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A BASIC IDENTITY FOR KOLMOGOROV OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS RELATED TO RDES WITH MULTIPLICATIVE NOISE

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We consider the Kolmogorov operator associated with a reaction–diffusion equation having polynomially growing reaction coefficient and perturbed by a noise of multiplicative type, in the Banach space E of continuous functions. By analyzing the smoothing properties of the associated transition semigroup, we prove a modification of the classical *identité du carré des champs* that applies to the present non-Hilbertian setting. As an application of this identity, we construct the Sobolev space $W^{1,2}(E;\mu)$, where μ is an invariant measure for the system, and we prove the validity of the Poincaré inequality and of the spectral gap.

1. Introduction. In the present paper we are concerned with the analysis of the Kolmogorov operator associated with the following reaction–diffusion equation in the interval (0, 1), perturbed by a noise of multiplicative type

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,\xi) = \dfrac{\partial^2 u}{\partial \xi^2}(t,\xi) + f\big(\xi, u(t,\xi)\big) + g\big(\xi, u(t,\xi)\big)\dfrac{\partial w}{\partial t}(t,\xi), \\
\qquad t \geq 0, \xi \in [0,1], \\
u(t,0) = u(t,1) = 0, \qquad u(0,\xi) = x(\xi), \\
\qquad \xi \in [0,1].
\end{cases}
\tag{1.1}
$$

Here $\partial w/\partial t(t,\xi)$ is a space–time white noise. The nonlinear terms $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are both continuous, the mapping $g(\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, uniformly with respect to $\xi \in [0,1]$, and the mapping $f(\xi,\cdot)$ has polynomial growth, is locally Lipschitz-continuous and satisfies suitable dissipativity conditions, uniformly with respect to $\xi \in [0,1]$. The example of $f(\xi,\cdot)$ we have in mind is an odd-degree polynomial, having negative leading coefficient.

In [4], the well posedness of equation (1.1) has been studied, and it has been proved that for any initial datum $x \in E := C_0([0,1])$ there exists a unique *mild* solution $u^x \in L^p(\Omega; C([0,T]; E))$, for any $T > 0$ and $p \geq 1$. This allows us to

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introduce the *Markov transition semigroup* P_t associated with equation (1.1), by setting for any Borel measurable and bounded function $\varphi : E \to \mathbb{R}$

$$P_t\varphi(x) = \mathbb{E}\varphi\big(u^x(t)\big), \qquad t \geq 0, x \in E.$$

As is known (see [2]) the semigroup P_t is *not* strongly continuous in $C_b(E)$. Nevertheless, it is *weakly continuous*, so that we can define the *weak generator* $\mathcal{K}$ associated with the semigroup P_t in terms of the Laplace transform of P_t,

$$(\lambda - \mathcal{K})^{-1}\varphi(x) = \int_0^\infty e^{-\lambda t} P_t\varphi(x)\,dt, \qquad \varphi \in C_b(E).^2 \tag{1.2}$$

For all definitions and details we refer to our previous work [2] and to Appendix B in [3].

In this paper we are going to study some important properties of the Kolmogorov operator $\mathcal{K}$ in $C_b(E)$. If we write equation (1.1) in the abstract form

$$du(t) = \big[Au(t) + F\big(u(t)\big)\big]\,dt + G\big(u(t)\big)\,dw(t)$$

(see Section 2 below for all notations), then $\mathcal{K}$ reads formally as

$$\mathcal{K}\varphi = \frac{1}{2}\sum_{k=1}^{\infty} D^2\varphi\big(G(x)e_k, G(x)e_k\big) + \big\langle Ax + F(x), D\varphi(x)\big\rangle_E \tag{1.3}$$

(here $D\varphi$ and $D^2\varphi$ represent the first and the second derivatives of a twice differentiable function $\varphi : E \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_E$ is the duality between E and its topological dual $E^\star$). Notice, however, that it is not easy to decide whether a given function belongs to the domain of $\mathcal{K}$ or not, as it is defined in an abstract way by formula (1.2). Our main concern here is studying some relevant properties of $\mathcal{K}$, such as the possibility to define the Sobolev space $W^{1,2}(E, \mu)$, with respect to the invariant measure μ for equation (1.1), or the validity of the Poincaré inequality and of the spectral gap, which, as is well known, implies the exponential convergence to equilibrium.

In the case of additive noise, that is, when $G(x)$ is constant, it is possible to study equation (1.1) in the Hilbert space $H = L^2(0, 1)$ in a generalized sense, so that the associated transition semigroup and the Kolmogorov operator can be introduced. In this case, it has been proved that the so-called *identité du carré des champs*

$$\mathcal{K}(\varphi^2) = 2\varphi\mathcal{K}\varphi + \big|G^\star D\varphi\big|_H^2 \tag{1.4}$$

is valid for functions φ in a *core* of $\mathcal{K}$.[3] Identity (1.4) has several important consequences. Actually, if there exists an invariant measure μ for $u^x(t)$, identity (1.4) provides the starting point to define the Sobolev space $W^{1.2}(H, \mu)$. Moreover, under some additional conditions, it allows to prove the Poincaré inequality and the exponential convergence of $P_t\varphi$ to equilibrium (*spectral gap*).

[2]The space of all uniformly continuous and bounded real-valued mappings defined on E.

[3]A core of $\mathcal{K}$ is a subset of $D(\mathcal{K})$ which is dense in the graph norm of $\mathcal{K}$ (see [8]).

To this purpose, we should mention that the existence of an invariant measure μ for equation (1.1) has been proved in [4]. The problem of uniqueness is more delicate, in general. But here we are in a favorable situation, as we are assuming that g is uniformly bounded from below by a positive constant. Actually, as we are dealing with white noise in space and time, this implies that the transition semigroup P_t is strongly Feller and irreducible, so that we can apply the Doob and the Khasminskii theorems, and we can conclude that the invariant measure μ is unique and strongly mixing.

The case we are dealing with in the present paper is much more delicate, as we are considering a polynomial reaction term f combined with a multiplicative noise. Because of this, it seems better and more natural to work in the Banach space E of continuous functions vanishing at the boundary, instead of in H. Moreover, the space $C_b(E)$ is larger than the space $C_b(H)$, and working in $C_b(E)$ allows us to estimate some interesting functions as, for example, the evaluation functional $\mathbb{E}\delta_{\xi_0}(u) = \mathbb{E}u(\xi_0)$, for $\xi_0 \in [0, 1]$ fixed.

On the other hand, deciding to work in $C_b(E)$ instead of $C_b(H)$ has some relevant consequences, and there is a price to pay. In our case it means in particular that formula (1.4) has to be changed in a suitable way. Actually, if $\varphi \in C_b^1(E)$ and $x \in E$ we cannot say that $D\varphi(x) \in H$ and hence the term $|G^\star(\cdot)D\varphi|_H$ is no more meaningful. In fact, it turns out that formula (1.4) has to be replaced by the formula

$$\mathcal{K}(\varphi^2) = 2\varphi\mathcal{K}\varphi + \sum_{k=1}^{\infty} \left|\langle G(\cdot)e_k, D\varphi\rangle_E\right|^2, \tag{1.5}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal system given by the eigenfunctions of the second derivative, endowed with Dirichlet boundary conditions.

Notice that, in order to give a meaning to (1.5), for $\varphi \in D(\mathcal{K})$, we have to prove that:

(i) $D(\mathcal{K})$ is included in $C_b^1(E)$;
(ii) the series in (1.5) is convergent for any $\varphi \in D(\mathcal{K})$;
(iii) $\varphi^2 \in D(\mathcal{K})$, for any $\varphi \in D(\mathcal{K})$ and (1.5) holds.

The proof of each one of these steps is very delicate in the framework we are considering here and requires the use of different arguments and techniques, compared to [8] and [3], Chapters 6 and 7.

In order to approach (i), we have proved that the solution $u^x(t)$ of equation (1.1) is differentiable with respect to the initial datum $x \in E$. Moreover, we have proved that the second derivative equation is solvable and suitable bounds for its solution have been given. These results were not available in the existing literature and, in order to be proved, required some new arguments based on positivity, as the classical techniques did not apply, due to the fact that f' is not globally bounded, and the noise is multiplicative. Next, we had to prove that, as in the Hilbertian case, a Bismuth–Elworthy–Li formula holds for the derivative of the semigroup. This well-known formula provides the important gradient estimate

$$\sup_{x\in E}\big|D(P_t\varphi)\big|_{E^\star}\le c(t\wedge 1)^{-1/2}\sup_{x\in E}\big|\varphi(x)\big|,\qquad t>0,$$

which is crucial in order to prove that $D(\mathcal{K})$ is contained in $C_b^1(E)$.

In order to prove (ii), we couldn't proceed directly as in [3], Chapter 5, by using the mild formulation of the first derivative equation and the fact that e^{tA} is an Hilbert–Schmidt operator, for any $t>0$, again because of the presence of the polynomial nonlinearity f combined with the multiplicative noise. Nevertheless, by using a suitable duality argument, we could prove that

$$\sum_{k=1}^{\infty}\big|\langle G(x)e_k, D(P_t\varphi)(x)\rangle_E\big|^2\le c\big|G(x)\big|_E^2\|\varphi\|_0^2(t\wedge 1)^{-1},\qquad t>0$$

and this allowed us to prove that the series in (1.5) is convergent, for any $\varphi\in D(\mathcal{K})$. For this reason, we would like to mention the fact that our duality argument does work because we are dealing with the two concrete spaces $E=C_0([0,1])$ and $H=L^2(0,1)$ together, and hence we can use some nice approximation and duality arguments between the corresponding spaces of continuous functions $C_b(E)$ and $C_b(H)$ and the corresponding spaces of differentiable functions $C_b^1(E)$ and $C_b^1(H)$; see Lemma 2.1.

Finally, in order to prove (iii), we had to use a suitable modification of the Itô formula that applies to Banach spaces and a suitable approximation argument based on the use of the Ornstein–Uhlenbeck semigroup in the Banach space E.

As we mentioned before, as a consequence of the modified *identité du carré des champs* (1.5), we were able to construct the space $W^{1,2}(E;\mu)$ and prove the Poincaré inequality and the existence of a spectral gap. For this reason, we would like to stress that in spite of the fact that the *identité du carré des champs* has to be modified and we have to replace $|G^\star(\cdot)D\varphi|_H^2$ by the series

$$\sum_{k=1}^{\infty}\big|\langle G(\cdot)e_k, D\varphi\rangle_E\big|^2,$$

the Poincaré inequality proved is identical to what we have in the case of the Hilbert space H, with $|D\varphi|_H$ clearly replaced by $|D\varphi|_{E^\star}$, that is,

$$\int_E\big|\varphi(x)-\bar\varphi\big|^2\,d\mu(x)\le\rho\int_E\big|D\varphi(x)\big|_{E^\star}^2\,d\mu(x).$$

2. Preliminaries. We shall denote by H the Hilbert space $L^2(0,1)$, endowed with the usual scalar product $\langle\cdot,\cdot\rangle_H$ and the corresponding norm $|\cdot|_H$. Moreover, we shall denote by E the Banach space $C_0([0,1])$ of continuous functions on $[0,1]$, vanishing at 0 and 1, endowed with the sup-norm $|\cdot|_E$ and the duality $\langle\cdot,\cdot\rangle_E$ between E and its dual topological space $E^\star$.

Now, if we fix $x \in E$ there exists $\xi_x \in [0,1]$ such that $|x(\xi_x)| = |x|_E$. Then, if δ is any element of $E^\star$ having norm equal 1, the element $\delta_x \in E^\star$ defined by

$$\langle y, \delta_x\rangle_E := \begin{cases} \dfrac{x(\xi_x)y(\xi_x)}{|x|_E}, & \text{if } x \neq 0, \\ \langle y, \delta\rangle_E, & \text{if } x = 0, \end{cases} \tag{2.1}$$

belongs to the subdifferential $\partial|x|_E := \{x^\star \in E^\star; |x^\star|_{E^\star} = 1, \langle x, x^\star\rangle_E = |x|_E\}$; see, for example, [3], Appendix A, for all definitions and details.

Next, let X be a separable Banach space. $\mathcal{L}(X)$ shall denote the Banach algebra of all linear bounded operators in X and $\mathcal{L}^1(X)$ shall denote the subspace of trace-class operators. We recall that

$$\|T\| = \sup_{|x|_X \le 1} |Tx|_X, \qquad T \in \mathcal{L}(X).$$

For any other Banach space Y, we denote by $B_b(X,Y)$ the linear space of all bounded and measurable mappings $\varphi : X \to Y$ and by $C_b(X,Y)$ the subspace of continuous functions. Endowed with the sup-norm

$$\|\varphi\|_0 = \sup_{x \in X} |\varphi(x)|_Y,$$

$C_b(X,Y)$ is a Banach space. Moreover, for any $k \ge 1$, $C_b^k(X,Y)$ shall denote the subspace of all functions which are k-times Fréchet differentiable. $C_b^k(X,Y)$, endowed with the norm

$$\|\varphi\|_k = \|\varphi\|_0 + \sum_{j=1}^k \sup_{x \in X} |D^j\varphi(x)|_Y =: \|\varphi\|_0 + \sum_{j=1}^k [\varphi]_j$$

is a Banach space. In the case $Y = \mathbb{R}$, we shall set $B_b(X,Y) = B_b(X)$ and $C_b^k(X,Y) = C_b^k(X)$, $k \ge 0$.

In what follows, we shall denote by A the linear operator

$$Ax = \frac{\partial^2 x}{\partial \xi^2}, \qquad x \in D(A) = H^2(0,1) \cap H_0^1(0,1).$$

A is a nonpositive and self-adjoint operator which generates an analytic semigroup e^{tA}, with dense domain in $L^2(0,1)$. The space $L^1(0,1) \cap L^\infty(0,1)$ is invariant under e^{tA}, so that e^{tA} may be extended to a nonpositive one-parameter contraction semigroup e^{tA_p} on $L^p(0,1)$, for all $1 \le p \le \infty$. These semigroups are strongly continuous, for $1 \le p < \infty$, and are consistent, in the sense that $e^{tA_p}x = e^{tA_q}(t)x$, for all $x \in L^p(0,1) \cap L^q(0,1)$. This is why we shall denote all e^{tA_p} by e^{tA}. Finally, if we consider the part of A in E, it generates a strongly continuous analytic semigroup.

