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# EXISTENCE, UNIQUENESS AND COMPARISONS FOR BSDES IN GENERAL SPACES

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We present a theory of backward stochastic differential equations in continuous time with an arbitrary filtered probability space. No assumptions are made regarding the left continuity of the filtration, of the predictable quadratic variations of martingales or of the measure integrating the driver. We present conditions for existence and uniqueness of square-integrable solutions, using Lipschitz continuity of the driver. These conditions unite the requirements for existence in continuous and discrete time and allow discrete processes to be embedded with continuous ones. We also present conditions for a comparison theorem and hence construct time consistent nonlinear expectations in these general spaces.

**1. Introduction.** The theory of backward stochastic differential equations (BSDEs) has been extensively studied. Typically, results have been obtained only in the context of a filtration generated by a Brownian motion, possibly with the addition of Poisson jumps. Specifically, attention has been given to equations of the form

$$dY_t = F(\omega, t, Y_{t-}, Z_t)\,dt - Z_t^* \,dM_t, \qquad Y_T = Q,$$

where $M$ is the martingale generating the filtration (typically Brownian motion), $T$ is a fixed finite terminal time, $Q \in L^2(\mathcal{F}_T)$ is a stochastic terminal value, $F$ is a progressively measurable function, $[\cdot]^*$ denotes matrix/vector transposition (and hence $A^*B$ denotes the inner product of $A$ and $B$) and the solution is a square integrable pair of processes $(Y, Z)$, where $Y$ is adapted and $Z$ is predictable.

A notable exception to this is the work of El Karoui and Huang [12], where a general probability space is considered. In the case considered in [12], the martingale $M$ is specified a priori, and the equation considered is

$$dY_t = F(\omega, t, Y_{t-}, Z_t)\,dC_t - Z_t^* \,dM_t - dN_t; \qquad Y_T = Q, \tag{1}$$

where each term is as above, the filtration is quasi-left continuous, $C$ is a continuous process such that $d\langle M\rangle$ is absolutely continuous with respect to $dC$ and $N$ is a

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martingale strongly orthogonal to $M$, that is, $\langle M, N\rangle = 0$, where $\langle\cdot,\cdot\rangle$ denotes the predictable quadratic covariation process.

These equations depend heavily on the continuity of $C$ and, therefore, are unable to deal with any situation where martingales may jump at a point with positive probability. However, these situations may arise in various applications. For example, when using BSDEs in modeling dividend paying assets, the martingales involved may jump at the time of the dividend announcement. Similarly, if we consider embedding a discrete time process in continuous time, we obtain processes which jump with positive probability at every integer.

A significant use of these equations is to generate "nonlinear expectations" or "nonlinear evaluations," in the sense of [18]. These are operators

$$\mathcal{E}(\cdot|\mathcal{F}_t): L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t),$$

satisfying certain basic properties. They have important applications in mathematical finance and stochastic control. Given the results of [9] and [15], it is known that in the Brownian setting, under certain conditions, these operators are completely described by BSDEs. Furthermore, it is clear, given the comparison theorem in [8], BSDEs of the form of (1) in arbitrary spaces, under some conditions, also describe nonlinear expectations. However, it is not known how large a class of nonlinear expectations in a general space is given by a BSDE.

To establish such a result for BSDEs of the form of (1), one faces a significant problem. If $\mathcal{E}(Q|\mathcal{F}_t) = Y_t$ is given as the solution to (1) for some $F$ not dependent on $Y_{t-}$, once $M$ is fixed, for any martingale $N$ orthogonal to $M$ with $N_0 = 0$, we have the property

$$\mathcal{E}(Q + N_T|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t).$$

This property is clearly not true for most nonlinear expectations, whenever there are nontrivial examples of such processes $N$, which is not the case in the Brownian setting (as a martingale representation theorem holds). It follows that these equations cannot describe any nonlinear expectations which do not possess this property.

Furthermore, the fact that the martingale $M$ must be specified a priori is arguably unsatisfying. Conceptually, it may be preferable if, in some sense, the probability space itself dictated what martingales are needed for the BSDE. In this case, one could proceed either by specifying the probability space using a collection of martingales (which, given a representation theorem holds, will then describe all martingales in the space), or vice versa.

