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CONVERGENCE IN LAW OF THE MINIMUM OF A BRANCHING RANDOM WALK

BY ELIE AÏDÉKON[1]

Eindhoven University of Technology

We consider the minimum of a super-critical branching random walk. Addario-Berry and Reed [*Ann. Probab.* **37** (2009) 1044–1079] proved the tightness of the minimum centered around its mean value. We show that a convergence in law holds, giving the analog of a well-known result of Bramson [*Mem. Amer. Math. Soc.* **44** (1983) iv+190] in the case of the branching Brownian motion.

1. Introduction. We consider a branching random walk defined as follows. The process starts with one particle located at 0. At time 1, the particle dies and gives birth to a point process $\mathcal{L}$. Then, at each time $n \in \mathbb{N}$, the particles of generation n die and give birth to independent copies of the point process $\mathcal{L}$, translated to their position. If $\mathbb{T}$ is the genealogical tree of the process, we see that $\mathbb{T}$ is a Galton–Watson tree, and we denote by $|x|$ the generation of the vertex $x \in \mathbb{T}$ (the ancestor is the only particle at generation 0). For each $x \in \mathbb{T}$, we denote by $V(x) \in \mathbb{R}$ its position on the real line. With this notation, $(V(x), |x| = 1)$ is distributed as $\mathcal{L}$. The collection of positions $(V(x), x \in \mathbb{T})$ defines our branching random walk.

We assume that we are in the boundary case (in the sense of [8])

$$\mathbf{E}\Bigg[\sum_{|x|=1} 1\Bigg] > 1, \qquad \mathbf{E}\Bigg[\sum_{|x|=1} \mathrm{e}^{-V(x)}\Bigg] = 1, \qquad \mathbf{E}\Bigg[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\Bigg] = 0. \tag{1.1}$$

Every branching random walk satisfying mild assumptions can be reduced to this case by some renormalization. We assume that $\sum_{|x|=1} 1 < \infty$ almost surely, but we allow $\mathbf{E}[\sum_{|x|=1} 1] = \infty$. We are interested in the minimum at time n

$$M_n := \min\{V(x), |x| = n\},$$

where $\min\varnothing := \infty$. Writing for $y \in \mathbb{R} \cup \{\pm\infty\}$, $y_+ := \max(y, 0)$, we introduce the random variables

$$X := \sum_{|x|=1} \mathrm{e}^{-V(x)}, \qquad \tilde{X} := \sum_{|x|=1} V(x)_+ \mathrm{e}^{-V(x)}. \tag{1.2}$$

We assume throughout the remainder of the paper, including in the statements of theorems and lemmas, etc., that:

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- the distribution of $\mathcal{L}$ is nonlattice;
- we have

$$\mathbf{E}\Big[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\Big] < \infty, \tag{1.3}$$

$$\mathbf{E}\big[X(\ln_+ X)^2\big] < \infty, \qquad \mathbf{E}[\tilde{X} \ln_+ \tilde{X}] < \infty. \tag{1.4}$$

These assumptions are discussed after Theorem 1.1. Under (1.1), the minimum M_n goes to infinity, as it can be easily seen from the fact that $\sum_{|u|=n} \mathrm{e}^{-V(u)}$ goes to zero [22]. The law of large numbers for the speed of the minimum goes back to the works of Hammersley [16], Kingman [19] and Biggins [6], and we know that $\frac{M_n}{n}$ converges almost surely to 0 in the boundary case. The second order was recently found separately by Hu and Shi [17], and Addario-Berry and Reed [1], and is proved to be equal to $\frac{3}{2}\ln n$ in probability, though there exist almost sure fluctuations (Theorem 1.2 in [17]). In [1], the authors computed the expectation of M_n to within $O(1)$, and showed, under suitable assumptions, that the sequence of the minimum is tight around its mean. Through recursive equations, Bramson and Zeitouni [11] obtained the tightness of M_n around its median, when assuming some properties on the decay of the tail distribution. In the particular case where the step distribution is log-concave, the convergence in law of M_n around its median was proved earlier by Bachmann [4]. The aim of this paper is to get the convergence of the minimum M_n centered around $\frac{3}{2}\ln n$ for a general class of branching random walks. This is the analog of the seminal work from Bramson [10], to which our approach bears some resemblance. To state our result, we introduce the *derivative martingale*, defined for any $n \geq 0$ by

$$D_n := \sum_{|x|=n} V(x) \mathrm{e}^{-V(x)}. \tag{1.5}$$

From [7] (and Proposition A.3 in Appendix A), we know that the martingale converges almost surely to some limit D_∞, which is strictly positive on the set of nonextinction of $\mathbb{T}$. Notice that under (1.1), the tree $\mathbb{T}$ has a positive probability to survive.

THEOREM 1.1. *There exists a constant $C^* \in (0, \infty)$ such that for any real x*,

$$\lim_{n\to\infty} \mathbf{P}\Big(M_n \geq \frac{3}{2}\ln n + x\Big) = \mathbf{E}\big[\mathrm{e}^{-C^* \mathrm{e}^x D_\infty}\big]. \tag{1.6}$$

REMARK 1. We can see our theorem as the analog of the result of Lalley and Sellke [21] in the case of the branching Brownian motion: the minimum converges to a random shift of the Gumbel distribution.

REMARK 2. We assumed the number of children to be finite almost surely. We think that this assumption is superfluous, but our proof does not seem to work without this assumption; see equation (5.3). The condition of nonlattice distribution is necessary since it is hopeless to have a convergence in law around $\frac{3}{2}\ln n$ in general. We do not know if an analogous result holds in the lattice case. If (1.3) does not hold, we can expect, under suitable conditions, to have still a convergence in law, but centered around $\kappa \ln n$ for some constant $\kappa \neq 3/2$. This comes from the different behavior of the probability to remain positive for one-dimensional random walks with infinite variance. Finally, condition (1.4) appears naturally for D_∞ not being identically zero; see [7], Theorem 5.2.

The proof of the theorem is divided into three steps. First, we look at the tail distribution of the minimum M_n^{kill} of the branching random walk killed below zero, that is, $M_n^{\mathrm{kill}} := \min\{V(x), V(x_k) \geq 0, \forall 0 \leq k \leq |x|\}$, where x_k denotes the ancestor of x at generation k.

PROPOSITION 1.2. *There exists a constant $C_1 > 0$ such that*

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\mathrm{e}^{z}\mathbf{P}\left(M_n^{\mathrm{kill}} < \frac{3}{2}\ln n - z\right) - C_1\right| = 0.$$

This allows us to get the tail distribution of M_n in a second stage.

PROPOSITION 1.3. *We have*

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\frac{\mathrm{e}^{z}}{z}\mathbf{P}\left(M_n < \frac{3}{2}\ln n - z\right) - C_1 c_0\right| = 0,$$

where C_1 is the constant in Proposition 1.2, and $c_0 > 0$ is defined in (2.13).

Looking at the set of particles that cross a high level $A > 0$ for the first time, we then deduce the theorem for the constant $C^* = C_1 c_0$.

The paper is organized as follows. Section 2 introduces a useful and well-known tool, the many-to-one lemma. Then, Sections 3, 4 and 5 contain, respectively, the proofs of Propositions 1.2, 1.3 and Theorem 1.1. A sum-up of the notation used in the paper can be found in Appendix D.

Throughout the paper, $(c_i)_{i\geq 0}$ denote positive constants. We say that $a_n \sim b_n$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. We write $\mathbf{E}[f, A]$ for $\mathbf{E}[f \mathbf{1}_A]$, and we set $\sum_\varnothing := 0$, $\prod_\varnothing := 1$.

2. The many-to-one lemma. For $a \in \mathbb{R}$, we denote by $\mathbf{P}_a$ the probability distribution associated to the branching random walk starting from a, and $\mathbf{E}_a$ the corresponding expectation. Under (1.1), there exists a centered random walk $(S_n, n \geq 0)$ such that for any $n \geq 1$, $a \in \mathbb{R}$ and any measurable function $g : \mathbb{R}^n \to [0, \infty)$,

$$\mathbf{E}_a\left[\sum_{|x|=n} g\big(V(x_1), \dots, V(x_n)\big)\right] = \mathbf{E}_a\left[\mathrm{e}^{S_n - a} g(S_1, \dots, S_n)\right], \tag{2.1}$$

where, under $\mathbf{P}_a$, we have $S_0 = a$ almost surely. We will write $\mathbf{P}$ and $\mathbf{E}$ instead of $\mathbf{P}_0$ and $\mathbf{E}_0$ for brevity. In particular, under (1.3), S_1 has a finite variance $\sigma^2 := \mathbf{E}[S_1^2] = \mathbf{E}[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}]$. Equation (2.1) is called in the literature the many-to-one lemma and can be seen as a consequence of Proposition 2.2 below.

2.1. *Lyons's change of measure.* We introduce the *additive martingale*

$$W_n := \sum_{|u|=n} \mathrm{e}^{-V(u)}. \tag{2.2}$$

The fact that W_n is a martingale comes from the branching property together with the assumption that $\mathbf{E}[\sum_{|x|=1} \mathrm{e}^{-V(x)}] = 1$. From [22], we know that W_n converges almost surely as $n \to \infty$ to 0 under our assumption (1.1). For any $n \geq 0$, let $\mathscr{F}_n$ denote the σ-algebra generated by the positions $(V(x), |x| \leq n)$ up to time n, and $\mathscr{F}_\infty := \bigvee_{n\geq 0} \mathscr{F}_n$. For any $a \in \mathbb{R}$, the Kolmogorov extension theorem guarantees that there exists a probability measure $\hat{\mathbf{P}}_a$ on $\mathscr{F}_\infty$ such that for any $n \geq 0$,

$$\hat{\mathbf{P}}_a|_{\mathscr{F}_n} = \mathrm{e}^a W_n \bullet \mathbf{P}_a|_{\mathscr{F}_n}. \tag{2.3}$$

We will write $\hat{\mathbf{P}}$ instead of $\hat{\mathbf{P}}_0$. We associate to the probability $\hat{\mathbf{P}}_a$ the expectation $\hat{\mathbf{E}}_a$.

We introduce the point process $\hat{\mathcal{L}}$ with Radon–Nykodim derivative $\sum_{i\in\mathcal{L}} \mathrm{e}^{-V(i)}$ with respect to the law of $\mathcal{L}$, and we consider the following process. At time 0, the population is composed of one particle w_0 located at $V(w_0) = 0$. Then, at each step n, particles of generation n die and give birth to independent point processes distributed as $\mathcal{L}$, except for the particle w_n which generates a point process distributed as $\hat{\mathcal{L}}$. The particle w_{n+1} is chosen among the children of w_n with probability proportional to $\mathrm{e}^{-V(x)}$ for each child x of w_n. This defines a branching random walk $\hat{\mathcal{B}}$ with a marked ray $(w_n)_{n\geq 0}$, which we call the *spine*. On the space of marked branching random walks, let $\hat{\mathscr{F}}_n$ be the σ-algebra generated by the positions $(V(x), |x| \leq n)$ and the marked ray (or spine) $(w_k, k \leq n)$ up to time n. Then, $\hat{\mathcal{B}}$ is measurable with respect to $\hat{\mathscr{F}}_\infty := \bigvee_{n\geq 0} \hat{\mathscr{F}}_n$. We call $\mathcal{B}$ the natural projection of $\hat{\mathcal{B}}$ on the space of branching random walks without marked rays; in other words $\mathcal{B}$ is obtained from $\hat{\mathcal{B}}$ by forgetting the identity of the spine. In particular, $\mathcal{B}$ is measurable with respect to $\mathscr{F}_\infty$. Notice that $\mathcal{B}$ is a branching random walk with immigration. We use the notation $a + \hat{\mathcal{B}}$ or $a + \mathcal{B}$ to denote the branching random walk which positions are translated by a.

PROPOSITION 2.1 ([22]). *Under $\hat{\mathbf{P}}_a$, the branching random walk has the distribution of $a + \mathcal{B}$.*

Hence we will identify from now on our branching random walk under $\hat{\mathbf{P}}_a$ to the marked branching random walk $a + \hat{\mathcal{B}}$. Notice that by doing so, we introduce in our branching random walk a marked particle, the spine, and we extend the probability $\hat{\mathbf{P}}_a$ to $\hat{\mathscr{F}}_\infty$. We stress that in the filtration $(\mathscr{F}_n, n \geq 0)$, we do not know the identity of the spine. For $\ell \geq 1$, we call $\Omega(w_\ell)$ the siblings of the spine at generation ℓ: they are the vertices which share the same parent as w_ℓ. We will often use the σ-algebra

$$\hat{\mathscr{G}}_\ell := \sigma\{w_j, V(w_j), \Omega(w_j), (V(u))_{u\in\Omega(w_j)}, j \in [1, \ell]\}, \tag{2.4}$$

$$\hat{\mathscr{G}}_\infty := \sigma\{w_j, V(w_j), \Omega(w_j), (V(u))_{u\in\Omega(w_j)}, j \geq 1\} \tag{2.5}$$

associated to the positions of the spine and its siblings, respectively, up to time ℓ and up to time ∞.

PROPOSITION 2.2 ([22]). (i) *For any* $|x| = n$, *we have*

$$\hat{\mathbf{P}}_a\{w_n = x|\mathscr{F}_n\} = \frac{\mathrm{e}^{-V(x)}}{W_n}. \tag{2.6}$$

(ii) *The process of the positions of the spine* $(V(w_n), n \geq 0)$ *under* $\hat{\mathbf{P}}_a$ *has the distribution of the centered random walk* $(S_n, n \geq 0)$ *under* $\mathbf{P}_a$.

This change of probability was used in [22]. We refer to [23] for the case of the Galton–Watson tree, to [13] for the analog for the branching Brownian motion and to [7] for spine decompositions in various types of branching. Before closing this section, we collect some elementary facts about centered random walks with finite variance. We recall that we deal with nonlattice random walks.

There exists a constant $\alpha_1 > 0$ such that for any $x \geq 0$ and $n \geq 1$,

$$\mathbf{P}_x\Big(\min_{j\leq n} S_j \geq 0\Big) \leq \alpha_1(1+x)n^{-1/2}. \tag{2.7}$$

There exists a constant $\alpha_2 > 0$ such that for any $b \geq a \geq 0$, $x \geq 0$ and $n \geq 1$,

$$\mathbf{P}_x\Big(S_n \in [a, b], \min_{j\leq n} S_j \geq 0\Big) \leq \alpha_2(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.8}$$

Let $0 < \lambda < 1$. There exists a constant $\alpha_3 = \alpha_3(\lambda) > 0$ such that for any $b \geq a \geq 0$, $x, y \geq 0$ and $n \geq 1$

$$\begin{aligned} &\mathbf{P}_x\Big(S_n \in [y+a, y+b], \min_{j\leq n} S_j \geq 0, \min_{\lambda n\leq j\leq n} S_j \geq y\Big) \\ &\quad \leq \alpha_3(1+x)(1+b-a)(1+b)n^{-3/2}. \end{aligned} \tag{2.9}$$

Let $(a_n, n \geq 0)$ be a nonnegative sequence such that $\lim_{n\to\infty} \frac{a_n}{n^{1/2}} = 0$. There exists a constant $\alpha_4 > 0$ such that for any $a \in [0, a_n]$ and $n \geq 1$

$$\mathbf{P}\Big(S_n \in [a, a+1], \min_{j\leq n} S_j \geq 0, \min_{n/2<j\leq n} S_j \geq a\Big) \geq \alpha_4 n^{-3/2}. \tag{2.10}$$

Equation (2.7) is Theorem 1a, page 415 of [14]. Equations (2.8) and (2.9) are, for example, Lemmas 2.2 and 2.4 in [3]. Equation (2.10) is Lemma 4.3 of [2]: even if the uniformity in $a \in [0, a_n]$ is not stated there, it follows directly from the proof.

2.2. *A convergence in law for the one-dimensional random walk.* We recall that $(S_n)_{n\geq 0}$ is a nonlattice centered random walk under $\mathbf{P}$, with finite variance $\mathbf{E}[S_1^2] = \sigma^2 \in (0,\infty)$. We introduce its renewal function $R(x)$ which is zero if $x<0$, 1 if $x=0$ and for $x>0$,

$$R(x) := \sum_{k\geq 0} \mathbf{P}\Big(S_k \geq -x, S_k < \min_{0\leq j\leq k-1} S_j\Big). \tag{2.11}$$

If H_n denotes the nth strict descending ladder height (where by strict descending ladder height, we mean any S_k such that $S_k < \min_{0\leq j\leq k-1} S_j$), then we observe that for $x\geq 0$,

$$R(x) = \sum_{n\geq 0} \mathbf{P}(H_n \geq -x), \tag{2.12}$$

which is $\mathbf{E}$[number of strict descending ladder heights which are $\geq -x$]. Similarly, we define $R_-(x)$ as the renewal function associated to $-S$. Since $\mathbf{E}[S_1]=0$ and $\mathbf{E}[S_1^2]<\infty$, we have that $\mathbf{E}[|H_1|]<\infty$; see Theorem 1, Section XVIII.5, page 612 in [14]. Then the renewal theorem [14], page 360, implies that there exists $c_0>0$, such that

$$\lim_{x\to\infty} \frac{R(x)}{x} = c_0. \tag{2.13}$$

Moreover, there exist $C_-, C_+ > 0$ such that

$$\mathbf{P}\Big(\min_{1\leq i\leq n} S_i \geq 0\Big) \sim \frac{C_+}{\sqrt{n}}, \tag{2.14}$$

$$\mathbf{P}\Big(\max_{1\leq i\leq n} S_i \leq 0\Big) \sim \frac{C_-}{\sqrt{n}} \tag{2.15}$$

as $n\to\infty$ (Theorem 1a, Section XII.7, page 415 of [14]).

LEMMA 2.3. *Let $(r_n)_{n\geq 0}$ and $(\lambda_n)_{n\geq 0}$ be two sequences of numbers, respectively, in $\mathbb{R}_+$ and in $(0,1)$ and such that, respectively, $\lim_{n\to\infty} \frac{r_n}{n^{1/2}} = 0$, and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$. Let $F:\mathbb{R}_+\to\mathbb{R}$ be a Riemann integrable function. We suppose that there exists a nonincreasing function $\overline{F}:\mathbb{R}_+\to\mathbb{R}$ such that $|F(x)|\leq \overline{F}(x)$ for any $x\geq 0$ and $\int_{x\geq 0} x\overline{F}(x) < \infty$. Then, as $n\to\infty$,*

$$\begin{aligned} &\mathbf{E}\Big[F(S_n - y), \min_{k\in[0,n]} S_k \geq 0, \min_{k\in[\lambda_n n, n]} S_k \geq y\Big] \\ &\quad \sim \frac{C_- C_+ \sqrt{\pi}}{\sigma\sqrt{2}} n^{-3/2} \int_{x\geq 0} F(x) R_-(x)\,dx \end{aligned} \tag{2.16}$$

uniformly in $y\in[0,r_n]$.

PROOF. Let $\varepsilon > 0$. Since $|F(x)| \le \overline{F}(x)$ and $\overline{F}$ is nonincreasing, we have for any integer $M \ge 1$,

$$\begin{aligned}
&\mathbf{E}\Big[|F(S_n - y)|, \min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y, S_n \ge y + M\Big] \\
&\quad \le \sum_{j\ge M} \overline{F}(j)\mathbf{P}\Big(\min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y, S_n \in [y+j, y+j+1)\Big).
\end{aligned}$$

For $j \ge 1$, we have by (2.9) and the fact that $\limsup_{n\to\infty} \lambda_n < 1$,

$$\mathbf{P}\Big(\min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y, S_n \in [y+j, y+j+1)\Big) \le c_1 \frac{j}{n^{3/2}}.$$

This yields that

$$\mathbf{E}\Big[|F(S_n - y)|, \min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y, S_n \ge y + M\Big] \le \frac{c_1}{n^{3/2}} \sum_{j\ge M} \overline{F}(j) j,$$

which is less than $\varepsilon n^{-3/2}$ for $M \ge 1$ large enough by the assumption that $\int_{x\ge 0} x\overline{F}(x)\,dx < \infty$. Therefore, we can restrict to F with compact support. By approximating F by scale functions (F is Riemann integrable by assumption), we only prove (2.16) for $F(x) = \mathbf{1}_{\{x\in[0,a]\}}$, where $a \ge 0$. Let $a \ge 0$ be a fixed constant in the remainder of the proof. We have for such F,

$$\begin{aligned}
&\mathbf{E}\Big[F(S_n - y), \min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y\Big] \\
&\quad = \mathbf{P}\Big(\min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y, S_n \le y + a\Big).
\end{aligned}$$

Let

$$\phi_{y,a,n}(x) := \mathbf{P}_x\Big(\min_{k\in[0,(1-\lambda_n)n]} S_k \ge y, S_{(1-\lambda_n)n} \le y + a\Big).$$

For $F(x) = \mathbf{1}_{\{x\in[0,a]\}}$, applying the Markov property at time $\lambda_n n$ (we assume that $\lambda_n n$ is an integer for simplicity), we obtain that

$$\begin{aligned}
&\mathbf{E}\Big[F(S_n - y), \min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y\Big] \\
&\quad = \mathbf{E}\Big[\phi_{y,a,n}(S_{\lambda_n n}), \min_{k\in[0,\lambda_n n]} S_k \ge 0\Big].
\end{aligned} \tag{2.17}$$

We estimate $\phi_{y,a,n}(x)$. Reversing time, we notice that

$$\phi_{y,a,n}(x) = \mathbf{P}\Big(\min_{k\in[0,(1-\lambda_n)n]} (-S_k) \ge -S_{(1-\lambda_n)n} - (x - y) \ge -a\Big). \tag{2.18}$$

We introduce the strict descending ladder heights and times (H_ℓ^-, T_ℓ^-) of $-S$ defined by $H_0^- := 0$, $T_0^- := 0$, and for any $\ell \ge 0$,

$$\begin{aligned}
T_{\ell+1}^- &:= \min\{k \ge T_\ell^- + 1 : (-S_k) < H_\ell^-\}, \\
H_{\ell+1}^- &:= -S_{T_{\ell+1}^-}.
\end{aligned}$$

Since $\mathbf{E}[S_1]=0$ (and $\sigma>0$), we have $T_\ell^-<\infty$ for any $\ell\geq 0$ almost surely. Similarly to equation (2.12), we have now $R_-(x)=\sum_{\ell\geq 0}\mathbf{P}(H_\ell^-\geq -x)$. Splitting the right-hand side of (2.18), depending on the value of the time ℓ for which $H_\ell^-=\min_{k\in[0,(1-\lambda_n)n]}(-S_k)$, we then have

$$\phi_{y,a,n}(x)=\sum_{\ell\geq 0}\mathbf{P}\Big(T_\ell^-\leq (1-\lambda_n)n,\, H_\ell^-\geq -S_{(1-\lambda_n)n}-(x-y)\geq -a, \min_{k\in[T_\ell^-,(1-\lambda_n)n]}(-S_k)\geq H_\ell^-\Big). \tag{2.19}$$

By the strong Markov property at time T_ℓ^-, we see that for any $h\in[-a,0]$ and $t\in[0,(1-\lambda_n)n)]$,

$$\begin{aligned}&\mathbf{P}\Big(H_\ell^-\geq -S_{(1-\lambda_n)n}-(x-y)\geq -a,\\ &\qquad\min_{k\in[T_\ell^-,(1-\lambda_n)n]}(-S_k)\geq H_\ell^-\big|(H_\ell^-,T_\ell^-)=(h,t)\Big)\\ &\quad=\mathbf{1}_{\{h\geq -a\}}\mathbf{P}\Big(\min_{j\in[0,(1-\lambda_n)n-t]}(-S_j)\geq 0,\\ &\qquad\qquad -S_{(1-\lambda_n)n-t}\in\big[(x-y)-a-h,(x-y)\big]\Big).\end{aligned}$$

Let $\psi(x):=x\mathrm{e}^{-x^2/2}\mathbf{1}_{\{x\geq 0\}}$. By Theorem 1 of [12] and equation (2.15), we check that

$$\begin{aligned}&\mathbf{1}_{\{h\geq -a\}}\mathbf{P}\Big(\min_{j\in[0,(1-\lambda_n)n-t]}(-S_j)\geq 0,-S_{(1-\lambda_n)n-t}\in\big[(x-y)-a-h,(x-y)\big]\Big)\\ &\quad=\mathbf{1}_{\{h\geq -a\}}\frac{C_-}{\sigma(1-\lambda_n)n}(h+a)\psi\bigg(\frac{x}{\sigma\sqrt{(1-\lambda_n)n}}\bigg)+\mathbf{1}_{\{h\geq -a\}}o\big(n^{-1}\big)\end{aligned}$$

uniformly in $x\in\mathbb{R}$, $t\leq n^{1/2}$, $h\in[-a,0]$ and $y\in[0,r_n]$. Here we used the fact that $\limsup_{n\to\infty}\lambda_n<1$. We mention that the cut-off $t\leq n^{1/2}$ is arbitrary since the statement is valid for any $t=o(n)$. To deal with $t\in[n^{1/2},(1-\lambda_n)n]$, we see that

$$\begin{aligned}&\mathbf{1}_{\{h\geq -a\}}\mathbf{P}\Big(\min_{j\in[0,(1-\lambda_n)n-t]}(-S_j)\geq 0,\\ &\qquad\qquad -S_{(1-\lambda_n)n-t}\in\big[(x-y)-a-h,(x-y)\big]\Big)\\ &\quad=\mathbf{1}_{\{h\geq -a\}}O(h+a+1)\big((1-\lambda_n)n-t+1\big)^{-1}\end{aligned}$$

again by Theorem 1 of [12]. The last equation is valid uniformly in $x,y\in\mathbb{R}$, $t\in[0,(1-\lambda_n)n]$ and $h\in[-a,0]$. Going back to (2.19), this implies that, for any

$x \in \mathbb{R}$ and $y \in [0, r_n]$,

$$\phi_{y,a,n}(x) = o(n^{-1}) + \frac{C_-}{\sigma(1-\lambda_n)n}\psi\left(\frac{x}{\sigma\sqrt{(1-\lambda_n)n}}\right) \times \sum_{\ell\geq 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- \leq n^{1/2}\}}\big] + O(1)\sum_{\ell\geq 0}\mathbf{E}\left[\frac{H_\ell^- + a + 1}{(1-\lambda_n)n - T_\ell^- + 1}\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- \in (n^{1/2}, (1-\lambda_n)n]\}}\right],$$

where we used the fact that $\sum_{\ell\geq 0}\mathbf{P}(H_\ell^- \geq -a) = R_-(a) = O(1)$ since a is a constant. Observe that

$$\sum_{\ell\geq 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- > n^{1/2}\}}\big] \leq a\sum_{\ell\geq 0}\mathbf{P}\big(H_\ell^- \geq -a, T_\ell^- > n^{1/2}\big) = o(1)$$

as $n \to \infty$ by dominated convergence. Therefore,

$$\phi_{y,a,n}(x) = o(n^{-1}) + \frac{C_-}{\sigma(1-\lambda_n)n}\psi\left(\frac{x}{\sigma\sqrt{(1-\lambda_n)n}}\right)\sum_{\ell\geq 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^- \geq -a\}}\big] + O(1)\sum_{\ell\geq 0}\mathbf{E}\left[\frac{H_\ell^- + a + 1}{(1-\lambda_n)n - T_\ell^- + 1}\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- \in (n^{1/2}, (1-\lambda_n)n]\}}\right].$$

We want to show that the last term is $o(n^{-1})$ as well. We observe that

$$\mathbf{E}\left[\frac{H_\ell^- + a + 1}{(1-\lambda_n)n - T_\ell^- + 1}\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- \in (n^{1/2}, (1-\lambda_n)n]\}}\right] \leq (a+1)\mathbf{E}\left[\frac{\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- \in (n^{1/2}, (1-\lambda_n)n]\}}}{(1-\lambda_n)n - T_\ell^- + 1}\right].$$