For any $k \in \mathbb{N}$, we define

$$e_k(\xi) = \sqrt{2}\sin k\pi\xi, \qquad \xi \in [0,1]. \tag{2.2}$$

The family $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal system in H which diagonalizes A, so that

$$Ae_k = -k^2\pi^2 e_k, \qquad k \in \mathbb{N}.$$

Notice that for any $t > 0$ the semigroup e^{tA} maps $L^p(0,1)$ into $L^q(0,1)$, for any $1 \le p \le q \le \infty$ and for any $p \ge 1$ there exists $M_p > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}(L^p(0,1),L^q(0,1))} \le M_{p,q} e^{-\omega_{p,q}t} t^{-(q-p)/2pq}, \qquad t > 0. \tag{2.3}$$

Here we are assuming that $\partial w(t)/\partial t$ is a space–time white noise defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Thus, $w(t)$ can be written formally as

$$w(t) := \sum_{k=1}^{\infty} e_k \beta_k(t), \qquad t \ge 0,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal system in H which diagonalizes A and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard real Brownian motions on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. As is well known, the series above does not converge in H, but it does converge in any Hilbert space U containing H, with Hilbert–Schmidt embedding.

Concerning the nonlinearities f and g, we assume that they are both continuous. Moreover, they satisfy the following conditions:

HYPOTHESIS 1. (1) *For any $\xi \in [0,1]$, both $f(\xi,\cdot)$ and $g(\xi,\cdot)$ belong to $C^2(\mathbb{R})$.*

(2) *There exists $m \ge 1$ such that for $j = 0, 1, 2$*

$$\sup_{\xi\in[0,1]} \left|D^j_\rho f(\xi,\rho)\right| \le c_j\left(1 + |\rho|^{(m-j)^+}\right). \tag{2.4}$$

(3) *There exists $\lambda \in \mathbb{R}$ such that*

$$\sup_{(\xi,\rho)\in[0,1]\times\mathbb{R}} f'(\xi,\rho) \le \lambda. \tag{2.5}$$

(4) *The mapping $g(\xi,\cdot):\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $\xi \in [0,1]$, and*

$$\sup_{\xi\in[0,1]} \left|g(\xi,\rho)\right| \le c\left(1 + |\rho|^{1/m}\right). \tag{2.6}$$

(5) *If there exist $\alpha > 0$ and $\beta \ge 0$ such that*

$$\left(f(\xi,\rho+\sigma) - f(\xi,\rho)\right)\sigma \le -\alpha|\sigma|^{m+1} + \beta\left(1 + |\rho|^m\right)|\sigma|, \tag{2.7}$$

then no restriction is assumed on the linear growth of $g(\xi,\cdot)$.

In what follows, for any $x, y \in E$ and $\xi \in [0,1]$ we shall denote

$$F(x)(\xi) = f\big(\xi, x(\xi)\big), \qquad \big[G(x)y\big](\xi) = g\big(\xi, x(\xi)\big)y(\xi).$$

Due to (2.4), F is well defined and continuous from $L^p(0,1)$ into $L^q(0,1)$, for any $p, q \geq 1$ such that $p/q \geq m$. In particular, if $m \neq 1$, then F is not defined from H into itself. Moreover, due to (2.5), for $x, h \in L^{2m}(0,1)$,

$$\big\langle F(x+h) - F(x), h\big\rangle_H \leq \lambda |h|_H^2. \tag{2.8}$$

Clearly, F is also well defined in E, and it is possible to prove that it is twice continuously differentiable in E, with

$$\big[DF(x)y\big](\xi) = D_\rho f\big(\xi, x(\xi)\big)y(\xi),$$
$$\big[D^2F(x)(y_1, y_2)\big](\xi) = D_\rho^2 f\big(\xi, x(\xi)\big)y_1(\xi)y_2(\xi).$$

In particular, for any $x \in E$,

$$\big|D^j F(x)\big|_{\mathcal{L}^j(E)} \leq c\big(1 + |x|_E^{(m-j)^+}\big). \tag{2.9}$$

Moreover, for any $x, h \in E$,

$$\big\langle F(x+h) - F(x), \delta_h\big\rangle_E \leq \lambda |h|_E, \tag{2.10}$$

where δ_h is the element of $\partial|h|_E$ defined above in (2.1).

Finally, if (2.7) holds, we have

$$\big\langle F(x+h) - F(x), \delta_h\big\rangle_E \leq -\alpha |h|_E^m + \beta\big(1 + |x|_E^m\big). \tag{2.11}$$

Next, concerning the operator G, as the mapping $g(\xi, \cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous, uniformly with respect to $\xi \in [0,1]$, the operator G is Lipschitz-continuous from H into $\mathcal{L}(H; L^1(0,1))$, that is,

$$\|G(x) - G(y)\|_{\mathcal{L}(H;L^1(0,1))} \leq c|x - y|_H. \tag{2.12}$$

In the same way it is possible to show that the operator G is Lipschitz-continuous from H into $\mathcal{L}(L^\infty(0,1); H)$ and

$$\|G(x) - G(y)\|_{\mathcal{L}(L^\infty(0,1);H)} \leq c|x - y|_H. \tag{2.13}$$

By proceeding similarly as in [3], Proposition 6.1.5, it is possible to prove the following result.

LEMMA 2.1. *For any $\varphi \in C_b(E)$ there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_b(H)$ such that*

$$\begin{cases} \lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x), & x \in E, \\ \sup\limits_{x\in H} \big|\varphi_n(x)\big| \leq \sup\limits_{x\in E} \big|\varphi(x)\big|, & n \in \mathbb{N}. \end{cases} \tag{2.14}$$

Moreover, if $\varphi \in C_b^k(E)$, *we have* $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_b^k(H)$ *and for any* $j \le k$,

$$
(2.15)\qquad \begin{cases} \lim_{n\to\infty} D^j\varphi_n(x)(h_i,\dots,h_j) = D^j\varphi(x)(h_i,\dots,h_j), \\ \qquad x, h_1,\dots,h_j \in E, \\ \sup_{x\in H} \big|D^j\varphi_n(x)\big|_{\mathcal{L}^j(E)} \le \sup_{x\in E}\big|D^j\varphi(x)\big|_{\mathcal{L}^j(E)}, \\ \qquad n\in\mathbb{N}. \end{cases}
$$

PROOF. The sequence $\{\varphi_n\}$ has been already introduced in [3], Proposition 6.1.5, by setting

$$
\varphi_n(x) = \varphi(x_n), \qquad x \in H,
$$

where for any $x \in H$,

$$
x_n(\xi) = \frac{n}{2}\int_{\xi-1/n}^{\xi+1/n} \hat{x}(\eta)\,d\eta, \qquad \xi\in[0,1]
$$

and $\hat{x}(\eta)$ is the extension by oddness of $x(\eta)$, for $\eta\in(-1,0)$ and $\eta\in(1,2)$. Clearly, due to the boundary conditions, we have that $x_n \in E$, for any $n\in\mathbb{N}$, so that $\varphi_n(x)$ is well defined.

In [3], Proposition 6.1.5, we have already proved that (2.14) holds. In order to prove (2.15) (for $k=1$) we just notice that for any $n\in\mathbb{N}$ and $x,h\in H$ we have

$$
\varphi_n(x+h)-\varphi_n(x) = \varphi(x_n+h_n)-\varphi(x_n) = \langle h_n, D\varphi(x_n)\rangle_E + o\big(|h_n|_E\big).
$$

As $o(|h_n|_E) = o(|h|_H)$, we can conclude that φ_n is differentiable in H and

$$
\langle h, D\varphi_n(x)\rangle_H = \langle h_n, D\varphi(x_n)\rangle_E, \qquad x,h\in H.
$$

If $x,h\in E$, then x_n and h_n converge to x and h in E, respectively. This implies that (2.15) holds. $\square$

2.1. *The approximating Nemytskii operators*. Let γ be a function in $C^\infty(\mathbb{R})$ such that

$$
(2.16)\qquad \begin{aligned} \gamma(x) &= x, && |x|\le 1,\\ \big|\gamma(x)\big| &= 2, && |x|\ge 2,\\ \big|\gamma(x)\big| &\le |x|, && x\in\mathbb{R},\\ \gamma'(x) &\ge 0, && x\in\mathbb{R}. \end{aligned}
$$

For any $n\in\mathbb{N}$, we define

$$
f_n(\xi,\rho) = f\big(\xi, n\gamma(\rho/n)\big), \qquad (\xi,\rho)\in[0,1]\times\mathbb{R}.
$$

It is immediate to check that all functions f_n are in $C_b^2(\mathbb{R})$ and satisfy (2.4) and (2.5), so that the corresponding composition operators F_n satisfy inequalities (2.9) and (2.10), for constants c and λ independent of n. Namely

$$|D^j F_n(x)|_E \leq c\big(1 + |x|_E^{(m-j)^+}\big) \tag{2.17}$$

and

$$\langle F_n(x+h) - F_n(x), \delta_h \rangle_E \leq \lambda |h|_E. \tag{2.18}$$

Notice that all $f_n(\xi, \cdot)$ are Lipschitz continuous, uniformly with respect to $\xi \in [0, 1]$, so that all F_n are Lipschitz continuous in all $L^p(0, 1)$ spaces and in E.

According to (2.16), we can easily prove that for any $j = 0, 1, 2$ and $R > 0$,

$$\lim_{n\to\infty} \sup_{(\xi,\rho)\in[0,1]\times[-R,R]} |D^j f_n(\xi, \rho) - D^j f(\xi, \rho)| = 0$$

and then for any $R > 0$ and $j = 1, 2$

$$\begin{cases} \lim\limits_{n\to\infty} \sup\limits_{|x|_E\leq R} |F_n(x) - F(x)|_E = 0, \\ \lim\limits_{n\to\infty} \sup\limits_{\substack{|x|_E\leq R \\ |y_1|_E,\dots,|y_j|_E\leq R}} |D^j F_n(x)(y_1, \dots, y_j) \\ \qquad\qquad - D^j F(x)(y_1, \dots, y_j)|_E = 0. \end{cases} \tag{2.19}$$

We have already seen that the mappings F_n are Lipschitz-continuous in H. The differentiability properties of F_n in H are a more delicate issue. Actually, even if $f_n(\xi, \cdot)$ is assumed to be smooth, $F_n : H \to H$ is only Gateaux differentiable and its Gateaux derivative at $x \in H$ along the direction $h \in H$ is given by

$$[DF_n(x)h](\xi) = D_\rho f_n(\xi, x(\xi))h(\xi), \qquad \xi \in [0, 1].$$

Higher order differentiability is even more delicate, as the higher order derivatives do not exist along any direction in H, but only along more regular directions. For example, the second order derivative of F_n exists only along directions in $L^4(0, 1)$, and for any $x \in H$ and $h, k \in L^4(0, 1)$

$$[D^2 F_n(x)(h, k)](\xi) = D_\rho^2 f_n(\xi, x(\xi))h(\xi)k(\xi), \qquad \xi \in [0, 1].$$

3. The solution of (1.1). With the notation introduced in Section 2, equation (1.1) can be rewritten as the following abstract evolution equation:

$$du(t) = [Au(t) + F(u(t))]\,dt + G(u(t))\,dw(t), \qquad u(0) = x. \tag{3.1}$$

DEFINITION 3.1. An adapted process $u \in L^p(\Omega; C([0, T]; E))$ is a mild solution for equation (3.1) if

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(u(s))\,ds + \int_0^t e^{(t-s)A} G(u(s))\,dw(s).$$

Let $X = E$ or $X = H$. In what follows, for any $T > 0$ and $p \geq 1$ we shall denote by $C^w_{p,T}(X)$ the set of adapted processes in $L^p(\Omega; C([0,T]; X))$. Endowed with the norm

$$\|u\|_{C^w_{p,T}(X)} = \Big(\mathbb{E} \sup_{t\in[0,T]} \big|u(t)\big|_X^p\Big)^{1/p},$$

$C^w_{p,T}(X)$ is a Banach space. Furthermore, we shall denote by $L^w_{p,T}(X)$ the Banach space of adapted processes in $C([0,T]; L^p(\Omega; X))$, endowed with the norm

$$\|u\|_{L^w_{p,T}(X)} = \sup_{t\in[0,T]} \big(\mathbb{E}\big|u(t)\big|_X^p\big)^{1/p}.$$

In [4] it has been proved that, under Hypothesis 1, for any $T > 0$ and $p \geq 1$ and for any $x \in E$, equation (3.1) admits a unique mild solution u^x in $C^w_{p,T}(E)$. Moreover,

$$\big\|u^x\big\|_{C^w_{p,T}(E)} \leq c_{p,T}\big(1 + |x|_E\big). \tag{3.2}$$

One of the key steps in the proof of such an existence and uniqueness result, is given in [4], Theorem 4.2, where it is proved that the mapping

$$u \in C^x_{p,T}(E) \mapsto \left(t \mapsto \Gamma(u)(t) := \int_0^t e^{(t-s)A} G\big(u(s)\big)\, dw(s)\right) \in C^x_{p,T}(E)$$

is well defined and Lipschitz continuous. By adapting the arguments used in the proof of [4], Theorem 4.2, it is also possible to show that

$$\mathbb{E} \sup_{s\in[0,t]} \big|\Gamma(u)(s) - \Gamma(v)(s)\big|_E^p \leq c_p(t) \int_0^t \mathbb{E}\big|u(s) - v(s)\big|_E^p\, ds, \qquad t \geq 0 \tag{3.3}$$

for some continuous function $c_p(t)$, with $c_p(0) = 0$. In particular, there exists $T_p > 0$ such that

$$\big\|\Gamma(u) - \Gamma(v)\big\|_{L^w_{p,T}(E)} \leq \tfrac{1}{4}\|u - v\|_{L^w_{p,T}(E)}, \qquad T \leq T_p. \tag{3.4}$$

Now, for any $n \in \mathbb{N}$, we consider the approximating problem

$$du(t) = \big[Au(t) + F_n\big(u(t)\big)\big]\, dt + G\big(u(t)\big)\, dw(t), \qquad u(0) = x \tag{3.5}$$

and we denote by u^x_n its unique mild solution in $C^w_{p,T}(E)$. As all F_n satisfy (2.17) and (2.18), we have that

$$\big\|u^x_n\big\|_{C^w_{p,T}(E)} \leq c_p(T)\big(1 + |x|_E\big), \qquad n \in \mathbb{N} \tag{3.6}$$

for a function $c_p(T)$ independent of n.