In this paper we establish such a general result. We show that there is a sense in which the original BSDE can be interpreted in a general space, using only a separability assumption on $L^2(\mathcal{F}_T)$. We establish conditions on the existence and uniqueness of BSDEs in this setting, where the driver is integrated with respect to an arbitrary deterministic Stieltjes measure (Theorem 6.1). We also prove a comparison theorem for these solutions, which shows under which conditions they do indeed describe nonlinear expectations and evaluations.

A similar approach is taken in [14], where a form of BSDE is considered using generic maps from a space of semimartingales to the spaces of square-integrable martingales and of finite-variation processes integrable with respect to a given continuous increasing process. Using Browder's theorem, they demonstrate the existence of solutions to these equations on an infinite horizon. Our approach differs from theirs by considering a classical form of BSDE on a finite horizon and deriving an existence result using a contraction mapping technique. Because of this, our conditions for existence are a more straightforward extension of those in the classical case. More significantly, our approach does not require the driver of the BSDE to be integrated with respect to a continuous measure, which allows a unification of the discrete and continuous time theory of BSDEs.

**2. Martingale representations.** The key result used in the construction of BSDEs is the Martingale representation theorem. In the Brownian setting, this result is well known (see, e.g., [20], Chapter V.3, or [13], Theorem 12.33). In other cases, for example, when dealing with martingales generated by Markov chains, a similar result is available (see [4]); however it is also known that there exist probability spaces in which no finite-dimensional martingale representation theorem exists.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}, t \in [0, T]$, satisfying the usual conditions of completeness and right continuity. The time-interval $[0, T]$ is given the Borel $\sigma$-field $\mathcal{B}([0, T])$.

DEFINITION 2.1. For any nondecreasing process of finite variation $\mu$, we define the measure induced by $\mu$ to be the measure over $\Omega \times [0, T]$ given by

$$A \mapsto E\left[\int_{[0,T]} I_A(\omega, t)\, d\mu\right].$$

Here $A \in \mathcal{F} \otimes \mathcal{B}([0, T])$, and the integral is taken pathwise in a Stieltjes sense.

REMARK 2.1. If $\mu$ is a deterministic process, then this definition gives the product measure $\mu \times \mathbb{P}$. We can also consider these as measures on the space $(\Omega \times [0, T], \mathcal{P})$, where $\mathcal{P}$ is the predictable $\sigma$-algebra.

Under the assumption that the Hilbert space $L^2(\mathcal{F}_T)$ is separable, a paper of Davis and Varaiya [10] gives the following result (see also Malamud [17]).

THEOREM 2.1 (Martingale representation theorem; [10]). *Suppose that $L^2(\mathcal{F}_T)$ is a separable Hilbert space, with an inner product $(X, Y) = E[XY]$.*

*Then there exists a finite or countable sequence of square-integrable $\{\mathcal{F}_t\}$-martingales $M^1, M^2, \dots$ such that every square integrable $\{\mathcal{F}_t\}$-martingale $N$ has a representation*

$$N_t = N_0 + \sum_{i=1}^{\infty} \int_{]0,t]} Z_u^i \, dM_u^i$$

*for some sequence of predictable processes $Z^i$. This sequence satisfies*

$$E\left[\sum_{i=0}^{\infty} \int_{[0,T]} (Z_u^i)^2 \, d\langle M^i \rangle_u\right] < +\infty. \tag{2}$$

*These martingales are orthogonal (i.e., $E[M_T^i M_T^j] = 0$ for all $i \neq j$), and the predictable quadratic variation processes $\langle M^i \rangle$ satisfy*

$$\langle M^1 \rangle \succ \langle M^2 \rangle \succ \cdots,$$

*where $\succ$ denotes absolute continuity of the induced measures (Definition 2.1). Furthermore, these martingales are unique, in that if $N^i$ is another such sequence, then $\langle N^i \rangle \sim \langle M^i \rangle$, where $\sim$ denotes equivalence of the induced measures.*

COROLLARY 2.1.1. *For any predictable processes $Z^i$ satisfying (2), the process $\sum_i \int_{]0,t]} Z_u^i \, dM_u^i$ is well defined and is a square-integrable martingale.*

REMARK 2.2. When a finite-dimensional martingale representation theorem holds, as when the space is generated by a Brownian motion, then all but finitely many of the martingales $M^i$ given by Theorem 2.1 will be zero. We shall not, in general, assume that this is the case, but acknowledge that, in this situation, significant simplification of the equations considered is possible.