Since $\sum_{\ell\geq 0}\mathbf{P}(H_\ell^- \geq -a, T_\ell^- = k) \leq \mathbf{P}(S_k \in [0, a], \min_{j\leq k} S_j \geq 0)$, we obtain by (2.8) that

$$\sum_{\ell\geq 0}\mathbf{P}\big(H_\ell^- \geq -a, T_\ell^- = k\big) \leq \alpha_2(1+a)^2 k^{-3/2},$$

which yields that

$$\sum_{\ell\geq 0}\mathbf{E}\left[\frac{H_\ell^- + a + 1}{(1-\lambda_n)n - T_\ell^- + 1}\mathbf{1}_{\{H_\ell^- \geq -a, T_\ell^- \in (n^{1/2}, (1-\lambda_n)n]\}}\right] \leq \alpha_2(1+a)^3\sum_{k=\lfloor n^{1/2}\rfloor+1}^{\lfloor(1-\lambda_n)n\rfloor} k^{-3/2}\frac{1}{(1-\lambda_n)n - k + 1} = o(n^{-1})$$

as we require. Therefore,

$$\phi_{y,a,n}(x) = o(n^{-1}) + \frac{C_-}{\sigma(1-\lambda_n)n}\psi\left(\frac{x}{\sigma\sqrt{(1-\lambda_n)n}}\right)\sum_{\ell\ge 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^-\ge -a\}}\big]$$

uniformly in $x \ge 0$ and $y \in [0, r_n]$. By (2.7), we know that $\mathbf{P}(\min_{k\in[0,n]} S_k \ge 0) \le \alpha_1 n^{-1/2}$. It follows from equation (2.17) that

$$\begin{aligned}
&\mathbf{E}\Big[F(S_n - y), \min_{k\in[0,n]} S_k \ge 0, \min_{k\in[(1-\lambda_n)n,n]} S_k \ge y\Big] \\
&\quad = o(n^{-3/2}) + \frac{C_-}{\sigma(1-\lambda_n)n}\mathbf{E}\left[\psi\left(\frac{S_{\lambda_n n}}{\sigma\sqrt{(1-\lambda_n)n}}\right), \min_{k\in[0,\lambda_n n]} S_k \ge 0\right] \\
&\qquad \times \sum_{\ell\ge 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^-\ge -a\}}\big].
\end{aligned}$$

We know (see [9]) that $S_n/(\sigma n^{1/2})$ conditioned on $\min_{k\in[0,n]} S_k$ being nonnegative converges to the Rayleigh distribution. Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\mathbf{E}\left[\psi\left(\frac{S_{\lambda_n n}}{\sigma\sqrt{(1-\lambda_n)n}}\right)\Big| \min_{k\in[0,\lambda_n n]} S_k \ge 0\right] &= \int_{x\ge 0}\psi\left(x\sqrt{\frac{\lambda_n}{1-\lambda_n}}\right)\psi(x)\,dx \\
&= \sqrt{\lambda_n}(1-\lambda_n)\sqrt{\frac{\pi}{2}}.
\end{aligned}$$

In view of (2.14), we get that, as $n\to\infty$,

$$\mathbf{E}\left[\psi\left(\frac{S_{\lambda_n n}}{\sigma\sqrt{(1-\lambda_n)n}}\right), \min_{k\in[0,\lambda_n n]} S_k \ge 0\right] \sim \frac{C_+(1-\lambda_n)}{\sqrt{n}}\sqrt{\frac{\pi}{2}}.$$

We end up with

$$\begin{aligned}
&\mathbf{E}\Big[F(S_n - y), \min_{k\in[0,n]} S_k \ge 0, \min_{k\in[\lambda_n n,n]} S_k \ge y\Big] \\
&\quad = o(n^{-3/2}) + \frac{C_- C_+}{\sigma n^{3/2}}\sqrt{\frac{\pi}{2}}\sum_{\ell\ge 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^-\ge -a\}}\big]
\end{aligned}$$

uniformly in $y \in [0, r_n]$. We recall that $\sum_{\ell\ge 0}\mathbf{P}(H_\ell^- \ge -a) = R_-(a)$ by definition and we took $F(x) = \mathbf{1}_{[0,a]}(x)$. By Fubini's theorem, it follows that

$$\sum_{\ell\ge 0}\mathbf{E}\big[(H_\ell^- + a)\mathbf{1}_{\{H_\ell^-\ge -a\}}\big] = \int_{x\ge 0} F(x)R_-(x)\,dx,$$

which completes the proof. $\square$

3. The minimum of a killed branching random walk. It turns out to be useful to study first the killed branching random walk. Let

$$\mathbb{T}^{\mathrm{kill}} := \{u \in \mathbb{T} : V(u_k) \geq 0, \forall 0 \leq k \leq |u|\}$$

be the set of individuals that stay above 0. We investigate the behavior of the minimal position

$$M_n^{\mathrm{kill}} := \min\{V(u), |u|^{\mathrm{kill}} = n\}, \tag{3.1}$$

where we write $|u|^{\mathrm{kill}}$ to say that $u \in \mathbb{T}^{\mathrm{kill}}$ and $|u| = n$. If $M_n^{\mathrm{kill}} < \infty$; that is, if the killed branching random walk survives until time n, we denote by $m^{\mathrm{kill},(n)}$ a vertex chosen uniformly in the set $\{u : |u|^{\mathrm{kill}} = n, V(u) = M_n^{\mathrm{kill}}\}$ of the particles that achieve the minimum. It will be convenient to use the following notation: for $z \geq 0$,

$$a_n(z) := \tfrac{3}{2}\ln n - z, \tag{3.2}$$

$$I_n(z) := [a_n(z) - 1, a_n(z)), \tag{3.3}$$

and for $z \geq 0$, $0 \leq k \leq n$ and $\lambda \in (0, 1)$,

$$d_k(n, z, \lambda) := \begin{cases} 0, & \text{if } 0 \leq k \leq \lambda n, \\ \max(a_n(z+1), 0), & \text{if } \lambda n < k \leq n. \end{cases} \tag{3.4}$$

We will see later that, as $n \to \infty$, conditionally on being in $I_n(z)$, a particle that achieves the minimum at time n did not cross the curve $k \to d_k(n, z + L, \lambda)$ with probability tending to 1 when the constant L goes to ∞ [and λ is any constant in $(0, 1)$]. The section is devoted to the proof of the following proposition.

PROPOSITION 3.1. *For any $\varepsilon > 0$, there exist a real $A \geq 0$ and an integer $N \geq 1$ such that for any $n \geq N$ and $z \in [A, (3/2)\ln(n) - A]$,*

$$\left|e^z \mathbf{P}(M_n^{\mathrm{kill}} \in I_n(z)) - C_2\right| \leq \varepsilon,$$

where C_2 is some positive constant.

COROLLARY 3.2. *Let $C_1 := \frac{C_2}{1 - e^{-1}}$. For any $\varepsilon > 0$, there exist a real $A \geq 0$ and an integer $N \geq 1$ such that for any $n \geq N$ and $z \in [A, (3/2)\ln(n) - A]$,*

$$\left|e^z \mathbf{P}\left(M_n^{\mathrm{kill}} < \tfrac{3}{2}\ln n - z\right) - C_1\right| \leq \varepsilon.$$

Proposition 1.2 immediately follows from Corollary 3.2. Assuming that Proposition 3.1 holds, let us see how it implies the corollary.

PROOF OF COROLLARY 3.2. Let $\varepsilon > 0$. We have by equation (2.1), for any integer $n \geq 1$ and any real $r \geq 0$,

$$\mathbf{E}\Bigg[\sum_{|u|^{\mathrm{kill}}=n} \mathbf{1}_{\{V(u) \leq r\}}\Bigg] = \mathbf{E}\Big[e^{S_n}, S_n \leq r, \min_{0 \leq j \leq n} S_j \geq 0\Big]$$

$$\leq e^r \mathbf{P}\Big(S_n \leq r, \min_{0 \leq j \leq n} S_j \geq 0\Big).$$

By (2.8), we have $\mathbf{P}(S_n \le r, \min_{0\le j\le n} S_j \ge 0) \le c_2 \frac{(1+r)^2}{n^{3/2}}$. We deduce that

$$\mathbf{P}\big(M_n^{\text{kill}} \le r\big) \le c(r) n^{-3/2} \tag{3.5}$$

with $c(r) := c_2 e^r (1+r)^2$. Let A_1 and N_1 be as in Proposition 3.1. We have for $n \ge N_1$ and $z \in [A_1, (3/2)\ln(n) - A_1]$,

$$\big|\mathbf{P}\big(M_n^{\text{kill}} \in I_n(z)\big) - C_2 e^{-z}\big| \le \varepsilon e^{-z}.$$

Summing this equation over $z+k$ such that $z+k \in [A_1, (3/2)\ln(n) - A_1]$, we get that for any $n \ge N_1$ and $z \in [A_1, (3/2)\ln(n) - A_1]$,

$$\begin{aligned}\left|\mathbf{P}\big(M_n^{\text{kill}} \in \big[r_{n,z}, a_n(z)\big)\big) - C_2 e^{-z} \sum_{k=0}^{\lfloor a_n(z+A_1)\rfloor} e^{-k}\right| &\le \varepsilon e^{-z} \sum_{k=0}^{\lfloor a_n(z+A_1)\rfloor} e^{-k} \\ &\le \frac{\varepsilon}{1-e^{-1}} e^{-z},\end{aligned}$$

where $r_{n,z} := a_n(z) - \lfloor a_n(z+A_1)\rfloor \le A_1$. By (3.5), we get that

$$\left|\mathbf{P}\big(M_n^{\text{kill}} < a_n(z)\big) - C_2 e^{-z} \sum_{k=0}^{\lfloor a_n(z+A_1)\rfloor} e^{-k}\right| \le \frac{\varepsilon}{1-e^{-1}} e^{-z} + c(A_1+1) n^{-3/2}.$$

Let $A_2 \ge A_1$ large enough such that $\sum_{k > A_2 - A_1} e^{-k} \le \varepsilon$, and $c(A_1+1) \le \varepsilon e^{A_2}$. Then, for any $n \ge N_1$ and $z \in [A_1, (3/2)\ln(n) - A_2]$,

$$\begin{aligned}\left|\mathbf{P}\big(M_n^{\text{kill}} < a_n(z)\big) - \frac{C_2}{1-e^{-1}} e^{-z}\right| &\le \varepsilon\big(\big(1-e^{-1}\big)^{-1} + 1\big) e^{-z} + \varepsilon e^{A_2} n^{-3/2} \\ &\le \varepsilon\big(\big(1-e^{-1}\big)^{-1} + 1\big) e^{-z} + \varepsilon e^{-z},\end{aligned}$$

which completes the proof. □

3.1. *Tightness of the minimum.* Our aim is now to prove Proposition 3.1. In other words, we want to estimate the probability of the event $\{M_n^{\text{kill}} \in I_n(z)\}$. The first lemma gives information on the path of particles located in $I_n(z)$.

LEMMA 3.3. *Let* $0 < \lambda < 1$. *There exist constants* $c_3, c_4 > 0$ *such that for any* $n \ge 1$, $L \ge 0$, $x \ge 0$ *and* $z \ge 0$,

$$\begin{aligned}&\mathbf{P}_x\Big(\exists u \in \mathbb{T}^{\text{kill}} : |u| = n, V(u) \in I_n(z), \min_{k\in[\lambda n, n]} V(u_k) \in I_n(z+L)\Big) \\ &\quad \le c_3 (1+x) e^{-c_4 L} e^{-x-z}.\end{aligned} \tag{3.6}$$

PROOF. Let E be the event in (3.6), and write $d_k = d_k(n, z+L, \lambda)$ as defined in (3.4). Considering the time when the minimum $\min_{k\in[\lambda n,n]} V(u_k)$ is reached, we observe that $E \subset \bigcup_{k\in[\lambda n,n]} E_k$ where we define $E_k := \bigcup_{|u|=n} E_k(u)$ and for any $u \in \mathbb{T}$ with $|u| = n$,

$$E_k(u) := \{V(u_\ell) \geq d_\ell, \forall 0 \leq \ell \leq n, V(u) \in I_n(z), V(u_k) \in I_n(z+L)\}.$$

Similarly, let

$$E_k(S) := \{S_\ell \geq d_\ell, \forall 0 \leq \ell \leq n, S_n \in I_n(z), S_k \in I_n(z+L)\}.$$

We notice that $\mathbf{P}_x(E_k) \leq \mathbf{E}_x[\sum_{|u|=n} \mathbf{1}_{E_k(u)}]$ which is $\mathbf{E}_x[\mathrm{e}^{S_n - x}\mathbf{1}_{E_k(S)}]$ by (2.1). In particular,

$$\mathbf{P}_x(E_k) \leq n^{3/2}\mathrm{e}^{-x-z}\mathbf{P}_x(E_k(S)). \tag{3.7}$$

We need to estimate $\mathbf{P}_x(E_k(S))$. By the Markov property at time k,

$$\begin{aligned}\mathbf{P}_x(E_k(S)) &\leq \mathbf{P}_x(S_\ell \geq d_\ell, \forall 0 \leq \ell \leq k, S_k \in I_n(z+L)) \\ &\quad \times \mathbf{P}\Big(S_{n-k} \in [L-1, L+1], \min_{\ell\in[0,n-k]} S_\ell \geq -1\Big).\end{aligned}$$

For the second term of the right-hand side, we know from (2.8) that there exists a constant $c_5 > 0$ such that

$$\begin{aligned}&\mathbf{P}\Big(S_{n-k} \in [L-1, L+1], \min_{\ell\in[0,n-k]} S_\ell \geq -1\Big) \\ &\quad \leq c_5(n-k+1)^{-3/2}(1+L).\end{aligned} \tag{3.8}$$

To bound the first term, our argument depends on the value of k. Suppose that $\frac{\lambda+1}{2}n \leq k \leq n$. We have by (2.9),

$$\mathbf{P}_x(S_\ell \geq d_\ell, \forall 0 \leq \ell \leq k, S_k \in I_n(z+L)) \leq c_6\frac{(1+x)}{n^{3/2}}. \tag{3.9}$$

If $\lambda n \leq k < \frac{\lambda+1}{2}n$, we simply write

$$\begin{aligned}&\mathbf{P}_x(S_\ell \geq d_\ell, \forall 0 \leq \ell \leq k, S_k \in I_n(z+L)) \\ &\quad \leq \mathbf{P}_x\Big(S_k \in I_n(z+L), \min_{\ell\in[0,k]} S_\ell \geq 0\Big) \\ &\quad \leq c_7(1+x)\ln(n)n^{-3/2}\end{aligned} \tag{3.10}$$

by (2.8). From (3.8), (3.9) and (3.10), there exists a constant $c_8 > 0$ such that

$$\sum_{k\in[\lambda n,n-a]} \mathbf{P}_x(E_k(S)) \leq c_8(1+x)(1+L)n^{-3/2}\bigg(a^{-1/2} + \frac{\ln(n)}{\sqrt{n}}\bigg)$$

for any $a \geq 1$. By (3.7), this yields that

$$\sum_{k\in[\lambda n,n-a]} \mathbf{P}_x(E_k) \leq c_8(1+x)(1+L)\mathrm{e}^{-x-z}\bigg(a^{-1/2} + \frac{\ln(n)}{\sqrt{n}}\bigg). \tag{3.11}$$

It remains to bound $\mathbf{P}_x(E_k)$ for $n-a<k\le n$. We observe that

$$\mathbf{P}_x(E_k)\le \mathbf{P}_x\big(\exists |u|=k: V(u_\ell)\ge d_\ell, \forall 0\le \ell\le k, V(u)\in I_n(z+L)\big).$$

By an application of (2.1), we have

$$\mathbf{P}_x(E_k)\le n^{3/2}\mathrm{e}^{-x-z-L}\mathbf{P}_x\big(S_\ell\ge d_\ell, \forall 0\le \ell\le k, S_k\in I_n(z+L)\big),$$

which is $\le c_9\mathrm{e}^{-x-z-L}(1+x)$ by (2.9) [for $k\ge (1+\lambda)n/2$, e.g.]. It follows that, for $a\in[1,(1-\lambda)n/2]$,

$$\sum_{k\in[n-a,n]}\mathbf{P}_x(E_k)\le c_9(1+a)(1+x)\mathrm{e}^{-x-z-L}. \tag{3.12}$$

Equations (3.11) and (3.12) yield that, for any $n\ge 1$, $z, L\ge 0$ and $a\in[1,(1-\lambda)n/2]$,

$$\begin{aligned}\mathbf{P}_x(E) &\le \sum_{k\in[\lambda n,n]}\mathbf{P}_x(E_k)\\ &\le (1+x)\mathrm{e}^{-x-z}\left\{c_8(1+L)\left(a^{-1/2}+\frac{\ln(n)}{\sqrt{n}}\right)+c_9(1+a)\mathrm{e}^{-L}\right\}.\end{aligned} \tag{3.13}$$

Notice that (3.6) holds if $L>(3/2)\ln n$ since the left-hand side is 0. If $L\le(3/2)\ln n$, take $a=\max(1,\alpha\mathrm{e}^{\beta L})$ with $\alpha,\beta>0$ small enough and use (3.13) to complete the proof. $\square$

We deduce the following corollary.

COROLLARY 3.4. *We have for any $z, x\ge 0$ and any integer $n\ge 1$,*

$$\mathbf{P}_x\big(M_n^{\mathrm{kill}}\le a_n(z)\big)\le c_{10}(1+x)\mathrm{e}^{-x-z}.$$

Recall that $a_n(z):=\frac{3}{2}\ln n-z$, $I_n(z):=[a_n(z)-1,a_n(z))$ and $d_k(n,z+L,\lambda)$ is defined in (3.4).

DEFINITION 3.5. For $u\in\mathbb{T}$, we say that $u\in\mathcal{Z}_n^{z,L}$ if

$$|u|=n,\qquad V(u)\in I_n(z)\quad\text{and}\quad V(u_k)\ge d_k(n,z+L,1/2)\qquad\forall k\le n$$

(see Figure 1).

Notice that if $u\in\mathcal{Z}_n^{z,L}$, then necessarily $u\in\mathbb{T}^{\mathrm{kill}}$. In words, $u\in\mathcal{Z}_n^{z,L}$ means that a particle is located around $\frac{3}{2}\ln n-z$, and did not cross the curve $k\to d_k(n,z+L,1/2)$. We deduce from Lemma 3.3 that for any $\varepsilon>0$, there exists $L_0>0$ such that for any $n\ge 1$, $L\ge L_0$ and $z\ge 0$,

$$\mathbf{P}\big(\exists u\in\mathbb{T}^{\mathrm{kill}}: |u|=n, u\notin\mathcal{Z}_n^{z,L}, V(u)\in I_n(z)\big)\le\varepsilon\mathrm{e}^{-z}. \tag{3.14}$$

Equivalently, with high probability, any particle of the killed branching random walk located around $\frac{3}{2}\ln n-z$ stayed above the curve $k\to d_k(n,z+L,1/2)$. We show now that $\mathbf{P}(M_n^{\mathrm{kill}}\le\frac{3}{2}\ln n-z)$ has an exponential decay as $z\to\infty$. Corollary 3.4 gives an upper bound. The following lemma gives a lower bound.

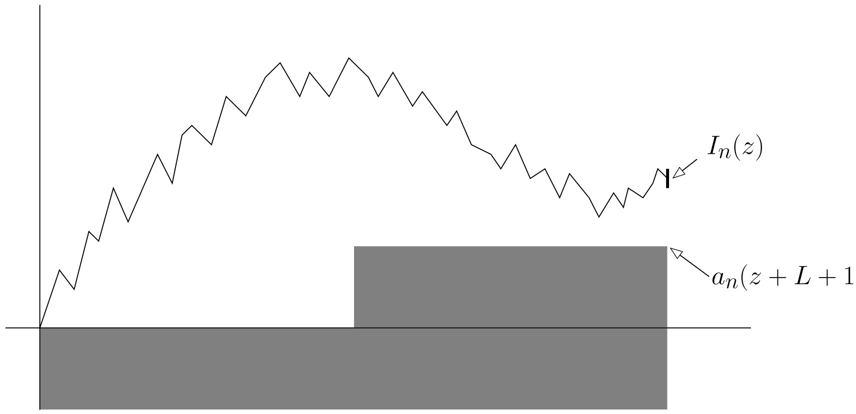

FIG. 1. *Path of a vertex in* $\mathcal{Z}_n^{z,L}$.

LEMMA 3.6. *There exists* $c_{11} > 0$ *such that for any* $n \geq 1$ *and* $z \in [0, (3/2)\ln n - 1]$

$$\mathbf{P}\big(M_n^{\mathrm{kill}} < \tfrac{3}{2}\ln n - z\big) \geq c_{11}\mathrm{e}^{-z}.$$

PROOF. The proof relies on a second moment argument. Let $z \in [0, (3/2)\ln n - 1]$ and $n \geq 1$. For $1 \leq k \leq n$, let

$$e_k = e_k^{(n)} := \begin{cases} k^{1/12}, & \text{if } 1 \leq k \leq \dfrac{n}{2}, \\ (n-k)^{1/12}, & \text{if } \dfrac{n}{2} < k \leq n \end{cases}$$

and write for brevity $d_k = d_k(n, z, 1/2)$. In order to have good bounds in our second moment argument, we will restrict to "good" vertices which do not have "too many" descendants. This leads us to the following definition. We say that $|u| = n$ is a z-good vertex if $u \in \mathcal{Z}_n^{z,0}$ and

$$\sum_{v \in \Omega(u_k)} \mathrm{e}^{-(V(v)-d_k)}\big(1 + \big(V(v) - d_k\big)_+\big) \leq B\mathrm{e}^{-e_k} \qquad \forall 1 \leq k \leq n, \tag{3.15}$$

where $\Omega(y)$ stands for the set of siblings of y, that is, the particles $x \neq y$ which share the same parent as y in the tree $\mathbb{T}$. The number $B > 0$ is a constant that we will fix later on. The reason for such a definition becomes clear in the computation of the second moment in (3.18). Such conditions on the behavior of the children off the path of the spine in a second moment argument are not new, and were already used in [15].

Remember the probability measure $\hat{\mathbf{P}}$ that we introduced in Section 2.1, which is associated to the expectation $\hat{\mathbf{E}}$. We recall that w_n is the spine at generation n, and we know from Proposition 2.2(ii) that $(V(w_k), k \geq 0)$ under $\hat{\mathbf{P}}$ has the law of the centered random walk $(S_k, k \geq 0)$ under $\mathbf{P}$. By (2.10) with $a_n = (3/2)\ln n$, there exists $c_{13} > 0$ such that $\hat{\mathbf{P}}(w_n \in \mathcal{Z}_n^{z,0}) \geq 2c_{13}n^{-3/2}$. Then, by Lemma C.1, we can choose $B > 0$ such that for any $n \geq 1$ and $z \in [0, (3/2)\ln n - 1]$,

$$\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex}) \geq c_{13}n^{-3/2}.$$

Let Good_n be the number of z-good vertices at generation n. We have by definition of the measure $\hat{\mathbf{P}}$ then Proposition 2.2(i),

$$\mathbf{E}[\mathrm{Good}_n] = \hat{\mathbf{E}}\left[\frac{1}{W_n}\sum_{|u|=n}\mathbf{1}_{\{u \text{ is a } z\text{-good vertex}\}}\right] = \hat{\mathbf{E}}\left[\mathrm{e}^{V(w_n)}, w_n \text{ is a } z\text{-good vertex}\right].$$

On the event that $w_n \in \mathcal{Z}_n^{z,0}$, we have that $V(w_n) \geq (3/2)\ln(n) - z - 1$. Therefore,

$$\mathbf{E}[\mathrm{Good}_n] \geq n^{3/2}\mathrm{e}^{-z-1}\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex}) \geq c_{13}\mathrm{e}^{-z-1}. \tag{3.16}$$

We look at the second moment. We use again Proposition 2.2(i) to see that

$$\begin{aligned}\mathbf{E}\left[(\mathrm{Good}_n)^2\right] &= \hat{\mathbf{E}}\left[\mathrm{e}^{V(w_n)}\mathrm{Good}_n, w_n \text{ is a } z\text{-good vertex}\right]\\ &\leq n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}[\mathrm{Good}_n, w_n \text{ is a } z\text{-good vertex}]\end{aligned}$$

since $V(w_n) \leq (3/2)\ln(n) - z$ when $w_n \in \mathcal{Z}_n^{z,0}$. Let Y_n be the number of vertices u such that $u \in \mathcal{Z}_n^{z,0}$. We notice that $Y_n \geq \mathrm{Good}_n$, hence

$$\mathbf{E}\left[(\mathrm{Good}_n)^2\right] \leq n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}[Y_n, w_n \text{ is a } z\text{-good vertex}].$$

We decompose Y_n along the spine. We get

$$Y_n = \mathbf{1}_{\{w_n \in \mathcal{Z}_n^{z,0}\}} + \sum_{k=1}^{n}\sum_{u \in \Omega(w_k)} Y_n(u),$$

where $Y_n(u)$ is the number of vertices v which are descendants of u and such that $v \in \mathcal{Z}_n^{z,0}$. Therefore,

$$\begin{aligned}\mathbf{E}\left[(\mathrm{Good}_n)^2\right] \leq n^{3/2}\mathrm{e}^{-z}\Biggl(&\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex})\\ &+ \sum_{k=1}^{n}\hat{\mathbf{E}}\left[\sum_{u \in \Omega(w_k)} Y_n(u), w_n \text{ is a } z\text{-good vertex}\right]\Biggr).\end{aligned} \tag{3.17}$$

Recall from (2.5) that $\hat{\mathcal{G}}_\infty$ is the σ-algebra generated by the spine and its siblings. Recall that the branching random walk rooted at $u \in \Omega(w_k)$ has the same law under $\mathbf{P}$ and $\hat{\mathbf{P}}$. For $u \in \Omega(w_k)$, we have $Y_n(u) = 0$ if there exists $j \leq |u|$ such that $V(u_j) < d_j$. Otherwise, we have by (2.1),

$$\begin{aligned}\hat{\mathbf{E}}\left[Y_n(u)|\hat{\mathcal{G}}_\infty\right] &= \mathbf{E}_{V(u)}\left[\sum_{|v|=n-k}\mathbf{1}_{\{V(v_j)\geq d_{k+j}, \forall 0\leq j\leq n-k, V(v)\in I_n(z)\}}\right]\\ &= \mathrm{e}^{-V(u)}\mathbf{E}_{V(u)}\left[\mathrm{e}^{S_{n-k}}, S_j \geq d_{k+j}, \forall 0 \leq j \leq n-k, S_{n-k} \in I_n(z)\right].\end{aligned}$$

Consequently,

$$\hat{\mathbf{E}}[Y_n(u)|\hat{\mathcal{G}}_\infty] \le n^{3/2}\mathrm{e}^{-z-V(u)}\mathbf{P}_{V(u)}(S_j \ge d_{k+j}, \forall 0 \le j \le n-k, S_{n-k} \in I_n(z))$$
$$=: n^{3/2}\mathrm{e}^{-z-V(u)}\mathrm{p}(V(u), k, n, z),$$

the latter inequality consisting of the definition of $\mathrm{p}(V(u), k, n, z)$. Hence, equation (3.17) gives that

$$\begin{aligned}
&\mathbf{E}[(\mathrm{Good}_n)^2] \\
&\quad\le n^{3/2}\mathrm{e}^{-z}\Bigg(\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex}) \\
&\qquad + n^{3/2}\mathrm{e}^{-z}\sum_{k=1}^{n}\hat{\mathbf{E}}\Bigg[\sum_{u\in\Omega(w_k)}\mathrm{e}^{-V(u)}\mathrm{p}(V(u), k, n, z), \\
&\qquad\qquad w_n \text{ is a } z\text{-good vertex}\Bigg]\Bigg).
\end{aligned} \tag{3.18}$$

We want to bound $\mathrm{p}(r, k, n, z)$ for $r \in \mathbb{R}$. We have to split the cases $k \le n/2$ and $n/2 < k \le n$. Suppose first that $k \le n/2$. Then $\mathrm{p}(r, k, n, z) = 0$ if $r < 0$. If $r \ge 0$, we apply (2.9) to see that for any $n \ge 1$, $k \le n/2$, $r \ge 0$ and $z \ge 0$,

$$\mathrm{p}(r, k, n, z) \le c_{14}(r+1)n^{-3/2}.$$

This implies that, for any $n \ge 1$, $k \le n/2$, $r \ge 0$ and $z \ge 0$,

$$\begin{aligned}
&\sum_{k=1}^{\lfloor n/2\rfloor}\hat{\mathbf{E}}\Bigg[\sum_{u\in\Omega(w_k)}\mathrm{e}^{-V(u)}\mathrm{p}(V(u), k, n, z), w_n \text{ is a } z\text{-good vertex}\Bigg] \\
&\quad\le c_{14}n^{-3/2}\sum_{k=1}^{\lfloor n/2\rfloor}\hat{\mathbf{E}}\Bigg[\sum_{u\in\Omega(w_k)}\mathrm{e}^{-V(u)}(1+V(u)_+), w_n \text{ is a } z\text{-good vertex}\Bigg] \\
&\quad\le c_{14}Bn^{-3/2}\sum_{k=1}^{\lfloor n/2\rfloor}\mathrm{e}^{-e_k}\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex}),
\end{aligned}$$

where the last inequality comes from the property (3.15) satisfied by a good vertex. When $n/2 < k \le n$, we simply write $\mathrm{p}(r, k, n, z) \le 1$ and we get

$$\begin{aligned}
&\sum_{k=\lfloor n/2\rfloor+1}^{n}\hat{\mathbf{E}}\Bigg[\sum_{u\in\Omega(w_k)}\mathrm{e}^{-V(u)}\mathrm{p}(V(u), k, n, z), w_n \text{ is a } z\text{-good vertex}\Bigg] \\
&\quad\le \sum_{k=\lfloor n/2\rfloor+1}^{n}\hat{\mathbf{E}}\Bigg[\sum_{u\in\Omega(w_k)}\mathrm{e}^{-V(u)}, w_n \text{ is a } z\text{-good vertex}\Bigg]
\end{aligned}$$

$$= n^{-3/2} e^{z+1} \sum_{k=\lfloor n/2\rfloor+1}^{n} \hat{\mathbf{E}}\Bigg[\sum_{u\in\Omega(w_k)} e^{-(V(u)-d_k)}, w_n \text{ is a } z\text{-good vertex}\Bigg]$$

$$\leq B n^{-3/2} e^{z+1} \sum_{k=\lfloor n/2\rfloor+1}^{n} e^{-e_k} \hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex})$$

by (3.15). Going back to (3.18), we deduce that for any $z \geq 0$ and $n \geq 1$,

$$\mathbf{E}\big[(\mathrm{Good}_n)^2\big] \leq n^{3/2} e^{-z} \Bigg\{1 + c_{15} \sum_{k=1}^{n} e^{-e_k}\Bigg\} \hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex})$$

$$\leq c_{16} n^{3/2} e^{-z} \hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex}).$$

Now, observe that $\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex}) \leq \hat{\mathbf{P}}(w_n \in \mathcal{Z}_n^{z,0}) \leq c_{17} n^{-3/2}$ by Definition 3.5 and equation (2.9). Hence

$$\mathbf{E}\big[(\mathrm{Good}_n)^2\big] \leq c_{18} e^{-z}. \tag{3.19}$$

By the Paley–Zygmund inequality, we have $\mathbf{P}(\mathrm{Good}_n \geq 1) \geq \frac{\mathbf{E}[\mathrm{Good}_n]^2}{\mathbf{E}[(\mathrm{Good}_n)^2]}$ which is greater than $c_{19} e^{-z}$ by (3.16) and (3.19). We conclude by observing that if $\mathrm{Good}_n \geq 1$, then $M_n^{\mathrm{kill}} < \frac{3}{2} \ln n - z$. □

3.2. *Proof of Proposition* 3.1. Corollary 3.4 and Lemma 3.6 already give the right rate of decay, but we want to strengthen it into an asymptotic as $z \to \infty$. We recall that $m^{\mathrm{kill},(n)}$ is chosen uniformly among the particles in $\mathbb{T}^{\mathrm{kill}}$ that achieve the minimum. We introduced the notation $\mathcal{Z}_n^{z,L}$ in Definition 3.5. By (3.14), we have that with high probability $m^{\mathrm{kill},(n)} \in \mathcal{Z}_n^{z,L}$ whenever $M_n^{\mathrm{kill}} \in I_n(z)$, where L is a large constant. The first step of the proof is to give a representation of the probability $\mathbf{P}(M_n^{\mathrm{kill}} \in I_n(z), m^{\mathrm{kill},(n)} \in \mathcal{Z}_n^{z,L})$ in terms of the spine decomposition presented in Section 2.1. Recall that the notation $|u|^{\mathrm{kill}} = n$ is a short way to say that $u \in \mathbb{T}^{\mathrm{kill}}$ and $|u| = n$.