As proved in [4], Section 3, the mapping

$$u \in C^x_{p,T}(H) \mapsto \left(t \mapsto \Gamma(u)(t) := \int_0^t e^{(t-s)A} G\big(u(s)\big)\, dw(s)\right) \in C^x_{p,T}(H)$$

is well defined and Lipschitz continuous. Then, as the mapping $F_n : H \to H$ is Lipschitz-continuous, we have that for any $x \in H$ and for any $T > 0$ and $p \geq 1$, problem (3.5) admits a unique mild solution $u_n^x \in C^x_{p,T}(H)$ such that

$$\|u_n^x\|_{C^w_{p,T}(H)} \leq c_{n,p}(T)\big(1 + |x|_H\big). \tag{3.7}$$

LEMMA 3.2. *Under Hypothesis 1, for any $T, R > 0$ and $p \geq 1$, we have*

$$\lim_{n\to\infty} \sup_{|x|_E \leq R} \|u_n^x - u^x\|_{C^w_{p,T}(E)} = 0. \tag{3.8}$$

PROOF. Since $F_n(x) = F(x)$, for $|x|_E \leq n$, we have

$$\Big\{ \sup_{t\in[0,T]} |u^x(t)|_E \leq n \Big\} \subset \Big\{ \sup_{t\in[0,T]} |u_n^x(t) - u^x(t)|_E = 0 \Big\}.$$

This implies

$$\begin{aligned}\|u_n^x - u^x\|^p_{C^w_{p,T}(E)} &= \mathbb{E}\Big(\sup_{t\in[0,T]} |u_n^x(t) - u^x(t)|_E^p;\ \sup_{t\in[0,T]} |u^x(t)|_E > n\Big) \\ &\leq \big(\|u_n^x\|^p_{C^w_{2p,T}(E)} + \|u^x\|^p_{C^w_{2p,T}(E)}\big)\mathbb{P}\Big(\sup_{t\in[0,T]} |u^x(t)|_E > n\Big)^{1/2}.\end{aligned}$$

Therefore, thanks to (3.2) and (3.6), we get

$$\|u_n^x - u^x\|_{C^w_{p,T}(E)} \leq c_p(T)\frac{(1 + |x|_E^2)}{n^2},$$

which implies (3.8). □

4. The first derivative.

For any $x \in E$, $u \in L^w_{p,T}(E)$ and $n \in \mathbb{N}$, we define

$$\Lambda_n(x,u)(t) = e^{tA}x + \int_0^t e^{(t-s)A} F_n\big(u(s)\big)\,ds + \int_0^t e^{(t-s)A} G\big(u(s)\big)\,dw(s), \qquad t \geq 0.$$

Clearly, for any $x \in E$ the solution u_n^x of problem (3.5) is the unique fixed point of $\Lambda_n(x,\cdot)$. The mapping $F_n : E \to E$ is Lipschitz continuous, then due to (3.4), there exists $T_p = T_p(n) > 0$ such that

$$\|\Lambda_n(x,u) - \Lambda_n(x,v)\|_{L^w_{p,T}(E)} \leq \tfrac{1}{2}\|u - v\|_{L^w_{p,T}(E)}, \qquad T \leq T_p.$$

Therefore, if we show that the contraction mapping Λ_n is of class C^1, we get that the mapping

$$x \in E \mapsto u_n^x \in L^w_{p,T}(E)$$

is differentiable, and for any $h \in E$

$$D_x u_n^x h = D_x\Lambda_n\big(x, u_n^x\big)h + D_u\Lambda_n\big(x, u_n^x\big)D_x u_n^x h \tag{4.1}$$

(for a proof see, e.g., [3]).

As $f_n(\xi, \cdot)$ is in $C^2(\mathbb{R})$, the mapping $F_n : E \to E$ is twice continuously differentiable, then it is possible to check that the mapping

$$u \in L^w_{p,T}(E) \mapsto \left(t \mapsto \int_0^t e^{(t-s)A} F_n\big(u(s)\big)\, ds\right) \in L^w_{p,T}(E)$$

is twice differentiable. Analogously, as the mapping $g(\xi, \cdot)$ is in $C^2(\mathbb{R})$, by using the stochastic factorization method as in [4], Theorem 4.2, it is not difficult to prove that the mapping

$$u \in L^w_{p,T}(E) \mapsto \left(t \mapsto \int_0^t e^{(t-s)A} G\big(u(s)\big)\, dw(s)\right) \in L^w_{p,T}(E)$$

is twice differentiable.

Moreover, for any $x \in E$ and $u, v \in L^w_{p,T}(E)$, we have

$$\begin{aligned} \big[D_u \Lambda_n(x, u)v\big](t) &= \int_0^t e^{(t-s)A} F_n'\big(u(s)\big)v(s)\, ds \\ &\quad + \int_0^t e^{(t-s)A} G'\big(u(s)\big)v(s)\, dw(s), \qquad t \geq 0, \end{aligned} \tag{4.2}$$

where, for any $x, y, z \in E$ and $\xi \in [0, 1]$

$$\begin{aligned} \big[F_n'(x)y\big](\xi) &= D_\rho f_n\big(\xi, x(\xi)\big)y(\xi), \\ \big[\big(G'(x)y\big)z\big](\xi) &= D_\rho g\big(\xi, x(\xi)\big)y(\xi)z(\xi) \end{aligned}$$

and $D_\rho f_n$ and $D_\rho g$ are the derivatives of f_n and g with respect to the second variable. Therefore, as, clearly,

$$\big[D_x \Lambda_n(x, u)h\big](t) = e^{tA}h,$$

from (4.1) we have that $\eta_n^h := D_x u_n^x h$ solves the linear equation

$$\begin{aligned} d\eta_n^h(t) &= \big[A\eta_n^h(t) + F_n'\big(u_n^x(t)\big)\eta_n^h(t)\big]\, dt + G'\big(u_n^x(t)\big)\eta_n^h(t)\, dw(t), \\ \eta_n^h(0) &= h. \end{aligned} \tag{4.3}$$

LEMMA 4.1. *Under Hypothesis 1, for any $T > 0$ and $p \geq 1$ the process u_n^x is differentiable with respect to $x \in E$ in $L^w_{p,T}(E)$. Moreover, the derivative $D_x u_n^x h =: \eta_n^h$ belongs to $C^w_{p,T}(E)$ and satisfies*

$$\|\eta_n^h\|_{C^w_{p,T}(E)} \leq M_p e^{\omega_p T} |h|_E \tag{4.4}$$

for some constants M_p and ω_p independent of $n \in \mathbb{N}$.

PROOF. To prove (4.4) we cannot use the Itô formula, due to presence of the white noise. Moreover we cannot use the same arguments used, for example, in [4] and [3], because of the unboundedness of f' and the presence of the noisy part. In view of what we have already seen, we have only to prove that (4.4) holds. To this purpose, the key remark here is that we can assume $h \geq 0$. Actually, in the general case we can decompose $h = h^+ - h^-$. As h^+ and h^- are nonnegative, both $\eta_n^{h^+}$ and $\eta_n^{h^-}$ verify the lemma and then, since by linearity $\eta_n^h = \eta_n^{h^+} - \eta_n^{h^-}$, we can conclude that the lemma is true also for η_n^h.

Let $\Gamma_n(t)$ be the mild solution of the problem

$$d\Gamma_n(t) = [A - I]\Gamma_n(t)\,dt + G'\big(u_n^x(t)\big)\eta_n^h(t)\,dw(t), \qquad \Gamma_n(0) = 0.$$

Since we are assuming that $D_\rho g(\xi, \cdot)$ is bounded uniformly with respect to $\xi \in [0, 1]$, we have that the argument of [4], Theorem 4.2 and Proposition 4.5, can be adapted to the present situation and

$$\mathbb{E} \sup_{s\in[0,t]} \big|\Gamma_n(s)\big|_E^p \leq c_p \int_0^t \mathbb{E}\big|\eta_n^h(s)\big|_E^p\,ds, \qquad t \in [0, T] \tag{4.5}$$

for some constant c_p independent of $T > 0$.

Next, if we set $z_n = \eta_n^h - \Gamma_n$, we have that z_n solves the equation

$$\frac{dz_n}{dt}(t) = [A - I]z_n(t) + \big[F_n'\big(u_n^x(t)\big) + I\big]\eta_n^h(t), \qquad z_n(0) = h.$$

Now, since we are assuming that $h \geq 0$ and equation (4.3) is linear, we have that

$$\mathbb{P}\big(\eta_n^h(t) \geq 0, t \in [0, T]\big) = 1;$$

see [10] for a proof and see also [5] for an analogous result for equations with non-Lipschitz coefficients. Therefore, as $f_n'(\xi, \rho) \leq \lambda$, for any $(\xi, \rho) \in [0, 1] \times \mathbb{R}$, and the semigroup e^{tA} is positivity preserving, we have

$$\begin{aligned} z_n(t) &= e^{t(A-I)}h + \int_0^t e^{(t-s)(A-I)}\big[F_n'\big(u_n^x(s)\big) + I\big]\eta_n^h(s)\,ds \\ &\leq e^{t(A-I)}h + (\lambda + 1)\int_0^t e^{(t-s)(A-I)}\eta_n^h(s)\,ds. \end{aligned}$$

This implies

$$0 \leq \eta_n^h(t) = z_n(t) + \Gamma_n(t) \leq e^{t(A-I)}h + (\lambda + 1)\int_0^t e^{(t-s)(A-I)}\eta_n^h(s)\,ds + \Gamma_n(t),$$

so that

$$\big|\eta_n^h(t)\big|_E \leq c|h|_E + c(\lambda + 1)^+ \int_0^t \big|\eta_n^h(s)\big|_E\,ds + \big|\Gamma_n(t)\big|_E.$$

Thanks to (4.5), this allows us to conclude that

$$\mathbb{E} \sup_{s\in[0,t]} \big|\eta_n^h(s)\big|_E^p \leq c_p|h|_E^p + c_p \int_0^t \mathbb{E} \sup_{r\in[0,s]} \big|\eta_n^h(r)\big|_E^p\,ds.$$

From the Gronwall lemma, this yields (4.4). $\square$

LEMMA 4.2. *Under Hypothesis 1, there exists $\eta^h \in C^w_{p.T}(E)$ such that for any $R>0$,*

$$\lim_{n\to\infty} \sup_{x,h\in B_R(E)} \|\eta^h_n - \eta^h\|_{C^w_{p.T}(E)} = 0. \tag{4.6}$$

Moreover, the limit η^h solves the equation

$$d\eta^h(t) = \big[A\eta^h(t) + F'\big(u^x(t)\big)\eta^h(t)\big]\,dt + G'\big(u^x(t)\big)\eta^h(t)\,dw(t), \qquad \eta^h(0) = h \tag{4.7}$$

and

$$\|\eta^h\|_{C^w_{p,T}(E)} \le M_p e^{\omega_p T}|h|_E. \tag{4.8}$$

PROOF. For any $n, k \in \mathbb{N}$ we have

$$\begin{aligned}\|\eta^h_{n+k} - \eta^h_n\|^p_{C^w_{p,T}(E)} &= \mathbb{E}\Big(\sup_{t\in[0,T]}|\eta^h_{n+k}(t) - \eta^h_n(t)|^p_E;\ \sup_{t\in[0,T]}|u^x(t)|_E \le n\Big)\\ &\quad + \mathbb{E}\Big(\sup_{t\in[0,T]}|\eta^h_{n+k}(t) - \eta^h_n(t)|^p_E;\ \sup_{t\in[0,T]}|u^x(t)|_E > n\Big).\end{aligned}$$

Since

$$\Big\{\sup_{t\in[0,T]}|u^x(t)|_E \le n\Big\} \subseteq \{\eta^h_{n+k}(t) = \eta^h_n(t), t\in[0,T]\},$$

thanks to (3.2) and (4.4) we get

$$\begin{aligned}&\|\eta^h_{n+k} - \eta^h_n\|^{2p}_{C^w_{p,T}(E)}\\ &\quad\le \mathbb{E}\Big(\sup_{t\in[0,T]}|\eta^h_{n+k}(t) - \eta^h_n(t)|^{2p}_E\Big)\mathbb{P}\Big(\sup_{t\in[0,T]}|u^x(t)|_E > n\Big)\\ &\quad\le \frac{c_p(T)}{n^{2p}}|h|^{2p}_E\big(1+|x|^{2p}_E\big)\end{aligned} \tag{4.9}$$

and this implies that $\{\eta^h_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^w_{p,T}(E)$.

Let η^h be its limit, and let $R>0$ and $n\in\mathbb{N}$. For any $m\ge n$ and $x,h\in B_R(E)$, due to (4.9) we have

$$\begin{aligned}\|\eta^h_n - \eta^h\|_{C^w_{p,T}(E)} &\le \|\eta^h_n - \eta^h_m\|_{C^w_{p,T}(E)} + \|\eta^h_m - \eta^h\|_{C^w_{p,T}(E)}\\ &\le \frac{c_p(T,R)}{n} + \|\eta^h_m - \eta^h\|_{C^w_{p,T}(E)}.\end{aligned}$$

Therefore, if we fix $\varepsilon>0$ and $\bar{m} = m(\varepsilon, x, h, \rho, T, p) \ge n$ such that

$$\|\eta^h_{\bar{m}} - \eta^h\|_{C^w_{p,T}(E)} < \varepsilon,$$

due to the arbitrariness of $\varepsilon > 0$ we get (4.6).

Moreover, as

$$\eta_n^h(t) = e^{tA}h + \int_0^t e^{(t-s)A} F_n'\big(u_n^x(s)\big)\eta_n^h(s)\,ds + \int_0^t e^{(t-s)A} G_n'\big(u_n^x(s)\big)\eta_n^h(s)\,dw(s)$$

and since, in addition to (4.6), (3.8) also holds, we can take the limit on both sides, and we get that the limit η^h is a mild solution of equation (4.7). $\square$

REMARK 4.3. In [3], Chapter 4, the differentiability of the mapping

$$x \in H \mapsto u_n^x \in L^x_{p,T}(H) \tag{4.10}$$

has been studied in the case $g(\xi, \rho) = 1$.

Since $f_n \in C_b^2(\mathbb{R})$, for any fixed $n \in \mathbb{N}$, we have that $DF_n : H \to \mathcal{L}(H)$ is bounded. Hence, the proof of [3], Proposition 4.2.1, can be adapted to the present situation of an equation with multiplicative noise, where the diffusion coefficient g is smooth, and g' is bounded. This means that the mapping in (4.10) is differentiable and the derivative $D_x u_n^x h$ satisfies equation (4.3). Moreover, by proceeding as in [3], Lemma 4.2.2, it is possible to prove that $D_x u_n^x(t)h \in L^p(0,1)$ for any $t > 0$, $\mathbb{P}$-a.s., and for any $p \geq 2$ and $q \geq 0$,

$$\sup_{x\in H} \mathbb{E}\big|D_x u_n^x(t)h\big|^q_{L^p(0,1)} \leq c_{p,q,n}(t \wedge 1)^{-((p-2)q)/4p}|h|_H^q. \tag{4.11}$$

Next, we show that we can estimate η^h in H.

