6] for the proof in the chordal case. The radial case is proved in the same way. □

For Hölder continuous driving terms, the existence of the curve and its regularity in the capacity parameterization is completely determined by the local behavior at the tip, that is, the growth of the derivative of the conformal map close to the preimage of the tip. The following result is a version of Proposition 3.9 of [6], but allows for a less regular driving term.

PROPOSITION 2.2. *Let* (f, W) *be a* $\mathbb{D}$*-Loewner pair and assume that* $W(t) = e^{i\xi(t)}$ *where* $\xi(t)$ *is Hölder-*α *on* $[0, T]$ *for some* $\alpha \leq 1/2$. *Then the following holds. Suppose there are* $c < \infty$, $d_0 > 0$, *and* $0 \leq \beta < 1$ *such that*

$$\sup_{t \in [0,T]} d\left|f_t'\big((1-d)W(t)\big)\right| \leq c d^{1-\beta} \qquad \forall d \leq d_0. \tag{8}$$

Then (f, W) *is generated by a curve that is Hölder-*$\alpha(1-\beta)$ *continuous on* $[0, T]$. *The analogous statement holds for* $\mathbb{H}$*-Loewner pairs.*

REMARK. At $t = 0$, we have $f_0'(z) = 1$ so we can never do better than $\beta = 0$ in (8). However, for $t \geq \varepsilon$, we can have $-1 \leq \beta < 0$ and in this case the curve will be Hölder-$\alpha(1-\beta)$ (which is then larger than α) for $t \in [\varepsilon, T]$ but only Hölder-α on $[0, T]$.

PROOF OF PROPOSITION 2.2. The bound on the derivative implies that the limit

$$\gamma(t) = \lim_{d \to 0+} f_t\big((1-d)W(t)\big)$$

exists for every $t \in [0,T]$ and since the convergence is uniform $\gamma(t)$ is a continuous function. Let $s>0$ and set $d=s^\alpha$. If $t, t+s \in [0,T]$, we have

$$\begin{aligned}\big|\gamma(t+s)-\gamma(t)\big| \le \big|&\gamma(t+s)-f_{t+s}\big((1-d)W(t+s)\big)\big|\\ &+\big|f_{t+s}\big((1-d)W(t+s)\big)-f_{t+s}\big((1-d)W(t)\big)\big|\\ &+\big|f_{t+s}\big((1-d)W(t)\big)-f_t\big((1-d)W(t)\big)\big|\\ &+\big|\gamma(t)-f_t\big((1-d)W(t)\big)\big|.\end{aligned}$$

If $t>0$, then the estimate (8) implies that the first and last terms are bounded by a constant times $d^{1-\beta}=s^{\alpha(1-\beta)}$. By assumption $|\xi(t+s)-\xi(t)| \le cs^\alpha = cd$, so the distortion theorem implies that

$$\big|f_{t+s}\big((1-d)W(t+s)\big)-f_{t+s}\big((1-d)W(t)\big)\big| \le cd^{1-\beta}.$$

Finally, since $s=d^{1/\alpha}$ and $\alpha \le 1/2$, (7) implies

$$\big|f_{t+s}\big((1-d)W(t)\big)-f_t\big((1-d)W(t)\big)\big| \le cd^{1-\beta}.$$

Since $d|f_0'((1-d)W(0))|=d$ and so cannot decay faster than linearly, we get the stated exponent on $[0,T]$. □

2.3. *An estimate for the reverse-time Loewner equation.* We want to compare solutions to the Loewner equation corresponding to driving terms which are close in the supremum norm. We will use the *reverse-time Loewner equation*: let $T<\infty$ and let (f_j, W_j), $j=1,2$, be Loewner pairs. Let $t_0 \in (0,T]$ be fixed. Consider solutions $h_j(t,z;t_0)=h_j(t,z)$ to the reverse-time Loewner equation

$$\partial_t h_j(t,z)=\Phi_X\big(h_j, U_j(t)\big), \qquad h_j(0,z)=z, \tag{9}$$

where X equals $\mathbb{D}$ and $\mathbb{H}$ in the radial and chordal case, respectively. We say that U_j is the driving term for (9). If we take $U_j(t)=W_j(t_0-t)$ we have the well-known identity

$$h_j(t_0,z;t_0)=f_j(t_0,z), \qquad z\in X,\ j=1,2,$$

where $f_j(t,z)$ solves the Loewner PDE (4) with $W_j(t)$ as driving term. These equalities only hold at the special time $t=t_0$; the families of conformal mappings $(h_j(\cdot,z))$ and $(f_j(\cdot,z))$ are in general different. Solutions $t\mapsto h(t,z)$ to (9) flow away from ∂X as t increases when $z\in X$ and this implies that if $z \in X$ is fixed then the solution $t\mapsto h(t,z)$ exists for all $t\ge 0$.

Let ε and ν be given nonnegative numbers. Let $z_1, z_2 \in X$ be given and suppose that

$$\sup_{t\in[0,T]}\big|W_1(t)-W_2(t)\big| \le \varepsilon, \qquad |z_1-z_2|\le \nu\varepsilon.$$

Set

$$H(t)=h_1(t,z_1)-h_2(t,z_2),$$

where the h_j are assumed to solve the reverse-time Loewner equations (9) driven by

$$\widetilde{W}_j(t) := W_j(t_0 - t), \qquad j = 1, 2.$$

Then $H(t_0) = f_1(t_0, z_1) - f_2(t_0, z_2)$. We differentiate with respect to t and use (9) to obtain the linear differential equation

$$\dot{H}(t) - H(t)\psi_X(t) = \big(\widetilde{W}_2(t) - \widetilde{W}_1(t)\big)\xi_X(t),$$

where

$$\psi_{\mathbb{D}}(t) = \frac{h_1 h_2 - \widetilde{W}_1\widetilde{W}_2 - (1/2)(h_1 + h_2)(\widetilde{W}_1 + \widetilde{W}_2)}{(h_1 - \widetilde{W}_1)(h_2 - \widetilde{W}_2)},$$

$$\xi_{\mathbb{D}}(t) = \frac{h_1^2 + h_2^2}{2(h_1 - \widetilde{W}_1)(h_2 - \widetilde{W}_2)}$$

and

$$\psi_{\mathbb{H}}(t) = \frac{2}{(h_1 - \widetilde{W}_1)(h_2 - \widetilde{W}_2)},$$

$$\xi_{\mathbb{H}}(t) = \psi_{\mathbb{H}}(t).$$

Here, we have suppressed the dependence on t in the right-hand sides. We can integrate the differential equation and with $u(t) = \exp\{-\int_0^t \psi_X(s)\,ds\}$ we find

$$H(t) = u(t)^{-1}\left(H(0) + \int_0^t (\widetilde{W}_2 - \widetilde{W}_1)u\xi_X\,ds\right).$$

Hence, for $0 \le t \le t_0$,

$$|H(t)| \le |H(0)|e^{\int_0^t \operatorname{Re}\psi_X(s)\,ds} + \int_0^t |\widetilde{W}_2 - \widetilde{W}_1| e^{\int_s^t \operatorname{Re}\psi_X(r)\,dr}|\xi_X|\,ds. \tag{10}$$

Consequently, since

$$\sup_{t\in[0,t_0]} \big|\widetilde{W}_1(t) - \widetilde{W}_2(t)\big| \le \varepsilon, \qquad \big|H(0)\big| = |z_1 - z_2| \le \nu\varepsilon,$$

recalling that $|f_1(t_0, z_1) - f_2(t_0, z_2)| = |H(t_0)|$, we get the estimate

$$\begin{aligned} &\big|f_1(t_0, z_1) - f_2(t_0, z_2)\big| \\ &\quad \le \varepsilon\left(\nu e^{\int_0^{t_0} \operatorname{Re}\psi_X(s)\,ds} + \int_0^{t_0} e^{\int_s^{t_0} \operatorname{Re}\psi_X(r)\,dr}|\xi_X|\,ds\right). \end{aligned} \tag{11}$$

The right-hand side in (11) can be estimated in different ways depending on what data is available. We would like an estimate that depends only on ε and $d = \operatorname{dist}(\{z_1, z_2\}, \partial X)$. Estimating naively, using only the fact that points flow away from ∂X under the reverse flow, gives a bound of order $\varepsilon e^{O(d^{-2})}$. (This kind of estimate was used in [4].) We shall see that we can do much better.

2.3.1. *The chordal case.* To give some intuition, let us first briefly discuss the easier chordal case which will be treated in greater detail in [8]. Assume $\nu = 1$ for simplicity. Write $z_j(t) = h_j(t, z_j) - \widetilde{W}_j(t)$. We can apply the Cauchy–Schwarz inequality to get

$$\begin{aligned}\int_0^t \operatorname{Re}\psi_{\mathbb{H}}(t)\,dt &\le \int_0^t \frac{2}{|z_1(t)z_2(t)|}\,dt\\ &\le \left(\int_0^t \frac{2}{|z_1(t)|^2}\,dt\right)^{1/2}\left(\int_0^t \frac{2}{|z_2(t)|^2}\,dt\right)^{1/2}.\end{aligned}$$

Since $\partial_t \log \operatorname{Im} z_j(t) = 2/|z_j(t)|^2$, this can now be used to show that the right-hand side of (11) is bounded by εd^{-1} times a constant depending only on T, if $\operatorname{Im} z_j(0) \ge d$, $j = 1, 2$. (Note that there is no logarithmic correction.)

REMARK. The estimate εd^{-1} is essentially sharp if no further assumptions are made. Indeed, consider a driving term $W_1(t)$ generating a Loewner chain such that for some fixed $p < 1$ very close to 1, $t_0 > 0$, there is a constant $c > 0$ such that $|f_1'(t_0, W_1(t_0) + id)| \ge cd^{-p}$ as $d \to 0$. (As shown in [14], one can take $W_1(t) = \kappa\sqrt{t_0 - t}$ with κ very close to but smaller than 4. The curve traces a kind of logarithmic spiral.) If we let $W_2(t) = W_1(t) + \varepsilon$, then $f_2(t, z) = f_1(t, z - \varepsilon) + \varepsilon$. Hence, for $\varepsilon \le d/2$, by Koebe's distortion theorem,

$$\begin{aligned}&\bigl|f_2\bigl(t_0, W_1(t_0) + id\bigr) - f_1\bigl(t_0, W_1(t_0) + id\bigr)\bigr|\\ &\quad\ge \bigl|f_1\bigl(t_0, W_1(t_0) + id - \varepsilon\bigr) - f_1\bigl(t_0, W_1(t_0) + id\bigr)\bigr| - \varepsilon\\ &\quad\ge c\varepsilon\bigl|f_1'\bigl(t_0, W_1(t_0) + id\bigr)\bigr| \ge c\varepsilon d^{-p}.\end{aligned}$$

A similar example can be constructed for the radial case.

If more information is available, one can do better. The reader may check that $\partial_t \operatorname{Re}\log h_j'(t, z) = \operatorname{Re}(2/z_j(t)^2)$. From this, one can see that the bound can be expressed in terms of the derivatives f_j'. In fact, in joint work with Rohde and Wong, [8], we show that

$$\begin{aligned}&\bigl|f_1(t_0, z) - f_2(t_0, z)\bigr|\\ &\quad\le \varepsilon\exp\left\{\frac{1}{2}\left[\log\frac{I_{t_0,y}|f_1'(t_0, z)|}{y}\log\frac{I_{t_0,y}|f_2'(t_0, z)|}{y}\right]^{1/2} + \log\log\frac{I_{t_0,y}}{y}\right\},\end{aligned}$$

where $I_{t,y} = \sqrt{4t + y^2}$. If a nontrivial power-law bound on the growth of the derivative at time t_0 holds, that is, if $c_j < \infty$ and $\beta_j < 1$ are such that for $j = 1, 2$,

$$\bigl|f_j'\bigl(t_0, W_j(t_0) + id\bigr)\bigr| \le c_j d^{-\beta_j}, \qquad d \le d_0, \tag{12}$$

then one gets a bound in (11) of order at most $c\varepsilon d^{-(1/2)[(1+\beta_1)(1+\beta_2)]^{1/2}}\log d^{-1}$, where c depends only on c_j, β_j, $j = 1, 2$.

2.3.2. *The radial case.* We now consider the radial setting $X = \mathbb{D}$. In order to bound the right-hand side of (11) we need to estimate $\int_s^{t_0} \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds$. The idea is to prove that for a constant q slightly larger than 1,

$$\operatorname{Re} \psi_{\mathbb{D}}(t) \le q \frac{\sqrt{1+|z_1(t)|}}{|1-z_1(t)|} \cdot \frac{\sqrt{1+|z_2(t)|}}{|1-z_2(t)|},$$

where for $t \in [0, t_0]$, we define

$$z_j(t) = h_j(t, z_j)\overline{\widetilde{W}_j(t)}.$$

Note that $|z_j(0)| = |z_j|$. Once we have this estimate, we can apply the Cauchy–Schwarz inequality to the corresponding bound on $\int_s^{t_0} \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds$ to decouple the two flows and then compare with

$$\frac{1+|z_j(t)|}{|1-z_j(t)|^2} = \partial_t \log\bigl(1 - |z_j(t)|\bigr). \tag{13}$$

This last identity follows from the reverse-time Loewner equation (9). This will give a bound in (11) of order εd^{-q}, where q can be taken arbitrarily close to 1. (Arguing as in the chordal case only gives a rough bound of order εd^{-4}, but we shall actually make use of this bound below.) This is essentially optimal in this general setting as we saw above.