LEMMA 3.7. *For any* $z \geq 0$, $L \geq 0$, *and* $n \geq 1$, *we have*

$$\mathbf{P}\big(M_n^{\mathrm{kill}} \in I_n(z), m^{\mathrm{kill},(n)} \in \mathcal{Z}_n^{z,L}\big) = \hat{\mathbf{E}}\Bigg[\frac{e^{V(w_n)} \mathbf{1}_{\{V(w_n)=M_n^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}}=n} \mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}\}}}, w_n \in \mathcal{Z}_n^{z,L}\Bigg]. \tag{3.20}$$

PROOF. We observe that

$$\mathbf{P}\big(M_n^{\mathrm{kill}} \in I_n(z), m^{\mathrm{kill},(n)} \in \mathcal{Z}_n^{z,L}\big) = \mathbf{E}\Bigg[\sum_{|u|=n} \mathbf{1}_{\{u=m^{\mathrm{kill},(n)}, u\in\mathcal{Z}_n^{z,L}\}}\Bigg]$$

$$= \mathbf{E}\Bigg[\frac{\sum_{|u|=n} \mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}, u\in\mathcal{Z}_n^{z,L}\}}}{\sum_{|u|^{\mathrm{kill}}=n} \mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}\}}}\Bigg].$$

Using the measure $\hat{\mathbf{P}}$, it follows from Proposition 2.2(i) that

$$\mathbf{E}\left[\frac{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}},u\in\mathcal{Z}_n^{z,L}\}}}{\sum_{|u|^{\mathrm{kill}}=n}\mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}\}}}\right] = \hat{\mathbf{E}}\left[\frac{\mathrm{e}^{V(w_n)}}{\sum_{|u|^{\mathrm{kill}}=n}\mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}\}}}\mathbf{1}_{\{V(w_n)=M_n^{\mathrm{kill}},w_n\in\mathcal{Z}_n^{z,L}\}}\right],$$

which completes the proof. □

We now study our branching random walk under $\hat{\mathbf{P}}$, which we identified with the branching random walk $\hat{\mathcal{B}}$ by the mean of Proposition 2.1. For $b\le n$ integers and $z\ge 0$, we define the event $\mathcal{E}_n(z,b)\in\hat{\mathscr{F}}_n$ by

$$\mathcal{E}_n(z,b):=\Big\{\forall k\le n-b,\forall v\in\Omega(w_k),\min_{u\ge v,|u|^{\mathrm{kill}}=n}V(u)\ge a_n(z)\Big\}, \tag{3.21}$$

where, as before, $\Omega(w_k)$ denotes the set of siblings of w_k. On the event $\mathcal{E}_n(z,b)\cap\{M_n^{\mathrm{kill}}\in I_n(z)\}$, we are sure that any particle located at the minimum separated from the spine after the time $n-b$. The following lemma will be proved in Section 3.3.

LEMMA 3.8. *Let $\eta>0$ and $L\ge 0$. There exist $A>0$ and $B\ge 1$ such that for any integers $n\ge b\ge B$ and any real $z\ge A$,*

$$\hat{\mathbf{P}}\big((\mathcal{E}_n(z,b))^c,w_n\in\mathcal{Z}_n^{z,L}\big)\le\eta n^{-3/2}. \tag{3.22}$$

Let, for $x\ge 0$, $L\ge 0$ and any integer $b\ge 1$

$$F_{L,b}(x):=\hat{\mathbf{E}}_x\left[\frac{\mathrm{e}^{V(w_b)-L}\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}},\ \min_{k\in[0,b]}V(w_k)\ge -1,\ V(w_b)\in[L-1,L)\right]. \tag{3.23}$$

We stress that M_b which appears in the definition of $F_{L,b}(x)$ is the minimum at time b of the *nonkilled* branching random walk. Then define

$$C_{L,b}:=\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\int_{x\ge 0}F_{L,b}(x)R_-(x)\,dx, \tag{3.24}$$

where C_-, C_+ and $R_-(x)$ were defined in Section 2.2. We recall that, by Proposition 2.2(ii), the spine has the law of $(S_n)_{n\ge 0}$. In (3.23), we see that $\frac{\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}$ is smaller than 1, and $\mathrm{e}^{V(w_b)-L}\le 1$. Hence, $|F_{L,b}(x)|\le\mathbf{P}(S_b\le L-x)=:\overline{F}(x)$

which is nonincreasing in x, and $\int_{x\ge 0}\overline{F}(x)x\,dx=\frac{1}{2}\mathbf{E}[(L-S_b)^2\mathbf{1}_{\{S_b\le L\}}]<\infty$. Moreover, changing the starting point from x to 0, we observe that

$$F_{L,b}(x)$$
$$=\mathrm{e}^x\hat{\mathbf{E}}\biggl[\frac{\mathrm{e}^{V(w_b)-L}\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}\mathbf{1}_{\{\min_{k\in[0,b]}V(w_k)\ge -x-1,V(w_b)\in[-x+L-1,-x+L)\}}\biggr].$$

The fraction in the expectation is smaller than 1. Using the identity $|\mathbf{1}_E-a\mathbf{1}_F|\le 1-a+|\mathbf{1}_E-\mathbf{1}_F|$ for $a\in(0,1)$, this yields that for $x\ge 0$, $\varepsilon>0$ and any $y\in[x,x+\varepsilon]$,

$$\begin{aligned}&\bigl|F_{L,b}(y)-F_{L,b}(x)\bigr|\\&\quad\le \mathrm{e}^y\mathbf{E}\bigl[\bigl|\mathbf{1}_{\{\min_{k\in[0,b]}S_k\ge -y-1,S_b+y-L\in[-1,0)\}}\\&\qquad\qquad -\mathrm{e}^{x-y}\mathbf{1}_{\{\min_{k\in[0,b]}S_k\ge -x-1,S_b+x-L\in[-1,0)\}}\bigr|\bigr]\\&\quad\le \mathrm{e}^y\bigl(1-\mathrm{e}^{-\varepsilon}\bigr)\\&\qquad +\mathrm{e}^y\mathbf{E}[\mathbf{1}_{\{\min_{k\in[0,b]}S_k+x+1\in[-\varepsilon,0)\}}+\mathbf{1}_{\{S_b+x-L\in[-1-\varepsilon,-1)\cup(-\varepsilon,0]\}}]\end{aligned}$$

from which we deduce that $x\to F_{L,b}(x)$ is Riemann integrable. Therefore, $F_{L,b}$ satisfies the conditions of Lemma 2.3 for any $L\ge 0$ and integer $b\ge 1$.

We want to prove that the expectation in (3.20) behaves like e^{-z} with some constant factor, as $z\to\infty$. By Lemma 3.8, we can restrict to the event $\mathcal{E}_n(z,b)$. The next lemma shows that the expectation on this event is then equivalent to $C_{L,b}\mathrm{e}^{-z}$.

LEMMA 3.9. *Let $L\ge 0$ and $\eta>0$. Let A and B be as in Lemma 3.8. For any integer $b\ge B$, we can find a constant $H>0$ such that for n large enough, and $z\in[A,(3/2)\ln(n)-L-H]$,*

$$\biggl|\mathrm{e}^z\hat{\mathbf{E}}\biggl[\frac{\mathrm{e}^{V(w_n)}\mathbf{1}_{\{V(w_n)=M_n^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}}=n}\mathbf{1}_{\{V(u)=M_n^{\mathrm{kill}}\}}},w_n\in\mathcal{Z}_n^{z,L},\mathcal{E}_n(z,b)\biggr]-C_{L,b}\biggr|\le 3\eta. \tag{3.25}$$

PROOF. Let L, η, A, B be as in the lemma. Throughout the proof, b is a fixed integer which is greater than B. We denote by $\hat{\mathbf{E}}_{(3.25)}$ the expectation in (3.25). Recall that under $\hat{\mathbf{P}}$, our process is identified with $\hat{\mathcal{B}}$. Applying the branching property at the vertex w_{n-b} to $\hat{\mathcal{B}}$, we have for any $n\ge b$ and $z\ge 0$,

$$\hat{\mathbf{E}}_{(3.25)}=\hat{\mathbf{E}}\bigl[F^{\mathrm{kill}}\bigl(V(w_{n-b})\bigr),V(w_\ell)\ge d_\ell,\forall\ell\le n-b,\mathcal{E}_n(z,b)\bigr],$$

where $d_\ell:=d_\ell(n,z+L,1/2)$ [see (3.4)] and F^{kill} is defined for $x\ge 0$ by

$$\begin{aligned}F^{\mathrm{kill}}(x):=\hat{\mathbf{E}}_x\biggl[\frac{\mathrm{e}^{V(w_b)}\mathbf{1}_{\{V(w_b)=M_b^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}}=b}\mathbf{1}_{\{V(u)=M_b^{\mathrm{kill}}\}}},\min_{k\in[0,b]}V(w_k)\ge a_n(z+L+1),\\ V(w_b)\in I_n(z)\biggr].\end{aligned}\tag{3.26}$$

Notice that $F^{\mathrm{kill}}(x) \le n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{P}}_x(\min_{k\in[0,b]} V(w_k) \ge a_n(z+L+1), V(w_b) \in I_n(z))$. Hence

$$\begin{aligned}
&\big|\hat{\mathbf{E}}_{(3.25)} - \hat{\mathbf{E}}\big[F^{\mathrm{kill}}\big(V(w_{n-b})\big), V(w_\ell) \ge d_\ell, \forall \ell \le n-b\big]\big| \\
&\quad = \hat{\mathbf{E}}\big[F^{\mathrm{kill}}\big(V(w_{n-b})\big), V(w_\ell) \ge d_\ell, \forall \ell \le n-b, (\mathcal{E}_n)^c\big] \\
&\quad \le n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}\Big[\hat{\mathbf{P}}_{V(w_{n-b})}\Big(\min_{k\in[0,b]} V(w_k) \ge a_n(z+L+1), V(w_b) \in I_n(z)\Big) \\
&\qquad\qquad \times \mathbf{1}_{\{V(w_\ell)\ge d_\ell, \forall \ell \le n-b\}\cap(\mathcal{E}_n)^c}\Big],
\end{aligned}$$

where we wrote $\mathcal{E}_n$ for $\mathcal{E}_n(z,b)$. By the Markov property, the term

$$\begin{aligned}
&\hat{\mathbf{E}}\Big[\hat{\mathbf{P}}_{V(w_{n-b})}\Big(\min_{k\in[0,b]} V(w_k) \ge a_n(z+L+1), V(w_b) \in I_n(z)\Big) \\
&\qquad \times \mathbf{1}_{\{V(w_\ell)\ge d_\ell, \forall \ell \le n-b\},(\mathcal{E}_n)^c}\Big]
\end{aligned}$$

is equal to $\hat{\mathbf{P}}(w_n \in \mathcal{Z}_n^{z,L}, (\mathcal{E}_n(z,b))^c)$ which is at most $\eta n^{-3/2}$ when $z \ge A$ and $n \ge b$ by Lemma 3.8 and our choice of A and B. Therefore, for any $n \ge b$ and $z \ge A$,

$$\big|\hat{\mathbf{E}}_{(3.25)} - \hat{\mathbf{E}}\big[F^{\mathrm{kill}}\big(V(w_{n-b})\big), V(w_\ell) \ge d_\ell, \forall \ell \le n-b\big]\big| \le \eta \mathrm{e}^{-z}. \tag{3.27}$$

Recall the definition of $F_{L,b}$ in (3.23). We would like to replace $F^{\mathrm{kill}}(x)$ by $n^{3/2}\mathrm{e}^{-z}F_{L,b}(x-a_n(z+L))$. We notice that

$$\begin{aligned}
&n^{3/2}\mathrm{e}^{-z}F_{L,b}\big(x-a_n(z+L)\big) \\
&\quad = \hat{\mathbf{E}}_x\left[\frac{\mathrm{e}^{V(w_b)}\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}, \min_{k\in[0,b]} V(w_k) \ge a_n(z+L+1), V(w_b) \in I_n(z)\right].
\end{aligned}$$

We observe that the only difference with (3.26) is that the branching random walk is not killed anymore. Since $\big|\frac{\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}} - \frac{\mathbf{1}_{\{V(w_b)=M_b^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}}=b}\mathbf{1}_{\{V(u)=M_b^{\mathrm{kill}}\}}}\big|$ is at most 1 and is equal to zero if no particle touched the barrier 0, we have that, for any $H \ge 0$ such that $H \le a_n(z+L)$,

$$\left|\frac{\mathbf{1}_{\{V(w_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}} - \frac{\mathbf{1}_{\{V(w_b)=M_b^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}}=b}\mathbf{1}_{\{V(u)=M_b^{\mathrm{kill}}\}}}\right| \le \mathbf{1}_{\{\exists |u|\le b:\, V(u)\le a_n(z+L+H)\}}.$$

Consequently,

$$\begin{aligned}
&\big|F^{\mathrm{kill}}(x) - n^{3/2}\mathrm{e}^{-z}F_{L,b}\big(x-a_n(z+L)\big)\big| \\
&\quad \le \hat{\mathbf{E}}_x\Big[\mathrm{e}^{V(w_b)}\mathbf{1}_{\{\exists |u|\le b:\, V(u)\le a_n(z+L+H)\}}, \\
&\qquad\qquad \min_{k\in[0,b]} V(w_k) \ge a_n(z+L+1), V(w_b) \in I_n(z)\Big]
\end{aligned}$$

$$\leq n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}_x\Big[\mathbf{1}_{\{\exists|u|\leq b:\,V(u)\leq a_n(z+L+H)\}},\\ \min_{k\in[0,b]} V(w_k)\geq a_n(z+L+1),\,V(w_b)\in I_n(z)\Big]$$

$$= n^{3/2}\mathrm{e}^{-z}G_H\big(x-a_n(z+L)\big)$$

with for any $y\geq 0$,

$$G_H(y):=\hat{\mathbf{P}}_y\Big(\{\exists|u|\leq b: V(u)\leq -H\}\\ \cap\Big\{\min_{k\in[0,b]} V(w_k)\geq -1,\,V(w_b)\in[L-1,L)\Big\}\Big).$$

We do not write the dependency on L and $b\geq B$ because they are fixed in this proof and so are considered as constants. This shows that, for any $z\geq 0$, $n\geq 1$ and $H\in[0,a_n(z+L)]$,

$$\hat{\mathbf{E}}\big[\big|F^{\mathrm{kill}}\big(V(w_{n-b})\big)-n^{3/2}\mathrm{e}^{-z}F_{L,b}\big(V(w_{n-b})-a_n(z+L)\big)\big|\mathbf{1}_{\{V(w_\ell)\geq d_\ell,\forall\ell\leq n-b\}}\big]\\ \leq n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}\big[G_H\big(V(w_{n-b})-a_n(z+L)\big)\mathbf{1}_{\{V(w_\ell)\geq d_\ell,\forall\ell\leq n-b\}}\big].$$

We choose H such that $\frac{C_-C_+\sqrt{\pi}}{\sigma\sqrt{2}}\int_{y\geq 0}G_H(y)R_-(y)\,dy\leq\eta/2$. We can check that the function G_H satisfies the conditions of Lemma 2.3 as we did for $F_{L,b}$. By Lemma 2.3, this yields that

$$\hat{\mathbf{E}}\big[\big|F^{\mathrm{kill}}\big(V(w_{n-b})\big)-n^{3/2}\mathrm{e}^{-z}F_{L,b}\big(V(w_{n-b})-a_n(z+L)\big)\big|\mathbf{1}_{\{V(w_\ell)\geq d_\ell,\forall\ell\leq n-b\}}\big]\\ \leq\eta\mathrm{e}^{-z}$$

for n large enough and $z\in[0,(3/2)\ln n-L-H]$. The cut-off at $(3/2)\ln(n)-L-H$ is here only to ensure that $H\leq a_n(z+L)$. Combined with (3.27), we get that for n large enough, and $z\in[A,(3/2)\ln(n)-L-H]$,

$$\big|\hat{\mathbf{E}}_{(3.25)}-n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}\big[F_{L,b}\big(V(w_{n-b})-a_n(z+L)\big),\,V(w_\ell)\geq d_\ell,\\ \forall 0\leq\ell\leq n-b\big]\big|\\ \leq 2\eta\mathrm{e}^{-z}. \tag{3.28}$$

Recall the definition of $C_{L,b}$ in (3.24). We apply again Lemma 2.3 to see that

$$\hat{\mathbf{E}}\big[F_{L,b}\big(V(w_{n-b})-a_n(z+L)\big),\,V(w_\ell)\geq d_\ell,\forall 0\leq\ell\leq n-b\big]\sim\frac{C_{L,b}}{n^{3/2}}$$

as $n\to\infty$ uniformly in $z\in[0,(3/2)\ln(n)-L]$. Consequently, we have for n large enough and $z\in[0,(3/2)\ln(n)-L]$,

$$\big|n^{3/2}\mathrm{e}^{-z}\hat{\mathbf{E}}\big[F_{L,b}\big(V(w_{n-b})-a_n(z+L)\big),\,V(w_\ell)\geq d_\ell,\forall 0\leq\ell\leq n-b\big]-\mathrm{e}^{-z}C_{L,b}\big|\\ \leq\eta\mathrm{e}^{-z}.$$

The lemma follows from (3.28). □

We now have the tools to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. Let $\hat{\mathbf{E}}_{(3.25)}$ be the expectation in the left-hand side of (3.25). We introduce for any $L \geq 0$ and any integer $b \geq 1$,

$$C_{L,b}^{-} := \liminf_{z\to\infty} \liminf_{n\to\infty} \mathrm{e}^{z} \hat{\mathbf{E}}_{(3.25)},$$

$$C_{L,b}^{+} := \limsup_{z\to\infty} \limsup_{n\to\infty} \mathrm{e}^{z} \hat{\mathbf{E}}_{(3.25)}.$$

In particular, taking the limits in $n \to \infty$ then $z \to \infty$ in (3.25), we have, for any $L \geq 0$, $\eta > 0$ and $b \geq B(L, \eta)$ [with $B(L, \eta)$ as in Lemma 3.8],

$$C_{L,b} - 3\eta \leq C_{L,b}^{-} \leq C_{L,b}^{+} \leq C_{L,b} + 3\eta. \tag{3.29}$$

Notice that $\mathcal{E}_n(z, b)$ (hence $\hat{\mathbf{E}}_{(3.25)}$) is increasing in b. This implies that $C_{L,b}^{-}$ and $C_{L,b}^{+}$ are both increasing in b. For any $L \geq 0$, let C_L^{-} and C_L^{+} be, respectively, the (possibly zero or infinite) limits of $C_{L,b}^{-}$ and $C_{L,b}^{+}$ when $b \to \infty$. By (3.29), we have for any $L \geq 0$ and $\eta > 0$,

$$\limsup_{b\to\infty} C_{L,b} - 3\eta \leq C_L^{-} \leq C_L^{+} \leq \liminf_{b\to\infty} C_{L,b} + 3\eta.$$

Letting η go to 0, this yields that $C_{L,b}$ has a limit as $b \to \infty$, that we denote by $C(L) = C_L^{-} = C_L^{+}$, this for any $L \geq 0$. Similarly, we see that $\hat{\mathbf{E}}_{(3.25)}$ is increasing in L. This gives that $C(L)$ admits a limit as $L \to \infty$, that we denote by C_2. Beware that at this stage, we do not know whether $C_2 \in (0, \infty)$. Let $\varepsilon > 0$. By (3.14), there exists $L_0 \geq 0$ such that for any $L \geq L_0$, $z \geq 0$ and $n \geq 1$,

$$\mathbf{P}\big(m^{\mathrm{kill},(n)} \notin \mathcal{Z}_n^{z,L}, M_n^{\mathrm{kill}} \in I_n(z)\big) \leq \varepsilon \mathrm{e}^{-z}.$$

By Lemma 3.7, this yields that for $L \geq L_0$, $z \geq 0$ and any $n \geq 1$,

$$\left| \mathbf{P}\big(M_n^{\mathrm{kill}} \in I_n(z)\big) - \hat{\mathbf{E}}\left[\frac{\mathrm{e}^{V(w_n)} \mathbf{1}_{\{V(w_n) = M_n^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}} = n} \mathbf{1}_{\{V(u) = M_n^{\mathrm{kill}}\}}}, w_n \in \mathcal{Z}_n^{z,L} \right] \right| \leq \varepsilon \mathrm{e}^{-z}.$$

Take again $\eta > 0$ and $L \geq L_0$, and let $B = B(L, \eta) \geq 1$ and $A = A(L, \eta) > 0$ as in Lemma 3.8. We have

$$\begin{aligned} &\hat{\mathbf{E}}\left[\frac{\mathrm{e}^{V(w_n)} \mathbf{1}_{\{V(w_n) = M_n^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}} = n} \mathbf{1}_{\{V(u) = M_n^{\mathrm{kill}}\}}}, w_n \in \mathcal{Z}_n^{z,L}, \mathcal{E}_n(z,b)^c \right] \\ &\qquad \leq n^{3/2} \mathrm{e}^{-z} \hat{\mathbf{P}}\big(w_n \in \mathcal{Z}_n^{z,L}, \mathcal{E}_n(z,b)^c\big) \leq \eta \mathrm{e}^{-z} \end{aligned}$$

for any $n \geq b \geq B$ and $z \geq A$. Consequently, for any $L \geq L_0$, $n \geq b \geq B$ and $z \geq A$,

$$\left| \mathbf{P}\big(M_n^{\mathrm{kill}} \in I_n(z)\big) - \hat{\mathbf{E}}\left[\frac{\mathrm{e}^{V(w_n)} \mathbf{1}_{\{V(w_n) = M_n^{\mathrm{kill}}\}}}{\sum_{|u|^{\mathrm{kill}} = n} \mathbf{1}_{\{V(u) = M_n^{\mathrm{kill}}\}}}, w_n \in \mathcal{Z}_n^{z,L}, \mathcal{E}_n(z,b) \right] \right| \leq (\varepsilon + \eta) \mathrm{e}^{-z}.$$

By Lemma 3.9, we get that for $L \geq L_0$, $b \geq B(L, \eta)$, n large enough and $z \in [A(L, \eta), (3/2)\ln(n) - L - H(L, \eta, b)]$,

$$\left|\mathrm{e}^z \mathbf{P}\big(M_n^{\mathrm{kill}} \in I_n(z)\big) - C_{L,b}\right| \leq (\varepsilon + 4\eta). \tag{3.30}$$

We stress that $C_{L,b}$ depends actually on η and ε through the choice of L_0 and $B(L, \eta)$. By (3.30) and Corollary 3.4, we know that for $L \geq L_0$ and $b \geq B(L, \eta)$, we have $C_{L,b} \leq c_{10} + \varepsilon + 4\eta$. Taking the limit $b \to \infty$, this implies that for any $L \geq L_0$, we have $C(L) \leq c_{10} + \varepsilon + 4\eta$. Taking the limit $L \to \infty$, we deduce that $C_2 \leq c_{10} + \varepsilon + 4\eta$ hence C_2 is finite. Let $L > L_0$ such that $|C_2 - C(L)| \leq \eta$ and $b \geq B(L)$ such that $|C_{L,b} - C(L)| \leq \eta$. Then, by (3.30), we have for n large enough and $z \in [A(L, \eta), (3/2)\ln(n) - L - H(L, \eta, b)]$,

$$\left|\mathrm{e}^z \mathbf{P}\big(M_n^{\mathrm{kill}} \in I_n(z)\big) - C_2\right| \leq \varepsilon + 6\eta \leq 2\varepsilon$$

if we take $\eta := \varepsilon/6$. It remains to show that $C_2 > 0$. We see that, necessarily,

$$\limsup_{z\to\infty} \limsup_{n\to\infty} \left| \mathrm{e}^z \mathbf{P}\left(M_n^{\mathrm{kill}} < \frac{3}{2}\log n - z\right) - \frac{C_2}{1 - \mathrm{e}^{-1}} \right| = 0.$$

We know then that $C_2 > 0$ by the lower bound obtained in Lemma 3.6. □

3.3. *Proof of Lemma* 3.8. We present here the postponed proof of Lemma 3.8.