LEMMA 4.4. *Under Hypothesis 1, for any $T > 0$ and $p \geq 1$ we have*

$$\|\eta^h\|_{C^w_{p,T}(H)} \leq c_p(T)|h|_H. \tag{4.12}$$

Moreover, for any $p \geq 1$ and $q \in [2, +\infty]$ such that $p(q-2)/4q < 1$, we have

$$\mathbb{E}\big|\eta^h(t)\big|^p_{L^q(0,1)} \leq c_{p,q}(t)t^{-(p(q-2))/4q}|h|_H^p, \qquad t > 0. \tag{4.13}$$

PROOF. If we denote by $\Gamma^h(t)$ the mild solution of

$$d\gamma(t) = (A+\lambda)\gamma(t)\,dt + G'\big(u^x(t)\big)\eta^h(t)\,dw(t), \qquad \gamma(0) = 0,$$

where λ is the constant introduced in (2.5), we have that $\rho(t) := \eta^h(t) - \Gamma^h(t)$ solves the problem

$$\frac{d\rho(t)}{dt} = (A+\lambda)\rho(t) + \big(F'\big(u^x(t)\big) - \lambda\big)\eta^h(t), \qquad \rho(0) = h.$$

As in the proof of Lemma 4.1, we decompose $\rho(t) = \rho^+(t) - \rho^-(t)$, where

$$\frac{d\rho^\pm(t)}{dt} = (A+\lambda)\rho^\pm(t) + \big(F'\big(u^x(t)\big) - \lambda\big)\eta^{h^\pm}(t), \qquad \rho(0) = h^\pm.$$

As

$$\mathbb{P}\big(\eta^{h^{\pm}}(t) \geq 0, t \in [0,T]\big) = 1$$

and $f' - \lambda \leq 0$, we have

$$\begin{aligned}\rho^{\pm}(t) &= e^{t(A+\lambda)}h^{\pm} + \int_0^t e^{(t-s)(A+\lambda)}\big(F'\big(u^x(s)\big) - \lambda\big)\eta^{h^{\pm}}(s)\,ds \\ &\leq e^{t(A+\lambda)}h^{\pm},\end{aligned} \tag{4.14}$$

so that

$$0 \leq \eta^{\pm}(t) = \rho^{\pm}(t) + \Gamma^{h^{\pm}}(t) \leq e^{t(A+\lambda)}h^{\pm} + \Gamma^{h^{\pm}}(t).$$

Therefore, since for any $q \in [2, +\infty]$ and $p \geq 0$

$$\mathbb{E} \sup_{t\in[0,T]} \big|\Gamma(t)\big|^p_{L^q(0,1)} \leq c_{p,q}(t) \int_0^t \mathbb{E}\big|\eta^h(s)\big|^p_{L^q(0,1)}\,ds, \tag{4.15}$$

we can conclude that

$$\mathbb{E} \sup_{s\in[0,t]} \big|\eta^h(s)\big|^p_H \leq c_p e^{\lambda p t}|h|^p_H + c_p(t)\int_0^t \mathbb{E} \sup_{r\in[0,s]} \big|\eta^h(r)\big|^p_H\,ds$$

and (4.12) follows from the Gronwall lemma.

In order to prove (4.13), we notice that, due to (4.14),

$$\big|\eta^h(t)\big| \leq e^{t(A+\lambda)}|h| + \big|\Gamma(t)\big|,$$

so that, in view of (2.3) and (4.15), we can conclude that

$$\begin{aligned}\mathbb{E}\big|\eta^h(t)\big|^p_{L^q(0,1)} &\leq c_p\big|e^{t(A+\lambda)}|h|\big|^p_{L^q(0,1)} + c_p\mathbb{E}\big|\Gamma(t)\big|^p_{L^q(0,1)} \\ &\leq c_{p,q}(t)t^{-(p(q-2))/4q}|h|^p_H + c_{p,q}(t)\int_0^t \mathbb{E}\big|\eta^h(s)\big|^p_{L^q(0,1)}\,ds.\end{aligned}$$

If $p(q-2)/4q < 1$, we can conclude by a comparison argument. □

5. The second derivative. Now, we investigate the second order differentiability of u_n^x with respect to $x \in E$. For any processes $z \in L^w_{p,T}(E)$ and $x \in E$, we define

$$\big[T_n(x)z\big](t) = \int_0^t e^{(t-s)A}F_n'\big(u_n^x(s)\big)z(s)\,ds + \int_0^t e^{(t-s)A}G'\big(u_n^x(s)\big)z(s)\,dw(s),$$

so that equation (4.3) can be rewritten as

$$\eta_n^h(t) = e^{tA}h + T_n(x)\eta_n^h(t).$$

Due to the boundedness of $D_\rho f_n(\xi,\cdot)$ and $D_\rho g(\xi,\cdot)$, we have that there exists $T_p = T_p(n) > 0$ such that for any $x \in E$,

$$\big\|T_n(x)\big\|_{\mathcal{L}(L^w_{p,T}(E))} \leq \tfrac{1}{2}, \qquad T \leq T_p,$$

so that

$$\eta_n^h = \big[I - T_n(x)\big]^{-1} e^{\cdot A} h. \tag{5.1}$$

Since f_n and g are twice differentiable with bounded derivatives and Lemma 4.1 holds, we have that the mapping

$$x \in E \mapsto T_n(x) z \in L^w_{p,T}(E)$$

is differentiable. Therefore, we can differentiate both sides in (5.1) with respect to $x \in E$ along the direction $k \in E$, and we obtain

$$D_x \eta_n^h k = \big[I - T_n(x)\big]^{-1} D_x\big[T_n(x)\eta_n^h\big]k,$$

so that

$$D_x \eta_n^h k - T_n(x) D_x \eta_n^h k = D_x\big[T_n(x)\eta_n^h\big]k.$$

Now it is immediate to check that for any $k \in E$,

$$\begin{aligned} D_x\big[T_n(x)z\big]k(t) &= \int_0^t e^{(t-s)A} F_n''\big(u_n^x(s)\big)\big(z(s), \eta_n^k(s)\big)\, ds \\ &\quad + \int_0^t e^{(t-s)A} G''\big(u_n^x(s)\big)\big(z(s), \eta_n^k(s)\big)\, dw(s) \end{aligned}$$

and then $\zeta_n^{h,k} := D_x \eta_n^h \cdot k = D_x^2 u_n^x(h,k)$ satisfies the equation

$$\begin{aligned} d\zeta_n^{h,k}(t) &= \big[A\zeta_n^{h,k}(t) + F_n'\big(u_n^x(t)\big)\zeta_n^{h,k}(t) + F_n''\big(u_n^x(t)\big)\big(\eta_n^h(t), \eta_n^k(t)\big)\big]\, dt \\ &\quad + \big[G'\big(u_n^x(t)\big)\zeta_n^{h,k}(t) + G''\big(u_n^x(t)\big)\big(\eta_n^h(t), \eta_n^k(t)\big)\big]\, dw(t), \\ &\qquad \zeta^{h,k}(0) = 0. \end{aligned} \tag{5.2}$$

Notice that, as the derivatives of F_n and G are bounded, thanks to (4.4) we have

$$\|\zeta_n^{h,k}\|_{C^w_{p,T}(E)} \le c_{n,p}(T)|h|_E|k|_E \tag{5.3}$$

for some continuous increasing function $c_{p,n}(T)$.

LEMMA 5.1. *Under Hypothesis 1, for any $T > 0$ and $p \ge 1$ the process u_n^x is twice differentiable in $L^w_{p,T}(E)$ with respect to $x \in E$. Moreover the second derivative $D_x^2 u_n^x(h,k) =: \zeta_n^{h,k}$ belongs to $C^w_{p,T}(E)$ and satisfies*

$$\|\zeta_n^{h,k}\|_{C^w_{p,T}(E)} \le c_p(T)\big(1 + |x|_E^{(m-1)}\big)|h|_E|k|_E \tag{5.4}$$

for some continuous increasing function $c_p(T)$ independent of $n \in \mathbb{N}$.

PROOF. We have already seen that u_n^x is twice differentiable in $L^w_{p,T}(E)$, and $D^2_x u_n^x(h,k)$ satisfies equation (5.2). Hence it only remains to prove estimate (5.4).

As we proved in Lemma 4.1, for any $x \in E$ and any $h \in L^p(\Omega; E)$ which is $\mathcal{F}_s$-measurable, the equation

$$d\eta(t) = \big[A\eta(t) + F_n'\big(u_n^x(t)\big)\eta(t)\big]\,dt + G'\big(u_n^x(t)\big)\eta(t)\,dw(t), \qquad \eta(s) = h, \tag{5.5}$$

admits a unique solution $\eta_n^h(s,\cdot) \in L^p(\Omega; C([s,T]; E))$ such that

$$\mathbb{E} \sup_{t\in[s,T]} \big|\eta_n^h(s,t)\big|_E^p \le M_p e^{\omega_p(T-s)} \mathbb{E}|h|_E^p.$$

Hence we can associate to equation (5.5) a stochastic evolution operator $\Phi_n(t,s)$ such that

$$\eta_n^h(s,t) = \Phi_n(t,s)h, \qquad h \in L^p(\Omega; E)$$

and such that

$$\mathbb{E} \sup_{r\in[s,t]} \big|\Phi_n(r,s)h\big|_E^p \le M_p e^{\omega_p(t-s)} \mathbb{E}|h|_E^p, \qquad 0 \le s \le t. \tag{5.6}$$

We claim that $\zeta_n^{h,k}$ can be represented in terms of the operator $\Phi_n(t,s)$ as

$$\zeta_n^{h,k}(t) = \Gamma_n^{h,k}(t) + \int_0^t \Phi_n(t,s)\Sigma_n^{h,k}(s)\,ds, \tag{5.7}$$

where $\Gamma_n^{h,k}$ is the solution of the problem

$$d\Gamma(t) = A\Gamma(t)\,dt + \big[G'\big(u_n^x(t)\big)\Gamma(t) + G''\big(u_n^x(t)\big)\big(\eta_n^h(t), \eta_n^k(t)\big)\big]\,dw(t), \qquad \Gamma(0) = 0 \tag{5.8}$$

and

$$\Sigma_n^{h,k}(t) = F_n'\big(u_n^x(t)\big)\Gamma_n^{h,k}(t) + F_n''\big(u_n^x(t)\big)\big(\eta_n^h(t), \eta_n^k(t)\big).$$

Clearly, in order to prove (5.7) we have to show that $\int_0^t \Phi_n(t,s)\Sigma_n^{h,k}(s)\,ds$ solves the problem

$$dz(t) = \big[Az(t) + F_n'\big(u_n^x(t)\big)z(t) + \Sigma_n^{h,k}(t)\big]\,dt + G'\big(u_n^x(t)\big)z(t)\,dw(t), \qquad z(0) = 0.$$

More generally, we have to prove that for any $\Sigma \in C^w_{p,T}(E)$ the mild solution of the problem

$$dz(t) = \big[Az(t) + F_n'\big(u_n^x(t)\big)z(t) + \Sigma(t)\big]\,dt + G'\big(u_n^x(t)\big)z(t)\,dw(t), \qquad z(0) = 0 \tag{5.9}$$

is given by

$$\hat{z}(t) := \int_0^t \Phi_n(t,s)\Sigma(s)\,ds.$$

We have

$$\begin{aligned}
&\int_0^t e^{(t-s)A} F_n'\big(u_n^x(s)\big)\hat{z}(s)\,ds \\
&\qquad = \int_0^t e^{(t-s)A} F_n'\big(u_n^x(s)\big) \int_0^s \Phi_n(s,r)\Sigma(r)\,dr\,ds \\
&\qquad = \int_0^t \int_r^t e^{(t-s)A} F_n'\big(u_n^x(s)\big)\Phi_n(s,r)\Sigma(r)\,ds\,dr
\end{aligned}$$

and analogously, by the stochastic Fubini theorem,

$$\begin{aligned}
&\int_0^t e^{(t-s)A} G'\big(u_n^x(s)\big)\hat{z}(s)\,dw(s) \\
&\qquad = \int_0^t e^{(t-s)A} G'\big(u_n^x(s)\big) \int_0^s \Phi_n(s,r)\Sigma(r)\,dr\,dw(s) \\
&\qquad = \int_0^t \int_r^t e^{(t-s)A} G_n'\big(u_n^x(s)\big)\Phi_n(s,r)\Sigma(r)\,dw(s)\,dr.
\end{aligned}$$

Now, recalling the definition of $\Phi_n(t,s)\Sigma$, we have

$$\begin{aligned}
&\int_0^t \bigg[\int_r^t e^{(t-s)A} F_n'\big(u_n^x(s)\big)\Phi_n(s,r)\Sigma(r)\,ds \\
&\qquad + \int_r^t e^{(t-s)A} G_n'\big(u_n^x(s)\big)\Phi_n(s,r)\Sigma(r)\,dw(s)\bigg]\,dr \\
&\qquad = \int_0^t \big[\Phi_n(t,r)\Sigma(r) - e^{(t-r)A}\Sigma(r)\big]\,dr \\
&\qquad = \hat{z}(t) - \int_0^t e^{(t-r)A}\Sigma(r)\,dr,
\end{aligned}$$

so that $\hat{z}$ is the mild solution of equation (5.9).

Once we have representation (5.7) for $\zeta_n^{h,k}$, we can proceed with the proof of estimate (5.4). As $\Gamma_n^{h,k}$ solves equation (5.8), we have

$$\Gamma_n^{h,k}(t) = \int_0^t e^{(t-s)A}\big[G'\big(u_n^x(s)\big)\Gamma_n^{h,k}(s) + G''\big(u_n^x(s)\big)\big(\eta_n^h(s), \eta_n^k(s)\big)\big]\,dw(s).$$

Therefore, due to the boundedness of $D_\rho g(\xi,\rho)$ and $D_\rho^2 g(\xi,\rho)$, from (4.4) and (3.3) we get

$$\mathbb{E} \sup_{s\in[0,t]} \big|\Gamma_n^{h,k}(s)\big|_E^p \le c_p(T)\int_0^t \mathbb{E}\big|\Gamma_n^{h,k}(s)\big|_E^p\,ds + c_p(T)|h|_E^p|k|_E^p,$$

so that, from the Gronwall lemma, we can conclude

$$\mathbb{E}\sup_{s\in[0,t]}\big|\Gamma_n^{h,k}(s)\big|_E^p \leq c_p(T)|h|_E^p|k|_E^p. \tag{5.10}$$

Next, as (3.6) and (4.4) hold and as the derivatives of F_n satisfy (2.17), due to (5.6) we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \Phi_n(t,s)\Sigma_n^{h,k}(s)\,ds\right|_E^p\\
&\qquad\leq c_p(T)\int_0^T \mathbb{E}\big|\Sigma_n^{h,k}(s)\big|_E^p\,ds\\
&\qquad\leq c_p(T)\big(1+|x|_E^{(m-1)p}\big)\big\|\Gamma_n^{h,k}\big\|_{C^w_{2p,T}(E)}^{1/2}+c_p(T)\big(1+|x|_E^{(m-2)p}\big)|h|_E^p|k|_E^p
\end{aligned}$$

and then, thanks to (5.10), we get

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \Phi_n(t,s)\Sigma_n^{h,k}(s)\,ds\right|_E^p \leq c_p(T)\big(1+|x|_E^{(m-1)p}\big)|h|_E^p|k|_E^p.$$

Together with (5.10), this implies (5.4). □

In view of the previous lemmas, by arguing as in the proof of Lemma 4.2, we get the following result.