PROPOSITION 2.3. *For $j = 1, 2$, let (f_j, W_j) be $\mathbb{D}$-Loewner pairs. For any $\rho > 1$, there exist $\varepsilon_0 = \varepsilon_0(\rho) > 0$, $d_0 = d_0(\rho) > 0$, and $c = c(\rho) < \infty$ such that the following holds. Let $T < \infty$ and suppose that*

$$\sup_{t\in[0,T]} \bigl|W_1(t) - W_2(t)\bigr| \le \varepsilon,$$

where $\varepsilon < \varepsilon_0$. Then for any $z_1, z_2 \in \mathbb{D}$ with $|z_1 - z_2| \le \varepsilon$ and $|z_1|, |z_2| \le 1 - d$ with $(4\varepsilon)^{1/\rho} \le d \le d_0$,

$$\bigl|f_1(T, z_1) - f_2(T, z_2)\bigr| \le c\varepsilon d^{-\rho}. \tag{14}$$

PROOF. By factoring out $\widetilde{W}_1\widetilde{W}_2$, we can write

$$\begin{aligned} &\operatorname{Re} \psi_{\mathbb{D}}(t) \\ &\quad= \operatorname{Re}\biggl(\frac{z_1(t)z_2(t) - 1 - (z_1(t) + z_2(t)) + O(\varepsilon)}{(1-z_1(t))(1-z_2(t))}\biggr) \qquad (15)\\ &\quad= \frac{\operatorname{Re}\{(z_1(t)z_2(t) - 1 - (z_1(t) + z_2(t)) + O(\varepsilon))(1-\overline{z_1(t)})(1-\overline{z_2(t)})\}}{|1-z_1(t)|^2|1-z_2(t)|^2}. \end{aligned}$$

This uses that $\overline{\widetilde{W}_1(t)}\widetilde{W}_2(t) = 1 + O(\varepsilon)$ in the sense that $|\overline{\widetilde{W}_1(t)}\widetilde{W}_2(t) - 1| \le c\varepsilon$ for a universal constant c. For $z, w \in \overline{\mathbb{D}}$ we now consider the function

$$R(z, w) = \frac{\operatorname{Re}\{(zw - 1 - (z+w))(1-\overline{z})(1-\overline{w})\}}{|1-z||1-w|\sqrt{(1+|z|)(1+|w|)}},$$

which is bounded and continuous on the closed bi-disk $\mathbb{D} \times \mathbb{D}$. We claim that $\sup_{z,w \in \partial\mathbb{D}} R(z,w) \le 1$. A computation shows that R simplifies when $|z| = |w| = 1$ so that

$$R(z,w) = \frac{(1-\operatorname{Re} z)(1-\operatorname{Re} w) + \operatorname{Im} z \operatorname{Im} w}{2\sqrt{(1-\operatorname{Re} z)(1-\operatorname{Re} w)}} \qquad \big(|z| = |w| = 1\big).$$

By changing coordinates $z = e^{i\theta}$ and $w = e^{i\mu}$, with $\theta, \mu \in [0, 2\pi]$, in the last expression we find

$$\big(R\big(e^{i\theta}, e^{i\mu}\big)\big)^2 = \cos^2\Big(\frac{\theta - \mu}{2}\Big) \le 1.$$

Let $\delta > 0$ be such that $\rho = 1 + 2\delta$; we assume that δ is small. By the last expression and the continuity of R, there exists $\varepsilon'(\delta) > 0$ such that if $1 - \varepsilon' \le |z|, |w| \le 1$ then $R(z,w) \le 1 + \delta/2$. We will fix ε' from now on. We can think of ε' as small but macroscopic compared to ε. Returning to the flows, by (15) and the bound on R, if ε is sufficiently small compared to δ, we have the estimate

$$\begin{aligned} \operatorname{Re} \psi_{\mathbb{D}}(t) &= \operatorname{Re}\Big(\frac{z_1(t)z_2(t) - 1 - (z_1(t) + z_2(t)) + O(\varepsilon)}{(1 - z_1(t))(1 - z_2(t))}\Big) \\ &\le (1+\delta)\frac{\sqrt{1 + |z_1(t)|}}{|1 - z_1(t)|} \cdot \frac{\sqrt{1 + |z_2(t)|}}{|1 - z_2(t)|}, \qquad 0 \le t \le \tau, \end{aligned} \tag{16}$$

where

$$\tau = \inf\big\{t \ge 0 : \min\big\{\big|z_1(t)\big|, \big|z_2(t)\big|\big\} \le 1 - \varepsilon'\big\}.$$

We will assume that $\tau > 0$ as there is nothing to prove otherwise. We split the integral

$$\int_0^T \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds = \int_0^\tau \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds + \int_\tau^T \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds.$$

We estimate the first integral using (16) and the Cauchy–Schwarz inequality. We get, for $0 \le s \le \tau$:

$$\int_s^\tau \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds \le (1+\delta)\Big(\int_0^\tau \frac{1 + |z_1(s)|}{|1 - z_1(s)|^2}\, ds\Big)^{1/2}\Big(\int_0^\tau \frac{1 + |z_2(s)|}{|1 - z_2(s)|^2}\, ds\Big)^{1/2}.$$

Using (13), we see that for $0 \le s \le \tau$,

$$\int_s^\tau \operatorname{Re} \psi_{\mathbb{D}}(s)\, ds \le (1+\delta)\Big(\log\Big(\frac{\varepsilon'}{1 - |z_1|}\Big)\Big)^{1/2}\Big(\log\Big(\frac{\varepsilon'}{1 - |z_2|}\Big)\Big)^{1/2}. \tag{17}$$

Thus, with $\max\{|z_1|, |z_2|\} = 1 - d$ we conclude that

$$\begin{aligned} \left|z_1(\tau) - z_2(\tau)\right| &\le \varepsilon\left(e^{\int_0^\tau \operatorname{Re}\psi_{\mathbb{D}}(s)\,ds} + \int_0^\tau e^{\int_s^\tau \operatorname{Re}\psi_{\mathbb{D}}(r)\,dr}|\xi_{\mathbb{D}}|\,ds\right) \\ &\le \varepsilon\left(\frac{\varepsilon'}{d}\right)^{1+\delta}\left(1 + \log\frac{\varepsilon'}{d}\right) \qquad (18) \\ &\le 2\varepsilon\left(\frac{\varepsilon'}{d}\right)^{1+\delta}\log\frac{1}{d}, \end{aligned}$$

if $d \le 1/e$. Here, we also used that

$$\left|\xi_{\mathbb{D}}(s)\right| \le \frac{\sqrt{1+|z_1(s)|}}{|1-z_1(s)|}\cdot\frac{\sqrt{1+|z_2(s)|}}{|1-z_2(s)|},$$

the integral of which is estimated using the Cauchy–Schwarz inequality as above. Recall that $1 + 2\delta = \rho$. There is a $d_0(\rho) > 0$ such that $d \le d_0$ implies that $d^\rho = d^{1+2\delta} \le d^{1+\delta}/\log(1/d)$. Consequently, if ε is sufficiently small we can choose d such that

$$4\varepsilon\left(\varepsilon'\right)^\delta \le 4\varepsilon \le d^{1+2\delta} \le d_0^{1+2\delta}$$

and then use (18) to get the estimate

$$\begin{aligned} \max\left\{\left|z_1(\tau)\right|, \left|z_2(\tau)\right|\right\} &\le 1 - \varepsilon' + \left|z_1(\tau) - z_2(\tau)\right| \\ &\le 1 - \varepsilon' + 2\varepsilon\left(\varepsilon'\right)^{1+\delta} d^{-(1+2\delta)} \qquad (19) \\ &\le 1 - \frac{\varepsilon'}{2}. \end{aligned}$$

Note the easy bound

$$\operatorname{Re}\psi_{\mathbb{D}}(t) \le \left|\psi_{\mathbb{D}}(t)\right| \le 4\frac{\sqrt{1+|z_1(t)|}}{|1-z_1(t)|}\cdot\frac{\sqrt{1+|z_2(t)|}}{|1-z_2(t)|}, \qquad 0 \le t \le T. \qquad (20)$$

Combining this with the Cauchy–Schwarz inequality, (13) and (19) gives

$$\int_\tau^T \operatorname{Re}\psi_{\mathbb{D}}(s)\,ds \le 4\log\frac{2}{\varepsilon'}.$$

Putting things together, we get

$$\begin{aligned} \left|f_1(T, z_1) - f_2(T, z_1)\right| &\le \varepsilon\left(e^{\int_0^T \operatorname{Re}\psi_{\mathbb{D}}(s)\,ds} + \int_0^T e^{\int_s^T \operatorname{Re}\psi_{\mathbb{D}}(r)\,dr}|\xi_{\mathbb{D}}|\,ds\right) \\ &\le 2\varepsilon\log\frac{1}{d}\exp\left\{(1+\delta)\log\frac{\varepsilon'}{d} + 4\log\frac{2}{\varepsilon'}\right\} \\ &\le c\varepsilon d^{-(1+2\delta)}, \end{aligned}$$

where $c = c(\rho) < \infty$. □

REMARK. We believe that the function $R(z, w)$ used in the last proof is bounded by 1 on the whole bi-disk, and with some work one should be able to verify this. [However, this is not true for $|R(z, w)|$.] This would allow for taking $\rho = 1$ in (14). This would not improve the resulting convergence rate in Theorem 4.3, so we will not pursue this here. However, we do expect a bound of type $\varepsilon d^{-(1/2)[(1+\beta_1)(1+\beta_2)]^{1/2}} \log d^{-1}$ to hold in the radial case, too. Having this estimate could slightly improve the resulting convergence rate in Theorem 4.3.

Suppose now that for $j = 1, 2$, f_j satisfies the derivative estimate (12) with $\beta = \beta_j$ and $c = c_j$. [In the radial case, we consider the radial version of (12) and take $\beta_j = 1$; indeed, it is a general fact about (normalized) conformal maps that (12) always holds with $\beta = 1$ for some constant universal constant $c < \infty$.] Set

$$\rho_0 = \rho_0(\beta_1, \beta_2) = \begin{cases} 1, & \text{if } X = \mathbb{D}; \\ \frac{1}{2}\sqrt{(1+\beta_1)(1+\beta_2)}, & \text{if } X = \mathbb{H}. \end{cases} \tag{21}$$

Suppose $\rho > \rho_0$ and $p \in (0, 1/\rho)$. Let $\varepsilon > 0$ and define

$$d_* = \varepsilon^p. \tag{22}$$

We have proved that for any z and w with $|z - w| \le \varepsilon$ at distance at least d_* from the boundary, if the driving terms satisfy $\sup |W_1(t) - W_2(t)| \le \varepsilon$, then there are $c = c(\rho, p) < \infty$ and $\varepsilon_0 = \varepsilon_0(\rho) > 0$ such that if $\varepsilon < \varepsilon_0$, then

$$\left| f_1(t_0, z) - f_2(t_0, w) \right| \le c\varepsilon^{1-\rho p}.$$

By estimating using Cauchy's integral formula, we also get a bound relating the derivatives: write $f_j(z) = f_j(z, t_0)$. Then with $d = \operatorname{dist}(z, \partial X)$,

$$\left| f_1'(z) - f_2'(z) \right| = \frac{1}{2\pi} \left| \oint_{|\zeta - z| = r} \frac{f_1(\zeta) - f_2(\zeta)}{(z - \zeta)^2} \, d\zeta \right| \le c\varepsilon d^{-\rho} r^{-1},$$

where $r \le d/2$. Taking $d = 2r = \varepsilon^p$ this estimate combined with the reverse triangle inequality shows that there is a constant $c = c(\rho, p, T) < \infty$ (recall that $t_0 \le T$) such that

$$\sup_{z : \operatorname{dist}(z, \partial X) \ge \varepsilon^p} \left| \left| f_1'(z) \right| - \left| f_2'(z) \right| \right| \le c\varepsilon^{1-(1+\rho)p}.$$

We have proved the radial part of the following result. (The chordal case is joint work with Rohde and Wong; see [8] for its complete proof.)

PROPOSITION 2.4. *Let $X \in \{\mathbb{D}, \mathbb{H}\}$ and $T > 0$. Let (f_j, W_j), $j = 1, 2$, be X-Loewner pairs so that f_j solve (4) with W_j as driving terms and assume that the f_j satisfiy (8) with $\beta = \beta_j$ and $c = c_j < \infty$. Suppose $\rho > \rho_0$, where ρ_0 is defined by (21). Assume that $z, w \in X$ and for $\varepsilon > 0$*

$$\sup_{t \in [0, T]} \left| W_1(t) - W_2(t) \right| \le \varepsilon, \qquad |z - w| \le \varepsilon$$

and for $p \in (0, 1/\rho)$ *define*

$$d_* = \varepsilon^p. \tag{23}$$

There exist $c = c(T, \rho, p, c_1, c_2) < \infty, \varepsilon_0 = \varepsilon_0(\rho, p) > 0, d_0 = d_0(\rho) > 0$ *such that if*

$$d_* \le \operatorname{dist}\bigl(\{z, w\}, \partial X\bigr) \le d_0$$

and $\varepsilon < \varepsilon_0$, *then*

$$\sup_{t\in[0,T]} \bigl|f_1(t,z) - f_2(t,w)\bigr| + \sup_{t\in[0,T]} \bigl|d_*\bigl|f_1'(t,z)\bigr| - d_*\bigl|f_2'(t,z)\bigr|\bigr| \le c\varepsilon^{1-\rho p}.$$

One way to interpret the last proposition is that information about the derivative of one of the conformal maps transfers to the other via the Loewner equation if they are evaluated sufficiently far away from the boundary. The proper scale (or resolution) is determined by the distance between the driving terms. Note that we make no assumptions about the regularity of the driving terms; the above results are consequences of the structure of the Loewner equation alone.

2.4. *Supremum distance between Loewner curves*. We will now consider two Loewner curves, $\gamma_j : [0, T] \to X$, $j = 1, 2$, generating the X-Loewner pairs (f_j, W_j) and suppose that

$$\sup_{t\in[0,T]} \bigl|W_1(t) - W_2(t)\bigr| \le \varepsilon. \tag{24}$$

We are interested in estimating the supremum distance $\sup_{t\in[0,T]} |\gamma_1(t) - \gamma_2(t)|$ when the curves are parameterized by capacity, in terms ε. We have the following estimate.

PROPOSITION 2.5. *Let* $X \in \{\mathbb{D}, \mathbb{H}\}$. *For* $j = 1, 2$, *let* (f_j, W_j) *be* X*-Loewner pairs generated by the curves* γ_j *and suppose that there are* $d_0 > 0$ *and* β_j, c_j *such that* f_j *satisfy* (8) *with* $\beta = \beta_j$ *and* $c = c_j$. *Let* $\rho > \rho_0$, *where* ρ_0 *is given by* (21). *Suppose that* $\varepsilon > 0$ *is such that*

$$\sup_{t\in[0,T]} \bigl|W_1(t) - W_2(t)\bigr| \le \varepsilon.$$

Let $p \in (1, 1/\rho)$ *and set* $d = \varepsilon^p$. *There exist* $c = c(T, \rho, p) < \infty$ *and* $\varepsilon_0 = \varepsilon_0(\rho, p) > 0$ *such that if* $\varepsilon < \varepsilon_0$, *then*

$$\begin{aligned} &\sup_{t\in[0,T]} \bigl|\gamma_1(t) - \gamma_2(t)\bigr| \\ &\quad \le c\varepsilon^{1-\rho p} + c \sup_{t\in[0,T]} \Bigl(\bigl|\gamma_1(t) - f_1\bigl(t, (1-d)W_1(t)\bigr)\bigr| \\ &\qquad + \bigl|\gamma_2(t) - f_2\bigl(t, (1-d)W_2(t)\bigr)\bigr|\Bigr) \end{aligned} \tag{25}$$

with $f_j(t, (1-d)W_j(t))$ *replaced by* $f_j(t, W_j(t) + id)$ *in the chordal case.*

PROOF. We will do the radial case. Write

$$\begin{aligned}|\gamma_1(t)-\gamma_2(t)| \le{}& |\gamma_1(t)-f_1(t,(1-d)W_1(t))|\\&+|f_1(t,(1-d)W_1(t))-f_1(t,(1-d)W_2(t))|\\&+|f_1(t,(1-d)W_2(t))-f_2(t,(1-d)W_2(t))|\\&+|f_2(t,(1-d)W_2(t))-\gamma_2(t)|.\end{aligned}$$

Denote by $b_1,\dots,b_4$ the four terms on the right-hand side in the last inequality in the order in which they appear. By the distortion theorem, since $d \ge \varepsilon$ we have that

$$b_2 \le c\,\mathrm{dist}(f_1(t,(1-d)W_1(t)), \partial f_1(t,\mathbb{D})) \le cb_1.$$

Finally, by Proposition 2.4, $b_3 \le c\varepsilon^{1-\rho p}$. □

COROLLARY 2.6. *For $j=1,2$, let (f_j, W_j) be $\mathbb{H}$-Loewner pairs generated by the curves γ_j and assume that* (24) *holds. Suppose that there exist $d_0>0$, $c<\infty$, and $\beta<1$ such that the f_j satisfy the estimate* (8). *Then for every*

$$r < 2\frac{1-\beta}{3-\beta},$$

there exist $c=c(r,T)<\infty$ and $\varepsilon_0=\varepsilon_0(r,T)>0$ such that if $\varepsilon<\varepsilon_0$, then

$$\sup_{t\in[0,T]} |\gamma_1(t)-\gamma_2(t)| \le c\varepsilon^r.$$

PROOF. Under our assumptions $\rho_0=(1+\beta)/2$. Let $\rho>\rho_0$ and $0<p<1/\rho$. We set $d=\varepsilon^p$, apply Proposition 2.5, and integrate the bound on the derivatives to see that for $\varepsilon>0$ sufficiently small,

$$\sup_{t\in[0,T]} |\gamma_1(t)-\gamma_2(t)| \le c(\varepsilon^{1-\rho p}+\varepsilon^{p(1-\beta)}).$$

We optimize over exponents to find the stated bound for r. □

The proof of the next corollary is an analog for Loewner curves of the well-known fact that the Riemann map onto a Hölder domain satisfies a power-law bound on the growth of the derivative.