PROOF OF LEMMA 3.8. We follow the same strategy as for Lemma 3.6. Let $\eta > 0$. To avoid superfluous notation, we prove the lemma for $L = 0$ (the general case works similarly). Recall the definition of $\mathcal{E}_n(z, b)$ in (3.21). We want to show that $\hat{\mathbf{P}}(\mathcal{E}_n(z, b)^c, w_n \in \mathcal{Z}_n^{z,0}) \leq \eta n^{-3/2}$ when b and z are large enough. Let $d_k = d_k(n, z, 1/2)$ as defined in (3.4) and

$$e_k = e_k^{(n)} := \begin{cases} k^{1/12}, & \text{if } 0 \leq k \leq \dfrac{n}{2}, \\ (n-k)^{1/12}, & \text{if } \dfrac{n}{2} < k \leq n. \end{cases} \tag{3.31}$$

We recall that $|u| = n$ is a z-good vertex if $u \in \mathcal{Z}_n^{z,0}$ and

$$\sum_{v \in \Omega(u_k)} \mathrm{e}^{-(V(v) - d_k)}\big\{1 + \big(V(v) - d_k\big)_+\big\} \leq B\mathrm{e}^{-e_k} \qquad \forall 1 \leq k \leq n$$

with B such that, for $n \geq 1$ and $z \geq 0$,

$$\hat{\mathbf{P}}\big(w_n \in \mathcal{Z}_n^{z,0}, w_n \text{ is not a } z\text{-good vertex}\big) \leq \frac{\eta}{n^{3/2}}; \tag{3.32}$$

see Lemma C.1. Recall that $\Omega(w_k)$ is the set of siblings of w_k and $\hat{\mathcal{G}}_\infty$ is defined in (2.5). Recall the law of the branching random walk under $\hat{\mathbf{P}}$ which we identified with $\hat{\mathcal{B}}$ by the mean of Proposition 2.1. For $\mathcal{E}_n(z, b)$ to happen, every sibling of the spine at generation less than $n-b$ must have all its descendants at time n at position greater than $a_n(z)$. In other words,

$$\hat{\mathbf{P}}\big((\mathcal{E}_n(z,b))^c, w_n \text{ is a } z\text{-good vertex}\big) = \hat{\mathbf{E}}\Bigg[1-\prod_{k=1}^{n-b}\prod_{u\in\Omega(w_k)}\big(1-\Phi_{k,n}^{\mathrm{kill}}(V(u),z)\big), w_n \text{ is a } z\text{-good vertex}\Bigg], \tag{3.33}$$

where $\Phi_{k,n}^{\mathrm{kill}}(V(u),z) := \mathbf{P}_{V(u)}(M_{n-k}^{\mathrm{kill}} < a_n(z))$ is the probability that the killed branching random walk rooted at u has its minimum greater than $a_n(z)$ at time $n-k$. By Corollary 3.4, we see that if $|u|\le n/2$ [hence $a_n(z)=a_{n-|u|}(z)+O(1)$], then

$$\Phi_{k,n}^{\mathrm{kill}}\big(V(u),z\big)\le c_{20}\big(1+V(u)_+\big)\mathrm{e}^{-z-V(u)}.$$

On the event that w_n is a z-good vertex, we have for $k\le n/2$ (hence $d_k=0$), $\sum_{u\in\Omega(w_k)}(1+V(u)_+)\mathrm{e}^{-V(u)}\le B\mathrm{e}^{-e_k}=B\mathrm{e}^{-k^{1/12}}$. Using the inequality $x\ge \mathrm{e}^{(x-1)/2}$ for x close enough to 1, we deduce that there exists $A_0\ge 0$ such that for $z\ge A_0$, $n\ge 1$ and $1\le k\le n/2$, on the event that w_n is a z-good vertex, we have

$$\prod_{u\in\Omega(w_k)}\big(1-\Phi_{k,n}^{\mathrm{kill}}(V(u),z)\big)\ge \exp\big(-c_{21}\mathrm{e}^{-z}\mathrm{e}^{-k^{1/12}}\big)$$

with $c_{21}:=c_{20}B/2$. This yields that

$$\prod_{k=1}^{\lfloor n/2\rfloor}\prod_{u\in\Omega(w_k)}\big(1-\Phi_{k,n}^{\mathrm{kill}}(V(u),z)\big)\ge \exp\Bigg(-c_{21}\mathrm{e}^{-z}\sum_{k=1}^{\lfloor n/2\rfloor}\mathrm{e}^{-k^{1/12}}\Bigg)\ge \exp\big(-c_{22}\mathrm{e}^{-z}\big).$$

Therefore, there exists $A_1>A_0$ such that for any $z\ge A_1$ and $n\ge 1$,

$$\prod_{k=1}^{\lfloor n/2\rfloor}\prod_{u\in\Omega(w_k)}\big(1-\Phi_{k,n}^{\mathrm{kill}}(V(u),z)\big)\ge (1-\eta)^{1/2}. \tag{3.34}$$

If $k>n/2$, we simply observe that if $M_\ell^{\mathrm{kill}}\le x$, a fortiori $M_\ell\le x$. Since W_n [defined in (2.2)] is a martingale, we have $1=\mathbf{E}[W_\ell]\ge \mathbf{E}[\mathrm{e}^{-M_\ell}]\ge \mathrm{e}^{-x}\mathbf{P}(M_\ell\le x)$ for any $\ell\ge 1$ and $x\in\mathbb{R}$. We get that

$$\Phi_{k,n}^{\mathrm{kill}}\big(V(u),z\big)\le \mathbf{P}\big(M_{n-|u|}<a_n(z)-V(u)\big)\le \mathrm{e}^{a_n(z)}\mathrm{e}^{-V(u)}.$$

We rewrite it $\Phi_{k,n}^{\mathrm{kill}}(V(u),z)\le \mathrm{e}^{-(V(u)-d_k)}$ for $n/2<k\le n$. On the event that w_n is a z-good vertex, we get that $\prod_{u\in\Omega(w_k)}(1-\Phi_{k,n}^{\mathrm{kill}}(V(u),z))\ge \mathrm{e}^{-c_{23}\mathrm{e}^{-e_k}}=\mathrm{e}^{-c_{23}(n-k)^{1/12}}$ for k greater than some constant b_1. Consequently, for any $b\ge b_1$,

$$\prod_{k=\lfloor n/2\rfloor+1}^{n-b}\prod_{u\in\Omega(w_k)}\big(1-\Phi_{k,n}^{\mathrm{kill}}(V(u),z)\big)\ge \mathrm{e}^{-c_{23}\sum_{k=\lfloor n/2\rfloor+1}^{n-b}\mathrm{e}^{-(n-k)^{1/12}}}.$$

This yields that there exists $B \geq 1$ such that for any $b \geq B$ and any $n \geq 1$, we have

$$\prod_{k=\lfloor n/2\rfloor+1}^{n-b} \prod_{u\in\Omega(w_k)} \bigl(1-\Phi_{k,n}^{\mathrm{kill}}\bigl(V(u),z\bigr)\bigr) \geq (1-\eta)^{1/2}. \tag{3.35}$$

In view of (3.34) and (3.35), we have for $b \geq B$, $z \geq A_1$ and $n \geq 1$,

$$\prod_{k=1}^{n-b} \prod_{u\in\Omega(w_k)} \bigl(1-\Phi_{k,n}^{\mathrm{kill}}\bigl(V(u),z\bigr)\bigr) \geq (1-\eta).$$

Plugging it into (3.33) yields that

$$\begin{aligned}\hat{\mathbf{P}}\bigl(\bigl(\mathcal{E}_n(z,b)\bigr)^c, w_n \text{ is a } z\text{-good vertex}\bigr) &\leq \eta\hat{\mathbf{P}}(w_n \text{ is a } z\text{-good vertex})\\ &\leq \eta\hat{\mathbf{P}}\bigl(w_n \in \mathcal{Z}_n^{z,0}\bigr).\end{aligned}$$

It follows from (3.32) that

$$\hat{\mathbf{P}}\bigl(\bigl(\mathcal{E}_n(z,b)\bigr)^c, w_n \in \mathcal{Z}_n^{z,0}\bigr) \leq \eta\bigl(\hat{\mathbf{P}}\bigl(w_n \in \mathcal{Z}_n^{z,0}\bigr)+n^{-3/2}\bigr).$$

Recall that the spine behaves as a centered random walk. Then apply (2.9) to see that $\hat{\mathbf{P}}(w_n \in \mathcal{Z}_n^{z,0}) \leq c_{24}n^{-3/2}$, which completes the proof of the lemma. □

4. Tail distribution of the minimum of the BRW.

We prove a slightly stronger version of Proposition 1.3.

PROPOSITION 4.1. *Let C_1 be as in Proposition 1.2 and c_0 as in (2.13). For any $\varepsilon > 0$, there exist $N \geq 1$ and $A > 0$ such that for any $n \geq N$ and $z \in [A, (3/2)\ln n - A]$,*

$$\left|\frac{\mathrm{e}^z}{z}\mathbf{P}\left(M_n < \frac{3}{2}\ln n - z\right) - C_1c_0\right| \leq \varepsilon.$$

We introduce some notation. To go from the tail distribution of M_n^{kill} to the one of M_n, we have to control excursions inside the negative axis that can appear at the beginning of the branching random walk. For any real r, we define the set

$$\mathcal{S}^r := \Bigl\{u \in \mathbb{T}: \min_{k\leq|u|-1} V(u_k) > V(u) \geq -r\Bigr\} \tag{4.1}$$

(see Figure 2). Notice that $\mathcal{S}^r = \varnothing$ when $r < 0$. Let for $|v| \geq 1$,

$$\xi(v) := \sum_{w\in\Omega(v)} \bigl(1+\bigl(V(w)-V(\overleftarrow{v})\bigr)_+\bigr)\mathrm{e}^{-(V(w)-V(\overleftarrow{v}))}, \tag{4.2}$$

where $\overleftarrow{v}$ denotes the parent of v [and $y_+ := \max(y,0)$]. Notice that $\xi(v)$ is stochastically smaller than $X + \tilde{X}$ as defined in (1.2). To avoid some extra integrability conditions, we are led to consider vertices $u \in \mathcal{S}^r$ which behave “nicely,”

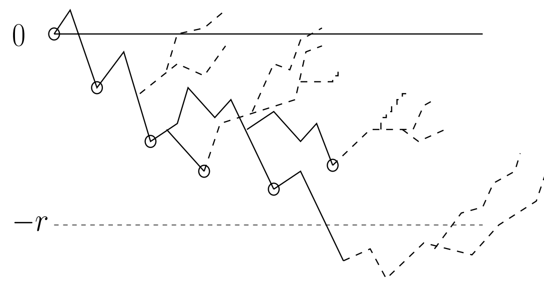

FIG. 2. *The set $\mathcal{S}^r$.*

meaning that $\xi(u_k)$ is not too big along the path $\{u_1, \ldots, u_{|u|} = u\}$. Hence, for any real $r \geq 0$, we introduce

$$\mathcal{T}^r := \{u \in \mathbb{T} : \forall 1 \leq k \leq |u| : \xi(u_k) < \mathrm{e}^{(V(u_{k-1})+r)/2}\}. \tag{4.3}$$

For any integer $k \geq 0$, we denote by $\mathcal{S}_k^r$, respectively, $\mathcal{T}_k^r$, the set $\mathcal{S}^r \cap \{|u| = k\}$, respectively, $\mathcal{T}^r \cap \{|u| = k\}$. Finally, for any integer $n \geq 1$, any $z \geq 0$ and any $u \in \mathbb{T}$, define

$$B_n^z(u) := \begin{cases} 1, & \text{if } \exists v \geq u : |v| = n, \min_{\ell \in [|u|, n]} V(v_\ell) \geq V(u), \\ & \text{and } V(v) < a_n(z), \\ 0, & \text{otherwise} \end{cases} \tag{4.4}$$

(see Figure 3). Notice that $B_n^z(u) = 0$ if $|u| > n$. In words, $B_n^z(u) = 1$ if there exists a descendant of u which stays above $V(u)$ and is below level $a_n(z)$ at time n. Observe that if $M_n < a_n(z)$, then necessarily we can find such vertices u and v. The first subsection controls the set $\mathcal{S}^r$. Proposition 1.3 is then proved in Section 4.2.

4.1. *The branching random walk at the beginning.* We will see that $\mathbf{P}(M_n < \frac{3}{2} \ln n - z)$ is comparable to the probability that there exists $u \in \mathcal{S}^z$ such that $B_n^z(u) = 1$. The lemmas in this section are used to give an asymptotic of this probability. As usual, we will use a second moment argument. Lemmas 4.2 and 4.3 give bounds, respectively, on the first moment and second moment of the number of such vertices u. We recall that M_n^{kill} is the minimum at time n of the branching random walk killed below zero. For any integers $n \geq 1$, $k \in [0, n]$, and any reals x, r, we recall that

$$\Phi_{k,n}^{\text{kill}}(x, r) := \mathbf{P}_x(M_{n-k}^{\text{kill}} < a_n(r)). \tag{4.5}$$

By Corollary 3.2, there exists $N_0 \geq 1$ and $A_0 \geq 0$ such that for any $n \geq N_0$, $k \leq n^{1/2}$ and $r \in [A_0, (3/2)\ln(n) - A_0]$,

$$|\mathrm{e}^r \Phi_{k,n}^{\text{kill}}(0, r) - C_1| \leq \varepsilon, \tag{4.6}$$

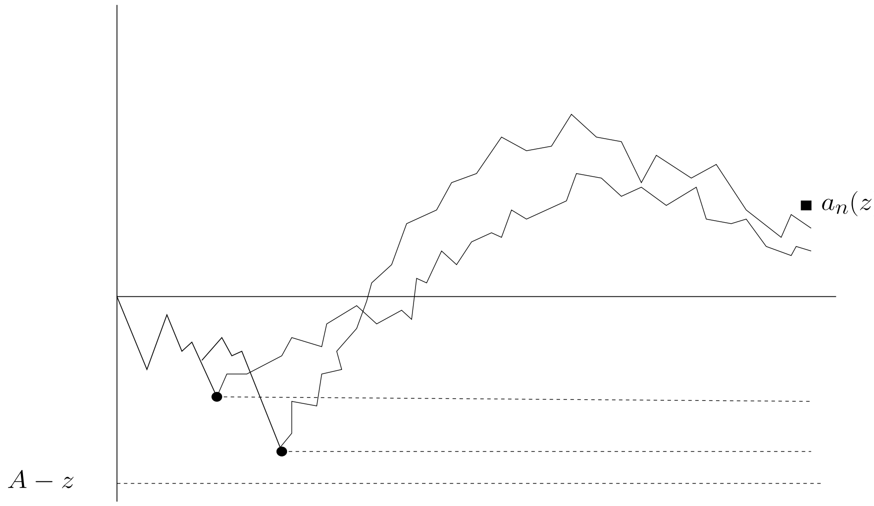

FIG. 3. *Particles in $\mathcal{S}^{z-A}$ such that $B_n^z(u)=1$.*

where we used the fact that $k=o(n)$, thus $\ln(n-k)=\ln(n)+o(1)$ [the same statement holds when replacing $n^{1/2}$ by any sequence $o(n)$]. Moreover, we know by Corollary 3.4 that for any integers $n\geq 1$, $k\in[0,n]$ and any reals $x,r\geq 0$,

$$\Phi_{k,n}^{\mathrm{kill}}(x,r)\leq c_{25}(1+x)\mathrm{e}^{-x-r}\left(\frac{n}{n-k+1}\right)^{3/2}. \tag{4.7}$$

LEMMA 4.2. (i) *Fix $\varepsilon>0$ and let C_1 be the constant in Proposition 1.2. There exists $A\geq 0$ such that for all n sufficiently large, and all $z\in[A,(3/2)\ln(n)-A]$,*

$$\left|\frac{\mathrm{e}^z}{R(z-A)}\mathbf{E}\left[\sum_{u\in\mathcal{S}^{z-A}}B_n^z(u)\mathbf{1}_{\{|u|\leq n^{1/2}\}}\right]-C_1\right|\leq\varepsilon. \tag{4.8}$$

(ii) *There exists a constant c such that for any $n\geq 1$ and any $z\in[0,(3/2)\ln(n)]$,*

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}^z}B_n^z(u)\mathbf{1}_{\{|u|>n^{1/2}\}}\right]\leq c\mathrm{e}^{-z}.$$

(iii) *Uniformly in $A\geq 0$ and $n\geq 1$, we have*

$$\mathbf{E}\left[\sum_{u\in\mathcal{S}^{z-A}\cap(\mathcal{T}^{z-A})^c}B_n^z(u)\mathbf{1}_{\{|u|\leq n/2\}}\right]=o(z)\mathrm{e}^{-z}$$

as $z\to\infty$, where the set $(\mathcal{T}^{z-A})^c$ denotes the complement of the set $\mathcal{T}^{z-A}$ in the set of vertices of $\mathbb{T}$.

REMARK. In (i) and (ii), we could replace $n^{1/2}$ by n^a with $a\in(0,1)$.

PROOF OF LEMMA 4.2. Let $k \le n$. By the Markov property at time k, we have

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A}} B_n^z(u)\Big] = \mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A}} \Phi_{k,n}^{\mathrm{kill}}(0, z+V(u))\Big] \tag{4.9}$$

with $\Phi_{k,n}^{\mathrm{kill}}$ as defined in (4.5). We want to apply equation (4.6) to $r = z + V(u)$. We observe that $z + V(u) \in [A, z]$ when $u \in \mathcal{S}^{z-A}$. Hence, equation (4.6) holds for $n \ge N_0$, $k \le n^{1/2}$ and $r = z + V(u)$, with $u \in \mathcal{S}^{z-A_0}$ and $z \in [A_0, (3/2)\ln(n) - A_0]$. It follows from (4.9) that for $n \ge N_0$, $k \le n^{1/2}$ and $z \in [A_0, (3/2)\ln(n) - A_0]$,

$$\Big|\mathrm{e}^z\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A_0}} B_n^z(u)\Big] - C_1\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A_0}} \mathrm{e}^{-V(u)}\Big]\Big| \le \varepsilon\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A_0}} \mathrm{e}^{-V(u)}\Big]. \tag{4.10}$$

From the definition of $\mathcal{S}_k^{z-A}$ and (2.1), we observe that, for any integer k, and any $z \ge A \ge 0$,

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A}} \mathrm{e}^{-V(u)}\Big] = \mathbf{P}(S_k \ge A - z, S_k < S_\ell, \forall 0 \le \ell < k-1). \tag{4.11}$$

Summing over $k \ge 0$ yields that

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}^{z-A}} \mathrm{e}^{-V(u)}\Big] = R(z-A). \tag{4.12}$$

In particular, summing equation (4.10) over $k \le n^{1/2}$ gives that for $n \ge N_0$ and $z \in [A_0, (3/2)\ln(n) - A_0]$,

$$\begin{aligned}&\Big|\mathrm{e}^z\mathbf{E}\Big[\sum_{u\in\mathcal{S}^{z-A_0}} B_n^z(u)\mathbf{1}_{\{|u|\le n^{1/2}\}}\Big] - C_1\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A_0}} \mathrm{e}^{-V(u)}\mathbf{1}_{\{|u|\le n^{1/2}\}}\Big]\Big|\\ &\quad\le \varepsilon R(z-A_0).\end{aligned} \tag{4.13}$$

Using the fact that $\mathbf{P}(S_k \ge -x, S_k < \min_{0\le j\le k-1} S_j) = \mathbf{P}((-S_k) \le x, \min_{0\le j\le k-1}(-S_j) \ge 0)$, we have by (2.8), for any integer $k \ge 0$ and any real $x \ge 0$,

$$\mathbf{P}\Big(S_k \ge -x, S_k < \min_{0\le j\le k-1} S_j\Big) \le \alpha_2'(1+x)^2(1+k)^{-3/2}. \tag{4.14}$$

Therefore, we have for n greater than some N_1 and $x \in [0, (3/2)\ln(n)]$,

$$\sum_{k>n^{1/2}} \mathbf{P}(S_k \ge -x, S_k < S_\ell, \forall 0 \le \ell < k-1) \le \varepsilon.$$

Going back to (4.11) with $A = A_0$, and summing over $k > n^{1/2}$, we obtain that for $n \ge N_1$ and $z \in [A_0, (3/2)\ln(n) - A_0]$,

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}^{z-A_0}} \mathrm{e}^{-V(u)}\mathbf{1}_{\{|u|>n^{1/2}\}}\Big] \le \varepsilon.$$

In view of (4.13) and (4.12), this yields that for any $n \geq \max(N_0, N_1)$ and any $z \in [A_0, (3/2)\ln(n) - A_0]$,

$$\left| e^z \mathbf{E}\left[\sum_{u \in \mathcal{S}^{z-A_0}} B_n^z(u) \mathbf{1}_{\{|u| \leq n^{1/2}\}}\right] - C_1 R(z - A_0)\right| \leq \varepsilon\big(R(z - A_0) + C_1\big).$$

Since $R(x) \geq 1$ for any $x \geq 0$, this completes the proof of (i). Let us prove (ii). The notation c denotes a constant whose value can change from line to line. Using (4.9) with $A = 0$, we find that

$$\mathbf{E}\left[\sum_{u \in \mathcal{S}^z} B_n^z(u) \mathbf{1}_{\{n/2 \geq |u| > n^{1/2}\}}\right] = \sum_{k=\lfloor n^{1/2}\rfloor+1}^{\lfloor n/2\rfloor} \mathbf{E}\left[\sum_{u \in \mathcal{S}_k^z} \Phi_{k,n}^{\mathrm{kill}}\big(0, z + V(u)\big)\right]. \tag{4.15}$$

Equation (4.7) yields that

$$\begin{aligned} &\mathbf{E}\left[\sum_{u \in \mathcal{S}^z} B_n^z(u) \mathbf{1}_{\{n/2 \geq |u| > n^{1/2}\}}\right] \\ &\quad \leq c_{25} e^{-z} \sum_{k=\lfloor n^{1/2}\rfloor+1}^{\lfloor n/2\rfloor} \left(\frac{n}{n-k+1}\right)^{3/2} \mathbf{E}\left[\sum_{u \in \mathcal{S}_k^z} e^{-V(u)}\right]. \end{aligned} \tag{4.16}$$

Equations (4.11) and (4.14) imply that

$$\begin{aligned} &\mathbf{E}\left[\sum_{u \in \mathcal{S}^z} B_n^z(u) \mathbf{1}_{\{n/2 \geq |u| > n^{1/2}\}}\right] \\ &\quad \leq c_{25} \alpha_2' e^{-z} (1+z)^2 \sum_{k=\lfloor n^{1/2}\rfloor+1}^{\lfloor n/2\rfloor} \left(\frac{n}{n-k+1}\right)^{3/2} (1+k)^{-3/2} \\ &\quad \leq c e^{-z} \end{aligned} \tag{4.17}$$

for any $n \geq 1$ and any $z \in [0, (3/2)\ln(n)]$. We deal now with vertices $u \in \mathcal{S}^z$ such that $|u| > n/2$, and split the case depending on whether $V(u)$ is greater or smaller than $-z + \frac{3}{2}\ln(\frac{n}{n-|u|+1})$. Using the fact that $B_n^z(u) \leq 1$, we get

$$\begin{aligned} &\mathbf{E}\left[\sum_{u \in \mathcal{S}^z} B_n^z(u) \mathbf{1}_{\{|u| > n/2\}}\right] \\ &\quad \leq \mathbf{E}\left[\sum_{u \in \mathcal{S}^z} B_n^z(u) \mathbf{1}_{\{|u| > n/2, V(u) \geq -z + (3/2)\ln(n/(n-|u|+1))\}}\right] \\ &\qquad + \mathbf{E}\left[\sum_{u \in \mathcal{S}^z} \mathbf{1}_{\{|u| > n/2, V(u) < -z + (3/2)\ln(n/(n-|u|+1))\}}\right]. \end{aligned} \tag{4.18}$$

We bound the first term of the right-hand side. Equation (4.7) shows that

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}^z} B_n^z(u)\mathbf{1}_{\{|u|>n/2,\, V(u)\ge -z+(3/2)\ln(n/(n-|u|+1))\}}\Big]$$

$$\le c_{25}\mathrm{e}^{-z}\sum_{k=\lfloor n/2\rfloor+1}^{n}\Big(\frac{n}{n-k+1}\Big)^{3/2}$$

$$\times\,\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^z}\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u)\ge -z+(3/2)\ln(n/(n-k+1))\}}\Big].$$

From (2.1), we observe that

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^z}\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u)\ge -z+(3/2)\ln(n/(n-k+1))\}}\Big]$$

$$=\mathbf{P}\Big(S_k\ge -z+\frac{3}{2}\ln\Big(\frac{n}{n-k+1}\Big), S_k<S_\ell, \forall 0\le \ell<k-1\Big),$$

which is 0 if $-z+\frac{3}{2}\ln(\frac{n}{n-k+1})\ge 0$. Using (4.14), we see that

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}^z} B_n^z(u)\mathbf{1}_{\{|u|>n/2,\, V(u)\ge -z+(3/2)\ln(n/(n-|u|+1))\}}\Big]$$

$$\le c\mathrm{e}^{-z}\sum_{k=\lfloor n/2\rfloor+1}^{n}\Big(\frac{n}{n-k+1}\Big)^{3/2}(1+k)^{-3/2}$$

$$\times\max\Big(z-\frac{3}{2}\ln\Big(\frac{n}{n-k+1}\Big),1\Big)^2.$$

Since $z\le\frac{3}{2}\ln(n)$, we get that $\max(z-\frac{3}{2}\ln(\frac{n}{n-k+1}),1)\le 1+\frac{3}{2}\ln(n-k+1)$. Consequently,

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}^z} B_n^z(u)\mathbf{1}_{\{|u|>n/2,\, V(u)\ge -z+(3/2)\ln(n/(n-|u|+1))\}}\Big]$$

$$\le c\mathrm{e}^{-z}\sum_{k=\lfloor n/2\rfloor+1}^{n}\Big(\frac{1}{n-k+1}\Big)^{3/2}\big(1+\ln(n-k+1)\big)^2\le c\mathrm{e}^{-z}.$$

Finally, let us consider the last term of (4.18). Equation (2.1) implies that, for any k,

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^z}\mathbf{1}_{\{|u|>n/2,\, V(u)<-z+(3/2)\ln(n/(n-|u|+1))\}}\Big]$$

$$=\mathbf{E}\big[\mathrm{e}^{S_k}\mathbf{1}_{\{S_k\in[-z,-z+(3/2)\ln(n/(n-k+1))),\, S_k<S_\ell,\forall 0\le\ell<k-1\}}\big] \tag{4.19}$$

$$\le\mathbf{E}\big[\mathrm{e}^{S_k}\mathbf{1}_{\{S_k<-z+(3/2)\ln(n/(n-k+1)),\, S_k<S_\ell,\forall 0\le\ell<k-1\}}\big].$$

We notice that

$$\mathbf{E}\big[\mathrm{e}^{S_k}\mathbf{1}_{\{S_k<-z+(3/2)\ln(n/(n-k+1)),S_k<S_\ell,\forall 0\le\ell<k-1\}}\big]$$
$$\le\sum_{y\ge 0}\mathrm{e}^{-y+1}\mathbf{P}(S_k\ge -y,\,S_k<S_\ell,\forall 0\le\ell<k-1)\mathbf{1}_{\{y>z-(3/2)\ln(n/(n-k+1))\}},$$

which, in view of (4.14) leads to

$$\begin{aligned}
&\mathbf{E}\big[\mathrm{e}^{S_k}\mathbf{1}_{\{S_k<-z+(3/2)\ln(n/(n-k+1)),S_k<S_\ell,\forall 0\le\ell<k-1\}}\big]\\
&\quad\le\alpha_2'\sum_{y\ge 0}\mathrm{e}^{-y+1}(1+y)^2(1+k)^{-3/2}\mathbf{1}_{\{y>z-(3/2)\ln(n/(n-k+1))\}}\\
&\quad\le c\mathrm{e}^{-z}\Big(\frac{n}{n-k+1}\Big)^{3/2}\max\Big(z-\frac{3}{2}\ln\Big(\frac{n}{n-k+1}\Big),1\Big)^2(1+k)^{-3/2}\\
&\quad\le c\mathrm{e}^{-z}\Big(\frac{1}{n-k+1}\Big)^{3/2}\Big(1+\frac{3}{2}\ln(n-k+1)\Big)^2
\end{aligned}$$

for $k\in[n/2,n]$ and $z\le\frac{3}{2}\ln(n)$. Going back to (4.19), it yields that

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{u\in\mathcal{S}^z}\mathbf{1}_{\{|u|>n/2,V(u)<-z+(3/2)\ln(n/(n-|u|+1))\}}\Big]\\
&\quad\le c\mathrm{e}^{-z}\sum_{k=\lfloor n/2\rfloor+1}^{n}\Big(\frac{1}{n-k+1}\Big)^{3/2}\Big(1+\frac{3}{2}\ln(n-k+1)\Big)^2\\
&\quad\le c\mathrm{e}^{-z}.
\end{aligned}$$