LEMMA 5.2. *Under Hypothesis 1, there exists $\zeta^{h,k}\in C^w_{p,T}(E)$ such that for any $R>0$,*

$$\lim_{n\to\infty}\sup_{x,h,k\in B_R(E)}\big\|\zeta_n^{h,k}-\zeta^{h,k}\big\|_{C^w_{p,T}(E)}=0. \tag{5.11}$$

Moreover, the limit $\zeta^{h,k}$ solves the equation

$$\begin{aligned}
d\zeta(t)&=\big[A\zeta(t)+F'\big(u^x(t)\big)\zeta(t)+F''\big(u^x(t)\big)\big(\eta^h(t),\eta^k(t)\big)\big]\,dt\\
&\qquad+\big[G'\big(u^x(t)\big)\zeta(t)+G''\big(u^x(t)\big)\big(\eta^h(t),\eta^k(t)\big)\big]\,dw(t),\qquad \zeta(0)=0.
\end{aligned}$$

In particular,

$$\big\|\zeta^{h,k}\big\|_{C^w_{p,T}(E)}\leq c_p(T)\big(1+|x|_E^{m-1}\big)|h|_E|k|_E. \tag{5.12}$$

REMARK 5.3. Concerning the second order differentiability of mapping (4.10), we can adapt again the arguments used in [3], Theorem 4.2.4, to the present situation, and thanks to (4.11) we have that mapping (4.10) is twice differentiable with respect to $x\in H$, and the derivative along the directions $h,k\in H$ satisfies equation (5.2). Moreover,

$$\big\|D_x^2u_n^x(h,k)\big\|_{C^w_{p,T}(H)}\leq c_{p,n}(T)|h|_H|k|_H. \tag{5.13}$$

As a consequence of Lemmas 3.2, 4.1, 4.2, 5.1 and 5.2, we have the following fact.

THEOREM 5.4. *Under Hypothesis 1, the mapping*

$$x \in E \mapsto u^x \in L^w_{p,T}(E)$$

is differentiable, and the derivative $D_x u^u h$ along the direction $h \in E$ solves the problem

$$d\eta(t) = \big[A\eta(t) + F'\big(u^x(t)\big)\eta(t)\big]\,dt + G'\big(u^x(t)\big)\eta(t)\,dw(t), \qquad \eta(0) = h. \tag{5.14}$$

PROOF. For any $n \in \mathbb{N}$ and $x, h \in E$ we have

$$u_n^{x+h} - u_n^x = D_x u_n^x h + \int_0^1 \int_0^1 D_x^2 u_n^{x+\rho\theta h}(h,h)\,d\theta\,d\rho.$$

Then, due to (3.8), (4.6) and (5.11), we can take the limit as $n \to \infty$, and we get

$$u^{x+h} - u^x = \eta^h + \int_0^1 \int_0^1 \zeta^{h,h}\,d\theta\,d\rho.$$

The mapping

$$h \in E \mapsto \eta^h \in L^w_{p,T}(E)$$

is clearly linear and according to (4.8) is bounded. Moreover, according to (5.12) we have

$$\left\| \int_0^1 \int_0^1 \zeta^{h,h}\,d\theta\,d\rho \right\|_{L^w_{p,T}(E)} \le c_p(T)\big(1 + |x|_E^{m-1} + |h|_E^{m-1}\big)|h|_E^2$$

and then we can conclude that u^x is differentiable in $L^w_{p,T}(E)$ with respect to $x \in E$, and its derivative along the direction $h \in E$ solves problem (5.14). $\square$

In view of Lemma 4.4 and Theorem 5.4, for any $T > 0$, $p \ge 1$ and $x, y \in E$ we have

$$\mathbb{E} \sup_{t \in [0,T]} \big|u^x(t) - u^y(t)\big|_H^p \le c_p(T)|x - y|_H^p. \tag{5.15}$$

Now, if $x \in H$ and $\{x_n\}_{n\in\mathbb{N}}$ is any sequence in E, converging to x in H, due to (5.15) we have that $\{u^{x_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^w_{p,T}(H)$, and then there exists a limit $u^x \in C^w_{p,T}(H)$, only depending on x, such that

$$\|u^x\|_{C^w_{p,T}(H)} \le c_p(T)\big(1 + |x|_H\big). \tag{5.16}$$

Such a solution will be called a *generalized solution*.

THEOREM 5.5. *Under Hypothesis 1, for any $x \in H$, equation (3.1) admits a unique generalized solution $u^x \in C^w_{p,T}(H)$, for any $T > 0$ and $p \ge 1$. Moreover estimate (5.16) holds.*

6. The transition semigroup. We define the transition semigroup associated with equation (3.1) as

$$P_t\varphi(x) = \mathbb{E}\varphi\big(u^x(t)\big), \qquad x \in E, t \geq 0$$

for any $\varphi \in B_b(E)$. In view of Theorem 5.4, we have that

$$\varphi \in C_b^1(E) \quad \Longrightarrow \quad P_t\varphi \in C_b^1(E), \qquad t \geq 0 \tag{6.1}$$

and there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|P_t\varphi\|_1 \leq Me^{\omega t}\|\varphi\|_1, \qquad t \geq 0.$$

We would like to stress that, in view of Theorem 5.5, the semigroup P_t can be restricted to $C_b(H)$. Actually, for any $\varphi \in B_b(H)$ we can define

$$P_t^H\varphi(x) = \mathbb{E}\varphi\big(u^x(t)\big), \qquad t \geq 0, x \in H,$$

where $u^x(t)$ is the unique generalized solution of (3.1) in $C^w_{p,T}(H)$ introduced in Theorem 5.5. Notice that if $x \in E$ and $\varphi \in B_b(H)$, then $P_t^H\varphi(x) = P_t\varphi(x)$.

Our first purpose here is to prove that the semigroup P_t has a smoothing effect in $B_b(E)$. Namely, we want to prove that P_t maps $B_b(E)$ into $C_b^1(E)$, for any $t > 0$. For this reason, we have to assume the following condition on the multiplication coefficient g in front of the noise.

HYPOTHESIS 2. *We have*

$$\inf_{(\xi,\rho)\in[0,1]\times\mathbb{R}} \big|g(\xi,\rho)\big| =: \beta > 0. \tag{6.2}$$

First of all, we introduce the transition semigroup P_t^n associated with the approximating equation (3.5) by setting

$$P_t^n\varphi(x) = \mathbb{E}\varphi\big(u_n^x(t)\big), \qquad x \in E, t \geq 0$$

for any $\varphi \in B_b(E)$. It is important to stress that, according to Lemmas 4.1 and 5.1 and to (5.3)

$$\varphi \in C_b^k(E) \quad \Longrightarrow \quad P_t^n\varphi \in C_b^k(E), \qquad t \geq 0, k = 0, 1, 2 \tag{6.3}$$

and

$$\big\|P_t^n\varphi\big\|_k \leq Me^{\omega t}\|\varphi\|_k, \qquad t \geq 0, k = 0, 1, 2$$

for some constants $M > 0$ and $\omega \in \mathbb{R}$, which are independent of $n \in \mathbb{N}$.

Notice that, as equation (3.5) is solvable in H, we can also consider the restriction of P_t^n to $B_b(H)$. In view of what we have seen in Remarks 4.3 and 5.3, we have that

$$\varphi \in C_b^k(H) \quad \Longrightarrow \quad P_t^n\varphi \in C_b^k(H), \qquad t \geq 0, k = 1, 2 \tag{6.4}$$

and there exist constant $M_n > 0$ and $\omega_n \in \mathbb{R}$ such that

$$\|P_t^n \varphi\|_k \le M_n e^{\omega_n t} \|\varphi\|_k, \qquad t \ge 0, k = 1, 2. \tag{6.5}$$

Now, due to Hypothesis 2, for any $x, y \in H$ we can define

$$[G^{-1}(x)y](\xi) = \frac{y(\xi)}{g(\xi, x(\xi))}, \qquad \xi \in [0, 1].$$

It is immediate to check that for any $p \in [1, +\infty]$,

$$G^{-1} : H \to \mathcal{L}(L^p(0,1), L^p(0,1))$$

and

$$G^{-1}(x)G(x) = G(x)G^{-1}(x), \qquad x \in H.$$

Therefore, we can adapt the proof of [3], Proposition 4.4.3 and Theorem 4.4.5, to the present situation and we can prove that P_t^n has a smoothing effect. Namely, we have

$$\varphi \in B_b(H) \quad \Longrightarrow \quad P_t^n \varphi \in C_b^2(H), \qquad t > 0$$

and the Bismut–Elworthy–Li formula holds

$$\langle h, D(P_t^n \varphi)(x)\rangle_H = \frac{1}{t} \mathbb{E}\varphi(u_n^x(t)) \int_0^t \langle G^{-1}(u_n^x(s)) D_x u_n^x(s) h, dw(s)\rangle_H, \qquad t > 0 \tag{6.6}$$

for any $\varphi \in C_b(H)$ and $x, h \in H$.

In view of all these results, by proceeding as in the proof of [3], Theorem 6.5.1, due to what we have proved in Sections 3, 4 and 5 we obtain the following fact.

THEOREM 6.1. *Under Hypotheses 1 and 2, we have*

$$\varphi \in B_b(E) \quad \Longrightarrow \quad P_t \varphi \in C_b^1(E), \qquad t > 0$$

and

$$\langle h, D(P_t \varphi)(x)\rangle_E = \frac{1}{t} \mathbb{E}\varphi(u^x(t)) \int_0^t \langle G^{-1}(u^x(s)) D_x u^x(s) h, dw(s)\rangle_H, \qquad t > 0. \tag{6.7}$$

In particular, for any $\varphi \in B_b(E)$,

$$\sup_{x \in E} |D(P_t \varphi)(x)|_{E^\star} \le c(t \wedge 1)^{-1/2} \|\varphi\|_0, \qquad t > 0. \tag{6.8}$$

Theorem 6.1 says that if $\varphi \in B_b(E)$, then $P_t\varphi \in C_b^1(E)$, for any $t > 0$. If we could prove that in fact $P_t\varphi \in C_b^1(H)$, then we would have

$$\sum_{i=1}^{\infty} \big|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_H\big|^2 = \big|G^\star(x)D(P_t\varphi)(x)\big|_H^2 < \infty.$$

But in general we have only $P_t\varphi \in C_b^1(E)$, and it is not clear in principle whether the sum

$$\sum_{i=1}^{\infty} \big|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_E\big|^2$$

is convergent or not. The next theorem provides a positive answer to this question, which will be of crucial importance for the statement and the proof of the *egalité du carré des champs* and for its application to the Poincaré inequality.

THEOREM 6.2. *Let $\{e_i\}_{i\in\mathbb{N}}$ be the complete orthonormal basis of H defined in* (2.2). *Then, under Hypotheses* 1 *and* 2, *for any $\varphi \in C_b(E)$ and $x \in E$ we have*

$$\sum_{i=1}^{\infty} \big|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_E\big|^2 \le c\big|G(x)\big|_E^2 \|\varphi\|_0^2 (t\wedge 1)^{-1}, \qquad t > 0. \tag{6.9}$$

Moreover, if $\varphi \in C_b^1(E)$, for any $x \in E$ we have

$$\sum_{i=1}^{\infty} \big|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_E\big|^2 \le c(t) P_t\big(\big|D\varphi(\cdot)\big|_{E^\star}^2\big)(x)\big|G(x)\big|_E^2 t^{-1/2}, \qquad t > 0 \tag{6.10}$$

for some continuous increasing function. If we also assume that there exists $\gamma > 0$ such that

$$\mathbb{E}\big|\eta^h(t)\big|_E^2 \le c e^{-\gamma t}(t\wedge 1)^{-1/2}|h|_H^2, \tag{6.11}$$

then there exists $\delta > 0$ such that

$$\sum_{i=1}^{\infty} \big|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_E\big|^2 \le c e^{-\delta t} P_t\big(\big|D\varphi(\cdot)\big|_{E^\star}^2\big)(x)\big|G(x)\big|_E^2 t^{-1/2}, \qquad t > 0. \tag{6.12}$$

PROOF. Assume $\varphi \in C_b(E)$ and $x, h \in E$. According to (4.12) and (6.7), for any $t \in (0, 1]$ we have

$$\begin{aligned}
\big|\langle h, D(P_t\varphi)(x)\rangle_E\big| &= \frac{1}{t}\left|\mathbb{E}\varphi\big(u^x(t)\big)\int_0^t \big\langle G^{-1}\big(u^x(s)\big)D_x u^x(s)h, dw(s)\big\rangle_H\right| \\
&\le \frac{\|\varphi\|_0}{t}\left(\int_0^t \mathbb{E}\big|G^{-1}\big(u^x(s)\big)D_x u^x(s)h\big|_H^2\,ds\right)^{1/2} \\
&\le \frac{c\|\varphi\|_0}{t}\left(\int_0^t c(s)\,ds\right)^{1/2}|h|_H \\
&\le c\|\varphi\|_0 t^{-1/2}|h|_H.
\end{aligned}$$

Due to the semigroup law, it follows that for any $t > 0$,

$$\big|\langle G(x)h, D(P_t\varphi)(x)\rangle_E\big| \le c\|\varphi\|_0(t \wedge 1)^{-1/2}\big|G(x)\big|_E|h|_H. \tag{6.13}$$

This implies in particular that for any $t > 0$ and $x \in E$, there exists $\Lambda_\varphi(t,x) \in H$ such that

$$\langle G(x)h, D(P_t\varphi)(x)\rangle_E = \langle \Lambda_\varphi(t,x), h\rangle_H, \qquad h \in E.$$

Therefore, in view of (6.13)

$$\begin{aligned}\sum_{i=1}^{\infty}\big|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_E\big|^2 &= \sum_{i=1}^{\infty}\big|\langle \Lambda_\varphi(t,x), e_i\rangle_H\big|^2 \\ &= \big|\Lambda_\varphi(t,x)\big|_H^2 \\ &\le c\|\varphi\|_0^2(t \wedge 1)^{-1}\big|G(x)\big|_E^2\end{aligned}$$

and (6.9) holds.