COROLLARY 2.7. *For $j=1,2$, let (f_j,W_j) be $\mathbb{H}$-Loewner pairs generated by the curves γ_j and assume that* (24) *holds. Suppose that both curves are Hölder-α continuous in the capacity parameterization, where $\alpha>0$. Then for every*

$$r < \frac{2\alpha}{1+\alpha},$$

there exist $c = c(r, T) < \infty$ *and* $\varepsilon_0 = \varepsilon_0(r, T) > 0$ *such that if* $\varepsilon < \varepsilon_0$, *then*

$$\sup_{t\in[0,T]} \left|\gamma_1(t) - \gamma_2(t)\right| \le c\varepsilon^r.$$

PROOF. We will prove a bound on the growth of the derivative and then apply the previous corollary. It is enough to consider $f(t,z) := f_1(t,z)$ since we made the same assumptions on both Loewner chains. Write $\gamma = \gamma_1$ and $W = W_1$ and for $t, t+s \in [0, T]$, let

$$\tilde{\gamma} = f^{-1}\bigl(t, \gamma[t, t+s]\bigr).$$

Then $\tilde{\gamma}$ is a curve in $\mathbb{H}$ "rooted" at $W(t)$. Set $d = \operatorname{diam}\tilde{\gamma}$. Let $z \in \tilde{\gamma}$ be a point such that $|z - W(t)| = d/2$ and let Γ be the hyperbolic geodesic in $\mathbb{H}$ connecting $W(t)$ with z. Then Γ contains a point w with $\operatorname{Im} w \ge d/4$. Note that by the distortion theorem, $|f'(t,w)| \asymp |f'(t, W(t) + id)|$ so that Koebe's 1/4 theorem implies that there is a universal constant $c > 0$ such that

$$\mathcal{B}\bigl(f(t,w), cd\bigl|f'\bigl(t, W(t)+id\bigr)\bigr|\bigr) \subset f\bigl(t, \mathcal{B}(w, d/8)\bigr).$$

[Here, and in the sequel $\mathcal{B}(z,r) = \{w : |w - z| < r\}$.] Consequently,

$$\operatorname{diam} f(t,\Gamma) \ge cd\bigl|f'\bigl(t, W(t)+id\bigr)\bigr|. \tag{26}$$

On the other hand, by the Gehring–Hayman theorem (see Chapter 4 of [20]) and the assumption on γ, we have that there are constants $c, c' < \infty$, depending only on the constant in the modulus of continuity for γ, such that

$$\operatorname{diam} f(t,\Gamma) \le c \operatorname{diam}\gamma[t, t+s] \le c's^{\alpha}.$$

Hence, using (26), there is a constant $c < \infty$ such that

$$d\bigl|f'\bigl(t, W(t)+id\bigr)\bigr| \le cs^{\alpha} \le c'd^{2\alpha},$$

where the last inequality follows since $\operatorname{hcap}\tilde{\gamma} = 2s$ so that there is a universal constant $c < \infty$ such that $s \le cd^2$. The diameter d depended on s, but every d sufficiently small can be written like this since $s \mapsto d$ is an increasing continuous function. $\square$

REMARK. If $\gamma(t)$ is Hölder-α continuous in the capacity parameterization, then its driving term is at least Hölder-$\alpha/2$: using the notion of the proof of Corollary 2.7, we note that by the Beurling estimate, $\operatorname{diam}\tilde{\gamma} \le cs^{\alpha/2}$ and by Lemma 2.1 of [13], we have $|W(t+s) - W(t)| \le c \operatorname{diam}\tilde{\gamma} \le c's^{\alpha/2}$.

3. Geometric conditions. This section develops a geometric condition that we will use in place of a bound on the growth of the derivative of the conformal map in order to measure the regularity of a Loewner curve locally at the tip. As pointed out in the Introduction, several similar conditions have appeared in the literature. We will work in the radial setting, but the results hold also in the chordal setting with minor modifications in their statements and proofs.

Let $D \ni 0$ be a simply connected domain. Let $\psi : D \to \mathbb{D}$ be the uniformizing conformal map. We consider a radial Loewner curve $\gamma : [0, T] \to D$, that is, the conformal image of γ in $\mathbb{D}$ using the conformal map ψ is a $\mathbb{D}$-Loewner curve. In this section we write D_t for the connected component of $D \setminus \gamma[0, t]$ containing the origin.

3.1. *Tip structure modulus.* For $s, t \in [0, T]$ with $s \le t$, we let $\gamma_{s,t}$ denote the curve determined by $\gamma(r), r \in [s, t]$. For a crosscut $\mathcal{C}$ of D_t, we write $J_{\mathcal{C}}$ for the component of $D_t \setminus \mathcal{C}$ of smaller diameter.

For each $0 \le t \le T$ and $\delta > 0$, let $S_{t,\delta}$ be the collection of crosscuts of D_t of diameter at most δ that separate $\gamma(t)$ from 0 in D_t. For a crosscut $\mathcal{C} \in S_{t,\delta}$, define

$$s_{\mathcal{C}} = \inf\{s > 0 : \gamma[t - s, t] \cap \overline{\mathcal{C}} \neq \varnothing\}, \qquad \gamma_{\mathcal{C}} = \big(\gamma(r), r \in [t - s_{\mathcal{C}}, t]\big).$$

(We set $s_{\mathcal{C}} = t$ if γ never intersects $\overline{\mathcal{C}}$.) For $\delta > 0$, we define the *tip structure modulus* of $(\gamma(t), t \in [0, T])$ in D, written $\eta_{\text{tip}}(\delta)$, to be the maximum of δ and

$$\sup_{t\in[0,T]} \sup_{\mathcal{C}\in S_{t,\delta}} \operatorname{diam} \gamma_{\mathcal{C}}. \tag{27}$$

REMARK. In the chordal setting, we consider instead crosscuts separating $\gamma(t)$ from ∞ in H_t in the definition of the structure modulus. The remaining construction is the same.

It is useful to introduce some more terminology. Given $0 < \delta \le \eta$, we will say that the curve γ has a (δ, η)-*bottleneck* in D if there exist $t \in [0, T]$ and $\zeta \in \partial D_t$ such that $\gamma(t)$ and ζ can be connected by a crosscut $\mathcal{C}_t$ of D_t and $\operatorname{diam} J_{\mathcal{C}_t} \ge \eta$ while $\operatorname{diam} \mathcal{C}_t \le \delta$. This definition is similar to the one for "quasi-loops" given by Schramm in [22]. We say that the bottleneck is at z_0 if the points ζ and $\gamma(t)$ in the previous definition are contained in the disk $\mathcal{B}(z_0, \eta/4)$.

Similarly, given $0 < \delta \le \eta$ we will say that the curve γ has a *nested* (δ, η)-*bottleneck* in D if there exist $t \in [0, T]$ and $\mathcal{C} \in S_{t,\delta}$ with

$$\operatorname{diam} \gamma_{\mathcal{C}} \ge \eta.$$

That $\gamma(t), t \in [0, T]$ has no nested (δ, η)-bottleneck in D is clearly equivalent to having the inequality $\eta_{\text{tip}}(\delta) \le \eta$.

REMARK. The definition of nested bottleneck is independent of the particular chosen parameterization of the curve in the sense that any increasing reparameterization would do in the definition. The definition is not, however, symmetric with respect to reversibility of the curve.

The term "structure modulus" is borrowed from Warschawski [26] who used it in the following sense: the "structure modulus of the boundary of D" is defined by the function

$$\eta_W(\delta) = \sup_{\mathcal{C}} \operatorname{diam} J_{\mathcal{C}},$$

where the supremum is over all crosscuts (of D) of diameter at most δ and $J_{\mathcal{C}} \subset \partial D$ is the subarc of smaller diameter separated from 0 by $\mathcal{C}$. Intuitively, the decay rate of η_W places a restriction on bottlenecks/outward-pointing cusps in the boundary and this gives estimates on the regularity of the Riemann mapping from $\mathbb{D}$. For example, D is a John domain if and only if $\eta_W(\delta) \le A\delta$ for some constant $A < \infty$. One can use this to show (see [26]) that if $h < 2/(A^2\pi^2)$, then the Riemann map from $\mathbb{D}$ is Hölder-h on the closed unit disk. The tip structure modulus is the natural analogue to η_W for Loewner curves; see Theorem 3.5 below. Moreover, and importantly, the tip structure modulus is related to annuli crossing events (see Figure 1), the probabilities of which are often known how to control for discrete-model

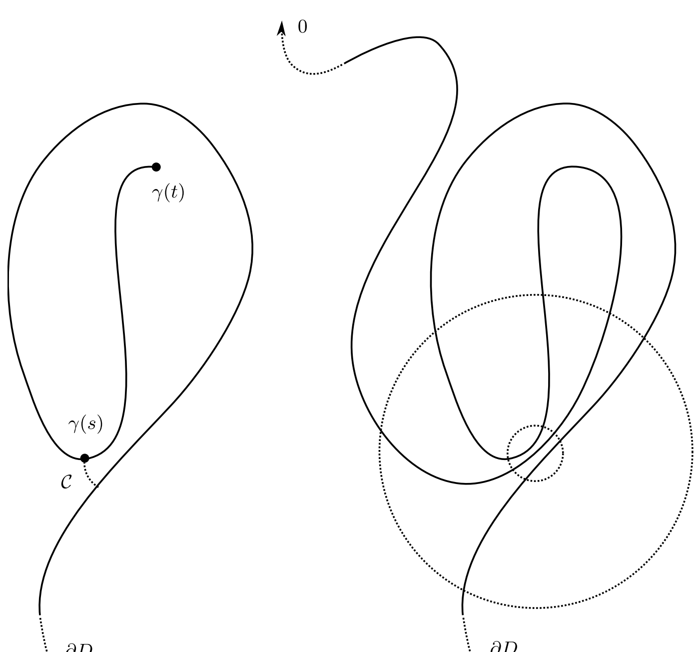

FIG. 1. *A nested* (δ, η)*-bottleneck with* $\operatorname{diam} \mathcal{C} = \delta$ *and* $\operatorname{diam} \gamma_{\mathcal{C}} \ge \eta$, *where* $\gamma_{\mathcal{C}} = \gamma[s, t]$. *A* 6*-crossing event of a* (δ, η)*-annulus for the whole curve.*

curves; the connection between annuli crossings and regularity of curves is well known; see, for example, [1].

3.2. *Distance to the tip*. Let (f, W) be a $\mathbb{D}$-Loewner pair and assume it is generated by a curve γ. We use the notation

$$\Delta_t(d) = \operatorname{dist}\big(f_t\big((1-d)W_t\big), D_t\big),$$

where $W_t = e^{i\xi_t}$ is the driving term for (f_t). Note that Koebe's distortion theorem implies that

$$\Delta_t(d) \asymp d\big|f_t'\big((1-d)W_t\big)\big|.$$

Recall also that for each t, the tip of the curve is given by taking the radial limit

$$\gamma(t) = \lim_{d \to 0+} f_t\big((1-d)W_t\big).$$

We saw in Section 2.4 that we need to obtain uniform (in t) bounds on

$$\big|\gamma(t) - f_t\big((1-d)W_t\big)\big|.$$

A lower bound on this quantity is clearly given by $\Delta_t(d)$ and if we have a bound for $\eta_{\text{tip}}(\delta)$ in terms of δ, then we can also give an estimate from above in terms of $\Delta_t(d)$. We need the following lemma. (See Figure 2 for a sketch illustrating the proof.)

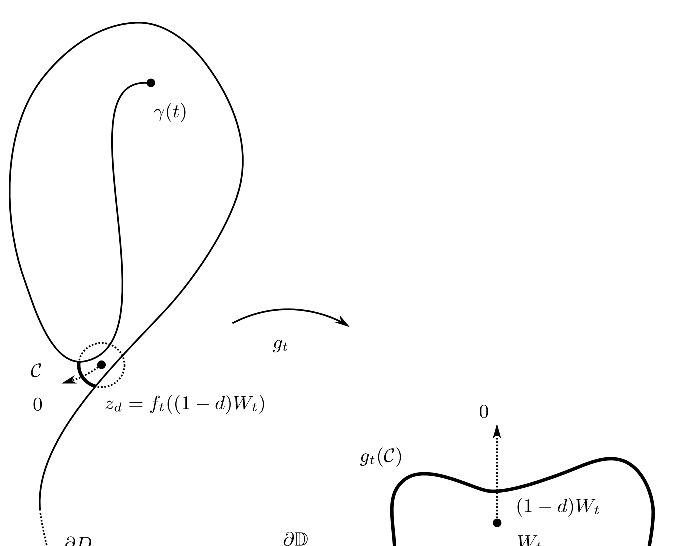

FIG. 2. *Sketch for the proof of Lemma 3.1. The crosscut $g_t(\mathcal{C})$ separates $(1-d)W_t$ and $g_t(\mathcal{E}) \subset \partial\mathbb{D}$ from 0 in $\mathbb{D}$. The harmonic measure of $g_t(\mathcal{E})$ from $(1-d)W_t$ is at least $1/2$. Hence, $W_t \in g_t(\mathcal{E})$.*

LEMMA 3.1. *Let $T < \infty$ be given. There exist constants $0 < \rho_1, c_1 < \infty$ with ρ_1 universal and $c_1 = c_1(T)$ such that the following holds. Let γ be a curve in $\mathbb{D}$ generated by the Loewner pair (f, W). Let $t \in [0, T]$. If $\Delta_t(d) < c_1$ then there is a crosscut $\mathcal{C} = \mathcal{C}_t$ of D_t that separates $f_t((1-d)W_t)$ and $\gamma(t)$ from 0 in D_t while*

$$\operatorname{diam} \mathcal{C} \le \rho_1 \Delta_t(d).$$

Moreover, $\mathcal{C}$ can be taken to be a subarc of $\mathcal{B}(f_t((1-d)W_t), \rho_1 \Delta_t(d)/2)$.