Finally, by (4.18), $\mathbf{E}[\sum_{u\in\mathcal{S}^z}B_n^z(u)\mathbf{1}_{\{|u|>n/2\}}]\le c\mathrm{e}^{-z}$ which, combined with (4.17), proves (ii). We prove now (iii). We have by the Markov property at time k,

$$\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A}\cap(\mathcal{T}^{z-A})^c}B_n^z(u)\Big]=\mathbf{E}\Big[\sum_{u\in\mathcal{S}_k^{z-A}\cap(\mathcal{T}^{z-A})^c}\Phi_{k,n}^{\mathrm{kill}}(0,z+V(u))\Big],$$

where $\Phi_{k,n}^{\mathrm{kill}}$ is defined in (4.5). By (4.7), this implies that

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{u\in\mathcal{S}^{z-A}\cap(\mathcal{T}^{z-A})^c}B_n^z(u)\mathbf{1}_{\{|u|\le n/2\}}\Big]\\
&\quad\le c_{26}\mathrm{e}^{-z}\mathbf{E}\Big[\sum_{u\in\mathcal{S}^{z-A}\cap(\mathcal{T}^{z-A})^c}\mathrm{e}^{-V(u)}\mathbf{1}_{\{|u|\le n/2\}}\Big].
\end{aligned}\tag{4.20}$$

At this stage, we make use of the measure $\hat{\mathbf{P}}$, introduced in Section 2.1. We recall that under $\hat{\mathbf{P}}$, we identified our branching randomly with $\hat{\mathcal{B}}$. By definition of $\hat{\mathbf{P}}$ then

Proposition 2.2(i), we have for any $k \le n/2$,

$$
\begin{aligned}
\mathbf{E}\Big[\sum_{u\in \mathcal{S}_k^{z-A}\cap(\mathcal{T}^{z-A})^c} \mathrm{e}^{-V(u)}\Big] &= \hat{\mathbf{E}}\Big[\frac{1}{W_k}\sum_{u\in \mathcal{S}_k^{z-A}\cap(\mathcal{T}^{z-A})^c} \mathrm{e}^{-V(u)}\Big] \\
&= \hat{\mathbf{P}}\big(w_k \in \mathcal{S}_k^{z-A}\cap\big(\mathcal{T}^{z-A}\big)^c\big).
\end{aligned}
\tag{4.21}
$$

The right-hand side is equal to 0 when $k = 0$ since $w_0 \in \mathcal{T}^{z-A}$ by definition. For $k \ge 1$, we observe that $\mathbf{1}_{\{w_k\in(\mathcal{T}^{z-A})^c\}} \le \sum_{\ell=1}^k \mathbf{1}_{\{\xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}}$. It follows that

$$
\hat{\mathbf{P}}\big(w_k \in \mathcal{S}_k^{z-A}\cap\big(\mathcal{T}^{z-A}\big)^c\big) \le \sum_{\ell=1}^{k}\hat{\mathbf{P}}\big(w_k\in\mathcal{S}^{z-A}, \xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big).
$$

Together with equations (4.20) and (4.21), this gives that

$$
\begin{aligned}
&\mathbf{E}\Big[\sum_{u\in\mathcal{S}^{z-A}\cap(\mathcal{T}^{z-A})^c} B_n^z(u)\mathbf{1}_{\{|u|\le n/2\}}\Big] \\
&\quad\le c_{26}\mathrm{e}^{-z}\sum_{\ell=1}^{\lfloor n/2\rfloor}\sum_{k=\ell}^{\lfloor n/2\rfloor}\hat{\mathbf{P}}\big(w_k\in\mathcal{S}^{z-A}, \xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big).
\end{aligned}
$$

In order to prove (iii), it is enough to show that

$$
\sum_{\ell\ge1}\sum_{k\ge\ell}\hat{\mathbf{P}}\big(w_k\in\mathcal{S}^{z-A}, \xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big) = o(z)
\tag{4.22}
$$

uniformly in $A \ge 0$ as $z\to\infty$. The left-hand side of (4.22) is 0 if $z < A$. Therefore, we will assume that $z \ge A$. For $k \ge \ell$, notice that if $w_k \in \mathcal{S}^{z-A}$, then necessarily $\min_{j\le\ell} V(w_j) \ge A - z$, $V(w_k) \ge A - z$ and $V(w_k) < \min_{\ell\le j\le k-1} V(w_j)$ [in particular, k is a ladder epoch for the random walk started at $V(w_\ell)$]. This implies that

$$
\begin{aligned}
&\hat{\mathbf{P}}\big(w_k\in\mathcal{S}^{z-A}, \xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big) \\
&\quad\le \hat{\mathbf{P}}\Big(\xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}, \min_{j\le\ell} V(w_j)\ge A-z, \\
&\qquad\qquad A-z\le V(w_k) < \min_{\ell\le j\le k-1} V(w_j)\Big).
\end{aligned}
$$

Summing over $k \ge \ell$, we get

$$
\begin{aligned}
&\sum_{k\ge\ell}\hat{\mathbf{P}}\big(w_k\in\mathcal{S}^{z-A}, \xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big) \\
&\quad\le \hat{\mathbf{E}}\Big[\mathbf{1}_{\{\xi(w_\ell)\ge \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}}\mathbf{1}_{\{\min_{j\le\ell} V(w_j)\ge A-z\}} \\
&\qquad\qquad \times\sum_{k\ge\ell}\mathbf{1}_{\{A-z\le V(w_k)<\min_{\ell\le j\le k-1} V(w_j)\}}\Big].
\end{aligned}
$$

By the Markov property at time ℓ, we recognize in the term

$$\sum_{k\geq \ell} \mathbf{1}_{\{A-z\leq V(w_k)<\min_{\ell\leq j\leq k-1} V(w_j)\}}$$

the number of strict descending ladder heights above level $A-z$ when starting from $V(w_\ell)$. Consequently,

$$\sum_{k\geq \ell} \hat{\mathbf{P}}\big(w_k \in \mathcal{S}^{z-A}, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big)$$
$$\leq \hat{\mathbf{E}}\big[\mathbf{1}_{\{\xi(w_\ell)\geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j\leq \ell} V(w_j)\geq A-z\}} R\big(z-A+V(w_\ell)\big)\big].$$

We know from (2.13) that there exists $c_{27}>0$ such that $R(x)\leq c_{27}(1+x)_+$ for any real x. Thus, $R(z-A+V(w_\ell)) \leq c_{27}(1+z-A+V(w_{\ell-1}))_+ + c_{27}(V(w_\ell) - V(w_{\ell-1}))_+$. Also, we obviously have $\min_{j\leq \ell} V(w_j) \leq \min_{j\leq \ell-1} V(w_j)$. This yields that

$$\sum_{k\geq \ell} \hat{\mathbf{P}}\big(w_k \in \mathcal{S}^{z-A}, \xi(w_\ell) \geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\big) \leq c_{27}\big(f(\ell)+g(\ell)\big),$$

where

$$f(\ell) := \hat{\mathbf{E}}\big[\mathbf{1}_{\{\xi(w_\ell)\geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j\leq \ell-1} V(w_j)\geq A-z\}} \big(1+z-A+V(w_{\ell-1})\big)\big],$$
$$g(\ell) := \hat{\mathbf{E}}\big[\mathbf{1}_{\{\xi(w_\ell)\geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j\leq \ell-1} V(w_j)\geq A-z\}} \big(V(w_\ell)-V(w_\ell-1)\big)_+\big].$$

Equation (4.22) boils down to

$$\sum_{\ell\geq 1} \big(f(\ell)+g(\ell)\big) = o(z). \tag{4.23}$$

Let (ξ, Δ) be a generic random variable independent of all the random variables used so far, and distributed as $(\xi(w_1), V(w_1))$ (under $\hat{\mathbf{P}}$). Using the Markov property at time $\ell-1$ in $f(\ell)$, we get

$$f(\ell) = \hat{\mathbf{E}}\big[\mathbf{1}_{\{\xi\geq \mathrm{e}^{(V(w_{\ell-1})+z-A)/2}\}} \mathbf{1}_{\{\min_{j\leq \ell-1} V(w_j)\geq A-z\}} \big(1+z-A+V(w_{\ell-1})\big)\big].$$

Summing over ℓ (and replacing $\ell-1$ by ℓ) yields that

$$\sum_{\ell\geq 1} f(\ell) = \hat{\mathbf{E}}\Big[\sum_{\ell\geq 0} \mathbf{1}_{\{V(w_\ell)+z-A\leq 2\ln(\xi)\}} \mathbf{1}_{\{\min_{j\leq \ell} V(w_j)\geq A-z\}} \big(1+z-A+V(w_\ell)\big)\Big].$$

By Lemma B.2(i), there exists $c_{28}>0$ such that for any $x\geq 0$

$$\hat{\mathbf{E}}\Big[\sum_{\ell\geq 0} \mathbf{1}_{\{V(w_\ell)+z-A\leq x\}} \mathbf{1}_{\{\min_{j\leq \ell} V(w_j)\geq A-z\}} \big(1+z-A+V(w_\ell)\big)\Big]$$
$$\leq (1+x) \sum_{\ell\geq 0} \hat{\mathbf{P}}\Big(V(w_\ell)+z-A\leq x, \min_{j\leq \ell} V(w_j)\geq A-z\Big)$$
$$\leq c_{28}(1+x)^2\big(1+\min(x, z-A)\big)$$
$$\leq c_{28}(1+x)^2\big(1+\min(x,z)\big).$$

We deduce that, with the notation of (1.2),

$$\sum_{\ell\ge 1} f(\ell) \le c_{28}\hat{\mathbf{E}}\big[(1+2\ln_+\xi)^2\big(1+\min(2\ln_+\xi, z)\big)\big]$$

$$(4.24)\qquad \le c_{28}\mathbf{E}\big[X\big(1+2\ln_+(X+\tilde X)\big)^2\big(1+\min\big(2\ln_+(X+\tilde X), z\big)\big)\big] = o(z)$$

under (1.4) by Lemma B.1(ii). We now consider $g(\ell)$. We have similarly

$$\sum_{\ell\ge 1} g(\ell) = \hat{\mathbf{E}}\Big[\Delta_+ \sum_{\ell\ge 0} \mathbf{1}_{\{V(w_\ell)+z-A\le 2\ln(\xi)\}}\mathbf{1}_{\{\min_{j\le\ell} V(w_j)\ge A-z\}}\Big].$$

From Lemma B.2(i), we get

$$\sum_{\ell\ge 1} g(\ell) \le c_{28}\hat{\mathbf{E}}\big[\Delta_+(1+2\ln_+\xi)\big(1+\min(2\ln_+\xi, z)\big)\big]$$

$$(4.25)\qquad \le c_{28}\mathbf{E}\big[\tilde X\big(1+2\ln_+(X+\tilde X)\big)\big(1+\min\big(2\ln_+(X+\tilde X), z\big)\big)\big] = o(z)$$

by Lemma B.1(ii). Equations (4.24) and (4.25) imply (4.23), and so complete the proof of (ii). □

REMARK. Equations (4.9), (4.7) and (4.12) imply that for any $n\ge 1$ and $z\ge 0$,

$$(4.26)\qquad \mathbf{E}\Big[\sum_{u\in\mathcal{S}^z} B_n^z(u)\mathbf{1}_{\{|u|\le n^{1/2}\}}\Big] \le c_{25}2^{3/2}R(z)\mathrm{e}^{-z}.$$

We compute the second moment in the following lemma.

LEMMA 4.3. *There exists a constant $c_{29}>0$ such that for any $z\ge A\ge 0$, and any integer $n\ge 1$,*

$$(4.27)\qquad \mathbf{E}[U^2]-\mathbf{E}[U] \le c_{29}\mathrm{e}^{-z}\mathrm{e}^{-A},$$

where $U := \sum_{u\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}} B_n^z(u)\mathbf{1}_{\{|u|\le n/2\}}$.

PROOF. Let U be as in the lemma. We observe that

$$U^2-U = \sum_{u\ne v} B_n^z(u)B_n^z(v)\mathbf{1}_{\{u,v\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{u,v\in\mathcal{T}^{z-A}\}}\mathbf{1}_{\{|u|,|v|\le n/2\}}.$$

It follows that

$$\mathbf{E}[U^2-U] = \mathbf{E}\Big[\sum_{u\ne v} B_n^z(u)B_n^z(v)\mathbf{1}_{\{u,v\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{u,v\in\mathcal{T}^{z-A}\}}\mathbf{1}_{\{|u|,|v|\le n/2\}}\Big]$$

$$\le 2\mathbf{E}\Big[\sum_{u\ne v, |u|\ge|v|} B_n^z(u)B_n^z(v)\mathbf{1}_{\{u,v\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{u\in\mathcal{T}^{z-A}\}}\mathbf{1}_{\{|u|\le n/2\}}\Big].$$

For $|u| \geq |v|$, and $u \neq v$, notice that $B_n^z(u)$ depends on the branching random walk rooted at u, whereas $B_n^z(v)\mathbf{1}_{\{u,v\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{u\in\mathcal{T}^{z-A}\}}$ is independent of it [even if v is a (strict) ancestor of u]. Therefore, by the branching property,

$$\mathbf{E}[U^2-U] \leq 2\mathbf{E}\Bigl[\sum_{u\neq v,|u|\geq|v|}\Phi^{\mathrm{kill}}_{|u|,n}\bigl(0,z+V(u)\bigr)B_n^z(v)\mathbf{1}_{\{u,v\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{u\in\mathcal{T}^{z-A}\}}\mathbf{1}_{\{|u|\leq n/2\}}\Bigr],$$

where $\Phi^{\mathrm{kill}}_{k,n}$ is defined in (4.5). By (4.7), we have $\Phi^{\mathrm{kill}}_{|u|,n}(z+V(u)) \leq c_{26}\mathrm{e}^{-z-V(u)}$ for $|u|\leq n/2$. This gives that

$$\begin{aligned}\mathbf{E}[U^2-U] &\leq c_{26}\mathrm{e}^{-z}\mathbf{E}\Bigl[\sum_{u\neq v,|u|\geq|v|}\mathrm{e}^{-V(u)}B_n^z(v)\mathbf{1}_{\{u,v\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{u\in\mathcal{T}^{z-A}\}}\mathbf{1}_{\{|u|\leq n/2\}}\Bigr]\\ &\leq c_{26}\mathrm{e}^{-z}\sum_{k=1}^{\lfloor n/2\rfloor}\mathbf{E}\Bigl[\sum_{u\in\mathcal{S}_k^{z-A}\cap\mathcal{T}^{z-A}}\mathrm{e}^{-V(u)}\sum_{v\neq u,|v|\leq k}B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Bigr].\end{aligned}\tag{4.28}$$

The weight $\mathrm{e}^{-V(u)}$ hints at a change of measure from $\mathbf{P}$ to $\hat{\mathbf{P}}$. For any $k\in[0,n/2]$, we have by Proposition 2.2(i),

$$\begin{aligned}&\mathbf{E}\Bigl[\sum_{u\in\mathcal{S}_k^{z-A}\cap\mathcal{T}^{z-A}}\mathrm{e}^{-V(u)}\sum_{v\neq u,|v|\leq k}B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Bigr]\\ &\quad=\hat{\mathbf{E}}\Bigl[\mathbf{1}_{\{w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\}}\sum_{v\neq w_k,|v|\leq k}B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Bigr].\end{aligned}\tag{4.29}$$

We have to split the cases depending on the location of the vertex v with respect to w_k. We say that $u\nsim v$ if neither v nor u is an ancestor of the other. If $v\neq w_k$ and $|v|\leq k$, then either $v\nsim u$, or $v=w_\ell$ for some $\ell<k$. In view of (4.28) and (4.29), the lemma will be proved once the following two estimates are shown:

$$\sum_{k=1}^{\lfloor n/2\rfloor}\hat{\mathbf{E}}\Bigl[\sum_{v\nsim w_k}B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}},w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\Bigr]\leq c_{31}\mathrm{e}^{-A},\tag{4.30}$$

$$\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\ell=0}^{k-1}\hat{\mathbf{E}}\bigl[B_n^z(w_\ell),w_\ell\in\mathcal{S}^{z-A},w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\bigr]\leq c_{32}\mathrm{e}^{-A}.\tag{4.31}$$

Proof of equation (4.30). Decomposing the sum $\sum_{v\nsim w_k}$ along the spine, we see that for any $k\in[1,n/2]$,

$$\sum_{v\nsim w_k}B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}=\sum_{\ell=1}^{k}\sum_{x\in\Omega(w_\ell)}\sum_{v\geq x}B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}.\tag{4.32}$$

The branching random walk rooted at $x \in \Omega(w_\ell)$ has the same law under $\mathbf{P}$ and $\hat{\mathbf{P}}$. Recall the definition of $\hat{\mathcal{G}}_\infty$ in (2.5). We have for $\ell \le n/2$ and $x \in \Omega(w_\ell)$,

$$\hat{\mathbf{E}}\Big[\sum_{v\ge x} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Big|\hat{\mathcal{G}}_\infty\Big] = \hat{\mathbf{E}}\Big[\sum_{v\ge x} \Phi_{|v|,n}^{\mathrm{kill}}\big(0, z+V(v)\big)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Big|\hat{\mathcal{G}}_\infty\Big]$$

with the notation of (4.5), and (4.7) implies that

$$\hat{\mathbf{E}}\Big[\sum_{v\ge x} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Big|\hat{\mathcal{G}}_\infty\Big] \le c_{26}\mathrm{e}^{-z}\hat{\mathbf{E}}\Big[\sum_{v\ge x} \mathrm{e}^{-V(v)}\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Big|\hat{\mathcal{G}}_\infty\Big]. \tag{4.33}$$

We observe now that if $v \ge x$ and $v \in \mathcal{S}^{z-A}$, then $\min_{|x|\le j\le |v|-1} V(v_j) > V(v) \ge A - z$. Therefore, by the Markov property,

$$\hat{\mathbf{E}}\Big[\sum_{v\ge x} \mathrm{e}^{-V(v)}\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Big|\hat{\mathcal{G}}_\infty\Big] \le \mathbf{E}_{V(x)}\Big[\sum_{v\in\mathbb{T}} \mathrm{e}^{-V(v)}\mathbf{1}_{\{\min_{j\le|v|-1} V(v_j)>V(v)\ge A-z\}}\Big].$$

By (2.1) and the definition of the renewal function $R(x)$ in (2.11), we observe that

$$\mathbf{E}_{V(x)}\Big[\sum_{v\in\mathbb{T}} \mathrm{e}^{-V(v)}\mathbf{1}_{\{\min_{j\le|v|-1} V(v_j)>V(v)\ge A-z\}}\Big] = \mathrm{e}^{-V(x)}R\big(z-A+V(x)\big).$$

Going back to (4.33), we get that for any $\ell \le n/2$ and $x \in \Omega(w_\ell)$,

$$\hat{\mathbf{E}}\Big[\sum_{v\ge x} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}\Big|\hat{\mathcal{G}}_\infty\Big] \le c_{26}\mathrm{e}^{-z}\mathrm{e}^{-V(x)}R\big(z-A+V(x)\big).$$

In view of (4.32), we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\lfloor n/2\rfloor} \hat{\mathbf{E}}\Big[\sum_{v\nsim w_k} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}, w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\Big] \\
&\quad \le c_{26}\mathrm{e}^{-z}\sum_{k\ge1}\sum_{\ell=1}^{k}\hat{\mathbf{E}}\Big[\sum_{x\in\Omega(w_\ell)} \mathrm{e}^{-V(x)}R\big(z-A+V(x)\big), \\
&\qquad\qquad w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\Big].
\end{aligned} \tag{4.34}$$

We look at $R(z-A+V(x))$ for $x \in \Omega(w_\ell)$. If $V(x) \le V(w_{\ell-1})$ and $z - A + V(w_{\ell-1}) \ge 0$, we have

$$R\big(z-A+V(x)\big) \le R\big(z-A+V(w_{\ell-1})\big) \le c_{27}\big(1+z-A+V(w_{\ell-1})\big).$$

If $V(x) > V(w_{\ell-1})$ and $z - A + V(w_{\ell-1}) \ge 0$, we write that

$$\begin{aligned}
R\big(z-A+V(x)\big) &\le c_{27}\big(1+z-A+V(x)\big) \\
&\le c_{27}\big(1+z-A+V(w_{\ell-1})\big)\big(1+V(x)-V(w_\ell-1)\big).
\end{aligned}$$

Therefore, for any $\ell \le k$, we have on the event that $w_k \in \mathcal{S}^{z-A}$,

$$\sum_{x\in\Omega(w_\ell)} \mathrm{e}^{-V(x)} R\big(z-A+V(x)\big) \le c_{27}\big(1+z-A+V(w_{\ell-1})\big) \sum_{x\in\Omega(w_\ell)} \big(1+\big(V(x)-V(w_{\ell-1})\big)_+\big)\mathrm{e}^{-V(x)} = c_{27}\big(1+z-A+V(w_{\ell-1})\big)\mathrm{e}^{-V(w_{\ell-1})}\xi(w_\ell)$$

by definition (4.2). On the event that $w_k \in \mathcal{S}^{z-A} \cap \mathcal{T}^{z-A}$, we conclude that

$$\sum_{x\in\Omega(w_\ell)} \mathrm{e}^{-V(x)} R\big(z-A+V(x)\big) \le c_{27}\mathrm{e}^{(z-A)/2}\big(1+z-A+V(w_{\ell-1})\big)\mathrm{e}^{-V(w_{\ell-1})/2}.$$

Therefore, we have by (4.34),

$$\sum_{k=1}^{\lfloor n/2\rfloor} \hat{\mathbf{E}}\Big[\sum_{v\nsim w_k} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}}, w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\Big] \le c_{26}c_{27}\mathrm{e}^{-(z+A)/2}\sum_{k\ge1}\sum_{\ell=1}^{k}\hat{\mathbf{E}}\big[\big(1+z-A+V(w_{\ell-1})\big)\mathrm{e}^{-V(w_{\ell-1})/2}, w_k\in\mathcal{S}^{z-A}\big]. \tag{4.35}$$

Proposition 2.2(ii) says that

$$\hat{\mathbf{E}}\big[\big(z-A+V(w_{\ell-1})+1\big)\mathrm{e}^{-V(w_{\ell-1})/2}, w_k\in\mathcal{S}^{z-A}\big] = \mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1}), \min_{j\le k-1} S_j > S_k \ge A-z\Big].$$

We observe that

$$\sum_{k\ge1}\sum_{\ell=1}^{k}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1}), \min_{j\le k-1} S_j > S_k \ge A-z\Big] = \sum_{\ell\ge1}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1})\sum_{k\ge\ell}\mathbf{1}_{\{\min_{j\le k-1} S_j > S_k\ge A-z\}}\Big].$$

Since

$$\sum_{k\ge\ell}\mathbf{1}_{\{\min_{j\le k-1} S_j > S_k\ge A-z\}} \le \mathbf{1}_{\{\min_{j\le\ell-1} S_j\ge A-z\}}\sum_{k\ge\ell}\mathbf{1}_{\{\min_{j\in[\ell-1,k-1]} S_j > S_k\ge A-z\}},$$

we deduce by the Markov property at time $\ell - 1$ that

$$\sum_{k\ge 1}\sum_{\ell=1}^{k}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1}),\ \min_{j\le k-1} S_j > S_k \ge A-z\Big]$$
$$\le \sum_{\ell\ge 1}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1})R(S_{\ell-1}+z-A),\ \min_{j\le \ell-1} S_j \ge A-z\Big]$$
$$\le c_{27}\sum_{\ell\ge 1}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1})^2,\ \min_{j\le \ell-1} S_j \ge A-z\Big].$$

Using this bound in (4.35) yields that

$$\sum_{k=1}^{\lfloor n/2\rfloor}\hat{\mathbf{E}}\Big[\sum_{v\not\sim w_k} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}},\ w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\Big]$$
$$\le c_{26}c_{27}^2\mathrm{e}^{-(z+A)/2}\sum_{\ell\ge 1}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1})^2,\ \min_{j\le \ell-1} S_j \ge A-z\Big]. \tag{4.36}$$

By Lemma B.2(iii), we have

$$\sum_{\ell\ge 1}\mathbf{E}\Big[\mathrm{e}^{-S_{\ell-1}/2}(1+z-A+S_{\ell-1})^2,\ \min_{j\le \ell-1} S_j \ge A-z\Big]\le c_{33}\mathrm{e}^{(z-A)/2}.$$

Consequently, by (4.36)

$$\sum_{k=1}^{\lfloor n/2\rfloor}\hat{\mathbf{E}}\Big[\sum_{v\not\sim w_k} B_n^z(v)\mathbf{1}_{\{v\in\mathcal{S}^{z-A}\}},\ w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\Big]\le c_{34}\mathrm{e}^{-(z+A)/2}\mathrm{e}^{(z-A)/2}$$
$$= c_{34}\mathrm{e}^{-A}.$$

Equation (4.30) follows.