We shall use this result to construct a form of BSDE on this general space.

DEFINITION 2.2. We denote by $\mathbb{R}^{K \times \infty}$ the space of infinite $\mathbb{R}^K$-valued sequences. We note that the predictable processes $Z^i$ in Theorem 2.1 can be written as a vector process $Z$, which takes values in $\mathbb{R}^{1 \times \infty}$.

**3. BSDEs in general spaces: A definition.** We seek to construct BSDEs, assuming only the usual properties of the filtration and that $L^2(\mathcal{F}_T)$ is a separable Hilbert space. For simplicity, we shall also assume that $\mathcal{F}_0$ is trivial, which, by right continuity, ensures that, almost surely, no martingale has a jump at $t = 0$.

DEFINITION 3.1. Let $\mu$ be a deterministic signed Stieltjes measure. For $K \in \mathbb{N}$, a BSDE is an equation of the form

$$Q = Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, Z_u) \, d\mu_u + \sum_{i=1}^{\infty} \int_{]t,T]} Z_u^i \, dM_u^i, \tag{3}$$

where $Z_t(\omega)$ is the (countably infinite) vector with entries $\{Z_t^i(\omega) \in \mathbb{R}^K\}_{i\in\mathbb{N}}$. For a terminal value $Q \in L^2(\mathbb{R}^K; \mathcal{F}_T)$, a predictable *driver* function $F : \Omega \times [0, T] \times \mathbb{R}^K \times \mathbb{R}^{K\times\infty} \to \mathbb{R}^K$, a solution is a pair of processes $(Y, Z)$ taking values in $\mathbb{R}^K \times \mathbb{R}^{K\times\infty}$, where $Z$ is predictable, and $Y$ is adapted. We shall restrict our attention to the case when $Y$ is square integrable, and $Z$ satisfies (2).

REMARK 3.1. We note that this type of equation encompasses most previously studied forms of BSDEs. When the filtration is Brownian, we can take $M^i$ to be the $i$th component of the generating Brownian motion, $\mu = t$, and the equation is standard. When the filtration is generated by a Poisson random measure over a separable space and a Brownian motion, as in [2, 22] and others, or by a Markov chain, as in [4, 5], we have a similar reduction. When we consider the analogous equations in discrete time, we can form the discrete-time filtration embedded in this continuous time context (see [16], Chapter 1f) and hence obtain the backward stochastic difference equations considered in [6] and [7].

Comparing with the work of [12], we see that if $F$ depends only on the projection of $Z$ into a finite-dimensional subspace of $\mathbb{R}^{K\times\infty}$, then it is possible to reduce the equation to a form similar to (1).

We shall present a result (Theorem 6.1) demonstrating conditions under which there exists a unique solution to such an equation.

## 4. Inequalities for Stieltjes integrals.

To give conditions under which solutions to a BSDE exist, we must first establish the following results regarding integrals with respect to Stieltjes measures. These results are standard whenever the measures are continuous.

### 4.1. *Stieltjes exponentials.*

DEFINITION 4.1. For any càdlàg function of finite variation $\nu : [0, \infty[ \to \mathbb{R}$, we write

$$\mathfrak{E}(\nu_t) := e^{\nu_t} \prod_{0\le s\le t} (1 + \Delta\nu_s)e^{-\Delta\nu_s},$$

and call this the *Stieltjes exponential* of $\nu$. Note that this is also a càdlàg function.

Note that $\mathfrak{E}(\nu_t)$ should be more properly written as $\mathfrak{E}(\nu_{(\cdot)}; t)$, as it is a function of $\{\nu_s; s \le t\}$ not just of $\nu_t$. We use the former notation purely for compactness, whenever this does not lead to confusion. We note the following useful bound.