PROOF. Let $t \in [0, T]$ and set

$$z_d = f_t\big((1-d)W_t\big).$$

We will write

$$\Delta = \Delta_t(d) = \operatorname{dist}(z_d, \partial D_t).$$

For $\rho > 1$, consider $(\partial \mathcal{B}(z_d, \rho\Delta)) \cap D_t$. The components of this set form crosscuts of D_t and we let C_0 be the subset of those crosscuts that separate z_d from 0 in D_t. (Since the inner radius of D_t from 0 is bounded below by $e^{-T}/4$, C_0 is nonempty whenever $\rho\Delta$ is smaller than, say, $e^{-T}/16$.) Let $\mathcal{C}_\rho$ be the unique crosscut in C_0 with the property that it separates every other member in C_0 from 0 in D_t. Let $\mathcal{O}_\rho$ be the component of $D_t \setminus \mathcal{C}_\rho$ that contains z_d and let $\mathcal{E}_\rho = \partial\mathcal{O}_\rho \setminus \mathcal{C}_\rho$. By Beurling's projection theorem and the maximum principle, there exists a universal $\rho_0 < \infty$ and for each $\rho > \rho_0$ a constant $c_0 = c_0(\rho, T) > 0$ such that if $\Delta < c_0$ then we have the following lower bound on harmonic measure:

$$\omega(z_d, \mathcal{E}_\rho, \mathcal{O}_\rho) > 1/2. \tag{28}$$

Let $\mathcal{O} := \mathcal{O}_{2\rho_0}$, $\mathcal{C} := \mathcal{C}_{2\rho_0}$ and $\mathcal{E} := \mathcal{E}_{2\rho_0}$. Let $c_1 = c_1(T) < \infty$ be such that if $\Delta < c_1$, then the diameter of the preimage of $\mathcal{C}$ in $\mathbb{D}$ is at most $1/2$ and (28) holds with ρ replaced by $2\rho_0$. (Existence of such a c_1 follows from Beurling's projection theorem.) We shall assume that $\Delta < c_1$ in the sequel. We claim that the preimage of $\mathcal{E}$ in $\partial\mathbb{D}$ is an arc containing the point W_t. Indeed, it is clear that it is an arc of $\partial\mathbb{D}$. If $g_t = f_t^{-1}$ then $g_t(\mathcal{C})$ is a crosscut of $\mathbb{D}$ separating $g_t(\mathcal{E})$ and $(1-d)W_t$ from 0. By conformal invariance, the maximum principle and (28), the harmonic measure of $g_t(\mathcal{E})$ from $(1-d)W_t$ is strictly bigger than $1/2$. Write $W_t = e^{i\xi_t}$. Note that by symmetry, the harmonic measure from $(1-d)W_t$ of $\{e^{i(\xi_t+\theta)} : 0 \le \theta \le \pi\}$ in $\mathbb{D}$ is exactly $1/2$. Therefore, if $W_t = e^{i\xi_t} \notin g_t(\mathcal{E})$, then the arc $g_t(\mathcal{E})$ must contain the point $e^{i(\xi_t+\pi)}$. Since $g_t(\mathcal{C})$ separates $(1-d)W_t$ and $e^{i(\xi_t+\pi)}$ from 0, this would imply that $\operatorname{diam} g_t(\mathcal{C}) > 1/2$ and this is a contradiction. $\square$

PROPOSITION 3.2. *Let $T < \infty$ be given. There exist constants $0 < c_1$, c_2, $c_3 < \infty$ with c_1 depending only on T and c_2, c_3 universal such that the following holds. Let γ be a curve in $\mathbb{D}$ generating the Loewner pair (f, W) and let $\eta_{\mathrm{tip}}(\delta)$ be the tip structure modulus for $(\gamma(t), t \in [0, T])$. Then if $t \in [0, T]$ and $\Delta_t(d) < c_1$, we have*

$$\big|\gamma(t) - f_t\big((1-d)W_t\big)\big| \le c_2 \eta_{\mathrm{tip}}\big(c_3 \Delta_t(d)\big). \tag{29}$$

PROOF. We use the notation from the proof of Lemma 3.1. Set

$$\delta_0 = \rho_1 \Delta/2,$$

where ρ_1 is as in Lemma 3.1. Then by Lemma 3.1 (if $\Delta < c_1$, where c_1 is the constant of that lemma) there is a crosscut $\mathcal{C} \subset \mathcal{B}(z_d, \delta_0)$ separating z_d and $\gamma(t)$ from 0 in D_t while $\operatorname{diam} \mathcal{C} \le 2\delta_0$. By the definition of tip structure modulus, $\operatorname{dist}(\gamma(t), \mathcal{C}) \le \eta_{\text{tip}}(2\delta_0)$ and consequently, $|z_d - \gamma(t)| \le \eta_{\text{tip}}(2\delta_0) + \delta_0$. □

One can also estimate the distance to the tip directly in terms of d, the distance to the boundary in $\mathbb{D}$.

PROPOSITION 3.3. *There is a constant $c < \infty$ such that the following holds. Let $T < \infty$ be given. Let γ be a curve in $\mathbb{D}$ generating the Loewner pair (f, W) and let $\eta_{\text{tip}}(\delta)$ be the tip structure modulus for $(\gamma(t), t \in [0, T])$. Then for every $t \in [0, T]$ and $d < 1/2$,*

$$|\gamma(t) - f_t((1-d)W_t)| \le c\eta_{\text{tip}}((2\pi A/(\log 1/d))^{1/2}), \tag{30}$$

where A may be chosen as $\min\{\pi(\operatorname{diam} \gamma_{0,T})^2, \pi\}$.

PROOF. The needed estimate is a consequence of a classical result due to J. Wolff. We will give a short proof using extremal length. Consider $\mathcal{A} = \mathcal{A}(r, R) \cap \mathbb{D}$ centered around W_t, the preimage of $\gamma(t)$ in $\partial\mathbb{D}$. Let E and F be the two boundary components of $\mathcal{A}$ which are contained in $\partial\mathbb{D}$. By comparing with a half-annulus and mapping to a rectangle, using also the comparison principle for extremal length, we see that the extremal distance between E and F in $\mathcal{A}$ is at most $\pi/\log(R/r)$. Hence, by conformal invariance and the definition of extremal length,

$$\frac{\pi}{\log(R/r)} \ge \frac{L^2}{A},$$

where L is the euclidean length of the curve-family connecting $f(E)$ with $f(F)$ in $f(\mathcal{A})$ and A is the Euclidean area of $f(\mathcal{A})$. The number A is clearly bounded above by the minimum of $\pi(\operatorname{diam} \gamma_{0,T})^2$ and π. Consequently, by taking $r = d$ and $R = \sqrt{d}$ we see that there exists a crosscut $\mathcal{C}'$ of D_t separating $\gamma(t)$ and $z_d = f_t((1-d)W_t)$ from 0 and the diameter of $\mathcal{C}'$ is at most $l(d) := (2\pi A/(\log 1/d))^{1/2}$. Hence, $\operatorname{dist}(\gamma(t), \mathcal{C}') \le \eta_{\text{tip}}(l(d))$ and an argument using the Gehring–Hayman theorem (see, e.g., Theorem 4.20 of [20], and also below) now shows that $\operatorname{dist}(z_d, \gamma(t)) \le c(\eta_{\text{tip}}(l(d)) + l(d)) \le c'\eta_{\text{tip}}(l(d))$. □

We end the section with a lemma that combines some of the previous work in this section and that of Section 2. It is tailored for the situation where a discrete model Loewner curve approaches an SLE curve in the scaling limit. We will use it in the proof of Theorem 4.3 in Section 4.

LEMMA 3.4. *For $j = 1, 2$, let (f_j, W_j) be $\mathbb{D}$-Loewner pairs generated by the curves γ_j. Fix $T < \infty$ and $\rho > 1$. Assume that there exist $\beta < 1$, $r \in (0, 1)$, $p \in (0, \frac{1}{\rho})$ and $\varepsilon > 0$ such that the following holds with*

$$d_* = \varepsilon^p.$$

(i) *The driving terms satisfy*

$$\sup_{t\in[0,T]} \big|W_1(t) - W_2(t)\big| \le \varepsilon;$$

(ii) *There exists a constant $c < \infty$ such that the tip structure modulus for $(\gamma_1(t), t \in [0, T])$ in $\mathbb{D}$ satisfies*

$$\eta_{\text{tip}}(d_*) \le c d_*^r;$$

(iii) *There exists a constant $c' < \infty$ such that the derivative estimate*

$$\sup_{t\in[0,T]} d\big|f_2'\big(t, (1-d)W_2(t)\big)\big| \le c' d^{1-\beta} \qquad \forall d \le d_*,$$

holds.

Then there is a constant $c'' = c''(T, \beta, r, p, c, c') < \infty$ such that

$$\sup_{t\in[0,T]} \big|\gamma_1(t) - \gamma_2(t)\big| \le c'' \max\big\{\varepsilon^{p(1-\beta)r}, \varepsilon^{(1-\rho p)r}\big\}.$$

The analogous statement holds for $\mathbb{H}$-Loewner pairs.

PROOF. The proof is immediate from the assumptions using Proposition 2.5 combined with Proposition 3.2. □

3.3. *Hölder regularity.* We shall now see that the John-type condition $\eta_{\text{tip}}(\delta) \le A\delta, \delta < \delta_0$, forces a curve driven by a Hölder continuous function to be Hölder continuous in the capacity parameterization, with exponent depending only on A and the exponent for the driving term. Note that we must have $A \ge 1$. We will derive a bound on the growth of the derivative as in (8) from the bound on η_{tip}. Hölder regularity then follows from Proposition 2.2. The proof uses the length–area principle. The situation is different from the classical one; see, for example, [26] or [20], in that our assumptions do not prevent large bottlenecks to form.

THEOREM 3.5. *Suppose that the radial Loewner pair $(f, e^{i\xi})$ is generated by a curve γ. Assume that ξ is Hölder continuous and that there exist $A < \infty$ and $\delta_0 > 0$ such that the tip structure modulus for $(\gamma(t), t \in [0, T])$ in $\mathbb{D}$ satisfies $\eta_{\text{tip}}(\delta) \le A\delta$, $\delta < \delta_0$. Then γ is Hölder continuous on $[0, T]$ with Hölder exponent depending only on A and the Hölder exponent for ξ.*

REMARK. A bound on the tip structure modulus alone cannot imply Hölder regularity of the path in the capacity parameterization; it is necessary to have some regularity of the driving term. Indeed, consider the chordal setting and take γ to be the graph of $e^{-1/x}$, $x \in [0, 1]$. For this curve, the tip structure modulus clearly decays linearly, uniformly in t. On the other hand, parameterize by half-plane capacity and note that there is a universal constant c such that

$$2t = \operatorname{hcap} \gamma[0, t] \le c \text{ height } \gamma[0, t] \cdot \operatorname{diam} \gamma[0, t].$$

(This follows, e.g., from a harmonic measure estimate.) Hence,

$$t \le c e^{-1/\operatorname{Re} \gamma(t)} \operatorname{Re} \gamma(t),$$

which shows that γ is not Hölder continuous at $t = 0$. (By precomposing with slit map $\sqrt{z^2 - 4T}$, a similar example can be constructed with the "singularity" occurring at an arbitrary $T > 0$.) Moreover, if W is the driving term for γ, then

$$\operatorname{diam} \gamma[0, t] \asymp \sqrt{t} + \sup_{s \in [0,t]} \left|W(s)\right|,$$

so W is also not Hölder continuous. (In fact, a similar argument shows that if the driving term is Hölder-α, $\alpha \le 1/2$, at $t = 0$, then so is the curve.)

It is possible to take this example as a starting point to formulate a geometric condition that implies Hölder continuity for the driving term. We shall not, however, pursue this further here.

Before giving the proof of Theorem 3.5, we need a simple lemma.

LEMMA 3.6. *Let $f : \mathbb{D} \to D$ be a conformal map with $f(0) = 0$. Define the Stolz cone*

$$S_r = \left\{1 - \rho e^{i\theta} : 0 \le \rho \le r, -\pi/4 \le \theta \le \pi/4\right\}.$$

There is a universal constant $c < \infty$ such that

$$\operatorname{diam} f(S_r) \le c \operatorname{diam} f(\sigma_r),$$

where $\sigma_r = [1 - r, 1)$ is the line segment connecting $1 - r$ and 1.

PROOF. Let $u = 1 - \rho e^{i\theta}$ be an arbitrary point in S_r. By Koebe's distortion theorem, there is a universal constant c such that

$$\left|f(u) - f(1 - \rho)\right| \le c\rho \left|f'(1 - \rho)\right|.$$

Hence, by Koebe's estimate there is a universal constant c' such that

$$\begin{aligned} \left|f(u) - f(1 - \rho)\right| &\le c' \operatorname{dist}\bigl(f(1 - \rho), \partial D\bigr) \\ &\le c' \operatorname{diam} f(\sigma_r) \end{aligned}$$

and this completes the proof. $\square$

PROOF OF THEOREM 3.5. Let $t \in [0, T]$ and write $W_t = e^{i\xi_t}$. Without loss of generality, we may assume that $t > 0$ and that $W_t = 1$. We suppress the dependence on t and write f for f_t and D for D_t, etc. throughout the proof. Set $z_r = f(1-r)$ and $\Delta_r = \operatorname{dist}(z_r, \partial D)$. By Proposition 3.3, there is an r_0 depending only on A and δ_0 such that $\Delta_r \le \delta_0$ for all $r \le r_0$. By taking r_0 smaller if necessary, depending only on T, we can guarantee that the assumptions of Lemma 3.1 are satisfied so that there will exist a universal $\rho_0 < \infty$ and a crosscut $\mathcal{C}$ contained in $\partial\mathcal{B}(z_r, \rho_0\Delta_r)$ that separates z_r and $\gamma(t)$ from 0 in D. Let $\sigma_r = [1 - r, 1]$. We claim that $f(\sigma_r)$, which connects z_r with $\gamma(t)$ in D, satisfies

$$\operatorname{diam} f(\sigma_r) \le c\rho_0 A\Delta_r, \tag{31}$$

where c is a universal constant. To prove this, note that since $\mathcal{C}$ separates $\gamma(t)$ and z_r from 0, the hyperbolic geodesic $f(\sigma_1) \supset f(\sigma_r)$ which connects $\gamma(t)$ and 0 must intersect $\mathcal{C}$. [Since γ is a Loewner curve, $\gamma(t)$ is always on the boundary of the simply connected domain $D_t \ni 0$.] Let Γ'' be the curve obtained by tracing $f(\sigma_1)$ from 0 to $\gamma(t)$ until $\mathcal{C}$ is first hit. Let $\Gamma' = f(\sigma_1) \setminus \Gamma''$. Then Γ' is a hyperbolic geodesic connecting a point on $\mathcal{C}$ with $\gamma(t)$ in D_t and $f(\sigma_r) \subset \Gamma'$. By the bound on the structure modulus, there is a curve Γ connecting $\gamma(t)$ with $\mathcal{C}$ in D_t and

$$\operatorname{diam}\Gamma \le 2A\operatorname{diam}\mathcal{C} \le 4\rho_0 A\Delta_r.$$

The Gehring–Hayman theorem (see, e.g., Chapter 4 of [20]) now implies that there is a universal constant c such that

$$\operatorname{diam} f(\sigma_r) \le \operatorname{diam}\Gamma' \le c(\operatorname{diam}\Gamma + \operatorname{diam}\mathcal{C})$$

and this gives (31).