Next, in order to prove (6.10), we notice that if $\varphi \in C_b^1(E)$, then

$$\langle G(x)h, D(P_t\varphi)(x)\rangle_E = \mathbb{E}\langle Du^x(t)G(x)h, D\varphi\big(u^x(t)\big)\rangle_E.$$

According to (4.13), with $p = 2$ and $q = +\infty$, for any $t > 0$, we have

$$\begin{aligned}\big|\langle G(x)h, D(P_t\varphi)(x)\rangle_E\big|^2 &\le \mathbb{E}\big|D\varphi\big(u^x(t)\big)\big|_{E^\star}^2\mathbb{E}\big|Du^x(t)G(x)h\big|_E^2 \\ &\le P_t\big(\big|D\varphi(\cdot)\big|_{E^\star}^2\big)(x)c_{2,\infty}(t)t^{-1/2}\big|G(x)\big|_E^2|h|_H^2.\end{aligned}$$

As above, this implies that for any $t > 0$ and $x \in E$ there exists $\hat{\Lambda}_\varphi(t,x) \in H$ such that

$$\langle G(x)h, D(P_t\varphi)(x)\rangle_E = \langle \hat{\Lambda}_\varphi(t,x), h\rangle_H$$

and as above we can conclude that (6.10) holds.

Finally, in order to get (6.12), we have to proceed exactly in the same way, by using (6.11) instead of (4.13). $\square$

REMARK 6.3. Condition (6.11) is satisfied if we assume that there exists $\alpha > 0$ such that

$$\sup_{(\xi,\rho)\in[0,1]\times\mathbb{R}} D_\rho f(\xi,\rho) = -\alpha$$

and if

$$\beta_g := \sup_{(\xi,\rho)\in[0,1]\times\mathbb{R}} \big|D_\rho g(\xi,\rho)\big|$$

is sufficiently small, compared to α.

Actually, by adapting the arguments used in [6], Lemma 7.1, it is possible to prove that there exists some $\bar{p} > 1$ such that for any $p \geq \bar{p}$, $0 < \delta < \alpha$ and $v \in C^w_{p,T}(E)$,

$$\sup_{s \leq t} e^{\delta p s} \mathbb{E} \left| \int_0^s e^{(s-r)(A-\alpha)} G'(u^x(r)) v(r)\, dw(r) \right|_E^p$$
$$\leq c_{1,p} \frac{\beta_g^p}{(\alpha - \delta)^{c_{2,p}}} \sup_{s \leq t} e^{\delta p s} \mathbb{E} |v(s)|_E^p$$

for two positive constants $c_{1,p}$ and $c_{2,p}$ independent of δ. This implies that if $z(t)$ solves the linear problem

$$dz(t) = (A - \alpha) z(t)\, dt + G'(u^x(t)) z(t)\, dw(t), \qquad z(0) = h,$$

then

$$\sup_{s \leq t} e^{\delta p s} \mathbb{E} |z(t)|_E^p \leq |h|_E^p + c_{1,p} \frac{\beta_g^p}{(\alpha - \delta)^{c_{2,p}}} \sup_{s \leq t} e^{\delta p s} \mathbb{E} |z(s)|_E^p.$$

Therefore, if we pick α and β_g such that

$$c_{1,p} \frac{\beta_g^p}{\alpha^{c_{2,p}}} < 1,$$

we can conclude that

$$\mathbb{E} |z(t)|_E^p \leq c_p e^{-\delta p t} |h|_E^p \leq c_p e^{-\delta p t} (t \wedge 1)^{-p/4} |h|_H^p \tag{6.14}$$

for every $\delta > 0$ small enough, so that

$$c_{1,p} \frac{\beta_g^p}{(\alpha - \delta)^{c_{2,p}}} < 1.$$

Finally, as we have $D_\rho f + \alpha \leq 0$, by using a comparison argument as in [7], Example 4.4, we can show that if $h \geq 0$, then

$$0 \leq \eta^h(t) \leq z(t), \qquad t \geq 0.$$

Therefore, by linearity, thanks to (6.14), we can conclude that (6.11) holds true.

7. Kolmogorov operator. We define the Komogorov operator $\mathcal{K}$ in $C_b(E)$ associated with P_t, by proceeding as in [2] and [3]. The operator $\mathcal{K}$ is defined through its resolvent by

$$(\lambda - \mathcal{K})^{-1} \varphi(x) = \int_0^{+\infty} e^{-\lambda t} P_t \varphi(x)\, dt, \qquad x \in E \tag{7.1}$$

for all $\lambda > 0$ and $\varphi \in C_b(E)$; see also [11].

We notice that, by Theorem 6.1, we have

$$D(\mathcal{K}) \subset C_b^1(E), \tag{7.2}$$

where $D(\mathcal{K})$ is the domain of $\mathcal{K}$. In fact, this stronger property holds.

THEOREM 7.1. *Let $\{e_i\}_{i\in\mathbb{N}}$ be the complete orthonormal basis of H defined in* (2.2). *Then, under Hypotheses* 1 *and* 2, *for any $\varphi \in D(\mathcal{K})$ and $x \in E$ we have*

$$\sum_{i=1}^{\infty} \left|\langle G(x)e_i, D\varphi(x)\rangle_E\right|^2 \le c\left|G(x)\right|_E^2 \left(\|\varphi\|_0^2 + \|\mathcal{K}\varphi\|_0^2\right). \tag{7.3}$$

PROOF. Due to the Hölder inequality, for any $\varepsilon \in (0,1)$ and $\psi \in C_b(E)$ we have

$$\begin{aligned}&\left|\langle G(x)e_i, D\big((1-\mathcal{K})^{-1}\psi\big)(x)\rangle_E\right|^2\\ &\qquad \le \int_0^\infty e^{-t}(t\wedge 1)^{-(1-\varepsilon)}\,dt \int_0^\infty e^{-t}(t\wedge 1)^{1-\varepsilon}\left|\langle G(x)e_i, D(P_t\psi)(x)\rangle_E\right|^2 dt\end{aligned}$$

and then, according to (6.9), we get

$$\sum_{i=1}^{\infty}\left|\langle G(x)e_i, D\big((1-\mathcal{K})^{-1}\psi\big)(x)\rangle_E\right|^2 \le c_\varepsilon \int_0^\infty e^{-t}(t\wedge 1)^{-\varepsilon}\,dt\left|G(x)\right|_E^2\|\psi\|_0^2.$$

Therefore, if we take $\psi = (1-\mathcal{K})\varphi$, we get (7.3). □

Our goal is to prove the following result:

THEOREM 7.2. *Assume Hypotheses* 1 *and* 2. *Then, for any $\varphi \in D(\mathcal{K})$ we have $\varphi^2 \in D(\mathcal{K})$ and the following identity holds*:

$$\mathcal{K}\varphi^2 = 2\varphi\mathcal{K}\varphi + \sum_{i=1}^{\infty}\left|\langle G(\cdot)e_i, D\varphi\rangle_E\right|^2. \tag{7.4}$$

In order to prove identity (7.4), we need suitable approximations of problem (1.1) in addition to (3.5). For any $m \in \mathbb{N}$, we denote by $u_{n,m}^x$ the unique mild solution in $C_{p,T}^w(E)$ of the problem

$$du(t) = \left[Au(t) + F_n\big(u(t)\big)\right]dt + G\big(u(t)\big)P_m\,dw(t), \qquad u(0) = x,$$

where $P_m x = \sum_{i=1}^m \langle x, e_k\rangle e_k$, $x \in H$. Moreover for any $k \in \mathbb{N}$ we denote by $u_{n,m,k}^x$ the unique solution in $C_{p,T}^w(E)$ of the problem

$$du(t) = \left[A_k u(t) + F_n\big(u(t)\big)\right]dt + G\big(u(t)\big)P_m\,dw(t), \qquad u(0) = x, \tag{7.5}$$

where $A_k = kA(k-A)^{-1}$ are the Yosida approximations of A. The following result is straightforward.

LEMMA 7.3. *Under Hypotheses* 1 *and* 2, *for any $x \in E$ and $T > 0$ we have*

$$\lim_{m\to\infty}\left|u_{n,m}^x(t) - u_n^x(t)\right|_E = 0 \qquad \textit{uniformly on } [0,T].$$

Moreover for any $x \in E$, $m \in \mathbb{N}$ and $T > 0$ we have

$$\lim_{k\to\infty}\left|u_{n,m,k}^x(t) - u_{n,m}^x(t)\right|_E = 0 \qquad \textit{uniformly on } [0,T].$$

Let us introduce the approximating Kolmogorov operators. If $\varphi \in C_b(E)$ and $\lambda > 0$, they are defined as above throughout their resolvents

$$(\lambda - \mathcal{K}_n)^{-1}\varphi(x) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}\varphi\big(u_n^x(t)\big)\,dt,$$

$$(\lambda - \mathcal{K}_{n,m})^{-1}\varphi(x) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}\varphi\big(u_{n,m}^x(t)\big)\,dt$$

and

$$(\lambda - \mathcal{K}_{n,m,k})^{-1}\varphi(x) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}\varphi\big(u_{n,m,k}^x(t)\big)\,dt$$

for any $\varphi \in C_b(E)$ and $x \in E$. From Lemmas 3.2 and 7.3, we get the following approximation results.

LEMMA 7.4. *Assume Hypotheses 1 and 2. Then, for any $\lambda > 0$ and $x \in E$, we have*

$$\lim_{n\to\infty}\big|(\lambda - \mathcal{K}_n)^{-1}\varphi(x) - (\lambda - \mathcal{K})^{-1}\varphi(x)\big|_E = 0.$$

If moreover $m \in \mathbb{N}$,

$$\lim_{m\to\infty}\big|(\lambda - \mathcal{K}_{n,m})^{-1}\varphi(x) - (\lambda - \mathcal{K}_n)^{-1}\varphi(x)\big|_E = 0.$$

If finally $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty}\big|(\lambda - \mathcal{K}_{n,m,k})^{-1}\varphi(x) - (\lambda - \mathcal{K}_{n,m})^{-1}\varphi(x)\big|_E = 0.$$

LEMMA 7.5. *Assume Hypotheses 1 and 2. Then, for any $n, m, k \in \mathbb{N}$ we have $C_b^2(E) \subset D(\mathcal{K}_{n,m,k})$, and for any $\varphi \in C_b^2(E)$ we have*

$$\mathcal{K}_{n,m,k}\varphi^2 = 2\varphi\mathcal{K}_{n,m,k}\varphi + \sum_{i=1}^m\big|\langle G(\cdot)e_i, D\varphi\rangle_E\big|^2. \tag{7.6}$$

PROOF. Since the stochastic equation (7.5) has regular coefficients and a finite-dimensional noise term, the conclusion follows from Itô's formula in the Banach space E; see Appendix. $\square$

COROLLARY 7.6. *Let $\varphi_{n,m,k} = (\lambda - \mathcal{K}_{n,m,k})^{-1}\psi$, for $n, m, k \in \mathbb{N}$ and $\psi \in C_b^2(E)$. Then, under Hypotheses 1 and 2, the following identity holds:*

$$\varphi_{n,m,k}^2 = (2\lambda - \mathcal{K}_{n,m,k})^{-1}\left(2\varphi_{n,m,k}\psi + \sum_{i=1}^m\big|\langle G(\cdot)e_i, D\varphi_{n,m,k}\rangle_E\big|^2\right). \tag{7.7}$$

PROOF. As

$$\lambda\varphi_{n,m,k} - \mathcal{K}_{n,m,k}\varphi_{n,m,k} = \psi, \tag{7.8}$$

since $\psi \in C_b^2(E)$ we have $\varphi_{n,m,k} \in C_b^2(E)$. Now, multiplying (7.8) by $\varphi_{n,m,k}$ and taking into account (7.6), we get

$$\lambda\varphi_{n,m,k}^2 - \frac{1}{2}\mathcal{K}_{n,m,k}(\varphi_{n,m,k}^2) - \frac{1}{2}\sum_{i=1}^{m}\left|\langle G(\cdot)e_i, D\varphi_{n,m,k}\rangle_E\right|^2 = \psi\varphi_{n,m,k}$$

and the conclusion follows. □

LEMMA 7.7. *Let $\varphi = (\lambda - \mathcal{K})^{-1}\psi$, for $\psi \in C_b^2(E)$ and $\lambda > 0$. Then, under Hypotheses 1 and 2, the following identity holds*:

$$\varphi^2 = (2\lambda - \mathcal{K})^{-1}\left(2\varphi\psi + \sum_{i=1}^{\infty}\left|\langle G(\cdot)e_i, D\varphi\rangle_E\right|^2\right). \tag{7.9}$$

Consequently, $\varphi^2 \in D(\mathcal{K})$ and (7.4) holds.

PROOF. The conclusion follows from Theorem 7.1, Lemma 7.4 and Corollary 7.6, by letting $n, m, k \to \infty$. □

We are now in a position to prove Theorem 7.2.

PROOF OF THEOREM 7.2. Let $\varphi \in D(\mathcal{K})$, $\lambda > 0$ and $\psi = \lambda\varphi - \mathcal{K}\varphi$. If we assume that $\psi \in C_b^2(E)$, then, due to Lemma 7.7, we know that (7.9) holds. Now assume $\psi \in C_b(E)$. It is well known that we cannot find a uniform approximation of ψ because $C_b^2(E)$ is not dense in $C_b(E)$. Thus we define

$$R_t\psi(x) = \int_H \psi(e^{tA}x + y)N_{Q_t}(dy),$$

where N_{Q_t} is the Gaussian measure in H with mean 0 and covariance $Q_t = -\frac{1}{2}A^{-1}(1 - e^{2tA})$ for $t \geq 0$. As N_{Q_t} is the law of the solution of the linear equation

$$du(t) = Au(t)\,dt + dw(t), \qquad u(0) = 0,$$

which takes values in E and $e^{tA}x \in E$, for any $x \in H$ and $t > 0$, we have that $R_t\psi \in B_b(H)$. Moreover, as proved in [9], we have that for each $t > 0$, $R_t\psi$ belongs to $C_b^\infty(H)$ and consequently to $C_b^\infty(E)$.