LEMMA 4.1. *If $\nu$ is a càdlàg function, then $\mathfrak{E}(\nu_t) \le e^{\nu_t}$, where $e^{\nu_t}$ is the classical exponential of $\nu_t$.*

PROOF. As $e^x \geq 1 + x$, it is clear that $(1 + \Delta\nu_t)e^{-\Delta\nu_t} \leq 1$ for all $t$. The result follows. □

LEMMA 4.2. *For any càdlàg function of finite variation, the Stieltjes exponential is well defined. Furthermore, if $\Delta\nu_t \geq -1$, then $\mathfrak{E}(\nu_t) \geq 0$. If $\Delta\nu_s > -1$, then $\mathfrak{E}(\nu_t) > 0$, and $\mathfrak{E}(\nu_t)^{-1}$ is well defined. In this case, the process $u_t = u_s\mathfrak{E}(\nu_t)\mathfrak{E}(\nu_s)^{-1}$ is the solution to the Lebesgue–Stieltjes integral equation,*

$$u_t = u_s + \int_{]s,t]} u_{r-}\, d\nu_r.$$

PROOF. As the process $\nu_t$ is càdlàg and of finite variation, it is a (deterministic) semimartingale. $\mathfrak{E}(\nu_t)$ is then the standard Doléans–Dade exponential of this process, and so its existence and basic properties can be seen in [13], Theorem 13.5 ff. This guarantees the convergence of the infinite products considered and solves the desired integral equation. The nonnegativity result is clear by inspection.

For the positivity result, we need only show that $\prod_{0\leq s\leq t}(1 + \Delta\nu_s) > 0$. By continuity of the logarithm, this is equivalent to showing that

$$-\sum_{0\leq s\leq t} \log(1 + \Delta\nu_s) < \infty.$$

We then note that we can consider three cases. First, if $\Delta\nu_s \geq 0$, then $-\log(1 + \Delta\nu_s) \leq 0$, and hence

$$-\Bigg(\sum_{\{0\leq s\leq t\}\cap\{\Delta\nu_s\geq 0\}} \log(1 + \Delta\nu_s)\Bigg) \leq 0 < \infty.$$

Second, we note that $\sum_{0<s\leq t} |\Delta\nu_s|$ is finite, as $\nu$ is of finite variation, and hence there are only finitely many $s$ such that $\Delta\nu_s \leq -0.7$. Therefore

$$-\Bigg(\sum_{\{0\leq s\leq t\}\cap\{\Delta\nu_s\leq -0.7\}} \log(1 + \Delta\nu_s)\Bigg) < \infty.$$

Finally, we know that $2x < \log(1 + x) < 0$ for $-0.7 < x < 0$. Hence, we have

$$-\Bigg(\sum_{\{0\leq s\leq t\}\cap\{-0.7<\Delta\nu_s<0\}} \log(1 + \Delta\nu_s)\Bigg) < \Bigg(\sum_{\{0\leq s\leq t\}\cap\{-0.7<\Delta\nu_s<0\}} 2|\Delta\nu_s|\Bigg) < \infty.$$

Combining these three sums gives the desired constraint on the logarithm, and hence the strict positivity of the desired product. □

LEMMA 4.3. *For $\nu$ a càdlàg function of finite variation with $\Delta\nu_t > -1$, we have the stronger result*

$$\inf_{0\leq t\leq T}\Bigg\{\prod_{0\leq s\leq t}(1 + \Delta\nu_s)\Bigg\} > 0.$$

PROOF. By the same argument as in Lemma 4.2, we have

$$-\left(\sum_{\{0\le s\le T\}\cap\{\Delta\nu_s<0\}}\log(1+\Delta\nu_s)\right)<\infty.$$

It follows that

$$-\sum_{0\le s\le t}\log(1+\Delta\nu_s)<-\left(\sum_{\{0\le s\le T\}\cap\{\Delta\nu_s<0\}}\log(1+\Delta\nu_s)\right)<\infty$$

for all $t$. Hence

$$\inf_{0\le t\le T}\left\{\prod_{0\le s\le t}(1+\Delta\nu_s)\right\}>\left(\prod_{\{0\le s\le T\}\cap\{\Delta\nu_s<0\}}(1+\Delta\nu_s)\right)>0. \qquad \square$$