Using Lemma 3.6, the remainder of the proof now proceeds by a standard length–area type argument (see, e.g., Chapter 5 of [20]). Define

$$\varphi(r) = \int_0^r |f'(1-r)|^2 r\,dr.$$

Then by Koebe's distortion theorem, there is a universal constant c_0 such that

$$r^2|f'(1-r)|^2 \le c_0\int_{r/2}^r r|f'(1-r)|^2\,dr \le c_0\varphi(r). \tag{32}$$

This theorem also implies that there is a constant c_1 depending only on c_0 such that

$$\varphi(r) \le c_1\int_0^r \int_{-\pi/4}^{\pi/4} |f'(1-re^{i\theta})|^2 r\,dr\,d\theta = c_1 \operatorname{area} f(S_r),$$

where S_r is the Stolz cone defined in the statement of Lemma 3.6. Now, by (31) and Lemma 3.6 we have that

$$\operatorname{area} f(S_r) \le \frac{\pi^2}{4}(\operatorname{diam} f(S_r))^2 \le c_2\Delta_r^2.$$

Hence,

$$\varphi(r) \le c_1 \operatorname{area} f(S_r) \le c_3 r^2 \left|f'(1-r)\right|^2.$$

Consequently, since $\varphi'(r) = r|f'(1-r)|^2$, we have for $r_0 > r$ and a constant c_4 depending only on A

$$\log\left(\frac{\varphi(r_0)}{\varphi(r)}\right) = \int_r^{r_0} \frac{\varphi'(r)}{\varphi(r)}\, dr \ge c_4^{-1} \log\left(\frac{r_0}{r}\right).$$

Taking exponentials, using (32), gives for $0 < r \le r_0$

$$r^2 \left|f'(1-r)\right|^2 \le c_5 r^{1/c_4},$$

where c_5 depends only on r_0. Hence, if $\beta = 1 - 1/(2c_4) < 1$ we see that

$$r\left|f'(1-r)\right| \le c_6 r^{1-\beta}.$$

By Proposition 2.2, since the estimates were uniform in t, this implies Hölder regularity with an exponent depending only on A and the exponent for W. □

4. Loop-erased random walk and $\mathbf{SLE_2}$. This section proves a convergence rate result for loop-erased random walk using the setup detailed in the previous sections.

4.1. *Definitions*. The radial *Schramm–Loewner evolution*, radial SLE_κ, is defined by taking $W(t) = e^{i\sqrt{\kappa}B(t)}$ as driving term for the radial Loewner equation, where B is standard Brownian motion. It is a fact that this Loewner chain is almost surely generated by a curve—the SLE_κ path. This is a random fractal curve which is simple when $0 \le \kappa \le 4$, has double points when $4 < \kappa$ and is space filling when $\kappa \ge 8$. See [21] for proofs of these results. In Appendix A, we discuss a derivative estimate for radial SLE_κ that we will state and use in this section when $\kappa = 2$. For technical reasons, we need a stopping time σ for the radial SLE path $\tilde{\gamma}$ further discussed in Appendix A. Fix a small constant $\varepsilon > 0$. We then define

$$\sigma = \sigma(\varepsilon, T) = \inf\left\{t \ge 0 : \left|g_t(-1) - W(t)\right| \le \varepsilon\right\} \wedge T, \tag{33}$$

where $g_t = f_t^{-1}$ is the forward Loewner SLE_2 flow and $W(t)$ is the driving term for f_t.

PROPOSITION 4.1. *Let $\varepsilon > 0$ and $T < \infty$ be fixed and let $(f_s), 0 \le s \le \sigma$, be the stopped radial SLE_2 Loewner chain with $\sigma = \sigma(\varepsilon, T)$ defined by* (33). *For every $\beta \in (2(\sqrt{10} - 1)/9, 1)$ and $q < q(\beta)$, there exists a constant $c = c(\beta, q, \varepsilon, T) < \infty$ such for all $d_* \le 1$*

$$\mathbb{P}\left\{\forall d \le d_*, \sup_{s\in[0,\sigma]} d\left|f_s'\big((1-d)W(s)\big)\right| \le d^{1-\beta}\right\} \ge 1 - c d_*^q,$$

where

$$q(\beta) = -1 + 2\beta + \frac{\beta^2}{4(1+\beta)}.$$

PROOF. See Appendix A. $\square$

Let $D \ni 0$ be a simply connected domain and assume that the inner radius of D with respect to 0 equals 1. We will assume, for simplicity, that D is a Jordan domain with $C^{1+\alpha}$ boundary, where $\alpha > 0$. We shall consider a particular discretization of D. A *grid-domain* with respect to $n^{-1}\mathbb{Z}^2$ is a simply connected domain whose boundary is a subset of the edge set of the graph $n^{-1}\mathbb{Z}^2$. We define $D_n = D_n(D)$, the $n^{-1}\mathbb{Z}^2$ grid-domain approximation of D, as the component of 0 of $\mathbb{C}$ minus those closed $n^{-1}\mathbb{Z}^2$ lattice faces that intersect ∂D. Then clearly D_n is a grid-domain contained in D. Let $\psi_n : D_n \to \mathbb{D}$ be the normalized conformal map.

Suppose $S = S(j)$, $j = 0, 1, \dots, m$, is a finite nearest-neighbor walk on (the vertices of $n^{-1}\mathbb{Z}^2$ contained in) D_n. We define the loop-erasure $\mathcal{L}\{S\} \subset S$ in the following way. If S is already self-avoiding, set $\mathcal{L}\{S\} = S$. Otherwise, let $s_0 = \max\{j : S(j) = S(0)\}$, and for $i > 0$, let $s_i = \max\{j : S(j) = S(s_{i-1} + 1)\}$. If we let $n = \min\{i : s_i = m\}$, then $\mathcal{L}\{S\} = \{S(s_0), S(s_1), \dots, S(s_n)\}$. Notice that $\mathcal{L}\{S\}(0) = S(0)$ and $\mathcal{L}\{S\}(s_n) = S(m)$, that is, the loop-erased walk has the same end points as the original walk S. *Loop-erased random walk* (LERW) from 0 to ∂D_n in D_n is the random self-avoiding walk γ_n obtained by taking S to be a simple random walk on $n^{-1}\mathbb{Z}^2$ started from 0 and stopped when reaching ∂D_n, and then setting $\gamma_n = \mathcal{L}\{S\}$. For a nearest-neighbor walk S, let S^R be the time-reversed walk. It is known that LERW has the following symmetry with respect to time-reversal: the distribution of $(\mathcal{L}\{S\})^R$ is equal to that of $\mathcal{L}\{S^R\}$. Sometimes it is more convenient to consider $\mathcal{L}\{S^R\}$, and when we do we will call it the *time-reversed LERW* (or time-reversal of LERW) and usually assume that the path is traced from the boundary toward 0; we always add edges in the obvious way to discrete walks to make them curves.

4.2. *Convergence rate for the LERW path.* Lawler, Schramm and Werner proved in [13] that, as $n \to \infty$, the image of the time-reversed LERW path in $\mathbb{D}$, $\psi_n(\mathcal{L}\{S^R\})$, traced from ∂D toward 0, converges weakly with respect to a natural metric on curves modulo increasing reparameterization toward the radial SLE_2 path started uniformly on ∂D. (See Theorem 3.9 of [13] for a precise statement.) The goal of this section is to prove Theorem 4.3, which can be viewed as a quantitative version of Theorem 3.9 of [13].

Let D be a simply connected $C^{1+\alpha}$ domain with grid-domain approximation $D_n = D_n(D)$. Let γ_n be the time-reversal of LERW on $n^{-1}\mathbb{Z}^2$ from 0 to ∂D_n and let $\tilde{\gamma}_n = \psi_n(\gamma_n)$ be its image in $\mathbb{D}$ traced from the boundary and parameterized by capacity. (Since γ_n is a simple curve that intersects ∂D_n at only one point it follows that $\tilde{\gamma}_n$ is a $\mathbb{D}$-Loewner curve for each n.) Let $W_n(t)$ be the Loewner driving term for $\tilde{\gamma}_n$. Fix $s \in (0, 1/24)$, and define

$$\varepsilon_n = n^{-s}.$$

THEOREM 4.2 ([4]). *For every $T > 0$, there exists $n_0 = n_0(T, s) < \infty$ such that the following holds. For each $n \geq n_0$, there is a coupling of γ_n with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that*

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} |W_n(t) - W(t)| > \varepsilon_n\Big\} < \varepsilon_n, \tag{34}$$

where $W(t) = e^{iB(2t)}$.

REMARK. The coupling(s) of $W_n = e^{i\theta_n}$ and $W = e^{iB}$ in Theorem 4.2 are via Shorokhod embedding of θ_n into B.

We can now state a precise version of the main result of the paper.

THEOREM 4.3. *There exists $n_1 = n_1(\varepsilon, T, s) < \infty$ such that if $n \geq n_1$, then in the coupling of Theorem 4.2, if $\tilde{\gamma}$ denotes the radial SLE_2 path in $\mathbb{D}$ driven by W,*

$$\mathbb{P}\Big\{\sup_{t\in[0,\sigma]} |\tilde{\gamma}_n(t) - \tilde{\gamma}(t)| > \varepsilon_n^m\Big\} < \varepsilon_n^m, \tag{35}$$

where both curves are parameterized by capacity,

$$m = 1/41$$

and $\sigma = \sigma(\varepsilon, T)$ is the stopping time defined by (33).

REMARK. The proof of Theorem 4.3 (with minor modifications) would also work under the weaker assumption that D is a quasidisk. (The class of quasidisks includes, e.g., the von Koch snowflake.) In this case, the rate would depend on the constant in the Ahlfors three-point condition satisfied by ∂D; see Appendix B. We may also note that the conclusion (and proof) of Theorem 4.3 holds true in any coupling like the one of Theorem 4.2, with the proviso that ε_n decays slower than $n^{-1/2}$.

REMARK. By Lemma 4.7 below, the preimages of the curves (parameterized by capacity) in D_n satisfy a similar estimate as in (35), namely,

$$\mathbb{P}\Big\{\sup_{t\in[0,\sigma]} |\gamma_n(t) - \psi_n^{-1}(\tilde{\gamma}(t))| > \varepsilon_n^m\Big\} < \varepsilon_n^m, \qquad m = 1/41.$$

In order to apply the work from previous sections, we need to verify that the assumptions of these results hold with large probability. In Section 4.3, we will first estimate the probability of the existence of a certain power-law bound for the tip structure modulus for the LERW path in D_n. We show in Appendix B that if ∂D is sufficiently smooth ($\mathcal{C}^{1+\alpha}$), then the image of the LERW path in $\mathbb{D}$ enjoys the same tip structure modulus up to constants. This uses a convergence rate result for grid-domain approximations of quasidisks that we derive from a result of Warshawski's. In Appendix A, we prove the needed estimate on the derivative of the SLE_2 conformal maps. These results are combined to prove Theorem 4.3 in Section 4.5.

4.3. *Tip structure modulus for LERW in a grid domain*. An important tool to get quantitative estimates for LERW is the *Beurling estimate* for simple random walk; see, for example, [11]. There are many ways to formulate this result and we state only one version here.

LEMMA 4.4. *There exists a constant $c < \infty$ such that the following holds. Let $A \subset \mathbb{Z}^2$ be an infinite connected set. Let S be simple random walk on $\mathbb{Z}^2$ started from z and stopped at the time τ_A at which S hits A. Then for $r > 1$*

$$\mathbb{P}\{|S(\tau_A) - z| \geq r \operatorname{dist}(z, A)\} \leq cr^{-1/2}.$$

We can now formulate the main estimate of this section.

PROPOSITION 4.5. *Let D_n be a grid domain with respect to $n^{-1}\mathbb{Z}^2$ and assume that $1 \leq \operatorname{inrad}(D_n) \leq 2$ and that $\operatorname{diam} D_n \leq R < \infty$, where R is given. Let γ_n be the time-reversal of loop-erased random walk from 0 to ∂D_n. Let $\eta_{\mathrm{tip}}^{(n)}(\delta)$ be the tip structure modulus for γ_n (traced from ∂D_n) stopped when first reaching distance $\varepsilon > 0$ from 0. Let $r \in (0, 1/11)$. There exists a universal constant $c_0 > 0$ and $c = c(R, r, \varepsilon) < \infty$ such that if n is sufficiently large and $\delta > c_0/n$, then*

$$\mathbb{P}\{\eta_{\mathrm{tip}}^{(n)}(\delta) \leq \delta^r\} \geq 1 - c\delta^{1/5 - 11r/5}|\log \delta|. \tag{36}$$

REMARK. When we apply Proposition 4.5, we will choose $\delta = \delta(n) \in \omega(n^{-1})$ (in the sense of Landau notation) so that $\delta > c_0/n$ is automatically satisfied for n sufficiently large.

REMARK. The Beurling estimate implies that there is a constant $c < \infty$ such that

$$\mathbb{P}\{\operatorname{diam} \gamma_n > R\} \leq cR^{-1/2}$$

for large R. This means that one can formulate and prove Proposition 4.5 with an estimate independent of the diameter of D_n.

4.4. *Proof of Proposition 4.5*. The result was formulated for the time-reversal of LERW but in the proof we shall consider the LERW generated by erasing the loops of simple random walk from 0 to ∂D_n (without the time-reversal). By time-reversal symmetry, this is sufficient.

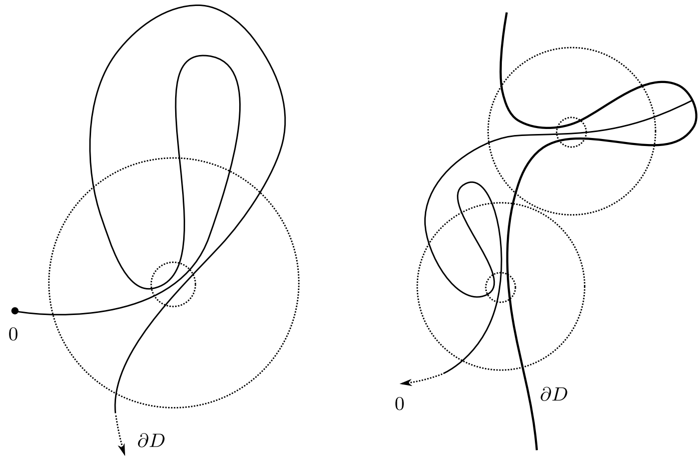

FIG. 3. *A 6-crossing and crossings close to ∂D.*

The strategy of the proof is based on that of the proof of Lemma 3.4 in [22], but see also the related Lemma 3.12 of [13]. See Figure 3 for a sketch of different crossing configurations that may occur. Let w be a fixed point in D_n. Let $\mathcal{A} = \mathcal{A}(w; \delta, \eta) = \{z : \delta < |z - w| < \eta\}$ be the (δ, η)-annulus about w and assume (for now) that $\delta > 10/n$ and we think of η as much larger than δ but still small compared to inrad D; eventually, we want to choose $\eta = \delta^r$ for some $r \in (0, 1)$. Let γ be a curve in D_n. We say that γ has a k-crossing of the annulus $\mathcal{A}$ if the number of components of $\gamma \cap \mathcal{A}$ that connect the two boundary components of $\mathcal{A}$ is at least k. Recall that $\eta(\delta)$ is a bound for the tip structure modulus for γ in D_n if and only if γ has no nested $(\delta, \eta(\delta))$-bottleneck in D_n. Now consider γ_n, the LERW path in D_n traced from ∂D_n toward 0 and the event that there is a nested $(\delta, 2\eta)$-bottleneck in γ_n stopped when reaching $\partial\mathcal{B}(0, \varepsilon)$. We claim that this event is contained in the union of the following two events:

$\mathcal{E}_5 = \{$There is a $w \in D_n$ with $|w| > \varepsilon$ such that γ_n has a 5-crossing of a (δ, η)-annulus about $w\}$.