Now let $\varphi_t = (\lambda - \mathcal{K})^{-1}R_t\psi$. Since $R_t\psi \in C_b^2(E)$, we have by (7.9)

$$\varphi_t^2 = (2\lambda - \mathcal{K})^{-1}\left(2\varphi_t R_t\psi + \sum_{i=1}^{\infty}\left|\langle G(\cdot)e_i, D\varphi_t\rangle_E\right|^2\right). \tag{7.10}$$

Therefore, the conclusion follows letting $t \to 0$. Actually, if for any $x \in E$ we have

$$\lim_{t\to 0} h_t(x) = h(x), \qquad \sup_{t\in[0,1]} \sup_{x\in E} |h_t(x)| < \infty,$$

then it is immediate to check that

$$\lim_{t\to 0} (\lambda - \mathcal{K})^{-1} h_t(x) = (\lambda - \mathcal{K})^{-1} h(x), \qquad x \in E.$$

Therefore, as for any $x \in E$

$$\lim_{t\to 0} \varphi_t^2(x) = \varphi^2(x), \qquad \lim_{t\to 0} \varphi_t(x) R_t \psi(x) = \varphi(x)\psi(x),$$

we get (7.9) by taking the limit as $t \downarrow 0$ in both sides of (7.10) if we show that

$$\lim_{t\to 0} \sum_{i=1}^{\infty} \left| \langle G(x)e_i, D\varphi_t(x)\rangle_E \right|^2 = \sum_{i=1}^{\infty} \left| \langle G(x)e_i, D\varphi(x)\rangle_E \right|^2. \tag{7.11}$$

Thus, in order to complete the proof of Theorem 7.2, it remains to prove (7.11). Since

$$\lim_{t\to 0} R_t \psi(x) = \psi(x), \qquad \|R_t \psi\|_0 \le \|\psi\|_0,$$

according to (6.7) we have

$$\lim_{t\to 0} \langle G(x)e_i, D\varphi_t(x)\rangle_E = \langle G(x)e_i, D\varphi(x)\rangle_E \tag{7.12}$$

for any $i \in \mathbb{N}$. By proceeding as in the proof of Theorem 6.2, we see that for any $\varphi \in C_b(E)$, $x \in E$ and $t > 0$, there exists $\Lambda(t, x) \in H$ such that

$$\frac{1}{t} \mathbb{E} \int_0^t \langle G^{-1}(u^x(s)) Du^x(s) G(x)h, dw(s)\rangle_H = \langle \Lambda(t, x), h\rangle_H$$

and

$$|\Lambda(t, x)|_H \le c t^{-1/2} |G(x)|_E. \tag{7.13}$$

By arguing as in the proof of Theorem 7.1, with $\varepsilon = 1/2$, this implies that

$$\begin{aligned}
\left| \langle G(x)e_i, D\varphi_t(x)\rangle_E \right|^2 &\le c \int_0^\infty e^{-\lambda s} (s \wedge 1)^{1/2} \left| \langle G(x)e_i, D(P_s(R_t\psi))(x)\rangle_E \right|^2 ds \\
&\le c \|R_t \varphi\|_0 \int_0^\infty e^{-\lambda s} (s \wedge 1)^{1/2} \left| \langle \Lambda(s, x), e_i\rangle_H \right|^2 ds \\
&\le c \|\varphi\|_0 \int_0^\infty e^{-\lambda s} (s \wedge 1)^{1/2} \left| \langle \Lambda(s, x), e_i\rangle_H \right|^2 ds.
\end{aligned}$$

Therefore, due to (7.13),

$$\sum_{i=1}^{\infty} \int_0^\infty e^{-\lambda s} (s \wedge 1)^{1/2} \left| \langle \Lambda(s, x), e_i\rangle_H \right|^2 ds < \infty$$

and (7.12) holds. From Fatou's lemma we get (7.11), and (7.4) follows for a general $\varphi \in D(\mathcal{K})$. $\square$

8. Invariant measures. In [4] it has been proved that there exists an invariant probability measure μ on $(E, \mathcal{B}(E))$ for the semigroup P_t. In particular, if $\varphi \in D(\mathcal{K})$, we have

$$\int_E \mathcal{K}\varphi \, d\mu = 0. \tag{8.1}$$

From now on we shall assume that the following condition is satisfied:

HYPOTHESIS 3. *There exists* $\alpha > 0$ *such that*

$$\sup_{(\xi,\rho)\in[0,1]\times\mathbb{R}} D_\rho f(\xi, \rho) = -\alpha$$

and $g(\xi, \rho)$ *is uniformly bounded on* $[0, 1] \times \mathbb{R}$.

In [7], Proposition 4.1, we have proved that under Hypothesis 3 there exists $\delta > 0$ such that for any $p \ge 1$ and $x \in E$

$$\mathbb{E}|u^x(t)|_E^p \le c_p\big(1 + e^{-\delta p t}|x|_E^p\big), \qquad t \ge 0. \tag{8.2}$$

As a consequence of this, we have that for any $p \ge 1$,

$$\int_E |x|_E^p \mu(dx) < \infty. \tag{8.3}$$

Actually, due to the invariance of μ, for any $t \ge 0$ it holds

$$\int_E |x|_E^p \mu(dx) = \int_E \mathbb{E}|u^x(t)|_E^p \mu(dx) \le c_p\left(1 + e^{-\delta p t}\int_E |x|_E^p \mu(dx)\right).$$

Therefore, if we choose t_0 such that $c_p e^{-\delta p t_0} < 1/2$, we have that (8.3) follows.

REMARK 8.1. In order to have (8.2) it is not necessary to assume that g is uniformly bounded. Actually, (2.6) is what we need to prove (8.2). In Hypothesis 3 we are assuming that g is bounded in view of the proof of the Poincaré inequality, where we need an estimate, that is, uniform with respect to $x \in E$.

Now, as μ is invariant, it is well known that P_t can be uniquely extended to a semigroup of contractions on $L^2(E, \mu)$ which we shall still denote by P_t, whereas we shall denote by $\mathcal{K}_2$ its infinitesimal generator.

LEMMA 8.2. *Assume Hypotheses* 1, 2 *and* 3. *Then*, $D(\mathcal{K})$ *is a core for* $\mathcal{K}_2$.

PROOF. Let $\psi := \lambda\varphi - \mathcal{K}_2\varphi$, for $\varphi \in D(\mathcal{K}_2)$ and $\lambda > 0$. Since $C_b(E)$ is dense in $L^2(E, \mu)$, there exists a sequence $(\psi_n) \subset C_b(E)$ convergent to ψ in $L^2(E, \mu)$. If we set $\varphi_n := (\lambda - \mathcal{K}_2)^{-1}\psi_n$, then $\varphi_n \in D(\mathcal{K})$ and

$$\varphi_n \to \varphi, \qquad \mathcal{K}_2\varphi_n \to \mathcal{K}\varphi \qquad \text{in } L^2(E, \mu),$$

which shows that $D(\mathcal{K})$ is a core for $\mathcal{K}_2$. $\square$

8.1. *Consequences of the "egalité du carré des champs".* Our first result is the so called *egalité du carré des champs*; see [1].

PROPOSITION 8.3. *Assume that Hypotheses 1, 2 and 3 hold. Then for any $\varphi \in D(\mathcal{K})$ we have*

$$\int_E \mathcal{K}\varphi(x)\varphi(x)\,d\mu(x) = -\frac{1}{2}\int_E \sum_{i=1}^{\infty}\left|\langle G(x)e_i, D\varphi(x)\rangle_E\right|^2 d\mu(x). \tag{8.4}$$

PROOF. Let $\varphi \in D(\mathcal{K})$. Then, by Theorem 7.2, $\varphi^2 \in D(\mathcal{K})$, and identity (7.4) holds. According to (7.3) and (8.3), we can integrate both sides of (7.4) with respect to μ and taking into account that, in view of (8.1), $\int_E \mathcal{K}(\varphi^2)\,d\mu = 0$, and we get the conclusion. □

Let us show a similar identity for the semigroup P_t.

PROPOSITION 8.4. *Let $\varphi \in C_b^1(E)$, and set $v(t,x) = P_t\varphi(x)$. Then, under Hypotheses 1, 2 and 3, we have*

$$v \in L^\infty\big(0,T;L^2(E,\mu)\big), \qquad \sum_{i=1}^{\infty}\left|\langle G(\cdot)e_i, D_x v\rangle_E\right|^2 \in L^1\big(0,T;L^1(E,\mu)\big)$$

for any $T>0$. Moreover

$$\begin{aligned}\int_E (P_t\varphi)^2\mu(dx) + \int_0^t ds \int_E \sum_{i=1}^{\infty}\left|\langle G(x)e_i, D(P_s\varphi)(x)\rangle_E\right|^2\mu(dx)\\ = \int_H \varphi^2(x)\mu(dx).\end{aligned} \tag{8.5}$$

PROOF. If we assume that $\varphi \in D(\mathcal{K})$, we have $P_t\varphi \in D(\mathcal{K})$ and $\mathcal{K}P_t\varphi = P_t\mathcal{K}\varphi$; for a proof see [3], Lemma B.2.1. According to (7.3), this yields

$$\begin{aligned}&\sum_{i=1}^{\infty}\left|\langle G(x)e_i, D_x v(t,x)\rangle_E\right|^2\\ &\qquad\le c\left|G(x)\right|_E^2\big(\|P_t\varphi\|_0^2 + \|\mathcal{K}P_t\varphi\|_0^2\big)\\ &\qquad\le c\left|G(x)\right|_E^2\big(\|\varphi\|_0^2 + \|\mathcal{K}\varphi\|_0^2\big),\end{aligned}$$

so that

$$\sum_{i=1}^{\infty}\left|\langle G(\cdot)e_i, D_x v(t,\cdot)\rangle_E\right|^2 \in L^1\big(0,T;L^1(E,\mu)\big) \tag{8.6}$$

for any $T > 0$. Now, as $D_t v(t,x) = \mathcal{K}v(t,x)$ (see [3], Proposition B.2.2), multiplying both sides by $v(t,x)$ and integrating over E with respect to μ, due to (8.4) we get

$$\frac{1}{2}\frac{d}{dt}\int_E v^2(t,x)\mu(dx) = \int_E \mathcal{K}v(t,x)v(t,x)\mu(dx)$$
$$= -\frac{1}{2}\int_E \sum_{i=1}^{\infty} \left|\langle G(x)e_i, D_x v(t,x)\rangle_E\right|^2 \mu(dx).$$

Thus, integrating with respect to t, (8.5) follows when $\varphi \in D(\mathcal{K})$.

Now, assume $\varphi \in C_b^1(E)$. Clearly, the mapping $(t,x) \mapsto P_t\varphi(x)$ is in $L^\infty(0,T; L^2(E,\mu))$. Moreover, according to (6.10),

$$\begin{aligned}\sum_{i=1}^{\infty} \left|\langle G(x)e_i, D(P_t\varphi)(x)\rangle_E\right|^2 &\le c(t)P_t\left(\left|D\varphi(\cdot)\right|^2_{E^\star}\right)(x)\left|G(x)\right|_E^2 t^{-1/2} \\ &\le c(t)\sup_{x\in E}\left|D\varphi(x)\right|_{E^\star}\left|G(x)\right|_E^2 t^{-1/2}\end{aligned} \tag{8.7}$$

and then (8.6) holds. Next, for any $n \in \mathbb{N}$ we define $\varphi_n := n(n-\mathcal{K})^{-1}\varphi$. Clearly, $\varphi_n \in D(\mathcal{K})$, and for $x \in E$

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \qquad \|\varphi_n\|_0 \le \|\varphi\|_0, n \in \mathbb{N}. \tag{8.8}$$

Moreover, thanks to (4.8), we have

$$\begin{aligned}&\lim_{n\to\infty}\left|D\varphi_n(x) - D\varphi(x)\right|_{E^\star} = 0, \\ &\sup_{x\in E}\left|D\varphi_n(x)\right|_{E^\star} \le \sup_{x\in E}\left|D\varphi(x)\right|_{E^\star}, \qquad n \in \mathbb{N}.\end{aligned} \tag{8.9}$$

As (8.5) holds for $\varphi \in D(\mathcal{K})$, if we set $v_n(t,x) = P_t\varphi_n(x)$, we have for each $n \in \mathbb{N}$

$$\int_E v_n^2(t,x)\mu(dx) + \int_0^t ds \int_E \sum_{i=1}^{\infty} \left|\langle G(x)e_i, D_x v_n(s,x)\rangle_E\right|^2 \mu(dx)$$
$$= \int_H \varphi_n^2(x)\mu(dx).$$

Due to (8.7), (8.8) and (8.9), by arguing as in the proof of Lemma 7.7, we can take the limit in both sides above, as $n \to \infty$, and we get (8.5) for $\varphi \in C_b^1(E)$. $\square$

8.2. *The Sobolev space* $W^{1,2}(E,\mu)$. We are going to show that the derivative operator D is closable in $L^2(E,\mu)$ so that we can introduce the Sobolev space $W^{1,2}(E,\mu)$.

PROPOSITION 8.5. *Assume Hypotheses 1, 2 and 3. Then the derivative operator*

$$D : C_b^1(E) \to L^2(E, \mu; E^\star), \qquad \varphi \mapsto D\varphi$$

is closable in $L(E, \mu)$.

PROOF. Let $(\varphi_n) \subset C_b^1(E)$ such that

$$\varphi_n \to 0 \qquad \text{in } L^2(E, \mu),$$
$$D\varphi_n \to F \qquad \text{in } L^2(E, \mu; E^\star).$$

We have to show that $F = 0$. We first prove that for any $t > 0$, we have

$$\lim_{n\to\infty} D(P_t\varphi_n)(x) = \mathbb{E}\big[\big(Du^x(t)\big)^\star F\big(u^x(t)\big)\big] \qquad \text{in } L^2(E, \mu; E^\star). \tag{8.10}$$

In fact, recalling Theorem 5.4 and (4.8), we have

$$\begin{aligned}
&\int_E \big|D(P_t\varphi_n)(x) - \mathbb{E}\big(Du^x(t)\big)^\star F\big(u^x(t)\big)\big|_{E^\star}^2 \mu(dx) \\
&\quad = \int_E \big|\mathbb{E}Du^x(t)^*\big(D\varphi_n\big(u^x(t)\big) - F\big(u^x(t)\big)\big)\big|_{E^\star}^2 \mu(dx) \\
&\quad \le Me^{\omega t} \int_E \mathbb{E}\big|D\varphi_n\big(u^x(t)\big) - F\big(u^x(t)\big)\big|_{E^\star}^2 \mu(dx) \\
&\quad = Me^{\omega t} \int_E \big|D\varphi_n(x) - F(x)\big|_{E^\star}^2 \mu(dx),
\end{aligned}$$

the last inequality following from the invariance of μ. This implies (8.10).