DEFINITION 4.2. Let $\nu$ be a càdlàg function of finite variation with $\Delta\nu_t>-1$ for all $t$. Then the *left-jump inversion* of $\nu$ is defined by

$$\bar{\nu}_t=\nu_t-\sum_{0\le s\le t}\frac{(\Delta\nu_s)^2}{1+\Delta\nu_s}.$$

Similarly if $\Delta\nu_t<1$ for all $t$, the *right-jump inversion* is defined by

$$\tilde{\nu}_t=\nu_t+\sum_{0\le s\le t}\frac{(\Delta\nu_s)^2}{1-\Delta\nu_s}.$$

LEMMA 4.4. *For $\nu$ a function as in Definition 4.2, the left- and right-jump inversions are finite* (*whenever they are defined*), *and satisfy*

$$\mathfrak{E}(\nu_t)^{-1}=\mathfrak{E}(-\bar{\nu}_t)$$

*and*

$$\mathfrak{E}(-\nu_t)=\mathfrak{E}(\tilde{\nu}_t)^{-1}.$$

PROOF. Consider first the left-jump-inversion. We know that $\Delta\nu_s>-1$ and $\sum|\Delta\nu_s|<\infty$. Hence it follows that $\Delta\nu_s$ has only finitely many values in any neighborhood not containing zero and hence is bounded away from $-1$. That is, there exists some $\varepsilon>0$ such that $\Delta\nu_s>\varepsilon-1$ for all $s$. To show finiteness, write

$$\sum_{\{0\le s\le t\}\cap\{\Delta\nu_s\ge 0\}}\frac{(\Delta\nu_s)^2}{1+\Delta\nu_s}\le\sum_{\{0\le s\le t\}\cap\{\Delta\nu_s\ge 0\}}|\Delta\nu_s|<\infty$$

and

$$\begin{aligned}\sum_{\{0\le s\le t\}\cap\{\Delta\nu_s<0\}}\frac{(\Delta\nu_s)^2}{1+\Delta\nu_s}&\le\varepsilon^{-1}\left(\sum_{\{0\le s\le t\}\cap\{\Delta\nu_s<0\}}(\Delta\nu_s)^2\right)\\&<\varepsilon^{-1}\left(\sum_{\{0\le s\le t\}\cap\{\Delta\nu_s<0\}}|\Delta\nu_s|\right)\\&<\infty.\end{aligned}$$

Combining these sums gives the desired finiteness result.

We now note that, algebraically,

$$(1-\Delta\bar{\nu}_s)^{-1} = \left(1 - \Delta\nu_s + \frac{(\Delta\nu_s)^2}{1+\Delta\nu_s}\right)^{-1} = 1 + \Delta\nu_s.$$

Hence

$$\begin{aligned}
\mathfrak{E}(\nu_t)^{-1} &= e^{-\nu_t} \prod_{0\le s\le t} (1+\Delta\nu_s)^{-1} e^{\Delta\nu_s} \\
&= e^{-\nu_t + \sum_{0<s\le t}((\Delta\nu_s)^2/(1+\Delta\nu_s))} \prod_{0\le s\le t} (1+\Delta\nu_s)^{-1} e^{\Delta\nu_s - (\Delta\nu_s)^2/(1+\Delta\nu_s)} \\
&= e^{-\bar{\nu}_t} \prod_{0\le s\le t} (1-\Delta\bar{\nu}_s) e^{\Delta\bar{\nu}_s} \\
&= \mathfrak{E}(-\bar{\nu}_t).
\end{aligned}$$