$\mathcal{E}_B = \{$The random walk generating γ_n travels more than distance η before hitting ∂D_n, after the first time it has come within distance δ from $\partial D_n\}$.

Indeed, suppose that a nested $(\delta, 2\eta)$-bottleneck occurs in γ_n stopped when reaching $\partial\mathcal{B}(0, \varepsilon)$. Then if we choose some parameterization of γ_n traced from ∂D_n to 0, by definition there exist t_0 and a crosscut $\mathcal{C}$ of $D' = D_n \setminus \gamma[0, t_0]$ such that $\operatorname{diam} \mathcal{C} \le \delta$ and $\operatorname{diam} \gamma_{\mathcal{C}} \ge 2\eta$. Consider first the case when $\overline{\mathcal{C}} \cap \partial D_n \neq \varnothing$. Then since γ_n connects ∂D_n with 0 and $\mathcal{C}$ separates a piece of γ_n from 0 we must have that γ_n intersects $\mathcal{C}$. Consequently, the random walk that generates γ_n intersects $\mathcal{C}$, and if $\mathcal{C}$ is to separate a piece of γ_n of diameter at least 2η the event $\mathcal{E}_B$ must occur.

Now suppose that $\overline{\mathcal{C}} \cap \partial D_n = \varnothing$. We will show that this implies that $\mathcal{E}_5$ must occur. Notice that $D' \setminus \mathcal{C}$ consists of two simply connected components, one of which has no part of its boundary in common with ∂D_n. Call this component $\mathcal{O}$. There are two cases: first, assume that $0 \notin \mathcal{O}$. Then $\gamma_{\mathcal{C}} \subset \mathcal{O}$ and so $\operatorname{diam} \mathcal{O} \geq 2\eta$. By considering $\partial\mathcal{O} \setminus (\mathcal{C} \cup \gamma_{\mathcal{C}})$ (giving two crossings) and $\gamma_{\mathcal{C}}$ traced from $\mathcal{C}$ to $\gamma_n(t_0)$ and then continued along γ_n to 0 (giving three crossings) we see that γ_n indeed contains a 5-crossing of (δ, η)-annulus. On the other hand, if $0 \in \mathcal{O}$ we have that $\mathcal{B}(0, \varepsilon) \subset \mathcal{O}$ so $\operatorname{diam} \mathcal{O} \geq 2\eta$ if $\eta < \varepsilon/2$. In this case, $\gamma_{\mathcal{C}} \subset D' \setminus \mathcal{O}$ and again considering $\partial\mathcal{O} \setminus (\mathcal{C} \cup \gamma_{\mathcal{C}})$ and $\gamma_{\mathcal{C}}$ traced from $\mathcal{C}$ to $\gamma_n(t_0)$ and then continued along γ_n to 0, we see that γ_n contains a 5-crossing of a (δ, η)-annulus.

We will estimate the probabilities of the two events $\mathcal{E}_5$ and $\mathcal{E}_B$, starting with the last. In this case, the Beurling estimate immediately implies that there is a constant $c < \infty$ such that

$$\mathbb{P}(\mathcal{E}_B) \leq c\left(\frac{\delta}{\eta}\right)^{1/2}. \tag{37}$$

We proceed to bound $\mathbb{P}(\mathcal{E}_5)$. Fix a point $w \in D_n$ with $|w| > \varepsilon$. Set

$$d_0 = \operatorname{dist}(w, \partial D_n) > 0$$

and define

$$\mathcal{B}_1 = \mathcal{B}(w, \eta/4), \qquad \mathcal{B}_2 = \mathcal{B}(w, \eta/2).$$

For a curve $\gamma \subset D_n$, we let $\mathcal{Q}^3(\gamma; w, \delta, \eta)$ denote the event that γ has a 3-crossing of a (δ, η)-annulus whose smaller boundary component is contained in $\mathcal{B}_1$. Similarly, let $\mathcal{Q}^5(\gamma; w, \delta, \eta)$ denote the event that γ has a 5-crossing of a (δ, η)-annulus whose smaller boundary component is contained in $\mathcal{B}_1$. Clearly, the latter event is contained in the former. We will first estimate the probability of

$$\mathcal{Q}^5 := \mathcal{Q}^5(\gamma_n; w, \delta, \eta).$$

Let $S(t) = S_n(t), t = 0, 1, \ldots, \tau$, be the simple random walk generating γ_n; it is started from 0 and stopped at

$$\tau = \min\{t \geq 0 : S(t) \in \partial D_n\},$$

when ∂D_n is hit. Define

$$s_1 = \min\{t \geq 0 : S(t) \in \mathcal{B}_1\}, \qquad t_1 = \min\{t > s_1 : S(t) \notin \mathcal{B}_2\}$$

and recursively for $j = 2, 3, \ldots,$

$$s_j = \min\{t > t_{j-1} : S(t) \in \mathcal{B}_1\}, \qquad t_j = \min\{t > s_j : S(t) \notin \mathcal{B}_2\}.$$

Note that we have $s_1 = 0$ if $|w| \leq \eta/4$ and $s_1 > 0$ otherwise. We will write

$$\mathcal{Q}_j^5 := \mathcal{Q}^5(\mathcal{L}\{S[0, t_j]\}; w, \delta, \eta), \qquad \mathcal{Q}_j^3 := \mathcal{Q}^3(\mathcal{L}\{S[0, t_j]\}; w, \delta, \eta).$$

Clearly, $\mathcal{Q}_j^5 \subset \mathcal{Q}_j^3$, but it does not necessarily hold that $\mathcal{Q}_{j+1}^5 \subset \mathcal{Q}_j^5$ or $\mathcal{Q}_{j+1}^3 \subset \mathcal{Q}_j^3$ because part of the curve forming a crossing may be erased. Note that for $m \geq 1$

$$\mathbb{P}(\mathcal{Q}^5) \leq \mathbb{P}\{\tau > t_{m+1}\} + \mathbb{P}\left(\bigcup_{j=1}^{m} \mathcal{Q}_j^5\right).$$

We estimate $\mathbb{P}\{\tau > t_{m+1}\}$ in Lemma 4.6 below.

We have

$$\mathbb{P}\left(\bigcup_{j=1}^{m} \mathcal{Q}_j^5\right) \leq \sum_{j=1}^{m} \mathbb{P}(\mathcal{Q}_j^5, \neg \mathcal{Q}_{j-1}^5).$$

To get the last estimate, we split the event on the left-hand side according to the first time a 5-crossing has occurred; here and in the sequel, for an event A the symbol "$\neg A$" means the complement of A. To bound $\mathbb{P}(\mathcal{Q}_j^5, \neg \mathcal{Q}_{j-1}^5)$, let us first discuss the analogous quantity for a 3-crossing. In the proof of Lemma 3.4 of [22] [on p. 241, after equation (3.4)], it was essentially shown that there is a (nonrandom) constant $c < \infty$ such that

$$\mathbb{P}(\mathcal{Q}_j^3 | \neg \mathcal{Q}_{j-1}^3, S[0, t_{j-1}]) \leq c(j-1)\left(\frac{\delta}{\eta}\right)^{1/2}. \tag{38}$$

The exponent in the right-hand side of (38) was not specified in [22] so let us sketch the proof and explain how one gets the exponent $1/2$. Let $\{C_k\}_k$ be the components of $\mathcal{L}\{S[0, s_j]\} \cap \mathcal{B}_2$ intersecting $\mathcal{B}_1$ but not containing $S(s_j)$. By construction, there are at most $j-1$ such components. Conditionally, on $S[0, t_{j-1}]$, if $\mathcal{L}\{S[0, t_j]\}$ is to contain a 3-crossing which was not there in $\mathcal{L}\{S[0, t_{j-1}]\}$, then $S[s_j, t_j]$ has to come within distance δ of $C_k \cap \mathcal{B}_1$ for some k and then exit $\mathcal{B}_2$ without hitting that same C_k. (It may hit other components.) For each component C_k, we can use the strong Markov property and the Beurling estimate to see that this conditional probability of exiting $\mathcal{B}_2$ without hitting C_k is bounded above by $c(\delta/\eta)^{1/2}$. Summing over the $j-1$ components gives (38).

From (38),

$$\mathbb{P}(\mathcal{Q}_j^3 | \neg \mathcal{Q}_{j-1}^3) \leq c(j-1)\left(\frac{\delta}{\eta}\right)^{1/2}. \tag{39}$$

And this implies that

$$\mathbb{P}(\mathcal{Q}_j^3) \leq \sum_{k=1}^{j} \mathbb{P}(\mathcal{Q}_k^3, \neg \mathcal{Q}_{k-1}^3) \leq c j^2 \left(\frac{\delta}{\eta}\right)^{1/2}. \tag{40}$$

We now turn to $\mathbb{P}(\mathcal{Q}_j^5, \neg \mathcal{Q}_{j-1}^5)$. Since $(\mathcal{Q}_j^5 \cap \neg \mathcal{Q}_{j-1}^5) \subset \mathcal{Q}_{j-1}^3$, (40) implies

$$\begin{aligned}\mathbb{P}(\mathcal{Q}_j^5, \neg \mathcal{Q}_{j-1}^5) &= \mathbb{P}(\mathcal{Q}_j^5, \neg \mathcal{Q}_{j-1}^5 | \mathcal{Q}_{j-1}^3)\mathbb{P}(\mathcal{Q}_{j-1}^3) \\ &\leq c\mathbb{P}(\mathcal{Q}_j^5, \neg \mathcal{Q}_{j-1}^5 | \mathcal{Q}_{j-1}^3) j^2 \left(\frac{\delta}{\eta}\right)^{1/2}.\end{aligned}$$

We continue to write

$$\mathbb{P}(\mathcal{Q}_j^5, \neg\mathcal{Q}_{j-1}^5 | \mathcal{Q}_{j-1}^3) \le \mathbb{P}(\mathcal{Q}_j^5 | \neg\mathcal{Q}_{j-1}^5, \mathcal{Q}_{j-1}^3).$$

We can estimate the last expression by observing that

$$\mathbb{P}(\mathcal{Q}_j^5 | \neg\mathcal{Q}_{j-1}^5, \mathcal{Q}_{j-1}^3, S[0, t_{j-1}]) \le c(j-1)\left(\frac{\delta}{\eta}\right)^{1/2}.$$

Indeed, this estimate is proved in exactly the same way as (39) using the Beurling estimate.

Combining our bounds, we get

$$\mathbb{P}\left(\bigcup_{j=1}^{m} \mathcal{Q}_j^5\right) \le cm^4 \frac{\delta}{\eta}. \tag{41}$$

We now take $\nu > 0$ and let $m = \lfloor \delta^{-\nu} \rfloor$. We then use Lemma 4.6 (here we write the estimate for $d_0 > \eta/4$; in the case $d_0 \le \eta/4$ we use the second bound of Lemma 4.6) to get

$$\begin{aligned}\mathbb{P}(\mathcal{Q}^5) &\le \left(1 - \frac{c_3}{|\log(16 d_0/\eta)|}\right)^{\lfloor \delta^{-\nu} \rfloor} + c\frac{\delta^{1-4\nu}}{\eta} \\ &\le c\delta^{\nu}\left|\log(16 d_0/\eta)\right| + c\frac{\delta^{1-4\nu}}{\eta}.\end{aligned} \tag{42}$$

This bound is for a fixed w. To conclude, note that there is a universal $c < \infty$ such that we can (deterministically) cover D_n using at most $cR^2\eta^{-2}$ overlapping disks $\mathcal{B}(w_k, \eta/4)$ in such a way for every w such that γ_n has a 5-crossing of $\mathcal{A}(w; \delta, \eta)$, the smaller boundary component of $\mathcal{A}(w; \delta, \eta)$ is contained in $\mathcal{B}(w_k, \eta/4)$ for some k. Consequently, for $c = c(R) < \infty$,

$$\mathbb{P}(\mathcal{E}_5) \le c\eta^{-2}\delta^{\nu}\left|\log(16 d_0/\eta)\right| + c\eta^{-3}\delta^{1-4\nu}. \tag{43}$$

For any $r \in (0, 1/11)$, if $\eta = \delta^r$, we can take $\nu = (1-r)/5$ in (43) which makes both terms in the bound of the same ("polynomial") order so that the right-hand side of (43) decays like $\delta^{1/5-11r/5}$ with a logarithmic correction. Since this term is always larger than the one coming from $\mathcal{E}_B$, this completes the proof of Proposition 4.5, assuming Lemma 4.6.

LEMMA 4.6. *There exist constants* $0 < c_1, c_2 < 1$ *such that*

$$\mathbb{P}\{\tau > t_{m+1}\} \le \begin{cases} \left(1 - \dfrac{c_1}{|\log(16 d_0 \eta^{-1})|}\right)^m, & \text{if } d_0 > \eta/4; \\ (1 - c_2)^m, & \text{if } d_0 \le \eta/4. \end{cases}$$

PROOF. We first assume that $d_0 > \eta/4$. Using, for example, Proposition 6.4.1 of [11], we see that the probability that a simple random walk started just outside of $\mathcal{B}_2$ exits $\mathcal{B}(z_0, 8d_0)$ before hitting $\mathcal{B}_1$ is bounded below by

$$\frac{|\log 2| - O((\eta n)^{-1})}{|\log(16d_0\eta^{-1})|} \geq \frac{|\log 2|}{2|\log(16d_0\eta^{-1})|}$$

if $\eta n > c_1$, where $c_1 < \infty$ is a universal constant. (This uses also that $d_0 > \eta/4$.) This estimate is a discrete version of the expression for the harmonic measure of one of the boundary components in an annulus. Moreover, there is a universal constant $c > 0$ such that the probability that simple random walk from (a vertex adjacent to) $\partial\mathcal{B}(z_0, 8d_0)$ separates $\mathcal{B}(z_0, d_0)$ from ∞ before hitting $\mathcal{B}(z_0, d_0)$ is bounded below by c. (Recall that our assumptions imply that $d_0 > c'/n$, where we can assume that c' is large.) Consequently, by the strong Markov property the probability that simple random walk started from $\partial\mathcal{B}_2$ exits D_n before hitting $\mathcal{B}_1$ is bounded below by $c_1/|\log(16d_0\eta^{-1})|$. By iterating this argument using the strong Markov property,

$$\mathbb{P}\{\tau > t_{m+1}\} \leq \left(1 - \frac{c_1}{|\log(16d_0\eta^{-1})|}\right)^m. \tag{44}$$

When $d_0 \leq \eta/4$ the Beurling estimate and the Markov property directly show that the right-hand side of (44) can be replaced by $(1-c_2)^m$, where $c_2 > 0$ is a universal constant. □

If the boundary of the domain D that is being approximated is sufficiently regular, then the structure modulus on a sufficiently large mesoscopic scale for the image curve in $\mathbb{D}$ is essentially the same as the one in D_n. The next lemma, proved in Appendix B, makes this precise.