Now, according to (8.5) we have

$$\begin{aligned}
&\int_E (P_t\varphi_n)^2 \mu(dx) + \int_0^t ds \int_E \sum_{i=1}^\infty \big|\langle G(x)e_i, D(P_s\varphi_n)(x)\rangle_E\big|^2 \mu(dx) \\
&\quad = \int_H \varphi_n^2(x) \mu(dx).
\end{aligned}$$

Then we can take the limit as $n \to \infty$ on both sides, and we get

$$\lim_{n\to\infty} \int_0^t ds \int_E \sum_{i=1}^\infty \big|\langle G(x)e_i, D(P_s\varphi_n)(x)\rangle_E\big|^2 \mu(dx) = 0.$$

Due to (8.10), this implies that for any $i \in \mathbb{N}$,

$$\mathbb{E}\big\langle Du^x(t)G(x)e_i, F\big(u^x(t)\big)\big\rangle_E = 0,$$

so that

$$
\begin{aligned}
P_t\big(\langle G(x)e_i, F(x)\rangle_E\big) &= \mathbb{E}\langle G\big(u^x(t)\big)e_i, F\big(u^x(t)\big)\rangle_E \\
&= \mathbb{E}\langle Du^x(t)G(x)e_i, F\big(u^x(t)\big)\rangle_E + \mathbb{E}\langle G(x)e_i - Du^x(t)G(x)e_i, F\big(u^x(t)\big)\rangle_E \\
&\quad + \mathbb{E}\langle\big(G\big(u^x(t)\big) - G(x)\big)e_i, F\big(u^x(t)\big)\rangle_E \\
&= \mathbb{E}\langle G(x)e_i - Du^x(t)G(x)e_i, F\big(u^x(t)\big)\rangle_E \\
&\quad + \mathbb{E}\langle\big(G\big(u^x(t)\big) - G(x)\big)e_i, F\big(u^x(t)\big)\rangle_E.
\end{aligned}
$$

Consequently, due to the continuity at $t = 0$ of $u^x(t)$ and $Du^x(t)$, we get

$$\lim_{t\to 0} P_t\big(\langle G(\cdot)e_i, F\rangle_E\big) = 0 \qquad \text{in } L^1(E, \mu).$$

Since P_t is a strongly continuous semigroup in $L^1(E, \mu)$, we deduce $\langle G(\cdot)e_i, F\rangle_E = 0$ for all $i \in \mathbb{N}$. As $G(x)$ is invertible and by Fejer's theorem for any $h \in E$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i\le n} \sum_{j\le i} \langle h, e_j\rangle_H e_j = h \qquad \text{in } E, \tag{8.11}$$

which implies $\langle h, F(x)\rangle_E = 0$, for any $x, h \in E$, and then $F = 0$. $\square$

Since D is closable in $L^2(E, \mu)$, we define as usual the Sobolev space $W^{1,2}(E, \mu)$ as the domain of the closure of D endowed with its graph norm. Notice that if $\{\varphi_n\} \subset C_b^1(E)$ approximates some $\varphi \in W^{1,2}(E, \mu)$ in the graph norm of D, then, according to (8.7), the series

$$\int_0^t ds \int_E \sum_{i=1}^{\infty} \big|\langle G(x)e_i, D(P_s\varphi_n)(x)\rangle_E\big|^2 \, d\mu(x)$$

converges uniformly with respect to $n \in \mathbb{N}$, so that (8.5) holds for any $\varphi \in W^{1,2}(E, \mu)$.

PROPOSITION 8.6. *Under Hypotheses 1, 2 and 3, for any $\varphi \in D(\mathcal{K}_2)$, we have*

$$\int_E \mathcal{K}_2(\varphi)(x)\varphi(x)\, d\mu(x) = -\frac{1}{2} \int_E \sum_{i=1}^{\infty} \big|\langle G(x)e_i, D\varphi(x)\rangle_E\big|^2 \, d\mu(x). \tag{8.12}$$

PROOF. The proof follows from Lemma 8.2 and Proposition 8.5. $\square$

8.3. *The Poincaré inequality.* In what follows we shall assume the following condition:

HYPOTHESIS 4. *There exists $\gamma > 0$ such that*

$$\mathbb{E}\big|\eta^h(t)\big|_E^2 \le c e^{-\gamma t} (t \wedge 1)^{-1/2} |h|_H^2. \tag{8.13}$$

In Remark 6.3 we discussed in detail cases when condition (8.13) holds. Actually, we have seen that if $F' \leq -\alpha$, for some $\alpha > 0$, as stated in Hypothesis 3, then (8.13) holds if $\|G'\|_{\mathcal{L}(E)}$ is sufficiently small compared to α; see also [7] and [6].

As a consequence of Hypothesis 4, we have that there exists some $\theta > 0$ such that

$$|DP_t\varphi(x)|_E \leq e^{-\theta t} \sup_{x\in E} |D\varphi(x)|_{E^\star}. \tag{8.14}$$

By a standard argument this implies that for any $x \in E$,

$$\lim_{t\to\infty} P_t\varphi(x) = \bar{\varphi} = \int_E \varphi \, d\mu. \tag{8.15}$$

PROPOSITION 8.7. *Under Hypotheses 1–4, there exist $\rho > 0$ such that for all $\varphi \in W^{1,2}(E,\mu)$,*

$$\int_E |\varphi(x) - \bar{\varphi}|^2 \, d\mu(x) \leq \rho \int_E |D\varphi(x)|^2_{E^\star} \, d\mu(x). \tag{8.16}$$

PROOF. We start from (8.5) for $\varphi \in W^{1,2}(E,\mu)$,

$$\begin{aligned} &\int_E (P_t\varphi)^2(x)\mu(dx) + \int_0^t ds \int_E \sum_{i=1}^\infty |\langle G(x)e_i, D(P_s\varphi)(x)\rangle_E|^2 \mu(dx) \\ &\qquad = \int_H \varphi^2(x)\mu(dx). \end{aligned}$$

Taking into account (6.12), this yields

$$\begin{aligned} &\int_E (P_t\varphi)^2 \mu(dx) + c\int_0^t ds\, e^{-\delta s} s^{-1/2} \int_E P_s(|D\varphi(\cdot)|^2_{E^\star})(x)\mu(dx) \\ &\qquad \geq \int_H \varphi^2(x)\mu(dx), \end{aligned}$$

which, by the invariance of μ, yields

$$\int_E (P_t\varphi)^2 \mu(dx) + c\int_0^t ds\, e^{-\delta s} s^{-1/2} \int_E |D\varphi(x)|^2_{E^\star}\mu(dx) \geq \int_H \varphi^2(x)\mu(dx).$$

Letting $t \to \infty$, and recalling (8.15), this implies that for some $\rho > 0$,

$$(\bar{\varphi})^2 + \rho \int_E |D\varphi(x)|^2_{E^\star}\mu(dx) \geq \int_H \varphi^2(x)\mu(dx),$$

which is equivalent to (8.16). $\square$

8.4. *Spectral gap and convergence to equilibrium.*

PROPOSITION 8.8. *Under Hypotheses 1–4, we have*

$$\sigma(\mathcal{K}_2) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \mathfrak{Re}\lambda \leq -\beta^2/\rho\},$$

where $\sigma(\mathcal{K}_2)$ denotes the spectrum of $\mathcal{K}_2$.

PROOF. Let us consider the space of all mean zero functions from $L^2(E,\mu)$

$$L^2_\pi(E,\mu) := \{\varphi \in L^2(E,\mu) : \bar{\varphi} = 0\}.$$

Clearly

$$L^2(E,\mu) = L^2_\pi(E,\mu) \oplus \mathbb{R}.$$

Moreover if $\bar{\varphi} = 0$, we have by the invariance of μ

$$\overline{(P_t\varphi)} = \int_H P_t\varphi(x)\,d\mu(x) = \int_H \varphi(x)\,d\mu(x) = 0,$$

so that $L^2_\pi(E,\mu)$ is an invariant subspace of P_t.

Denote by $\mathcal{K}_\pi$ the restriction of $\mathcal{K}_2$ to $L^2_\pi(E,\mu)$. Then we have clearly

$$\sigma(\mathcal{K}_2) = \{0\} \cup \sigma(\mathcal{K}_\pi).$$

Moreover, if $\varphi \in L^2_\pi(E,\mu)$ we see, using (8.4), that

$$\begin{aligned}\int_E \mathcal{K}_\pi\varphi(x)\varphi(x)\,d\mu(x) &= \int_E \mathcal{K}_2\varphi(x)\varphi(x)\,d\mu(x)\\ &= -\frac{1}{2}\int_E \sum_{i=1}^\infty \left|\langle G(x)e_i, D\varphi(x)\rangle_E\right|^2 d\mu(x).\end{aligned} \tag{8.17}$$

Now, due to (8.11), for any $x, h \in E$ we have

$$\begin{aligned}\left|\langle G(x)h, D\varphi(x)\rangle_E\right|^2 &= \left(\sum_{i=1}^\infty \langle G(x)e_i, D\varphi(x)\rangle_E \langle h, e_i\rangle_H\right)^2\\ &\leq \sum_{i=1}^\infty |\langle h, e_i\rangle_H|^2 \sum_{i=1}^\infty \left|\langle G(x)e_i, D\varphi(x)\rangle_E\right|^2,\end{aligned}$$

so that, as $|h|_E \leq 1$ implies $|h|_H \leq 1$,

$$\left|G^\star(x)D\varphi(x)\right|^2_{E^\star} \leq \sum_{i=1}^\infty \left|\langle G(x)e_i, D\varphi(x)\rangle_E\right|^2.$$

Due to Hypothesis 2, according to (8.17) this yields

$$\int_E \mathcal{K}_\pi\varphi\varphi\,d\mu \leq -\frac{\beta^2}{2}\int_E \left|D\varphi(x)\right|^2_{E^\star} d\mu(x)$$

and by Poincaré's inequality, we deduce

$$\begin{aligned}\int_E \mathcal{K}_\pi \varphi(x)\varphi(x)\,d\mu(x) &\le -\frac{\beta^2}{2}\int_E \big|D\varphi(x)\big|^2_{E^\star}\,d\mu(x)\\ &\le -\frac{\beta^2}{2\rho}\int_E \varphi^2(x)\,d\mu(x),\end{aligned}\tag{8.18}$$

which yields by the Hille–Yosida theorem

$$\sigma(\mathcal{K}_\pi) \subset \{\lambda \in \mathcal{C} : \mathfrak{Re}\lambda \le -\beta^2/2\rho\}. \qquad \square$$

REMARK 8.9. The spectral gap implies the exponential convergence of $P_t\varphi$ to $\bar\varphi$. In fact from

$$\int_E \mathcal{K}_\pi \varphi(x)\varphi(x)\,d\mu(x) \le -\frac{\beta^2}{2\rho}\int_E \varphi^2(x)\,d\mu(x),$$

we deduce that $\mathcal{K}_\pi + \beta^2/2\rho I$ is m-dissipative, so that by the Hille–Yosida theorem we have

$$\int_E \big|P_t\psi(x)\big|^2\,d\mu(x) \le e^{-\beta^2/\rho t}\int_E \big|\psi(x)\big|^2\,d\mu(x), \qquad \psi \in L^2_\pi(E,\mu).\tag{8.19}$$

Now given $\varphi \in L^2(E,\mu)$, setting in (8.19) $\psi := \varphi - \bar\varphi$, we get

$$\begin{aligned}\int_E \big|P_t\varphi(x) - \bar\varphi\big|^2\,d\mu(x) &\le e^{-\beta^2/\rho t}\int_E \big|\varphi(x) - \bar\varphi\big|^2\,d\mu(x)\\ &= e^{-\beta^2/\rho t}\left(\int_H \varphi^2(x)\,d\mu(x) - \bar\varphi^2\right)\\ &\le e^{-\beta^2/\rho t}\int_E \big|\varphi(x)\big|^2\,d\mu(x).\end{aligned}$$

APPENDIX: AN ITÔ FORMULA IN THE SPACE OF CONTINUOUS FUNCTIONS

Fix $k \in \mathbb{N}$, and let $b, \sigma_1, \dots, \sigma_k$ be mappings from H into H and from E into E, which are Lipschitz continuous both in H and in E. Let X be the solution to the stochastic differential equation

$$X(t) = x + \int_0^t b\big(X(s)\big)\,ds + \sum_{i=1}^k \int_0^t \sigma_i\big(X(s)\big)\,d\beta_i(s),\tag{A.1}$$

where $\beta_1, \dots, \beta_n$ are independent real Brownian motions.

If $\varphi \in C_b^2(H)$, then it is well known that the following Itô's formula holds:

$$\mathbb{E}\varphi\big(X(t)\big) = \varphi(x) + \mathbb{E}\int_0^t \mathcal{L}\varphi\big(X(s)\big)\,ds, \tag{A.2}$$

where $\mathcal{L}$ is the Kolmogorov operator given by

$$\mathcal{L}\varphi(x) = \frac{1}{2}\sum_{i=1}^k \langle D^2\varphi(x)\sigma_i(x), \sigma_i(x)\rangle_H + \langle D\varphi(x), b(x)\rangle_H, \qquad x \in H. \tag{A.3}$$

Now we see what happens when dealing with (A.2) for functions defined in E.

PROPOSITION A.1. *If $\varphi \in C_b^2(E)$, then it holds*

$$\mathbb{E}\varphi\big(X(t)\big) = \varphi(x) + \mathbb{E}\int_0^t \mathcal{L}_E\varphi\big(X(s)\big)\,ds, \tag{A.4}$$

where $\mathcal{L}_E$ is given by

$$\mathcal{L}_E\varphi(x) = \frac{1}{2}\sum_{i=1}^k \langle \sigma_i(x), D^2\varphi(x)\sigma_i(x)\rangle_E + \langle b(x), D\varphi(x)\rangle_E \tag{A.5}$$

and D_E represents the Frèchet derivative in E.

PROOF. In view of Lemma 2.1, if $\varphi \in C_b^2(E)$, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_b^2(H)$ such that

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \qquad x \in E,$$

$$\lim_{n\to\infty} \langle y, D\varphi_n(x)\rangle_H = \langle y, D\varphi(x)\rangle_E, \qquad x, y \in E,$$

$$\lim_{n\to\infty} \langle y, D^2\varphi_n(x)y\rangle_H = \langle y, D_E^2\varphi(x)y\rangle_E, \qquad x, y \in E.$$

Consequently,

$$\lim_{n\to\infty} \mathcal{L}\varphi_n(x) = \mathcal{L}_E\varphi(x), \qquad x \in E. \tag{A.6}$$

Now, by Itô's formula (A.2), we have for any $n \in \mathbb{N}$,

$$\mathbb{E}\varphi_n\big(X(t)\big) = \varphi_n(x) + \mathbb{E}\int_0^t \mathcal{L}\varphi_n\big(X(s)\big)\,ds \tag{A.7}$$

and then, letting $n \to \infty$, we get (A.4). $\square$

REMARK A.2. Let $\varphi \in C_b^2(E)$. Then $\varphi^2 \in C_b^2(E)$, and we have

$$\langle y, D_E\varphi^2(x)\rangle_E = 2\varphi(x)\langle y, D_E\varphi(x)\rangle_E$$

and

$$\langle y, D_E^2\varphi^2(x)y\rangle_E = 2\varphi(x)\langle y, D_E^2\varphi(x)y\rangle_E + 2\big|\langle y, D_E\varphi(x)\rangle_E\big|^2.$$

Consequently,

$$\mathcal{L}_E\varphi^2(x) = 2\varphi(x)\mathcal{L}_E\varphi^2(x) + \sum_{k=1}^{n} \left|\langle\sigma_k(y), D_E\varphi(x)\rangle_E\right|^2. \tag{A.8}$$

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