The proof for the right-jump inversion follows in the same way, where finiteness is because

$$\sum_{0\le s\le t} \frac{(\Delta\nu_s)^2}{1-\Delta\nu_s} = \sum_{0\le s\le t} \frac{(-\Delta\nu_s)^2}{1+(-\Delta\nu_s)},$$

and $-\nu_s$ satisfies the requirements given above for the left-jump inversion. The algebraic result is then that

$$(1+\Delta\tilde{\nu}_s)^{-1} = \left(1 + \Delta\nu_s + \frac{(\Delta\nu_s)^2}{1-\Delta\nu_s}\right)^{-1} = 1 - \Delta\nu_s,$$

and the result is as given. □

LEMMA 4.5. *For $\nu$ a càdlàg function of bounded variation with $\Delta\nu_s > -1$, the right-jump inversion of the left-jump inversion of $\nu$ is the original function, that is,*

$$\tilde{\bar{\nu}}_t = \nu_t.$$

*Similarly, if $\Delta\nu_s < -1$, then $\bar{\tilde{\nu}}_t = \nu_t$.*

PROOF. For simplicity, we decompose $\nu$ into a discontinuous part $\nu_t^d := \sum_{0\le s\le t} \Delta\nu_s$ and a continuous part $\nu_t^c = \nu_t - \nu_d$. Clearly, taking either the left- or right-jump inversion will not alter the continuous part $\nu^c$, and so it is sufficient to show that the discontinuous parts are equal, that is, $\Delta\tilde{\bar{\nu}}_t = \Delta\bar{\tilde{\nu}}_t = \Delta\nu_t$ for all $t$, whenever these terms are well defined. From Definition 4.2 we have

$$\Delta\bar{\nu}_t = \frac{\Delta\nu_t}{1+\Delta\nu_t}, \qquad \Delta\tilde{\nu}_t = \frac{\Delta\nu_t}{1-\Delta\nu_t}$$

and hence

$$\Delta\tilde{\bar{\nu}}_t = \frac{\Delta\bar{\nu}_t}{1-\Delta\bar{\nu}_t} = \frac{\Delta\nu_t/(1+\Delta\nu_t)}{1-\Delta\nu_t/(1+\Delta\nu_t)} = \Delta\nu_t,$$

and similarly $\Delta\bar{\tilde{\nu}}_t = \Delta\nu_t$, as desired. □

4.2. *Integrating factors.* It is useful to have some results relating to the solutions of equations of the form $du_t - u_{t-}\,d\nu_t = \cdots$. These are similar the the classical results on the use of integrating factors and Grönwall's inequality in the study of ordinary differential equations.

DEFINITION 4.3. Let $u$, $v$ be two measures on a $\sigma$-algebra $\mathcal{A}$. We write $du \le dv$ if, for any $A \in \mathcal{A}$, $u(A) \le v(A)$.

REMARK 4.1. When $v$ is a nonnegative measure, and $u$ is absolutely continuous with respect to $v$, this definition is equivalent to requiring that the Radon–Nikodym derivative satisfies $du/dv \le 1$, $dv$-a.e.

LEMMA 4.6. *Let $u$, $\nu$ and $w$ be signed Stieltjes measures on $\mathcal{B}([0,T])$, such that $\Delta\nu_t < 1$ for all $t$, and*

$$du_t \ge -u_{t-}\,d\nu_t + dw_t,$$

*then*

$$d(u_t\mathfrak{E}(\tilde{\nu}_t)) \ge (1-\Delta\nu_t)^{-1}\mathfrak{E}(\tilde{\nu}_{t-})\,dw_t,$$

*where $\tilde{\nu}$ is the right-jump inversion of $\nu$.*

PROOF. Applying the product rule for Stieltjes integrals we have

$$\frac{d(u_t\mathfrak{E}(\tilde{\nu}_t))}{\mathfrak{E}(\tilde{\nu}_{t-})} = du_t + u_{t-}\,d\tilde{\nu}_t + \Delta u_t\Delta\tilde{\nu}_t.$$

As $d\tilde{\nu}_t = d\nu/(1-\Delta\nu_t)$ and $\Delta u_t\Delta\nu_t = (\Delta\nu_t)\,du_t$, this gives

$$\begin{aligned}\frac{d(u_t\mathfrak{E}(\tilde{\nu}_t))}{\mathfrak{E}(\tilde{\nu}_{t-})} &= du_t + u_{t-}\frac{d\nu_t}{1-\Delta\nu_t} + \frac{\Delta\nu_t}{1-\Delta\nu_t}\,du_t \\ &= \left(1+\frac{\Delta\nu_t}{1-\Delta\nu_t}\right)du_t + u_{t-}\frac{d\nu_t}{1-\Delta\nu_t} \\ &= (1-\Delta\nu_t)^{-1}(du_t + u_{t-}\,d\nu_t) \\ &\ge (1-\Delta\nu_t)^{-1}\,dw_t.\end{aligned}$$