Proof of equation (4.31). We have

$$\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\ell=0}^{k-1}\hat{\mathbf{E}}\big[B_n^z(w_\ell),\ w_\ell\in\mathcal{S}^{z-A},\ w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\big]$$
$$= \sum_{\ell=0}^{\lfloor n/2\rfloor-1}\sum_{k=\ell+1}^{\lfloor n/2\rfloor}\hat{\mathbf{E}}\big[B_n^z(w_\ell),\ w_\ell\in\mathcal{S}^{z-A},\ w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\big] \tag{4.37}$$
$$= \sum_{\ell=0}^{\lfloor n/2\rfloor-1}\hat{\mathbf{E}}\Bigg[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}}\sum_{k=\ell+1}^{\lfloor n/2\rfloor}\mathbf{1}_{\{w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\}}\Bigg].$$

Let t_ℓ be the first time t after ℓ such that $V(w_t) < V(w_\ell)$. If $k > \ell$ and $w_k \in \mathcal{S}^{z-A}$, then $V(w_k) < V(w_\ell)$, which means that necessarily $k \geq t_\ell$. Moreover, if $k \geq t_\ell$ and $w_k \in \mathcal{T}^{z-A}$, then $w_{t_\ell} \in \mathcal{T}^{z-A}$. Thus, for any $\ell \geq 0$,

$$\sum_{k=\ell+1}^{\lfloor n/2\rfloor} \mathbf{1}_{\{w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\}} = \sum_{k=t_\ell}^{\lfloor n/2\rfloor} \mathbf{1}_{\{w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\}}$$
$$\leq \mathbf{1}_{\{w_{t_\ell}\in\mathcal{T}^{z-A}, t_\ell \leq n/2\}} \sum_{k\geq t_\ell} \mathbf{1}_{\{\min_{t_\ell \leq j < k} V(w_j) > V(w_k) \geq A-z\}}.$$

We observe that $B_n^z(w_\ell)$ is a function of the branching random walk killed below $V(w_\ell)$ and therefore is independent of the subtree rooted at w_{t_ℓ}. As a result, applying the branching property, we get that for any $\ell \in [0, n/2]$,

$$\hat{\mathbf{E}}\left[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}} \sum_{k=\ell+1}^{\lfloor n/2\rfloor} \mathbf{1}_{\{w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\}}\right]$$
$$\leq \hat{\mathbf{E}}\left[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{T}^{z-A}, t_\ell\leq n/2\}} \sum_{k\geq t_\ell} \mathbf{1}_{\{\min_{t_\ell\leq j<k} V(w_j) > V(w_k)\geq A-z\}}\right]$$
$$= \hat{\mathbf{E}}\left[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{T}^{z-A}, t_\ell\leq n/2\}} R\big(z - A + V(w_{t_\ell})\big)\right].$$

By equation (4.37), we deduce that

$$\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\ell=0}^{k-1} \hat{\mathbf{E}}\left[B_n^z(w_\ell), w_\ell \in \mathcal{S}^{z-A}, w_k \in \mathcal{T}^{z-A}\right]$$
$$\leq \sum_{\ell=0}^{\lfloor n/2\rfloor -1} \hat{\mathbf{E}}\left[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{T}^{z-A}, t_\ell\leq n/2\}} R\big(z - A + V(w_{t_\ell})\big)\right].$$

We have $V(w_{t_\ell}) < V(w_\ell)$. Since R is a nondecreasing function, we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\ell=0}^{k-1} \hat{\mathbf{E}}\left[B_n^z(w_\ell), w_\ell \in \mathcal{S}^{z-A}, w_k \in \mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\right] \\
&\quad\leq \sum_{\ell=0}^{\lfloor n/2\rfloor -1} \hat{\mathbf{E}}\left[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{T}^{z-A}, t_\ell\leq n/2\}} R\big(z - A + V(w_\ell)\big)\right].
\end{aligned} \tag{4.38}$$

Recall from (2.4) that $\hat{\mathcal{G}}_\ell$ is the σ-algebra generated by the spine and its siblings up to time ℓ. The Markov property at time ℓ shows that

$$\hat{\mathbf{E}}\left[B_n^z(w_\ell), w_{t_\ell}\in\mathcal{T}^{z-A}, t_\ell \leq n/2 \,|\, \hat{\mathcal{G}}_\ell\right] = \mathbf{1}_{\{w_\ell\in\mathcal{T}^{z-A}\}}\tilde{\Phi}^{\mathrm{kill}}_{\ell,n,A}\big(z + V(w_\ell)\big)$$
$$\leq \tilde{\Phi}^{\mathrm{kill}}_{\ell,n,A}\big(z + V(w_\ell)\big),$$

where, if $\tau_0^- := \min\{j \ge 0 : V(w_j) < 0\}$, then for any integers $n \ge 1$, $\ell \le n/2$, any $A, r \ge 0$, we defined

$$\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(r) := \hat{\mathbf{P}}\big(\tau_0^- \le (n/2) - \ell, M^{\text{kill}}_{n-\ell} < a_n(r), \xi(w_j) \le \mathrm{e}^{(r+V(w_{j-1})-A)/2}, \forall 1 \le j \le \tau_0^-\big).$$

We deduce that, for any $n \ge 1$, any $\ell < n/2$, any $z \ge A \ge 0$,

$$\begin{aligned}\hat{\mathbf{E}}&\big[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell \in \mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell} \in \mathcal{T}^{z-A}, t_\ell \le n/2\}} R\big(z - A + V(w_\ell)\big)\big]\\ &\le \hat{\mathbf{E}}\big[\mathbf{1}_{\{w_\ell \in \mathcal{S}^{z-A}\}} R\big(z - A + V(w_\ell)\big)\tilde{\Phi}^{\text{kill}}_{\ell,n,A}\big(z + V(w_\ell)\big)\big]\\ &= \hat{\mathbf{E}}\big[\mathbf{1}_{\{\min_{j<\ell} V(w_j) > V(w_\ell) \ge A - z\}} R\big(z - A + V(w_\ell)\big)\tilde{\Phi}^{\text{kill}}_{\ell,n,A}\big(z + V(w_\ell)\big)\big].\end{aligned}$$

By Proposition 2.2(ii), this implies that

$$\begin{aligned}\hat{\mathbf{E}}&\big[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell \in \mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell} \in \mathcal{T}^{z-A}, t_\ell \le n/2\}} R\big(z - A + V(w_\ell)\big)\big]\\ &\le \mathbf{E}\big[\mathbf{1}_{\{\min_{j<\ell} S_j > S_\ell \ge A - z\}} R(z - A + S_\ell)\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(z + S_\ell)\big].\end{aligned} \tag{4.39}$$

Let us estimate $\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(r)$ for $\ell < n/2$. We have to decompose along the spine. Notice that if $M^{\text{kill}}_{n-\ell} < a_n(r)$, and $\tau_0^- \le \lfloor n/2 \rfloor - \ell$, then there must be some $j < \tau_0^- \le \lfloor n/2 \rfloor - \ell$ and $x \in \Omega(w_j)$ such that there exists a line of descent from x which stays above 0 and ends below $a_n(r)$ at time $n - \ell$. Therefore, for any $n \ge 1$, $\ell < n/2$ and $A, r \ge 0$,

$$\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(r) \le \sum_{j=1}^{\lfloor n/2 \rfloor - \ell} \hat{\mathbf{E}}\Big[\sum_{x \in \Omega(w_j)} \Phi^{\text{kill}}_{\ell+j,n}\big(V(x), r\big), \xi(w_j) \le \mathrm{e}^{(r+V(w_{j-1})-A)/2}, j \le \tau_0^-\Big]$$

with the notation of (4.5). By (4.7), we get that

$$\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(r) \le c_{35}\mathrm{e}^{-r} \sum_{j=1}^{\lfloor n/2 \rfloor - \ell} \hat{\mathbf{E}}\Big[\sum_{x \in \Omega(w_j)} \big(1 + V(x)_+\big)\mathrm{e}^{-V(x)}, \xi(w_j) \le \mathrm{e}^{(r+V(w_{j-1})-A)/2}, j \le \tau_0^-\Big]. \tag{4.40}$$

We observe that

$$\begin{aligned}\sum_{x \in \Omega(w_j)} &\big(1 + V(x)_+\big)\mathrm{e}^{-V(x)}\\ &\le \big(1 + V(w_{j-1})_+\big)\mathrm{e}^{-V(w_{j-1})}\\ &\quad\times \sum_{x \in \Omega(w_j)} \big(1 + \big(V(x) - V(w_{j-1})\big)_+\big)\mathrm{e}^{-(V(x)-V(w_{j-1}))}\\ &= \big(1 + V(w_{j-1})_+\big)\mathrm{e}^{-V(w_{j-1})}\xi(w_j)\end{aligned}$$

by definition (4.2). We deduce from (4.40) that

$$\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(r) \le c_{35}\mathrm{e}^{-r}\sum_{j=1}^{\lfloor n/2\rfloor-\ell}\hat{\mathbf{E}}\big[\mathrm{e}^{-V(w_{j-1})}\big(1+V(w_{j-1})\big)\mathrm{e}^{(r+V(w_{j-1})-A)/2}, j\le\tau_0^-\big].$$

It follows that, for any $n\ge 1$, $\ell<n/2$ and $A, r\ge 0$,

$$\begin{aligned}\tilde{\Phi}^{\text{kill}}_{\ell,n,A}(r) &\le c_{35}\mathrm{e}^{-A}\mathrm{e}^{-(r-A)/2}\sum_{j\ge1}\mathbf{E}\big[\mathrm{e}^{-S_{j-1}/2}(1+S_{j-1}), j\le\tau_0^-\big]\\ &= c_{36}\mathrm{e}^{-A}\mathrm{e}^{-(r-A)/2},\end{aligned}$$

by Lemma B.2(ii). Going back to (4.39), we obtain that for any $n\ge1$, $\ell<n/2$, $z\ge A\ge 0$,

$$\begin{aligned}&\hat{\mathbf{E}}\big[B_n^z(w_\ell)\mathbf{1}_{\{w_\ell\in\mathcal{S}^{z-A}\}}\mathbf{1}_{\{w_{t_\ell}\in\mathcal{T}^{z-A},t_\ell\le n/2\}}R\big(z-A+V(w_\ell)\big)\big]\\ &\qquad\le c_{36}\mathrm{e}^{-A}\mathbf{E}\big[\mathbf{1}_{\{\min_{j<\ell}S_j>S_\ell\ge A-z\}}R(z-A+S_\ell)\mathrm{e}^{-(S_\ell+z-A)/2}\big].\end{aligned}$$

Equation (4.38) yields that for any $n\ge1$, and $z\ge A\ge0$,

$$\begin{aligned}&\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\ell=0}^{k-1}\hat{\mathbf{E}}\big[B_n^z(w_\ell), w_\ell\in\mathcal{S}^{z-A}, w_k\in\mathcal{S}^{z-A}\cap\mathcal{T}^{z-A}\big]\\ &\qquad\le c_{36}\mathrm{e}^{-A}\mathbf{E}\Big[\sum_{\ell\ge0}\mathbf{1}_{\{\min_{j<\ell}S_j>S_\ell\ge A-z\}}R(z-A+S_\ell)\mathrm{e}^{-(S_\ell+z-A)/2}\Big].\end{aligned}$$

Applying Lemma B.2(iii) implies (4.31) and thus completes the proof of the lemma. □

4.2. *Proof of Proposition* 4.1. We can now prove Proposition 4.1.

PROOF OF PROPOSITION 4.1. Let $\varepsilon>0$. For any $r\ge0$, we observe that by (2.1),

$$\begin{aligned}\mathbf{P}\big(\exists u\in\mathbb{T}: V(u)\le -r\big) &\le \sum_{n\ge0}\mathbf{E}\Big[\sum_{|u|=n}\mathbf{1}_{\{V(u)\le -r, V(u_k)>-r,\forall k<n\}}\Big]\\ &= \sum_{n\ge0}\mathbf{E}\big[\mathrm{e}^{S_n}, S_n\le -r, S_k>-r, \forall k<n\big]\\ &\le \mathrm{e}^{-r}.\end{aligned}$$

Therefore

$$\mathbf{P}\big(\exists u\in\mathbb{T}: V(u)\le A-z\big)\le \mathrm{e}^{A-z}.$$

For any $z \geq A \geq 0$, we observe that on the event $\{\forall u \in \mathbb{T}, V(u) \geq A - z\}$, we have $M_n < \frac{3}{2}\ln n - z$ if and only if $\sum_{u \in \mathcal{S}^{z-A}} B_n^z(u) \geq 1$ [recall the definition of $\mathcal{S}^r$ and B_n^z in (4.1) and in (4.4)]. Therefore, for $n \geq 1$ and $z \geq A$, we have

$$0 \leq \mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) - \mathbf{P}\Big(\sum_{u \in \mathcal{S}^{z-A}} B_n^z(u) \geq 1\Big) \leq \mathrm{e}^{A-z}.$$

We notice that $\mathbf{P}(\sum_{u \in \mathcal{S}^{z-A}} B_n^z(u) \geq 1) \leq \mathbf{E}[\sum_{u \in \mathcal{S}^{z-A}} B_n^z(u)]$. Hence,

$$\mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) \leq \mathrm{e}^{A-z} + \mathbf{E}\Big[\sum_{u \in \mathcal{S}^{z-A}} B_n^z(u)\Big].$$

Lemma 4.2(i) and (ii) implies that for $n \geq N_1$ and $z \in [A_1, (3/2)\ln(n) - A_1]$,

$$\frac{\mathrm{e}^z}{R(z - A_1)}\mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) - C_1 \leq \frac{\mathrm{e}^{A_1} + c}{R(z - A_1)} + \varepsilon.$$

Since $R(x) \sim c_0 x$ at infinity by (2.13), we have for $n \geq N_1$ and $z \in [A_2, (3/2)\ln(n) - A_1]$,

$$\frac{\mathrm{e}^z}{c_0 z}\mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) - C_1 \leq \frac{\mathrm{e}^{A_1} + c}{c_0 z} + 2\varepsilon.$$

We deduce that for $n \geq N_1$ and $z \in [A_3, (3/2)\ln(n) - A_1]$,

$$\frac{\mathrm{e}^z}{c_0 z}\mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) - C_1 \leq 3\varepsilon.$$

This proves the upper bound. Similarly, we have for the lower bound,

$$\begin{aligned}\mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) &\geq \mathbf{P}\Big(\sum_{u \in \mathcal{S}^{z-A}} B_n^z(u) \geq 1\Big)\\ &\geq \mathbf{P}\Big(\sum_{u \in \mathcal{S}^{z-A} \cap \mathcal{T}^{z-A}} B_n^z(u)\mathbf{1}_{\{|u| \leq n/2\}} \geq 1\Big).\end{aligned}$$

If we write $U(A) := \sum_{u \in \mathcal{S}^{z-A} \cap \mathcal{T}^{z-A}} B_n^z(u)\mathbf{1}_{\{|u| \leq n/2\}}$, then by the Paley–Zygmund formula, we have $\mathbf{P}(U(A) \geq 1) \geq \frac{\mathbf{E}[U(A)]^2}{\mathbf{E}[U(A)^2]}$. By Lemma 4.2, we know that $\frac{\mathrm{e}^z}{R(z-A_4)}\mathbf{E}[U(A_4)] \geq C_1 - \varepsilon$ for $n \geq N_2$ and $z \in [A_5, (3/2)\ln(n) - A_4]$. By Lemma 4.3, we have that $\mathbf{E}[U(A_4)^2] \leq (1 + \varepsilon)\mathbf{E}[U(A_4)]$ if A_5 is taken large enough. Hence, $\frac{\mathrm{e}^z}{R(z-A_4)}\mathbf{P}(U(A_4) \geq 1) \geq \frac{\mathrm{e}^z}{R(z-A_4)}(1 + \varepsilon)^{-1}\mathbf{E}[U(A_4)] \geq (1 + \varepsilon)^{-1}(C_1 - \varepsilon)$. This yields that

$$\frac{\mathrm{e}^z}{R(z - A_4)}\mathbf{P}\Big(M_n \leq \frac{3}{2}\ln n - z\Big) \geq (1 + \varepsilon)^{-1}(C_1 - \varepsilon).$$

From here, we proceed as before to see that for $n \geq N_2$ and $z \in [A_6, (3/2)\ln(n) - A_4]$,

$$\frac{e^z}{c_0 z}\mathbf{P}\left(M_n \leq \frac{3}{2}\ln n - z\right) \geq C_1 - c_{37}\varepsilon.$$

The proposition follows. □

5. Proof of Theorem 1.1. For $\beta \geq 0$, we look at the branching random walk killed below $-\beta$. The population at time n of this process is $\{|u| = n : V(u_k) \geq -\beta, \forall k \leq n\}$. We define the associated martingale (see Appendix A)

$$D_n^{(\beta)} := \sum_{|u|=n} R(\beta + V(u))e^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta, k\leq n\}}. \tag{5.1}$$

Since $D_n^{(\beta)}$ is nonnegative, it has a limit almost surely that we denote by $D_\infty^{(\beta)}$. Under (1.3) and (1.4), we know by Proposition A.3 that $D_\infty^{(\beta)} > 0$ almost surely on the event of nonextinction for the branching random walk killed below $-\beta$. For $A \geq 0$, let $\mathcal{Z}[A]$ denote the set of particles absorbed at level A, that is,

$$\mathcal{Z}[A] := \{u \in \mathbb{T} : V(u) \geq A, V(u_k) < A, \forall k < |u|\}.$$

In the words of Section 6 in [7], this set is a very simple optional line. By Theorem 6.1 (and Lemma 6.1) of [7], we know that $\sum_{u\in\mathcal{Z}[A]} R(\beta + V(u))e^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta\}}$ converges to $D_\infty^{(\beta)}$ almost surely as $A \to \infty$. Recall that $R(x) \sim c_0 x$ at infinity by (2.13). Recall from (2.2) that the martingale W_n is defined by

$$W_n := \sum_{|x|=n} e^{-V(x)}$$

and we know from [22] that W_n converges to 0 almost surely as $n \to \infty$ under (1.1). On the event $\{\min_{u\in\mathbb{T}} V(u) \geq -\beta\}$, we see that necessarily $D_\infty^{(\beta)} = c_0 D_\infty$ almost surely, and $\sum_{u\in\mathcal{Z}[A]} R(\beta + V(u))e^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta\}} \sim c_0 \sum_{u\in\mathcal{Z}[A]}(\beta + V(u))e^{-V(u)}$ as $A \to \infty$. Again by Theorem 6.1 (and Lemma 6.1) of [7], we have $\lim_{A\to\infty}\sum_{u\in\mathcal{Z}[A]} e^{-V(u)} = W_\infty = 0$ almost surely. We deduce that

$$\lim_{A\to\infty}\sum_{u\in\mathcal{Z}[A]} V(u)e^{-V(u)} = D_\infty \tag{5.2}$$

on the event $\{\min_{u\in\mathbb{T}} V(u) \geq -\beta\}$, and therefore almost surely by making $\beta \to \infty$. We can now prove the convergence in law.

PROOF OF THEOREM 1.1. Fix $x \in \mathbb{R}$, and let $\varepsilon > 0$. For any $A > 0$, we have for n large enough,

$$\begin{aligned} &\mathbf{P}(\exists u \in \mathcal{Z}[A] : |u| \geq n^{1/2}) \leq \varepsilon, \\ &\mathbf{P}(\exists u \in \mathcal{Z}[A] : V(u) \geq (3/2)\ln n - A) \leq \varepsilon. \end{aligned} \tag{5.3}$$

Again, we could replace $n^{1/2}$ by any $o(n)$. Take $A>0$. Let $\mathcal{Y}_A := \{\max_{u\in\mathcal{Z}[A]} |u| \le n^{1/2}, \max_{u\in\mathcal{Z}[A]} V(u) \le \ln n\}$. We observe that

$$\mathbf{P}\bigl(M_n \ge (3/2)\ln n + x\bigr) \ge \mathbf{P}\bigl(M_n \ge (3/2)\ln n + x, \mathcal{Y}_A\bigr)$$
$$= \mathbf{E}\Bigl[\prod_{u\in\mathcal{Z}[A]} \bigl(1-\Phi_{|u|,n}\bigl(V(u)-x\bigr)\bigr), \mathcal{Y}_A\Bigr],$$

where for any integers $n\ge 1$, $k\in[0,n]$ and any real $r\ge 0$,

$$\Phi_{k,n}(r) := \mathbf{P}\bigl(M_{n-k} < (3/2)\ln(n) - r\bigr).$$

By Proposition 4.1, there exists A large enough and $N\ge 1$ such that for any $n\ge N$, $k\le n^{1/2}$ and $z\in[A-x,(3/2)\ln(n)-A-x]$,

$$\left|\frac{\mathrm{e}^z}{z}\Phi_{k,n}(z) - C_1c_0\right| \le \varepsilon. \tag{5.4}$$

We get that

$$\mathbf{P}\bigl(M_n \ge (3/2)\ln n + x\bigr) \ge \mathbf{E}\Bigl[\prod_{u\in\mathcal{Z}[A]} \bigl(1-(C_1c_0+\varepsilon)\bigl(V(u)-x\bigr)\mathrm{e}^{x-V(u)}\bigr), \mathcal{Y}_A\Bigr].$$

Since $\mathbf{P}(\mathcal{Y}_A^c)\le 2\varepsilon$ for n large enough, we have for n large enough,

$$\mathbf{P}\bigl(M_n \ge (3/2)\ln n + x\bigr) \ge \mathbf{E}\Bigl[\prod_{u\in\mathcal{Z}[A]} \bigl(1-(C_1c_0+\varepsilon)\bigl(V(u)-x\bigr)\mathrm{e}^{x-V(u)}\bigr)\Bigr] - 2\varepsilon.$$

In particular,

$$\liminf_{n\to\infty} \mathbf{P}\bigl(M_n \ge (3/2)\ln n + x\bigr)$$
$$\ge \mathbf{E}\Bigl[\prod_{u\in\mathcal{Z}[A]} \bigl(1-(C_1c_0+\varepsilon)\bigl(V(u)-x\bigr)\mathrm{e}^{x-V(u)}\bigr)\Bigr] - 2\varepsilon.$$

We let A go to infinity. We have almost surely by (5.2) and the fact that $\sum_{u\in\mathcal{Z}[A]} \mathrm{e}^{-V(u)}$ vanishes,

$$\lim_{A\to\infty} \sum_{u\in\mathcal{Z}[A]} \ln\bigl(1-(C_1c_0+\varepsilon)\bigl(V(u)-x\bigr)\mathrm{e}^{x-V(u)}\bigr) = -(C_1c_0+\varepsilon)\mathrm{e}^x D_\infty. \tag{5.5}$$

By dominated convergence, we deduce that

$$\liminf_{n\to\infty} \mathbf{P}\bigl(M_n \ge (3/2)\ln n + x\bigr) \ge \mathbf{E}\bigl[\exp\bigl(-(C_1c_0+\varepsilon)\mathrm{e}^x D_\infty\bigr)\bigr] - 2\varepsilon,$$

which gives the lower bound by letting $\varepsilon\to 0$. The upper bounds works similarly. Let A be such that (5.4) is satisfied for $n\ge N$, $k\le n^{1/2}$ and $z\in[A-x,(3/2)\ln(n)-A-x]$. We observe that, for n large enough,

$$\mathbf{P}\bigl(M_n \ge (3/2)\ln n + x\bigr) \le \mathbf{P}\bigl(M_n \ge (3/2)\ln n + x, \mathcal{Y}_A\bigr) + 2\varepsilon$$
$$= \mathbf{E}\Bigl[\prod_{u\in\mathcal{Z}[A]} \bigl(1-\Phi_{|u|,n}\bigl(V(u)-x\bigr)\bigr), \mathcal{Y}_A\Bigr] + 2\varepsilon.$$

Using (5.4), we end up with

$$\limsup_{n\to\infty} \mathbf{P}\big(M_n \geq (3/2)\ln n + x\big)$$
$$\leq \mathbf{E}\biggl[\prod_{u\in\mathcal{Z}[A]} \bigl(1-(C_1c_0-\varepsilon)\bigl(V(u)-x\bigr)\mathrm{e}^{x-V(u)}\bigr)\biggr] + 2\varepsilon.$$

From here, we proceed as for the lower bound. □

APPENDIX A: THE DERIVATIVE MARTINGALE

We work under (1.1), (1.3) and (1.4), but we drop the assumption that $\mathcal{L}$ is nonlattice. We recall from (2.11) that the renewal function $R(x)$ is defined by

$$R(x) = \sum_{k\geq 0} \mathbf{P}\Bigl(S_k \geq -x, S_k < \min_{0\leq j\leq k-1} S_j\Bigr).$$

The duality lemma says that $R(x)$ is also the expected number of visits of the random walk $(S_n)_{n\geq 0}$ to the interval $(-x, 0]$ before hitting $[0,\infty)$ (after time 1). For any $\beta \geq 0$, we introduce for $n \geq 0$,

$$D_n^{(\beta)} := \sum_{|u|=n} R\bigl(V(u)+\beta\bigr)\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\geq -\beta, \forall k\leq n\}}.$$

The following lemma is Lemma 10.2 in [7]. The analog in the case of the Brownian motion is Theorem 9 in [20].

LEMMA A.1 ([7]). *For any $\beta \geq 0$, the process $(D_n^{(\beta)}, n\geq 0)$ is a nonnegative martingale with respect to $(\mathscr{F}_n, n\geq 0)$.*

PROOF. We recall that under $\mathbf{P}_a$, the branching random walk $(V(v), v\in\mathbb{T})$ and the one-dimensional random walk $(S_k, k\geq 0)$ start at a. By the Markov property, we have

$$\mathbf{E}\bigl[D_{n+1}^{(\beta)}|\mathscr{F}_n\bigr]$$
$$= \sum_{|u|=n} \mathbf{1}_{\{V(u_k)\geq -\beta,\forall k\leq n\}}\mathbf{E}_{V(u)}\biggl[\sum_{|v|=1} R\bigl(V(v)+\beta\bigr)\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v)\geq -\beta\}}\biggr].$$

By (2.1), we see that for any $u\in\mathbb{T}$ with $|u| = n$,

$$\mathbf{E}_{V(u)}\biggl[\sum_{|v|=1} R\bigl(V(v)+\beta\bigr)\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v)\geq -\beta\}}\biggr]$$
$$= \mathbf{E}_{V(u)}\bigl[R(S_1+\beta)\mathbf{1}_{\{S_1\geq -\beta\}}\bigr]\mathrm{e}^{-V(u)},$$

which is $R(V(u)+\beta)\mathrm{e}^{-V(u)}$ by Lemma 1 of [24]. Therefore,

$$\begin{aligned}&\mathbf{E}\big[D_{n+1}^{(\beta)}|\mathscr{F}_n\big]\\&\quad=\sum_{|u|=n}\mathbf{1}_{\{V(u_k)\ge-\beta,\forall k\le n\}}R\big(V(u)+\beta\big)\mathrm{e}^{-V(u)},\end{aligned}$$

which completes the proof. □

Since $(D_n^{(\beta)}, n\ge 0)$ is a nonnegative martingale, we can define for any $a\ge 0$ a probability measure $\hat{\mathbf{P}}_a^{(\beta)}$ on $\mathscr{F}_\infty$ such that for any $n\ge 1$,

$$\frac{d\hat{\mathbf{P}}_a^{(\beta)}}{d\mathbf{P}_a}\Big|_{\mathscr{F}_n}=\frac{D_n^{(\beta)}}{R(a+\beta)\mathrm{e}^{-a}} \tag{A.1}$$

and we write as usual $\hat{\mathbf{P}}^{(\beta)}$ for $\hat{\mathbf{P}}_0^{(\beta)}$, and $\hat{\mathbf{E}}_a^{(\beta)}$ (resp., $\hat{\mathbf{E}}^{(\beta)}$) for the expectation associated with $\hat{\mathbf{P}}_a^{(\beta)}$ (resp., $\hat{\mathbf{P}}^{(\beta)}$). Let $\hat{\mathcal{B}}_a^{(\beta)}$ be the branching random walk with a spine defined as follows: The spine $w_0^{(\beta)}$ starts at $V(w_0^{(\beta)})=a$. At time 1 it gives birth to a point process distributed as $(V(x), |x|=1)$ under $\hat{\mathbf{P}}_a^{(\beta)}$. Then the spine element$w_1^{(\beta)}$ at time 1 is chosen proportionally to $R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u)\ge-\beta\}}$ among the children u of $w_0^{(\beta)}$. At each time n, the spine element $w_n^{(\beta)}$ produces an independent point process distributed as $(V(x), |x|=1)$ under $\hat{\mathbf{P}}^{(\beta)}_{V(w_n^{(\beta)})}$, while the other particles $|u|=n$ generate independent point processes distributed as $(V(x), |x|=1)$ under $\mathbf{P}_{V(u)}$. The spine $w_{n+1}^{(\beta)}$ at time $n+1$ is chosen proportionally to the weight $R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{V(u_k)\ge-\beta,\forall k\le n\}}$ among the children of $w_n^{(\beta)}$. We write $\hat{\mathscr{F}}_n^{(\beta)}$ for the σ-algebra obtained from $\mathscr{F}_n$ by including the information on the spine up to time n. We write $\mathcal{B}_a^{(\beta)}$ for the (nonmarked) branching random walk obtained from $\hat{\mathcal{B}}_a^{(\beta)}$ by ignoring the location of the spine, and note that $\mathcal{B}_a^{(\beta)}$ is measurable with respect to $\mathscr{F}_\infty$.