LEMMA 4.7. *Suppose $D \ni 0$ is a simply connected domain Jordan domain with $C^{1+\alpha}$ boundary, where $\alpha > 0$. Let D_n be the $n^{-1}\mathbb{Z}^2$ grid-domain approximation of D and let γ_n be a Loewner curve in D_n connecting ∂D_n with 0. There is a constant c depending only on α and the diameter of D such that the following holds. Set $0 < r < 1/2$ and $d_n = n^{-r}$ and let $\eta_{\text{tip}}^{(n)}(\delta; D_n)$ be the tip structure modulus for γ_n in D_n. Then for all n sufficiently large (independently of γ_n) the tip structure modulus $\eta_{\text{tip}}^{(n)}(\delta; \mathbb{D})$ for $\psi_n(\gamma_n)$ in $\mathbb{D}$ satisfies*

$$\eta_{\text{tip}}^{(n)}(c^{-1}d_n; \mathbb{D}) \leq c\eta_{\text{tip}}^{(n)}(d_n; D_n).$$

4.5. *Proof of Theorem 4.3.* We write γ for the radial SLE_2 path in $\mathbb{D}$ corresponding to the Brownian motion in (34). We thus have a coupling of the radial SLE_2 path and the image of the LERW path $\tilde{\gamma}_n$ and we will estimate the distance between these curves in this coupling. Take $s \in (0, 1/24)$ and $n > n_0$ where n_0 is as in Theorem 4.2; fix $\rho > 1$ and for $p \in (0, 1/\rho)$, let

$$\varepsilon_n = n^{-s}, \qquad d_n = (\varepsilon_n)^p.$$

For each $n \geq n_0$, we shall define three events each of which occurs with large probability in our coupling. On the intersection of these events, we can apply our estimates from Sections 2 and 3.

(a) Let $\mathcal{A}_n = \mathcal{A}_n(s)$ be the event that the estimate

$$\sup_{t \in [0,T]} \left| W_n(t) - W(t) \right| \leq \varepsilon_n$$

holds. By Theorem 4.2, we know that there exists $n_0 < \infty$ such that if $n \geq n_0$ then

$$\mathbb{P}(\mathcal{A}_n) \geq 1 - \varepsilon_n.$$

(b) For $\beta \in (2(\sqrt{10} - 1)/9, 1)$, let $\mathcal{B}_n = \mathcal{B}_n(s, r, \beta, \varepsilon, T, c_B)$ be the event the radial SLE_2 Loewner chain (f_t) driven by $W(t)$ satisfies the estimate

$$\sup_{t \in [0,\sigma]} d \left| f'\big(t, (1-d) W(t)\big) \right| \leq c_B d^{1-\beta} \qquad \forall d \leq d_n.$$

(Recall that ε, T were used in the definition of the stopping-time $\sigma \leq T$.) Then by Proposition 4.1 there exist $c_B' < \infty$, independent of n, and $n_1 < \infty$ such that if $n \geq n_1$ then

$$\mathbb{P}(\mathcal{B}_n) \geq 1 - c_B' d_n^q,$$

where

$$q < q_2(\beta) = -1 + 2\beta + \frac{\beta^2}{4(1+\beta)}.$$

(c) For $r \in (0, 1/11)$, let $\mathcal{C}_n = \mathcal{C}_n(s, r, p, c_C, \alpha, \operatorname{diam} D)$ be the event that the tip structure modulus for $\tilde{\gamma}_n(t), t \in [0, T]$, in $\mathbb{D}$, $\eta_{\text{tip}}^{(n)}$, satisfies

$$\eta_{\text{tip}}^{(n)}(d_n) \leq c_C d_n^r.$$

We know from Proposition 4.5 and Lemma 4.7 that there exist $c_C, c_C' < \infty$, independent of n, and $n_2 < \infty$ such that if $n \geq n_2$ then

$$\mathbb{P}(\mathcal{C}_n) \geq 1 - c_C' d_n^{1/5 - 11r/5} |\log d_n|.$$

Consequently, there exist $c_B, c_C < \infty$ and $c < \infty$, all independent of n (but depending on $s, r, p, \varepsilon, T, \beta, \alpha, \operatorname{diam} D$), such that for all n sufficiently large,

$$\mathbb{P}(\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n) \geq 1 - c\big(\varepsilon_n + d_n^q + d_n^{1/5 - 11r/5} |\log d_n|\big) \tag{45}$$

and on the event $\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n$ we can apply Lemma 3.4 with constants $c = c_C$, $c' = c_B$ independent of n to see that there exists c'' independent of n (but depending on the above parameters) such that for all n sufficiently large,

$$\sup_{t\in[0,\sigma]} \left|\tilde{\gamma}_n(t) - \tilde{\gamma}(t)\right| \le c''\bigl(d_n^{r(1-\beta)} + \varepsilon_n^{(1-\rho p)r}\bigr). \tag{46}$$

We now wish to optimize over the parameters in the exponents. Since $d_n = \varepsilon_n^p$, we see that $d_n^{r(1-\beta)}$ dominates in (46) when $p \in (0, 1/(1+\rho-\beta)]$ and $\varepsilon_n^{r(1-\rho p)}$ whenever $p \in [1/(1+\rho-\beta), 1]$. Suppose $p \in (0, 1/(1+\rho-\beta)]$.

Set

$$\mu(\beta, r) = \min\left\{r(1-\beta), -1+2\beta+\frac{\beta^2}{4(1+\beta)}, \frac{1}{5}-\frac{11r}{5}\right\}.$$

The optimal rate is given by optimizing μ over β, r and then choosing p very close to $1/(1+\rho-\beta)$. (No improvement is obtained by considering $p \in [1/(1+\rho-\beta), 1]$.) Let $\beta_* \in (2(\sqrt{10}-1)/9, 1)$ be a solution to

$$45\beta^3 - 128\beta^2 - 84\beta + 68 = 0.$$

(One can check that $\beta_* = 0.497\ldots$.) Then if $r_* = 1/(16-\beta_*) \in (0, 1/11)$

$$\mu(r_*, \beta_*) = \max\left\{\mu(\beta, r) : \frac{2(\sqrt{10}-1)}{9} < \beta < 1, 0 < r < \frac{1}{11}\right\} = 0.037\ldots.$$

Consequently, for every

$$m < m_* = \frac{\mu(r_*, \beta_*)}{2-\beta_*},$$

we obtain bounds in (45) and (46) of order ε_n^m for all n sufficiently large. Since $1/41 < m_* = 0.024\ldots$, this completes the proof.

APPENDIX A: DERIVATIVE ESTIMATE FOR RADIAL SLE

This section proves a derivative estimate for both chordal and radial SLE. The radial case was needed in Section 4 in the case $\kappa = 2$. The chordal case is a direct consequence of an estimate from [6], but the radial case requires a little bit of work. In this case, our goal will be to estimate explicitly in terms of d_* and β the probability of the event that when $(f(t,z))$ is the radial SLE$_\kappa$ Loewner chain, the estimate $d|f'(t,(1-d)W(t))| \le cd^{1-\beta}$ for all $d \le d_*$ holds uniformly in $t \in [0, T]$. This will follow from a moment estimate for the chordal reverse flow in [6] after changing "coordinates" from radial to chordal SLE. See also Section 7 of [4] where a similar but nonequivalent situation is dealt with. We will use ideas from [24].

A.1. Change of coordinates. Let (f_s, W_s) be a radial Loewner pair generated by the curve $\gamma(s)$ with W_s continuous. Recall that $f_s : \mathbb{D} \to \mathbb{D} \setminus K_s = D_s$ and that K_s is the hull generated by $\gamma[0, s]$. Let $g_s = f_s^{-1}$ and set $z_s = g_s(-1)\overline{W_s}$. We will need to keep track of the "disconnection time" σ' when K_s first disconnects -1 from 0 in $\mathbb{D}$, in other words, the first time that z_s hits 1. Fix $\varepsilon > 0$ small and $T < \infty$, and define

$$\sigma = \sigma(\varepsilon, T) = \inf\{s \geq 0 : |1 - z_s| \leq \varepsilon\} \wedge T. \tag{47}$$

Clearly, $\sigma < \sigma'$.

LEMMA A.1. *There exists a constant $c = c(\varepsilon, T) > 0$ such that*

$$\inf_{s \in [0,T]} |g'_{s\wedge\sigma}(-1)| \geq c.$$

PROOF. The Loewner equation implies that with z_s as above,

$$|g'_s(-1)| = \exp\left\{\int_0^s \operatorname{Re} \frac{2}{(1 - z_s)^2} - 1 \, ds\right\}.$$

This shows that $|g'_s(-1)|$ is strictly decreasing in s and that $|g'_{T\wedge\sigma}(-1)| \geq c = c(\varepsilon, T) > 0$. $\square$

REMARK. Note that if g_s is the radial SLE$_\kappa$ forward flow, and if

$$\theta_s := -i \log z_s = -i \log g_s(-1) - \sqrt{\kappa} B_s, \qquad \theta_0 = \pi,$$

then by Itô's formula,

$$d\theta_s = \cot(\theta_s/2)\, ds - \sqrt{\kappa}\, dB_s.$$

If $\kappa < 4$, then it follows from [10], Lemma 1.27, that almost surely θ_s does not hit $\{0, 2\pi\}$ in finite time. Hence, for each $T < \infty$, if $\kappa < 4$, then almost surely,

$$\lim_{\varepsilon \to 0} \sigma(\varepsilon, T) = T.$$

Consider now the Mobius transformation

$$\varphi : \mathbb{H} \to \mathbb{D}, \qquad \varphi(z) = \frac{i - z}{i + z}.$$

Then $\varphi^{-1} \circ \gamma$ is a curve in $\mathbb{H}$ (for sufficiently small s) and for $s \geq 0$ we define

$$t(s) := \operatorname{hcap}(\varphi^{-1}(\gamma[0, s]))/2.$$

For each $s \in [0, \sigma]$, let $F_{t(s)} : \mathbb{H} \to H_{t(s)} := \varphi^{-1}(D_s)$ be the conformal mapping satisfying the hydrodynamical normalization $F_{t(s)}(z) = z - 2t(s)/z + o(1/|z|)$ at infinity. It is known (see, e.g., [24]) that $t(s)$ is a strictly increasing, continuous function of s up to the disconnection time and we will write $s(t)$ for the inverse of $t(s)$. One can write (see [24] and [4])

$$f_s = \varphi \circ F_{t(s)} \circ \Delta_s. \tag{48}$$

Here,

$$\Delta_s(z) : \mathbb{D} \to \mathbb{H}, \qquad \Delta_s(z) = \frac{z\overline{\mu_{t(s)}} - \lambda_s \mu_{t(s)}}{z - \lambda_s}, \tag{49}$$

where the reader may verify that if

$$G_{t(s)}(z) = F_{t(s)}^{-1}(z), \qquad g_s(z) = f_s^{-1}(z),$$

then

$$\mu_{t(s)} = G_{t(s)}(i), \qquad \lambda_s = g_s(-1).$$

In fact, by expanding G at infinity via (48),

$$\operatorname{Im} \mu_{t(s)} = -\frac{g_s'(-1)}{g_s(-1)} = |g_s'(-1)|. \tag{50}$$

This uses that

$$\operatorname{Re}\left(1 - \frac{g_s''(-1)}{g_s'(-1)}\right) = -\frac{g_s'(-1)}{g_s(-1)},$$

which holds because the left-hand side equals $\partial_\theta[\arg \partial_\theta g_s(e^{i\theta})]$ at $\theta = \pi$, and g_s maps the circle to the circle locally at -1 so that the change of the tangent is equal to the change of the argument which is what is represented by the right-hand side. By Lemma A.1 and (50) there exists $c_1 = c_1(\varepsilon, T) > 0$ such that

$$\operatorname{Im} \mu_{t(s)} \ge c_1, \qquad s \in [0, \sigma]. \tag{51}$$

Set

$$\tau := t(\sigma)$$

and consider the family $(F_t), t \in [0, \tau]$, with the half-plane capacity parameterization. It satisfies the chordal Loewner PDE in t and we let $U_t = \Delta_{s(t)}(W_{s(t)})$ be the corresponding chordal driving term. The estimate (51) implies that there is $T' = T'(\varepsilon, T) < \infty$ such that $\tau \le T'$. Indeed, in Theorem 3 of [24] it is shown that $s'(t) = 4(\operatorname{Im} \mu_{s(t)})^2/|\mu_{s(t)} - U_t|^4$ which is bounded away from 0 on $[0, \tau]$. Using (51) and that $|W_s - \lambda_s| \ge \varepsilon$ for $s \in [0, \sigma]$, we see that there exist constants $0 < c < \infty$ and $d_0 > 0$ depending only on ε and T such that for all $d \le d_0$, uniformly in $s \in [0, \sigma]$,

$$\left|\operatorname{Re}\left(\Delta_s\left((1-d)W_s\right)\right) - U_{t(s)}\right| \le cd, \qquad c^{-1} d \le \operatorname{Im}\left(\Delta_s\left((1-d)W_s\right)\right) \le cd.$$

In other words, the hyperbolic distance between $\Delta_s((1-d)W_s)$ and $U_{t(s)} + id$ is bounded by a constant depending only on ε and T. Therefore, we can use Koebe's distortion theorem to see that there exist $c, c' < \infty$ depending only on ε, T such that for all $s \in [0, \sigma]$

$$|f_s'((1-d)W_s)| \le c|F_{t(s)}'(\Delta_s((1-d)W_s))| \le c'|F_{t(s)}'(U_{t(s)} + id)|.$$

We have proved the following result.

PROPOSITION A.2. *Let $T < \infty$ and $\varepsilon > 0$ be given. Suppose that (f_s, W_s) is a radial Loewner pair generated by the curve $\gamma(s)$. Define $\sigma = \sigma(\varepsilon, T)$ by (47). Let (F_t, U_t) be the chordal Loewner pair generated by the curve $s \mapsto \varphi^{-1}(\gamma(s))$, $s \in [0, \sigma]$ reparameterized by half-plane capacity and let $\tau = t(\sigma)$. There exists $c = c(\varepsilon, T) < \infty$ and $d_0 = d_0(\varepsilon, T) > 0$ such that for all $d \le d_0$,*

$$\sup_{s\in[0,\sigma]} |f_s'((1-d)W_s)| \le c \sup_{t\in[0,\tau]} |F_t'(U_t + id)|.$$

Now assume that (f_s) is the radial SLE$_\kappa$ Loewner chain. Then σ is a stopping time for (f_s) and τ is a stopping time for (F_t). The law of the chordal driving term U_t stopped at τ is absolutely continuous with respect to the law of standard linear Brownian motion with speed κ, as shown in [24]. Moreover, by (51) the Girsanov density is uniformly bounded above by a constant depending only on κ, ε and T. Indeed, it is a product of powers of $|G_t'(i)|$, $\operatorname{Im} \mu_t$, and $|\mu_t - U_t|$, all which are bounded away from 0 and ∞ when $t \le \tau$. Since (F_t) is absolutely continuous with respect to a chordal SLE$_\kappa$ Loewner chain and since the Girsanov density is uniformly bounded (for fixed κ, ε, T), using Proposition A.2 we can estimate the behavior of $\sup_{s\in[0,\sigma]} |f_s'((1-d)W_s)|$ using standard chordal SLE.