LEMMA A.2 ([7]). *The branching random walk under $\hat{\mathbf{P}}_a^{(\beta)}$ is distributed as $\mathcal{B}_a^{(\beta)}$.*

PROOF. We give a sketch of the proof. Let $n\ge 1$ and T_n be a deterministic tree of height less than n. We denote by $\mathbb{T}_{|n}$ the (random) tree $\mathbb{T}$ truncated at level n. Let $\mathbf{P}_{\hat{\mathcal{B}}_a^{(\beta)}}$ be a probability measure associated with $\hat{\mathcal{B}}_a^{(\beta)}$. We want to prove that the projection of $\mathbf{P}_{\hat{\mathcal{B}}_a^{(\beta)}}$ on the space of nonmarked branching random walks is $\hat{\mathbf{P}}_a^{(\beta)}$. Given deterministic infinitesimal intervals $(dz_u, u\in T_n)$, we compute that

$$\begin{aligned}&\mathbf{P}_{\hat{\mathcal{B}}_a^{(\beta)}}\big(\mathbb{T}_{|n}=T_n, V(u)\in dz_u, \forall u\in T_n\big)\\&\quad=\sum_{u\in T_n,|u|=n}\mathbf{P}_{\hat{\mathcal{B}}_a^{(\beta)}}\big(\mathbb{T}_{|n}=T_n, V(u)\in dz_u, \forall u\in T_n, w_n^{(\beta)}=u\big).\end{aligned}$$

For any $u \in T_n$ with $|u| = n$, we check that, by construction of our process $\hat{\mathcal{B}}_a^{(\beta)}$,

$$\begin{aligned}\mathbf{P}_{\hat{\mathcal{B}}_a^{(\beta)}}\big(\mathbb{T}_{|n} = T_n, V(u) \in dz_u, \forall u \in T_n, w_n^{(\beta)} = u\big) \\ = \mathbf{P}_a\big(\mathbb{T}_{|n} = T_n, V(u) \in dz_u, \forall u \in T_n\big) \\ \times \frac{R(V(u_j)+\beta)\mathrm{e}^{-V(u_j)}\mathbf{1}_{\{\min_{j\le n} V(u_j)\ge -\beta\}}}{R(a+\beta)\mathrm{e}^{-a}},\end{aligned}$$

where u_j denotes the ancestor of u in T_n at generation j. Therefore,

$$\mathbf{P}_{\hat{\mathcal{B}}_a^{(\beta)}}\big(\mathbb{T}_{|n} = T_n, V(u) \in dz_u, \forall u \in T_n\big) = \mathbf{E}_a\bigg[\mathbf{1}_{\{\mathbb{T}_{|n}=T_n, V(u)\in dz_u, \forall u \in T_n\}}\frac{D_n^{(\beta)}}{D_0^{(\beta)}}\bigg],$$

which is $\hat{\mathbf{P}}_a^{(\beta)}(\mathbb{T}_{|n} = T_n, V(u) \in dz_u \forall u \in T_n)$ by definition. □

From now on, we will identify our branching random walk under $\hat{\mathbf{P}}_a^{(\beta)}$ with $\hat{\mathcal{B}}_a^{(\beta)}$. Notice that the proof shows that, for any vertex $u \in \mathbb{T}$ such that $|u| = n$,

$$\hat{\mathbf{P}}_a^{(\beta)}\big(w_n^{(\beta)} = u | \mathscr{F}_n\big) = \frac{R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{\min_{j\le n} V(u_j)\ge -\beta\}}}{D_n^{(\beta)}}.$$

For F a measurable function from $\mathbb{R}^{n+1}$ to $\mathbb{R}_+$, we notice that

$$\begin{aligned}&\hat{\mathbf{E}}_a^{(\beta)}\big[F\big(V\big(w_0^{(\beta)}\big), \dots, V\big(w_n^{(\beta)}\big)\big)\big] \\ &\quad = \hat{\mathbf{E}}_a^{(\beta)}\bigg[\sum_{|u|=n} F\big(V(u_0), \dots, V(u_n)\big)\frac{R(V(u)+\beta)\mathrm{e}^{-V(u)}\mathbf{1}_{\{\min_{j\le n} V(u_j)\ge -\beta\}}}{D_n^{(\beta)}}\bigg] \\ &\quad = \frac{1}{D_0^{(\beta)}}\mathbf{E}_a\bigg[\sum_{|u|=n} F\big(V(u_0), \dots, V(u_n)\big) \\ &\qquad\qquad \times R\big(V(u)+\beta\big)\mathrm{e}^{-V(u)}\mathbf{1}_{\{\min_{j\le n} V(u_j)\ge -\beta\}}\bigg].\end{aligned}$$

Therefore, (2.1) yields that

$$\begin{aligned}&\hat{\mathbf{E}}_a^{(\beta)}\big[F\big(V\big(w_0^{(\beta)}\big), \dots, V\big(w_n^{(\beta)}\big)\big)\big] \\ &\quad = \frac{1}{R(a+\beta)}\mathbf{E}_a\Big[F(S_0, \dots, S_n)R(S_n+\beta), \min_{k\le n} S_k \ge -\beta\Big].\end{aligned} \tag{A.2}$$

Under $\hat{\mathbf{P}}_y^{(\beta)}$, the spine process $(V(w_n), n \ge 0)$ is distributed as the random walk $(S_n)_{n\ge 0}$ conditioned to stay above $-\beta$, in the sense of [24] or [5]. It is the Markov chain with transition probabilities, for any $x \ge -\beta$,

$$\hat{p}^{(\beta)}(x, dy) := \frac{R(y+\beta)}{R(x+\beta)}\mathbf{1}_{\{y\ge -\beta\}}p(x, dy),$$

where $p(x, dy) = \mathbf{P}_x(S_1 \in dy)$. The fact that this defines a transition probability comes from the equality $\mathbf{E}_x[R(S_1)\mathbf{1}_{\{S_1 \geq 0\}}] = R(x)$ for any $x \geq 0$ by Lemma 1 in [24]. This Markov chain then never hits the region $(-\infty, -\beta)$, hence its name.

Since $(D_n^{(\beta)}, n \geq 0)$ is a (nonnegative) martingale, it has a limit that we denote by $D_\infty^{(\beta)}$. The question of the convergence in L^1 was addressed in [7], where the authors give almost optimal conditions for the convergence to hold. However, we deal with slightly weaker conditions, so we have to prove the convergence in our case.

PROPOSITION A.3. *Assume* (1.1), (1.3) *and* (1.4). *Then*:

(i) *For any* $\beta \geq 0$, $D_n^{(\beta)}$ *converges in* L^1 *to* $D_\infty^{(\beta)}$.

(ii) *We have* $D_\infty^{(\beta)} > 0$ *almost surely on the event of nonextinction of the branching random walk killed below* $-\beta$.

(iii) *We have* $D_\infty > 0$ *almost surely on the event of nonextinction of* $\mathbb{T}$.

PROOF. We adapt the proof of [7]; see [22] for the case of the additive martingale. We observe that if $\sup_{n\to\infty} D_n^{(\beta)} < \infty$, $\hat{\mathbf{P}}^{(\beta)}$-a.s, then the family $(D_n^{(\beta)})_{n\geq 0}$ under $\mathbf{P}$ is uniformly integrable, hence converges in L^1. Let

$$\hat{\mathcal{G}}_\infty^{(\beta)} := \sigma\{w_j^{(\beta)}, V(w_j^{(\beta)}), \Omega(w_j^{(\beta)}), (V(u))_{u\in\Omega(w_j^{(\beta)})}, j \geq 1\}$$

be the σ-algebra of the spine and its brothers. Using the martingale property of $D_n^{(\beta)}$ for the subtrees rooted at brothers of the spine, we have

$$\hat{\mathbf{E}}^{(\beta)}[D_n^{(\beta)} | \hat{\mathcal{G}}_\infty^{(\beta)}] = R(V(w_n^{(\beta)}) + \beta)\mathrm{e}^{-V(w_n^{(\beta)})} + \sum_{k=1}^{n} \sum_{x\in\Omega(w_k^{(\beta)})} R(V(x) + \beta)\mathrm{e}^{-V(x)}\mathbf{1}_{\{V(x_j)\geq -\beta, \forall j\leq k\}}.$$

It is well known (see, e.g., the construction available in [24] for the random walk conditioned to stay positive) that $V(w_n^{(\beta)}) \to \infty$ $\hat{\mathbf{P}}^{(\beta)}$-almost surely; therefore $R(V(w_n^{(\beta)}) + \beta)\mathrm{e}^{-V(w_n^{(\beta)})}$ goes to zero as $n \to \infty$. Furthermore, we see that $1/D_n^{(\beta)}$ is under $\hat{\mathbf{P}}^{(\beta)}$ a positive supermartingale, and therefore converges as $n \to \infty$. We still denote by $D_\infty^{(\beta)}$ the (possibly infinite) limit of $D_n^{(\beta)}$ under $\hat{\mathbf{P}}^{(\beta)}$. We already know that there exists $c_{27} > 0$ such that $R(x) \leq c_{27}(1 + x)_+ \leq c_{27}(1 + x_+)$ for any $x \in \mathbb{R}$. Then, by Fatou's lemma,

$$\text{(A.3)} \qquad \begin{aligned} \hat{\mathbf{E}}^{(\beta)}[D_\infty^{(\beta)} | \hat{\mathcal{G}}_\infty^{(\beta)}] &\leq \liminf_{n\to\infty} \hat{\mathbf{E}}^{(\beta)}[D_n^{(\beta)} | \hat{\mathcal{G}}_\infty^{(\beta)}] \\ &\leq c_{27} \sum_{k\geq 1} \sum_{x\in\Omega(w_k^{(\beta)})} (1 + (\beta + V(x))_+)\mathrm{e}^{-V(x)}. \end{aligned}$$

To prove (i), it remains to show that the right-hand side of the last inequality is finite $\hat{\mathbf{P}}^{(\beta)}$-almost surely (which implies that $D_\infty^{(\beta)}$ is finite $\hat{\mathbf{P}}^{(\beta)}$-a.s). We observe that

$$\sum_{k\geq 1}\sum_{x\in\Omega(w_k^{(\beta)})}\left(1+\left(\beta+V(x)\right)_+\right)\mathrm{e}^{-V(x)}\leq A_1+A_2 \tag{A.4}$$

with

$$A_1:=\sum_{k\geq 1}\left(1+\beta+V\left(w_{k-1}^{(\beta)}\right)\right)\mathrm{e}^{-V(w_{k-1}^{(\beta)})}\sum_{x\in\Omega(w_k^{(\beta)})}\mathrm{e}^{-(V(x)-V(w_{k-1}^{(\beta)}))}, \tag{A.5}$$

$$A_2:=\sum_{k\geq 1}\mathrm{e}^{-V(w_{k-1}^{(\beta)})}\sum_{x\in\Omega(w_k^{(\beta)})}\left(V(x)-V\left(w_{k-1}^{(\beta)}\right)\right)_+\mathrm{e}^{-(V(x)-V(w_{k-1}^{(\beta)}))}. \tag{A.6}$$

Let us consider A_1. We recall that $X:=\sum_{|x|=1}\mathrm{e}^{-V(x)}$, $\tilde{X}:=\sum_{|x|=1}V(x)_+\mathrm{e}^{-V(x)}$, and we introduce $X':=\sum_{|x|=1}R(\beta+V(x))\mathrm{e}^{-V(x)}\mathbf{1}_{\{V(x)\geq-\beta\}}$. We observe that, for any $a\geq-\beta$,

$$X'\leq c_{27}\sum_{|x|=1}\mathrm{e}^{-V(x)}\left((1+a+\beta)+\left(V(x)-a\right)_+\right).$$

Therefore, we have for any $z\in\mathbb{R}$ and $a\geq-\beta$,

$$\begin{aligned}
&\hat{\mathbf{P}}_a^{(\beta)}\left(\sum_{|x|=1}\mathrm{e}^{-(V(x)-a)}>z\right)\\
&\quad=\frac{1}{R(a+\beta)\mathrm{e}^{-a}}\mathbf{E}_a\left[X'\mathbf{1}_{\{\sum_{|x|=1}\mathrm{e}^{-(V(x)-a)}>z\}}\right]\\
&\quad\leq c_{40}\mathrm{e}^{a}\mathbf{E}_a\left[\sum_{|x|=1}\mathrm{e}^{-V(x)}\left(1+\frac{(V(x)-a)_+}{1+a+\beta}\right)\mathbf{1}_{\{\sum_{|x|=1}\mathrm{e}^{-(V(x)-a)}>z\}}\right] \qquad (A.7)\\
&\quad=c_{40}\mathbf{E}[X\mathbf{1}_{\{X>z\}}]+c_{40}\frac{1}{1+a+\beta}\mathbf{E}[\tilde{X}\mathbf{1}_{\{X>z\}}]\\
&\quad=: c_{40}h_1(z)+c_{40}\frac{1}{1+a+\beta}h_2(z),
\end{aligned}$$

where h_1 and h_2 are defined by the last equation. We deduce by the Markov property at time $k-1$ that

$$\begin{aligned}
&\hat{\mathbf{P}}^{(\beta)}\left(\sum_{x\in\Omega(w_k^{(\beta)})}\mathrm{e}^{-(V(x)-V(w_{k-1}^{(\beta)}))}\geq\mathrm{e}^{V(w_{k-1}^{(\beta)})/2}\right)\\
&\quad\leq c_{40}\hat{\mathbf{E}}^{(\beta)}\left[h_1\left(\mathrm{e}^{V(w_{k-1}^{(\beta)})/2}\right)+\frac{1}{1+V(w_{k-1}^{(\beta)})+\beta}h_2\left(\mathrm{e}^{V(w_{k-1}^{(\beta)})/2}\right)\right].
\end{aligned}$$

Hence,

$$\sum_{k\geq 1}\hat{\mathbf{P}}^{(\beta)}\Bigg(\sum_{x\in\Omega(w_k^{(\beta)})}\mathrm{e}^{-(V(x)-V(w_{k-1}^{(\beta)}))}\geq \mathrm{e}^{V(w_{k-1}^{(\beta)})/2}\Bigg)$$

$$\tag{A.8}\leq c_{40}\sum_{\ell\geq 0}\hat{\mathbf{E}}^{(\beta)}\big[h_1\big(\mathrm{e}^{V(w_\ell^{(\beta)})/2}\big)\big] + c_{40}\sum_{\ell\geq 0}\hat{\mathbf{E}}^{(\beta)}\bigg[\frac{1}{1+V(w_\ell^{(\beta)})+\beta}h_2\big(\mathrm{e}^{V(w_\ell^{(\beta)})/2}\big)\bigg].$$

We next estimate $\sum_{\ell\geq 0}\hat{\mathbf{E}}^{(\beta)}[h_1(\mathrm{e}^{V(w_\ell^{(\beta)})/2})]$. By (A.2), we have

$$\begin{aligned}\hat{\mathbf{E}}^{(\beta)}\big[h_1\big(\mathrm{e}^{V(w_\ell^{(\beta)})/2}\big)\big] &= \frac{1}{R(\beta)}\mathbf{E}\Big[R(\beta+S_\ell)h_1\big(\mathrm{e}^{S_\ell/2}\big), \min_{j\leq\ell}S_j\geq -\beta\Big]\\ &= \frac{1}{R(\beta)}\mathbf{E}\Big[R(\beta+S_\ell)X\mathbf{1}_{\{S_\ell\leq 2\ln X\}}, \min_{j\leq\ell}S_j\geq -\beta\Big],\end{aligned}$$

where X and the random walk $(S_n, n\geq 0)$ are taken independent. Conditioning on X, then using Lemma B.2(i), we get that

$$\tag{A.9}\begin{aligned}&\sum_{\ell\geq 0}\hat{\mathbf{E}}^{(\beta)}\big[h_1\big(\mathrm{e}^{V(w_\ell^{(\beta)})/2}\big)\big]\\ &\quad\leq \frac{1}{R(\beta)}\mathbf{E}\Big[XR\big(\beta+2\ln(X)\big)\sum_{\ell\geq 0}\mathbf{1}_{\{S_\ell\leq 2\ln X, \min_{j\leq\ell}S_j\geq -\beta\}}\Big]\\ &\quad\leq \frac{c_{41}}{R(\beta)}\mathbf{E}\big[X(1+\ln_+ X)^2\big],\end{aligned}$$

which is finite by (1.4). Similarly,

$$\hat{\mathbf{E}}^{(\beta)}\bigg[\frac{1}{1+V(w_\ell^{(\beta)})+\beta}h_2\big(\mathrm{e}^{V(w_\ell^{(\beta)})/2}\big)\bigg]\leq c_{42}\mathbf{E}\Big[\tilde{X}\mathbf{1}_{\{S_\ell\leq 2\ln X\}}, \min_{j\leq\ell}S_j\geq -\beta\Big].$$

Lemma B.2(i) implies that

$$\tag{A.10}\sum_{\ell\geq 0}\hat{\mathbf{E}}^{(\beta)}\bigg[\frac{1}{1+V(w_\ell^{(\beta)})+\beta}h_2\big(\mathrm{e}^{V(w_\ell^{(\beta)})/2}\big)\bigg]\leq c_{43}\mathbf{E}\big[\tilde{X}(1+\ln_+ X)\big]<\infty$$

under (1.4) by Lemma B.1(i). Equations (A.8), (A.9) and (A.10) give that

$$\tag{A.11}\sum_{k\geq 1}\hat{\mathbf{P}}^{(\beta)}\Bigg(\sum_{x\in\Omega(w_k^{(\beta)})}\mathrm{e}^{-(V(x)-V(w_{k-1}^{(\beta)}))}\geq \mathrm{e}^{V(w_{k-1}^{(\beta)})/2}\Bigg)<\infty.$$

By the Borel–Cantelli lemma, we obtain that

$$\begin{aligned}(1+\beta+V(w_{k-1}^{(\beta)}))\mathrm{e}^{-V(w_{k-1}^{(\beta)})} &\sum_{x\in\Omega(w_k^{(\beta)})} \mathrm{e}^{-(V(x)-V(w_{k-1}^{(\beta)})} \\ &\leq (1+\beta+V(w_{k-1}^{(\beta)}))\mathrm{e}^{-V(w_{k-1}^{(\beta)})/2}\end{aligned}$$

for k large enough almost surely. It is known that, for any $a \in (0, 1/2)$, we have $V(w_k^{(\beta)}) \geq k^a$ for k large enough. From (A.5), we deduce that $A_1 < \infty$. We proceed similarly for A_2, replacing in (A.7) $\mathbf{1}_{\{X>z\}}$ by $\mathbf{1}_{\{\tilde{X}>z\}}$. By analogy, we find that $A_2 < \infty$ if $\mathbf{E}[X(1+\ln_+ \tilde{X})^2]$ and $\mathbf{E}[\tilde{X}(1+\ln_+ \tilde{X})]$ are finite. This is the case by (1.4) and Lemma B.1(i). Equations (A.3) and (A.4) yield that $D_\infty^{(\beta)} < \infty$ $\hat{\mathbf{P}}^{(\beta)}$-a.s., which ends the proof of (i). We prove now (iii). We see that, for any $x \in \mathbb{T}$ with $|x| = 1$,

$$D_\infty \geq \mathrm{e}^{-V(x)} D_{\infty,x} \geq 0,$$

where for any $x \in \mathbb{T}$, $D_{n,x} := \sum_{|u|=n,u\geq x}(V(u) - V(x))\mathrm{e}^{-(V(u)-V(x))}$ and $D_{\infty,x} := \lim_{n\to\infty} D_{n,x}$. We used the fact that the martingale $\sum_{|u|=n,u\geq x} \mathrm{e}^{-V(x)}$ converges to 0 as $n \to \infty$. This implies that if $D_\infty = 0$, then $D_{\infty,x} = 0$. Notice that $D_{\infty,x}$ is distributed as D_∞. Therefore, writing $p := \mathbf{P}(D_\infty = 0)$, we have that $p > 0$ implies that $p \leq \mathbf{E}[p^{\sum_{|x|=1} 1}]$. Consequently, $p = 1$ or $p \leq \mathbf{P}(\text{extinction of } \mathbb{T})$. On the other hand, observe that $p \geq \mathbf{P}(\text{extinction of } \mathbb{T})$, since the sum in (1.5) is empty for large n when the tree $\mathbb{T}$ is finite. Finally, we get that $\mathbf{P}(D_\infty = 0)$ is $\mathbf{P}(\text{extinction of } \mathbb{T})$ or 1. Now, notice that $\mathbf{P}(D_\infty^{(0)} > 0) > 0$ by (i). Since $R(x) \leq c_{27}(1 + x_+)$, we see that $D_\infty^{(0)} \leq c_{27} D_\infty$, and therefore $\mathbf{P}(D_\infty > 0) > 0$. Hence, we have $D_\infty > 0$ $\mathbf{P}$-a.s. on the event of nonextinction. We can now prove (ii). Let $\beta \geq 0$. On the event of nonextinction of the branching random walk killed below β, we can find a vertex u (in the killed branching random walk) such that there is an infinite line of descent from u which stays above $V(u)$. For such a vertex u, we have

$$\sum_{v\geq u,|v|=n} R(V(v)+\beta)\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v_k)\geq-\beta,\forall k\leq n\}} = \sum_{v\geq u,|v|=n} R(V(v)+\beta)\mathrm{e}^{-V(v)}.$$

The sum $\sum_{v\geq u,|v|=n} R(V(v)+\beta)\mathrm{e}^{-V(v)}$ converges to $c_0\mathrm{e}^{-V(u)}D_{\infty,u}$ as $n \to \infty$. We know from (iii) that $D_{\infty,u} > 0$; hence

$$\sum_{v\geq u,|v|=n} R(\beta+V(v))\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v_k)\geq-\beta,\forall k\leq n\}}$$

has a positive limit as $n \to \infty$. Since

$$D_n^{(\beta)} \geq \sum_{v\geq u,|v|=n} R(\beta+V(v))\mathrm{e}^{-V(v)}\mathbf{1}_{\{V(v_k)\geq-\beta,\forall k\leq n\}},$$

we have that $D_\infty^{(\beta)} > 0$. $\square$

APPENDIX B: AUXILIARY ESTIMATES

LEMMA B.1. *Let X and $\tilde{X}$ be nonnegative random variables such that* (1.4) *holds*.

(i) *We have*

$$\mathbf{E}\big[X(\ln_+ \tilde{X})^2\big] < \infty, \qquad \mathbf{E}[\tilde{X}\ln_+ X] < \infty.$$

(ii) *As $z \to \infty$*,

$$\mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2 \min\big(\ln_+(X+\tilde{X}), z\big)\big] = o(z),$$

$$\mathbf{E}\big[\tilde{X}\ln_+(X+\tilde{X}) \min\big(\ln_+(X+\tilde{X}), z\big)\big] = o(z).$$

PROOF. We first prove (i). We claim that for any $x, \tilde{x} \ge 0$,

$$x(\ln_+ \tilde{x})^2 \le 4x(\ln_+ x)^2 + 2\tilde{x}\ln_+ \tilde{x}. \tag{B.1}$$

We can assume that $\tilde{x} \ge 1$. If $\tilde{x} < x^2$, then $x(\ln_+ \tilde{x})^2 \le 4x(\ln_+ x)^2$. If $\tilde{x} \ge x^2$, we check that $x(\ln_+ \tilde{x})^2 \le 2\tilde{x}\ln_+ \tilde{x}$ since $\ln(y) \le 2\sqrt{y}$ for any $y \ge 1$. This gives (B.1). It follows that

$$\mathbf{E}\big[X(\ln_+ \tilde{X})^2\big] \le 4\mathbf{E}\big[X(\ln_+ X)^2\big] + 2\mathbf{E}[\tilde{X}\ln_+ \tilde{X}],$$

which is finite under (1.4). Also, $\tilde{X}\ln_+ X \le \max(\tilde{X}\ln_+ \tilde{X}, X\ln_+ X)$, hence $\mathbf{E}[\tilde{X}\ln_+(X)] < \infty$. We turn to the proof of (ii). Let $\varepsilon > 0$. We observe that

$$\begin{aligned}
&\mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2 \min\big(\ln_+(X+\tilde{X}), z\big)\big]\\
&\quad = \mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2 \min\big(\ln_+(X+\tilde{X}), z\big), \ln_+(X+\tilde{X}) \ge \varepsilon z\big]\\
&\qquad + \mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2 \min\big(\ln_+(X+\tilde{X}), z\big), \ln_+(X+\tilde{X}) < \varepsilon z\big].
\end{aligned}$$

On one hand,

$$\begin{aligned}
&\mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2 \min\big(\ln_+(X+\tilde{X}), z\big), \ln_+(X+\tilde{X}) \ge \varepsilon z\big]\\
&\quad \le z\mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2, \ln_+(X+\tilde{X}) \ge \varepsilon z\big]\\
&\quad = z o_z(1)
\end{aligned}$$

since $\mathbf{E}[X(\ln_+(X+\tilde{X}))^2] < \infty$. On the other hand,

$$\begin{aligned}
&\mathbf{E}\big[X\big(\ln_+(X+\tilde{X})\big)^2 \min\big(\ln_+(X+\tilde{X}), z\big), \ln_+(X+\tilde{X}) < \varepsilon z\big]\\
&\quad \le \varepsilon z\mathbf{E}\big[X(\ln_+ X+\tilde{X})^2\big].
\end{aligned}$$

Thus $\mathbf{E}[X(\ln_+(X+\tilde{X}))^2 \min(\ln_+(X+\tilde{X}), z)] \le (1+\mathbf{E}[X(\ln_+ X+\tilde{X})^2])\varepsilon z$ for z large enough, and is therefore $o(z)$. We show similarly that $\mathbf{E}[\tilde{X}\ln_+(X+\tilde{X})\min(\ln_+(X+\tilde{X}), z)] = o(z)$. □

Let $(S_n)_{n\ge 0}$ be a one-dimensional random walk, with $\mathbf{E}[S_1] = 0$ and $\mathbf{E}[(S_1)^2] < \infty$.

LEMMA B.2. (i) *There exists a constant $c_{45} > 0$ such that for any $z \ge 0$ and $x \ge 0$*

$$\sum_{\ell \ge 0} \mathbf{P}_z\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge 0\Big) \le c_{45}(1+x)\big(1+\min(x,z)\big).$$

(ii) *Let $a > 0$. We have*

$$\mathbf{E}\Big[\sum_{\ell \ge 0} \mathrm{e}^{-aS_\ell} \mathbf{1}_{\{\min_{j \le \ell} S_j \ge 0\}}\Big] = c_{46}(a) < \infty.$$

(iii) *Let $a > 0$. There exists a constant $c_{47}(a) > 0$ such that for any $z \ge 0$,*

$$\mathbf{E}_z\Big[\sum_{\ell \ge 0} \mathrm{e}^{-aS_\ell} \mathbf{1}_{\{\min_{j \le \ell} S_j \ge 0\}}\Big] \le c_{47}(a).$$

PROOF. Suppose that $x < z$. If τ_x^- denotes the first passage time below level x of $(S_n)_{n \ge 0}$, we have

$$\begin{aligned}\sum_{\ell \ge 0} \mathbf{P}_z\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge 0\Big) &= \mathbf{E}_z\Big[\sum_{\ell \ge \tau_x^-} \mathbf{1}_{\{S_\ell \le x, \min_{j \le \ell} S_j \ge 0\}}\Big] \\ &\le \mathbf{E}\Big[\sum_{\ell \ge 0} \mathbf{1}_{\{S_\ell \le x, \min_{j \le \ell} S_j \ge -x\}}\Big],\end{aligned}$$

where we used the Markov property at time τ_x^-. We have

$$\begin{aligned}\sum_{\ell \ge 0} \mathbf{P}\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge -x\Big) &\le 1 + x^2 + \sum_{\ell > x^2} \mathbf{P}\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge -x\Big) \\ &\le 1 + x^2 + c_{48} \sum_{\ell > x^2} (1+x)^3 \ell^{-3/2} \qquad \text{(B.2)} \\ &\le c_{49}(1+x)^2\end{aligned}$$

by (2.8). Suppose now that $x \ge z$. Then

$$\begin{aligned}&\sum_{\ell \ge 0} \mathbf{P}_z\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge 0\Big) \\ &\quad \le \sum_{\ell \le x^2} \mathbf{P}_z\Big(\min_{j \le \ell} S_j \ge 0\Big) + \sum_{\ell > x^2} \mathbf{P}_z\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge 0\Big).\end{aligned}$$

From (2.7), we know that $\mathbf{P}_z(\min_{j \le \ell} S_j \ge 0) \le c_{50}(1+z)(1+\ell)^{-1/2}$, whereas, by (2.8),

$$\mathbf{P}_z\Big(S_\ell \le x, \min_{j \le \ell} S_j \ge 0\Big) \le c_{51}(1+z)(1+x)^2(1+\ell)^{-3/2}.$$

We get

$$\sum_{\ell\ge 0}\mathbf{P}_z\Big(S_\ell\le x,\min_{j\le\ell}S_j\ge 0\Big)$$

(B.3)
$$\le c_{50}\sum_{\ell\le x^2}\frac{1+z}{\sqrt{1+\ell}}+c_{51}\sum_{\ell>x^2}(1+z)(1+x)^2(1+\ell)^{-3/2}$$
$$\le c_{52}(1+z)(1+x).$$

From (B.2) when $x<z$ and (B.3) when $x\ge z$, we have for $x,z\ge 0$,

$$\sum_{\ell\ge 0}\mathbf{P}_z\Big(S_\ell\le x,\min_{j\le\ell}S_j\ge 0\Big)\le (c_{49}+c_{52})(1+x)\big(1+\min(x,z)\big).$$

This ends the proof of (i). We turn to the statement (ii). Without loss of generality, we assume that $a=1$ [in (ii) and in (iii)]. We have

$$\sum_{\ell\ge 0}\mathbf{E}[\mathrm{e}^{-S_\ell}\mathbf{1}_{\{\min_{j\le\ell}S_j\ge 0\}}]=\sum_{\ell\ge 0}\sum_{i\ge 0}\mathrm{e}^{-i}\mathbf{P}\Big(S_\ell\in[i,i+1),\min_{j\le\ell}S_j\ge 0\Big).$$

By (2.8), $\mathbf{P}(S_\ell\in[i,i+1),\min_{j\le\ell}S_j\ge 0)\le c_{53}(1+i)(1+\ell)^{-3/2}$, which completes the proof of (ii). Finally, we prove (iii). Let $(T_k,H_k,k\ge 0)$ be the strict descending ladder epochs and heights of $(S_n)_{n\ge 0}$, that is, $T_0:=0$, $H_0:=S_0$, and for any $k\ge 1$, $T_k:=\min\{j>T_{k-1}:S_j<H_{k-1}\}$, $H_k:=S_{T_k}$. By applying the Markov property at the times $(T_k,k\ge 0)$, we observe that

$$\mathbf{E}_z\Big[\sum_{\ell\ge 0}\mathrm{e}^{-S_\ell}\mathbf{1}_{\{\min_{j\le\ell}S_j\ge 0\}}\Big]=c_{46}\mathbf{E}_z\Big[\sum_{k\ge 0}\mathrm{e}^{-H_k}\mathbf{1}_{\{H_k\ge 0\}}\Big],$$

where c_{46} is the constant of (ii). The fact that $Z(z):=\mathbf{E}_z[\sum_{k\ge 0}\mathrm{e}^{-H_k}\mathbf{1}_{\{H_k\ge 0\}}]$ is bounded in $z\ge 0$ then comes from the renewal theorem: let $U(dy)$ denote the renewal measure of $(H_k,k\ge 0)$, that is, $U(dy):=\sum_{k\ge 0}\mathbf{P}(H_k\in dy)$. Then $Z(z)=\int_{-z}^{0}\mathrm{e}^{-(z+y)}U(dy)$. In Section XI.1 of [14], combine lemma, page 359, with the renewal theorem, page 363, to conclude that Z is bounded. This completes the proof of (iii). □

For $\alpha>0$, $a\ge 0$, $n\ge 1$ and $0\le i\le n$, we define

(B.4)
$$k_i:=\begin{cases} i^\alpha, & \text{if } 0\le i\le\lfloor n/2\rfloor,\\ a+(n-i)^\alpha, & \text{if } \lfloor n/2\rfloor<i\le n.\end{cases}$$

LEMMA B.3. *Let $\alpha\in(0,1/6)$ and $\varepsilon>0$.*

(i) *There exist $d>0$ and $c_{53}>0$ such that for any $u\ge 0$, $a\ge 0$ and any integer $n\ge 1$,*

(B.5)
$$\mathbf{P}\Big\{\exists 0\le i\le n:S_i\le k_i-d,\min_{j\le n}S_j\ge 0,\min_{\lfloor n/2\rfloor<j\le n}S_j\ge a,S_n\le a+u\Big\}$$
$$\le (1+u)^2\Big\{\frac{\varepsilon}{n^{3/2}}+c_{53}\frac{(n^\alpha+a)^2}{n^{2-\alpha}}\Big\},$$

where k_i is given by (B.4).