A.2. Derivative estimate for chordal SLE. We now derive the needed estimate on the growth of the derivative in chordal coordinates. The estimate is essentially a direct consequence of work in [6] and we will describe the modifications here. Let (F_t), $t \ge 0$, be the standard chordal SLE Loewner chain mapping $\mathbb{H}$ onto the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$. We write $\widehat{F}_t(z) = F_t(z + U_t)$, where U is the chordal driving term for (F_t). Recall that the chordal reverse SLE$_\kappa$ flow is the family of conformal mappings solving

$$\dot{h}_t = -\frac{2}{h_t - \sqrt{\kappa} B_t}, \qquad h_0(z) = z,$$

where B is standard Brownian motion. For fixed $t_0 > 0$, $|h_{t_0}'(z)|$ is equal to $|\widehat{F}_{t_0}'(z)|$ in distribution. Hence, (first) moment estimates for $|\widehat{F}_{t_0}'|$ are reduced to corresponding estimates for $|h_{t_0}'|$ and these are often more easily obtained. Note that scaling implies that for fixed $y > 0$, $|h_t'(iy)| \overset{d}{=} |h_{ty^{-2}}'(i)|$. Define

$$\zeta(\lambda) = \lambda + \frac{\sqrt{(4+\kappa)^2 - 8\lambda\kappa} - (4+\kappa)}{4}.$$

We will assume that

$$\lambda < \lambda_c = 1 + \frac{2}{\kappa} + \frac{3\kappa}{32}.$$

In this range, we quote the following estimate from [6]. See also [7] and the references therein.

LEMMA A.3. *Let h_t be the chordal reverse SLE$_\kappa$ flow, $\kappa > 0$. There exists a constant $c < \infty$ such that for $\lambda < \lambda_c$.*

$$\mathbb{E}[|h_t'(i)|^\lambda] \le ct^{-\zeta(\lambda)/2}, \qquad t \ge 1. \tag{52}$$

This result now implies the needed estimate which is a version of Proposition 4.2 of [6] with a decay rate; we will sketch the proof and refer the reader to [6] for more details. Let $\kappa > 0$ and define the function

$$\rho(\beta) = \beta + \frac{2(1+\beta)}{\kappa} + \frac{\beta^2\kappa}{8(1+\beta)}$$

and

$$q(\beta) = \min\{\lambda_c\beta, \rho(\beta) - 2\}, \qquad \beta_+ < \beta < 1,$$

where

$$\beta_+ = \max\left\{0, \frac{4(\kappa\sqrt{8+\kappa} - (4-\kappa))}{(4+\kappa)^2}\right\}.$$

Note that $q(\beta) > 0$ for β in the above range.

PROPOSITION A.4. *Let $T < \infty$ be fixed and let (F_t) be the chordal SLE$_\kappa$ Loewner chain, $\kappa \in (0, 8)$. Let $\beta \in (\beta_+, 1)$ and $q < q(\beta)$. There exists a constant $0 < c < \infty$ depending only on T, κ, q such that for every $y_* < 1$*

$$\mathbb{P}\left\{\forall y \le y_*, \sup_{t\in[0,T]} y|\widehat{F}_t'(iy)| \le cy^{1-\beta}\right\} \ge 1 - cy_*^q.$$

PROOF. (Sketch.) By the distortion theorem, scaling and the fact that Brownian motion is almost surely weakly Hölder-$(1/2)$, it is enough (see [6]) to show that for $\beta_+ < \beta < 1$ and $q < q(\beta)$

$$\sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} \mathbb{P}(|\widehat{F}_{j2^{-2n}}'(i2^{-n})| > 2^{\beta n}) \le c2^{-N_* q},$$

where $N_* = \lfloor \log y_*^{-1} \rfloor$. We have for $0 < \lambda < \lambda_c$ using scaling, Chebyshev's inequality and Lemma A.3,

$$\begin{aligned}
&\sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} \mathbb{P}\big(\big|\widehat{F}'_{j2^{-2n}}(i2^{-n})\big| > 2^{\beta n}\big) \\
&\quad \le \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda\beta} \mathbb{E}\big[\big|\widehat{F}'_{j2^{-2n}}(i2^{-n})\big|^{\lambda}\big] \le c \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda\beta} \mathbb{E}\big[\big|h'_j(i)\big|^{\lambda}\big] \\
&\quad \le c \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda\beta} j^{-\zeta/2} \le c \sum_{n=N_*}^{\infty} \sum_{j=1}^{2^{2n}} 2^{-n\lambda\beta}\big(1 + 2^{n(2-\zeta)}\big) \\
&\quad \le c\big(2^{-N_*\lambda\beta} + 2^{-N_*(\lambda\beta+\zeta-2)}\big).
\end{aligned}$$

Recall that $\lambda \in (0, \lambda_c)$. Note that $\zeta - 2 < 0$ if and only if $\kappa > 1$, so for these κ the smaller exponent is $\lambda\beta + \zeta - 2$. In this range, we find $q(\beta)$ by maximizing over $0 < \lambda < \lambda_c$ for β fixed so that $q(\beta) = \max_\lambda \lambda\beta + \zeta(\lambda) - 2$. The lower bound β_+ is the smallest $\beta > 0$ such that $\beta > \beta_+$ implies $q(\beta) > 0$. When $\kappa \le 1$, $\lambda\beta$ is the smaller exponent and we must restrict attention to $\beta > 0$. We pick the largest $\lambda = \lambda_c$. $\square$

From this and the work in the previous subsection, we immediately obtain the following proposition. Recall that the stopping time σ was defined in (47).

PROPOSITION A.5. *Let $\kappa \in (0, 8)$. Let $\varepsilon > 0$ be fixed and let (f_s), $0 \le s \le \sigma$, be the radial SLE$_\kappa$ Loewner chain stopped at σ as defined by* (47). *For every $\beta \in (\beta_+, 1)$ and $q < q(\beta)$, there exists a constant $c = c(\beta, \kappa, q, \varepsilon, T) < \infty$ such that for $d_* < 1$,*

$$\mathbb{P}\Big\{\forall d \le d_*, \sup_{s\in[0,\sigma]} d\big|f'_s\big((1-d)W_s\big)\big| \le cd^{1-\beta}\Big\} \ge 1 - cd_*^q.$$

We note that when $\kappa = 2$

$$q(\beta) = -1 + 2\beta + \frac{\beta^2}{4(1+\beta)}, \qquad \beta_+ = \frac{2(\sqrt{10}-1)}{9}.$$

APPENDIX B: MAPPING TO $\mathbb{D}$

When mapping conformally a curve into a reference domain, bounds on the tip structure modulus for the curve are not automatically preserved. In this section, we will consider a general case without reference to a specific discrete model. It seems that this general setting requires information about boundary regularity of the approximated domain (as opposed to information about the behavior of the discrete curve). In particular, we will need uniform control of the distortion of annuli on the scales of the structure modulus.

B.1. Grid domains. Recall the definition of a grid domain that was given in Section 4. Let $D \ni 0$ be simply connected, and assume that the inner radius with respect to 0 equals 1. Let $D_n = D_n(D)$ be the $n^{-1}\mathbb{Z}^2$ grid-domain approximation of D. Notice that every point on ∂D_n is within distance $\sqrt{2}/n$ of a point on ∂D, so that the inner Hausdorff distance between ∂D_n and ∂D is at most $\sqrt{2}/n$. Let $\psi : D \to \mathbb{D}$ be the conformal map normalized by $\psi(0) = 0$ and $\psi'(0) > 0$. Similarly, for $n = 1, 2, \dots$, let $\psi_n : D_n \to \mathbb{D}$ be conformal maps with the same normalization. The sequence of domains D_n converge to D in the Carathéodory sense, and so the ψ_n converge to ψ uniformly on compacts. Our goal will be to find a convergence rate for

$$\sup_{z \in D_n} \left|\psi_n(z) - \psi(z)\right|.$$

For this to be achievable, we need some information about the regularity of the boundary of D. We will here consider the class of quasidisks, although it will be clear that similar methods can be used to handle other classes of domains (e.g., John domains) where Euclidean geometric estimates on the behavior of the conformal mapping on the boundary are available.

B.2. Discrete approximation of a quasidisk. A quasicircle is the image of the unit circle under a quasiconformal mapping. A quasidisk is a (bounded) domain bounded by a quasicircle. See [20] for definitions and an overview from a conformal mapping point of view. A quasicircle is not necessarily rectifiable as the example of the von Koch snowflake shows.

We find it convenient to use an equivalent but more geometric definition, namely Ahlfors' three-point condition: the closed Jordan curve ∂D is a quasicircle if and only if there exists a constant $A < \infty$ such that for any two points $x, y \in \partial D$ it holds that

$$\operatorname{diam} J(x, y) \le A|x - y|, \tag{53}$$

where $J(x, y) \subset \partial D$ is the arc of smaller diameter connecting x with y. One can consider the smallest such A as a measure of regularity. This regularity implies some uniform regularity for the grid-domain approximation D_n and this allows us to estimate the convergence rate of ψ_n using a result from [26]. See also Section 5 of [16] where similar questions are discussed.

LEMMA B.1. *Let D be a quasidisk satisfying (53) and let D_n be the $n^{-1}\mathbb{Z}^2$ grid-domain approximation of D. Let ψ, ψ_n be the normalized conformal maps from D and D_n, respectively, onto $\mathbb{D}$. Then there exists a constant $c < \infty$ depending only on A and the diameter of D such that*

$$\sup_{z \in D_n} \left|\psi_n(z) - \psi(z)\right| \le c\frac{\log n}{\sqrt{n}}. \tag{54}$$

PROOF. We will first show that D_n satisfies (53) uniformly in n with a constant A' depending only on A. Let $x, y \in \partial D_n$. First, we consider the case when $|x - y| < 1/n$. Then since ∂D_n is a Jordan curve which is a subset of the edge set of $n^{-1}\mathbb{Z}^2$, we have that $\operatorname{diam} J(x, y) \le \sqrt{2}|x-y|$. Now assume that $|x-y| \ge 1/n$. Let ξ and η be points on ∂D closest to x and y, respectively. Clearly, $|x - \xi|$ and $|y - \eta|$ are both at most $\sqrt{2}/n$. Let α, β be the two line segments connecting x with ξ and y with η. First, assume that the curve $\Gamma = J(x, y) \cup \alpha \cup \beta$ separates $J(\xi, \eta)$ from 0 in D. Let Q_j, $j = 1, \dots, N$, be those lattice squares whose faces are outside of D_n but whose boundaries touch $J(x, y)$. By the construction of D_n and the Jordan curve theorem, since Γ separates 0 from $J(\xi, \eta)$, each Q_j is intersected by $\alpha \cup \beta \cup J(\xi, \eta)$. Consequently,

$$\operatorname{diam} \Gamma \le \operatorname{diam} J(\xi, \eta) + 2\sqrt{2}/n \le A|\xi - \eta| + 2\sqrt{2}/n.$$

Hence,

$$\operatorname{diam} J(x, y) \le \operatorname{diam} \Gamma \le A|x - y| + (2A + 2)\sqrt{2}/n.$$

Now, if Γ does not separate $J(\xi, \eta)$ from 0 in D, then since Γ is a crosscut of D, $(\partial D_n \setminus J(x, y)) \cup \alpha \cup \beta$ does separate $J(\xi, \eta)$ from 0 in D. Thus, in this case we can do the same argument as in the previous paragraph showing that $\operatorname{diam}(\partial D_n \setminus J(x, y)) \le \operatorname{diam} J(\xi, \eta) + 2\sqrt{2}/n$. But by definition, $\operatorname{diam} J(x, y) \le \operatorname{diam}(\partial D_n \setminus J(x, y))$.

Using also the estimate we obtained in the case when $|x - y| < 1/n$, we conclude that

$$\operatorname{diam} J(x, y) \le \big(A + (2A + 2)\sqrt{2}\big)|x - y|. \tag{55}$$

By (55), there is a constant c depending only on A and the diameter of D such the Warschawshi structure moduli $\eta_W^{(n)}$ of ∂D_n satisfy

$$\eta_W^{(n)}(\delta) \le c\delta, \qquad \delta \le 1.$$

Consequently, since $D_n \subset D$ and each point on ∂D_n is within distance $\sqrt{2}/n$ of a point on ∂D, part (a) of Theorem VII in [26] implies (54). □

For simplicity, we will now assume that ∂D is $C^{1+\alpha}$ for some $\alpha > 0$, that is, we assume that there is a parameterization of ∂D which has a Hölder-α derivative. By Kellogg's theorem; see, for example, [5], this assumption implies that the conformal map $\psi : D \to \mathbb{D}$ (and ψ^{-1}) is in $C^{1+\alpha}(\overline{D})$. (So we can take the conformal parameterization of ∂D.) In particular, ψ is bilipschitz on $\overline{D}$, that is, there is a constant $c < \infty$ depending only on α and the diameter of D such that

$$c^{-1}|z - w| \le \big|\psi(z) - \psi(w)\big| \le c|z - w|, \qquad z, w \in \overline{D}. \tag{56}$$

Similar uniform estimates, but of Hölder type, and corresponding versions of Lemma 4.7 (stated again below) hold if D is assumed to be a quasidisk. Indeed, the uniformizing conformal map and its inverse are then Hölder continuous on a neighborhood of ∂D with an exponent depending only on A; see [20]. From (56), we immediately get the required control over distortion of annuli up to constants on sufficiently large scales. We can now prove Lemma 4.7 which we state again.

LEMMA B.2. *Suppose $D \ni 0$ is a simply connected domain Jordan domain with $C^{1+\alpha}$ boundary, where $\alpha > 0$. Let D_n be the $n^{-1}\mathbb{Z}^2$ grid-domain approximation of D and let γ_n be a Loewner curve in D_n connecting ∂D_n with 0. There is a constant c depending only on α and the diameter of D such that the following holds. Set $0 < r < 1/2$ and $d_n = n^{-r}$ and let $\eta_{\mathrm{tip}}^{(n)}(\delta; D_n)$ be the tip structure modulus for γ_n in D_n. Then for all n sufficiently large (independently of γ_n) the tip structure modulus $\eta_{\mathrm{tip}}^{(n)}(\delta; \mathbb{D})$ for $\psi_n(\gamma_n)$ in $\mathbb{D}$ satisfies*

$$\eta_{\mathrm{tip}}^{(n)}(c^{-1}d_n; \mathbb{D}) \le c\eta_{\mathrm{tip}}^{(n)}(d_n; D_n).$$

PROOF. Let $\eta_n = \eta^{(n)}(d_n; D_n)$. We can assume that $\eta_n \ge 2d_n$. It is enough to verify that there exists a constant c independent of n such that for all annuli $\mathcal{A}(z) = \{w : d_n \le |w - z| \le \eta_n\}$, $z \in D_n$ we have

$$\psi_n(\mathcal{A}(z) \cap D_n) \subset \{w : c^{-1}d_n \le |w - \psi_n(z)| \le c\eta_n\} \cap \mathbb{D}.$$

But this follows immediately from Lemma B.1 with the assumption that d_n decays slower than $O(n^{-1/2})$ and (56). $\square$

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