PROOF. We treat $n/2$ as an integer. Let E be the event in (B.5). We have $\mathbf{P}(E) \le \sum_{i=1}^n \mathbf{P}(E_i)$ where

$$E_i := \Big\{S_i \le k_i - d, \min_{j\le n} S_j \ge 0, \min_{n/2<j\le n} S_j \ge a, S_n \le a+u\Big\}.$$

We first treat the case $i \le n/2$, so that $k_i = i^\alpha$. By the Markov property at time $i \ge 1$ and (2.9), we have

$$\mathbf{P}(E_i) \le \frac{c_{54}(1+u)^2}{n^{3/2}} \mathbf{E}\big[(1+S_i)\mathbf{1}_{\{S_i \le i^\alpha, \min_{j\le i} S_j \ge 0\}}\big],$$

which is smaller than $\frac{c_{55}(1+u)^2}{n^{3/2}} \frac{(1+i^\alpha)^3}{i^{3/2}}$ by (2.8). It yields that, if K is greater than some constant K_0 (which does not depend on d), we have

$$\sum_{i=K}^{n/2} \mathbf{P}(E_i) \le (1+u)^2 \frac{\varepsilon}{n^{3/2}}. \tag{B.6}$$

($\sum_{i=x}^{y} := 0$ if $x > y$.) We treat the case $n/2 < i \le n$. We have by the Markov property at time i and (2.8),

$$\mathbf{P}(E_i) \le \frac{c_{56}(1+u)^2}{(n-i+1)^{3/2}} \mathbf{E}\big[(1+S_i-a)\mathbf{1}_{\{S_i \le a+(n-i)^\alpha, \min_{j\le i} S_j \ge 0, \min_{n/2<j\le i} S_j \ge a\}}\big].$$

If $i \ge 2n/3$, we use (2.9) to see that $\mathbf{P}(E_i) \le c_{57}(1+u)^2 \frac{(1+n-i)^{3\alpha-3/2}}{n^{3/2}}$. Therefore, if $K \ge K_1$ (K_1 does not depend on d),

$$\sum_{i=\lfloor 2n/3\rfloor}^{n-K} \mathbf{P}(E_i) \le (1+u)^2 \frac{\varepsilon}{n^{3/2}}. \tag{B.7}$$

If $n/2 < i < 2n/3$, we simply write

$$\begin{aligned}
\mathbf{P}(E_i) &\le \frac{c_{56}(1+u)^2}{(n-i+1)^{3/2}} \mathbf{E}\big[(1+S_i-a)\mathbf{1}_{\{a\le S_i \le a+(n-i)^\alpha, \min_{j\le i} S_j \ge 0\}}\big] \\
&\le c_{59}(1+u)^2 \frac{(n-i)^\alpha}{(n-i+1)^{3/2}} \mathbf{P}\Big(a \le S_i \le a+(n-i)^\alpha, \min_{j\le i} S_j \ge 0\Big) \\
&\le c_{60}(1+u)^2 \frac{n^\alpha (a+n^\alpha)^2}{n^3}
\end{aligned}$$

by (2.8). We deduce that

$$\sum_{i=n/2}^{\lfloor 2n/3\rfloor} \mathbf{P}(E_i) \le c_{61}(1+u)^2 \frac{(n^\alpha+a)^2}{n^{2-\alpha}}. \tag{B.8}$$

Notice that our choice of K does not depend on the constant d. Thus, we are allowed to choose $d \geq K^{\alpha}$, for which $\mathbf{P}(E_i) = 0$ if $i \in [1, K] \cup [n-K, n]$. We obtain by (B.6), (B.7) and (B.8)

$$\sum_{i=1}^{n} \mathbf{P}(E_i) \leq (1+u)^2 \left\{ 2\frac{\varepsilon}{n^{3/2}} + c_{61} \frac{(n^{\alpha}+a)^2}{n^{2-\alpha}} \right\},$$

hence $\mathbf{P}(E) \leq (1+u)^2 \{2\frac{\varepsilon}{n^{3/2}} + c_{61}\frac{(n^{\alpha}+a)^2}{n^{2-\alpha}}\}$, indeed. □

APPENDIX C: THE GOOD VERTEX

Let $z \geq 0$ and $L \geq 0$. Let $d_k = d_k(n, z+L, 1/2)$ as defined in (3.4). Let also

$$e_k = e_k^{(n)} := \begin{cases} k^{1/12}, & \text{if } 0 \leq k \leq \dfrac{n}{2}, \\ (n-k)^{1/12}, & \text{if } \dfrac{n}{2} < k \leq n. \end{cases}$$

We recall from definition 3.5 that $u \in \mathcal{Z}_n^{z,L}$ if $|u| = n$, $V(u_k) \geq d_k$ for $k \leq n$ and $V(u) \in I_n(z)$. We say that u such that $|u| = n$ is a (z, L)-good vertex if $u \in \mathcal{Z}_n^{z,L}$ and for any $1 \leq k \leq n$,

$$\sum_{v \in \Omega(u_k)} \mathrm{e}^{-(V(v)-d_k)} \{1 + (V(v) - d_k)_+\} \leq B\mathrm{e}^{-e_k}. \tag{C.1}$$

Note that a $(z, 0)$-good vertex is a z-good vertex as introduced in Section 3.1. We defined the probability $\hat{\mathbf{P}}$ in (2.3) and the spine $(w_n, n \geq 0)$ in Section 2.1.

LEMMA C.1. *Fix $L \geq 0$. For any $\varepsilon > 0$, we can find B large enough in* (C.1) *such that $\hat{\mathbf{P}}(w_n$ is not a (z, L)-good vertex, $w_n \in \mathcal{Z}_n^{z,L}) \leq \varepsilon n^{-3/2}$ for any $n \geq 1$ and $z \geq 0$.*

PROOF. Fix $L \geq 0$ and let $\varepsilon > 0$. We have

$$\begin{aligned} &\hat{\mathbf{P}}\big(w_n \text{ is not a } (z, L)\text{-good vertex}, w_n \in \mathcal{Z}_n^{z,L}\big) \\ &\quad \leq \hat{\mathbf{P}}\Bigg(\exists k \in [1, n] \colon \sum_{v \in \Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)} \{1 + (V(v) - d_k)_+\} > B\mathrm{e}^{-e_k}, \\ &\qquad\qquad w_n \in \mathcal{Z}_n^{z,L}\Bigg). \end{aligned} \tag{C.2}$$

We want to show that we can find B large enough such that

$$\begin{aligned} &\hat{\mathbf{P}}\Bigg(\exists k \in [1, n] \colon \sum_{v \in \Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)} \{1 + (V(v) - d_k)_+\} > B\mathrm{e}^{-e_k}, \\ &\qquad\qquad w_n \in \mathcal{Z}_n^{z,L}\Bigg) \leq \frac{\varepsilon}{n^{3/2}}. \end{aligned} \tag{C.3}$$

We see that, for any $1 \le k \le n$,

$$\left\{ \sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}\{1+(V(v)-d_k)_+\} > B\mathrm{e}^{-e_k}, V(w_{k-1}) \ge d_k + 2e_k - c_{62} \right\}$$
$$\subset \left\{ \sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}\{1+(V(v)-d_k)_+\} > B\mathrm{e}^{-(V(w_{k-1})-d_k+c_{62})/2} \right\}.$$

By Lemma B.3, there exists $c_{62} = c_{62}(L) > 0$ and $N = N(L)$ such that for $n \ge N$ and $z \ge 0$

$$\hat{\mathbf{P}}(w_n \in \mathcal{Z}_n^{z,L}, \exists 0 \le j \le n-1 : V(w_j) \le d_{j+1} + 2e_{j+1} - c_{62}) \le \frac{\varepsilon}{n^{3/2}}.$$

Consequently, it is enough to show that for B large enough,

$$\sum_{k=1}^{n} \hat{\mathbf{P}}\Bigg(\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}\{1+(V(v)-d_k)_+\} > B\mathrm{e}^{-(V(w_{k-1})-d_k)/2}, \qquad w_n \in \mathcal{Z}_n^{z,L} \Bigg) \le \varepsilon n^{-3/2}. \tag{C.4}$$

We see that

$$\begin{aligned}
&\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}(1+(V(v)-d_k)_+) \\
&\quad \le \mathrm{e}^{-(V(w_{k-1})-d_k)} \sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-V(w_{k-1}))} \\
&\qquad\qquad \times \{1+(V(w_{k-1})-d_k)_+ + (V(v)-V(w_{k-1}))_+\} \\
&\quad \le \mathrm{e}^{-(V(w_{k-1})-d_k)}(1+(V(w_{k-1})-d_k)_+) \\
&\qquad \times \sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-V(w_{k-1}))}\{1+(V(v)-V(w_{k-1}))_+\}.
\end{aligned}$$

With the notation of (4.2), we have then

$$\begin{aligned}
&\sum_{v\in\Omega(w_k)} \mathrm{e}^{-(V(v)-d_k)}(1+(V(v)-d_k)_+) \\
&\qquad \le \mathrm{e}^{-(V(w_{k-1})-d_k)}(1+(V(w_{k-1})-d_k)_+)\xi(w_k).
\end{aligned}$$

Equation (C.4) boils down to showing that, for B large enough,

$$\sum_{k=1}^{n} \hat{\mathbf{P}}\left(\xi(w_k) > B\frac{\mathrm{e}^{(V(w_{k-1})-d_k)/2}}{1+(V(w_{k-1})-d_k)_+}, w_n \in \mathcal{Z}_n^{z,L}\right) \le \varepsilon n^{-3/2}.$$

Actually, we are going to show that, for B large enough,

$$\sum_{k=1}^{n} \hat{\mathbf{P}}(\xi(w_k) > B\mathrm{e}^{(V(w_{k-1})-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}) \le \varepsilon n^{-3/2}. \tag{C.5}$$

First, we deal with the case $k \in [1, 3n/4]$. We notice that

$$\hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{(V(w_{k-1})-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}\big) \le \hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, w_n \in \mathcal{Z}_n^{z,L}\big).$$

By the Markov property at time k, we get

$$\begin{aligned}&\hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, w_n \in \mathcal{Z}_n^{z,L}\big)\\ &\quad = \hat{\mathbf{E}}\big[\lambda\big(V(w_k), k, n\big)\mathbf{1}_{\{\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, V(w_j)\ge 0, \forall j\le k\}}\big],\end{aligned}$$

where $\lambda(r,k,n) := \hat{\mathbf{P}}_r(V(w_j) \ge d_{j+k}, \forall j \le n-k, V(w_{n-k}) \in I_n(z))$. We get by (2.9), $\lambda(r,k,n) \le c_{63} n^{-3/2}(1+r_+)$ (since $k \le 3n/4$). This yields that

$$\begin{aligned}&\hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, w_n \in \mathcal{Z}_n^{z,L}\big)\\ &\quad \le c_{63} n^{-3/2}\hat{\mathbf{E}}\big[\big(1+V(w_k)_+\big)\mathbf{1}_{\{\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, V(w_j)\ge 0, \forall j\le k\}}\big].\end{aligned} \tag{C.6}$$

On the other hand, we have

$$1 + V(w_k)_+ \le 1 + V(w_{k-1})_+ + \big(V(w_k) - V(w_{k-1})\big)_+.$$

Let (ξ, Δ) be generic random variables distributed as $(\xi(w_1), V(w_1)_+)$ under $\hat{\mathbf{P}}$, and independent of the other random variables. By the Markov property at time $k-1$, we obtain that

$$\begin{aligned}&\hat{\mathbf{E}}\big[\big(1+V(w_k)_+\big)\mathbf{1}_{\{\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, V(w_j)\ge 0, \forall j\le k\}}\big]\\ &\quad \le \hat{\mathbf{E}}\big[\kappa\big(V(w_{k-1})\big)\mathbf{1}_{\{V(w_j)\ge 0, \forall j\le k-1\}}\big]\end{aligned}$$

with, for $x \ge 0$, $\kappa(x) := (1+x)\mathbf{1}_{\{\xi > B\mathrm{e}^{x/3}\}} + \Delta_+\mathbf{1}_{\{\xi > B\mathrm{e}^{x/3}\}}$. In view of (C.6), it follows that

$$\sum_{k=1}^{3n/4}\hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{V(w_{k-1})/3}, w_n \in \mathcal{Z}_n^{z,L}\big) \le c_{63} n^{-3/2}(D_1 + D_2),$$

where

$$D_1 := \sum_{k\ge 0}\hat{\mathbf{E}}\Big[\big(1+V(w_k)\big)\mathbf{1}_{\{V(w_k)\le 3(\ln\xi - \ln B)\}}, \min_{j\le k} V(w_j) \ge 0\Big],$$

$$D_2 := \sum_{k\ge 0}\hat{\mathbf{E}}\Big[\Delta_+\mathbf{1}_{\{V(w_k)\le 3(\ln\xi - \ln B)\}}, \min_{j\le k} V(w_j) \ge 0\Big].$$

We recall that by Proposition 2.2 $(V(w_n), n \ge 0)$ is distributed as $(S_n, n \ge 0)$ (under $\mathbf{P}$). Notice that in the definition of D_1, the term inside the expectation is 0 if $B > \xi$. Therefore, we can add the indicator that $B \le \xi$. By Lemma B.2(i), we get that

$$D_1 \le c_{65}\hat{\mathbf{E}}\big[\mathbf{1}_{\{B\le\xi\}}\big(1 + (\ln\xi - \ln B)_+\big)^2\big] \le c_{65}\hat{\mathbf{E}}\big[\mathbf{1}_{\{B\le\xi\}}(1 + \ln_+\xi)^2\big].$$

Observe that $\xi \le X + \tilde{X}$ with the notation of (1.2). Going back to the measure $\mathbf{P}$, we get

$$D_1 \le c_{65}\mathbf{E}\big[X\mathbf{1}_{\{B\le X+\tilde{X}\}}\big(1+\ln_+(X+\tilde{X})\big)^2\big] \le \varepsilon$$

for B large enough since $\mathbf{E}[X(1+\ln_+(X+\tilde{X}))^2] < \infty$ by (1.4) and Lemma B.1(i). Similarly,

$$D_2 \le c_{66}\mathbf{E}\big[\tilde{X}\mathbf{1}_{\{B\le X+\tilde{X}\}}\big(1+\ln_+(X+\tilde{X})\big)\big] \le \varepsilon$$

for B large enough. Therefore, for B large enough,

$$\text{(C.7)} \qquad \sum_{k=1}^{3n/4} \hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{-(V(w_{k-1})-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}\big) \le 2\frac{\varepsilon}{n^{3/2}}.$$

In order to prove (C.5), it remains to treat the case $3n/4 \le k \le n$. We want to show that for B large enough,

$$\text{(C.8)} \qquad \sum_{k=3n/4}^{n} \hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{(V(w_k)-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}\big) \le \varepsilon.$$

We want to condition the point process $\mu(w_1) := \sum_{u\in\Omega(w_1)} \delta_{V(u)}$ on the value of $V(w_1)$. To do this, we make a disintegration; see, for example, 15.3.3, page 164 of [18]. This gives the existence of probabilities $\mathbf{Q}_r$ on the space of locally finite measures $\mathcal{M}$ on $\mathbb{R}$, such that:

- For any set A in the canonical σ-algebra of $\mathcal{M}$, the map $r \in \mathbb{R} \to \mathbf{Q}_r(A)$ is measurable with respect to the Borelian σ-algebra of $\mathbb{R}$. Here, the canonical σ-algebra of $\mathcal{M}$ refers to the one generated by the mappings $\mu \in \mathcal{M} \to \mu(I)$ for I intervals of $\mathbb{R}$; see Chapter 1 of [18].
- For any bounded measurable function F, we have

$$\hat{\mathbf{E}}\big[F\big(\mu(w_1), V(w_1)\big)\big] = \int_{\mathbb{R}} \hat{\mathbf{P}}\big(V(w_1) \in dr\big) \int_{\mathcal{M}} F(\mu, r)\mathbf{Q}_r(d\mu).$$

We deduce that

$$\begin{aligned}&\hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{(V(w_k)-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}\big)\\ &\qquad = \hat{\mathbf{P}}\big(\overline{\xi}\big(V(w_k) - V(w_{k-1})\big) > B\mathrm{e}^{(V(w_k)-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}\big),\end{aligned}$$

where, given $(V(w_k), k \le n)$, the random variable $\overline{\xi}(V(w_k) - V(w_{k-1})) \in \mathbb{R}$ has the distribution of $\int_{x\in\mathbb{R}}(1+x_+)\mathrm{e}^{-x}\mu(dx)$ under $\mathbf{Q}_{V(w_k)-V(w_{k-1})}(d\mu)$. The last line is equal to

$$\mathbf{P}\big(\overline{\xi}(S_k - S_{k-1}) > B\mathrm{e}^{(S_k-d_k)/3}, S_n \in I_n(z), \underline{S}_n \ge 0, \underline{S}_{(n/2,n]} \ge a_n(z+L+1)\big),$$

where $\underline{S}_n := \min\{S_k, k \le n\}$, $\underline{S}_{(\ell_1,\ell_2]} := \min\{S_k, \ell_1 < k \le \ell_2\}$, and, under $\mathbf{P}$, and conditionally on $(S_k, k \le n)$, the random variable $\overline{\xi}(S_k - S_{k-1})$ has the distribution of $\int_{x\in\mathbb{R}}(1+x_+)\mathrm{e}^{-x}\mu(dx)$ under $\mathbf{Q}_{S_k-S_{k-1}}(d\mu)$. We return time, that is, we replace S_k by $S_n - S_{n-k}$. We check that

$$\begin{aligned}
&\mathbf{P}\big(\overline{\xi}(S_k - S_{k-1}) > B\mathrm{e}^{(S_k-d_k)/3}, S_n \in I_n(z), \underline{S}_n \geq 0, \underline{S}_{(n/2,n]} \geq a_n(z+L+1)\big) \\
&\quad \leq \mathbf{P}\big(\overline{\xi}(S_{n-k+1} - S_{n-k}) > B\mathrm{e}^{L+1}\mathrm{e}^{-S_{n-k}/3}, S_n \in I_n(z), \underline{-S}_n \geq -a_n(z), \\
&\qquad\qquad \underline{-S}_{[0,n/2)} \geq -L-1\big),
\end{aligned}$$

where $\underline{-S}_n := \min\{-S_k, k \leq n\}$ and $\underline{-S}_{[\ell_1,\ell_2)} := \min\{-S_k, \ell_1 \leq k < \ell_2\}$. We use the Markov property at time $n-k+1$. There exists a constant $c_{67} > 0$ such that, for any $r \leq L+1$, any $n \geq 1$, and any $k \in [3n/4, n]$,

$$\begin{aligned}
&\mathbf{P}_r\big(S_{k-1} \in I_n(z), \underline{-S}_{k-1} \geq -a_n(z), \underline{-S}_{[0,k-1-n/2)} \geq -L-1\big) \\
&\quad \leq c_{67}(2+L-r)n^{-3/2}.
\end{aligned}$$

The last inequality comes from (2.9), after a time reversal. This yields that, for any $n \geq 1$ and $k \in [3n/4, n]$,

$$\begin{aligned}
&\mathbf{P}\big(\overline{\xi}(S_k - S_{k-1}) > B\mathrm{e}^{(S_k-d_k)/3}, S_n \in I_n(z), \underline{S}_n \geq 0, \underline{S}_{(n/2,n]} \geq a_n(z+L+1)\big) \\
&\quad \leq c_{67}n^{-3/2}\mathbf{E}\big[(2+L-S_{n-k+1})\mathbf{1}_{\{\overline{\xi}(S_{n-k+1}-S_{n-k}) > \tilde{B}\mathrm{e}^{-S_{n-k}/3}, \underline{-S}_{n-k+1} \geq -L-1\}}\big] \\
&\quad = c_{67}n^{-3/2}\hat{\mathbf{E}}\big[\big(2 + \big(L - V(w_{n-k+1})\big)_+\big) \\
&\qquad\qquad \times \mathbf{1}_{\{\xi(w_{n-k}) > \tilde{B}\mathrm{e}^{-V(w_{n-k})/3}, -V(w_j) \geq -L-1, \forall j \leq n-k+1\}}\big],
\end{aligned}$$

where $\tilde{B} := B\mathrm{e}^{L+1}$. Beware that we reintegrated the measures $(\mathbf{Q}_r, r \in \mathbb{R})$ in the last line. We find that, for any $k \in [3n/4, n]$,

$$\begin{aligned}
&\hat{\mathbf{P}}\big(\xi(w_k) > B\mathrm{e}^{(V(w_k)-d_k)/3}, w_n \in \mathcal{Z}_n^{z,L}\big) \\
&\quad \leq c_{67}n^{-3/2}\hat{\mathbf{E}}\big[\big(2 + \big(L - V(w_{n-k+1})\big)_+\big) \\
&\qquad\qquad \times \mathbf{1}_{\{\xi(w_{n-k}) > \tilde{B}\mathrm{e}^{-V(w_{n-k})/3}, -V(w_j) \geq -L-1, \forall j \leq n-k+1\}}\big].
\end{aligned}$$

This is the analog of (C.6), replacing there $V(w_j)$ by $-V(w_j)$, k by $n-k+1$ and $\{V(w_j) \geq 0, \forall j \leq k\}$ by $\{-V(w_j) \geq -L-1, \forall j \leq n-k+1\}$. Then (C.8) follows as in the case $k \in [1, 3n/4]$. This with (C.7) prove (C.5) and hence the lemma. □

APPENDIX D: NOTATION

Branching random walk:

$\mathcal{L}$: the point process;
X, $\tilde{X}$: defined in (1.2);
$\mathbb{T}$: the genealogical tree;

$V(x)$: position of particle x;
$|x|$: generation of vertex x;
x_k: ancestor at generation k of vertex x;
M_n: minimum at generation n of the *nonkilled* branching random walk;
$\mathscr{F}_n$: σ-algebra of the branching random walk up to time n;
$\xi(x)$: defined in (4.2);
$\Phi_{k,n}(r) := \mathbf{P}(M_{n-k} \geq (3/2)\ln(n) - r)$;
$\mathcal{Z}[A]$: set of particles frozen when going above level A.

Killed branching random walk:

$\mathbb{T}^{\text{kill}}$: the genealogical tree of the *killed* branching random walk;
$|u|^{\text{kill}}$: generation of a vertex u when $u \in \mathbb{T}^{\text{kill}}$;
M_n^{kill}: minimum at generation n of the *killed* branching random walk;
$m^{\text{kill},n}$: uniform particle in $\mathbb{T}^{\text{kill}}$ among those achieving M_n^{kill};
$\Phi_{k,n}^{\text{kill}}(x, r) := \mathbf{P}_x(M_{n-k}^{\text{kill}} \leq (3/2)\ln(n) - r)$.

Random walk:

$(S_n)_{n\geq 0}$: nonlattice centered random walk with finite variance, defined by (2.1). The nonlattice assumption is dropped in the Appendix;
σ^2: variance of S_1;
$R(x)$: renewal function of S;
$R_-(x)$: renewal function of $-S$;
Many-to-one lemma: equation (2.1);
H_k, T_k: strict descending ladder heights and epochs of S;
H_k^-, T_k^-: strict descending ladder heights and epochs of $-S$.

Martingales:

W_n: additive martingale at time n;
D_n: derivative martingale at time n;
$D_n^{(\beta)}$: martingale of the branching random walk killed below $-\beta$.

Probability measures:

$\mathbf{P}_a$: probability under which the branching random walk $(V(x))_{x\in\mathbb{T}}$ and the random walk $(S_n)_n$ starts at a ($\mathbf{P}_0 = \mathbf{P}$). Expectation $\mathbf{E}_a$;
$\hat{\mathbf{P}}_a$: tilted probability I defined by (2.3). Expectation $\hat{\mathbf{E}}_a$;
$\hat{\mathbf{P}}_a^{(\beta)}$: tilted probability II defined by (A.1). Expectation $\hat{\mathbf{E}}_a^{(\beta)}$.

Spine decomposition I:

w_n: spine at generation n;
$(V(w_n))_n$: centered random walk distributed as $(S_n)_n$;
$\hat{\mathcal{L}}$: Radon-Nykodim derivative $\sum_{i\in\mathcal{L}} \mathrm{e}^{-V(i)}$ with respect to $\mathcal{L}$;

$\hat{\mathcal{B}}$: branching random walk with a spine. Under $\mathbf{P}$, we identify $(V(x))_{x\in\mathbb{T}}$ with $\hat{\mathcal{B}}$;
$\hat{\mathscr{F}}_n$: σ-algebra of $\hat{\mathcal{B}}$ up to time n;
$\hat{\mathcal{G}}_n$: σ-algebra of the spine and its siblings up to time n ($\hat{\mathcal{G}}_n \subset \hat{\mathscr{F}}_n$).

Spine decomposition II:

$w_n^{(\beta)}$: spine at generation n;
$(V(w_n^{(\beta)}))_n$: random walk conditioned to stay above $-\beta$;
$\hat{\mathcal{B}}^{(\beta)}$: branching random walk with a spine;
$\hat{\mathscr{F}}_n^{(\beta)}$: σ-algebra of $\hat{\mathcal{B}}^{(\beta)}$ up to time n;
$\hat{\mathcal{G}}_n^{(\beta)}$: σ-algebra of the spine and its siblings up to time n ($\hat{\mathcal{G}}_n^{(\beta)} \subset \hat{\mathscr{F}}_n^{(\beta)}$).

Paths of particles:

$a_n(z) = \frac{3}{2}\ln(n) - z$;
$d_k(n, z, \lambda)$: defined in (3.4);
e_k: defined in (3.31);
$I_n(z) = [a_n(z) - 1, a_n(z))$;
$\mathcal{Z}_n^{z,L}$: in Definition 3.5, see Figure 1. Particles of generation n that stayed above $d_k(n, z + L, 1/2)$ and end in $I_n(z)$;
$\mathcal{S}^r$: defined in (4.1), see Figure 2. Set of particles that achieve a new minimum (on their ancestral line);
$B_n^z(u)$: defined in (4.4), see Figure 3. Equal to 1 if there is a line of descent from u to a vertex at generation n which stays above $V(u)$ and ends below $a_n(z)$;
$\mathcal{T}^r$: defined in (4.3);
z-good vertex: defined in (3.15);
$\mathcal{E}_n(z, b)$: defined in (3.21). Good event on which the particles at generation n which are located below $a_n(z)$ have a common ancestor with the spine at generation greater than $n - b$;
$F_{L,b}$: defined in (3.23);
$C_{L,b}$: defined in (3.24).

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
EINDHOVEN UNIVERSITY OF TECHNOLOGY
P.O. BOX 513
5600 MB EINDHOVEN
THE NETHERLANDS
E-MAIL: elie.aidekon@gmail.com