□

LEMMA 4.7 (Backward Grönwall inequality). *Let $u$ be a process such that, for $\nu$ a nonnegative Stieltjes measure with $\Delta\nu_t < 1$ and $\alpha$ a $\tilde{\nu}$-integrable process, $u$ is $\nu$-integrable and*

$$u_t \le \alpha_t + \int_{]t,T]} u_s\, d\nu_s,$$

*then*

$$u_t \le \alpha_t + \mathfrak{E}(-\nu_t) \int_{]t,T]} \mathfrak{E}(\tilde{\nu}_s)\alpha_s\, d\tilde{\nu}_s.$$

*If $\alpha_t = \alpha$ is constant, this simplifies to*

$$u_t \le \alpha \mathfrak{E}(\tilde{\nu}_T)\mathfrak{E}(\tilde{\nu}_t)^{-1} = \alpha \mathfrak{E}(-\nu_t)\mathfrak{E}(-\nu_T)^{-1}.$$

PROOF. First note that $d\nu_t = \frac{d\tilde{\nu}}{1+\Delta\tilde{\nu}_t}$ and that $\Delta\tilde{\nu}_t \Delta\nu_t = \Delta\tilde{\nu}_t\, d\nu_t$. Then let

$$w_t := \mathfrak{E}(\tilde{\nu}_t) \int_{]t,T]} u_s\, d\nu_s.$$

From the product rule for stochastic integrals, as $\nu$ is of finite variation,

$$\begin{aligned}
\frac{dw_t}{\mathfrak{E}(\tilde{\nu}_{t-})} &= \left(\int_{]t,T]} u_s\, d\nu_s\right) d\tilde{\nu}_s - u_t\, d\nu_t - u_t \Delta\nu_t \Delta\tilde{\nu}_t \\
&= -u_t(1+\Delta\tilde{\nu}_t)\, d\nu_t + \left(\int_{]t,T]} u_s\, d\nu_s\right) d\tilde{\nu}_t \\
&= -u_t\, d\tilde{\nu}_t + \left(\int_{]t,T]} u_s\, d\nu_s\right) d\tilde{\nu}_t \\
&= \left(-u_t + \int_{]t,T]} u_s\, d\nu_s\right) d\tilde{\nu}_t \\
&\ge -\alpha_t\, d\tilde{\nu}_t.
\end{aligned}$$

Note that $d\tilde{\nu}_t$ and $\mathfrak{E}(\tilde{\nu}_{t-})$ are both nonnegative. Therefore, by integration,

$$w_t = \mathfrak{E}(\tilde{\nu}_t) \int_{]t,T]} u_s\, d\nu_s \le \int_{]t,T]} \mathfrak{E}(\tilde{\nu}_{s-})\alpha_s\, d\tilde{\nu}_s.$$

Substitution yields

$$u_t \le \alpha_t + \mathfrak{E}(\tilde{\nu}_t)^{-1} \int_{]t,T]} \mathfrak{E}(\tilde{\nu}_{s-})\alpha_s\, d\tilde{\nu}_s,$$

and the desired inequalities follow from $\mathfrak{E}(\tilde{\nu}_t)^{-1} = \mathfrak{E}(-\nu_t)$. If $\alpha_t = \alpha$, then this simplifies to

$$\begin{aligned}
u_{t-} &\le \alpha\left[1 + \mathfrak{E}(\tilde{\nu}_t)^{-1} \int_{]t,T]} \mathfrak{E}(\tilde{\nu}_{s-})\, d\tilde{\nu}_s\right] \\
&= \alpha\left[1 + \mathfrak{E}(\tilde{\nu}_t)^{-1}\big(\mathfrak{E}(\tilde{\nu}_T) - \mathfrak{E}(\tilde{\nu}_t)\big)\right] \\
&= \alpha \mathfrak{E}(\tilde{\nu}_T)\mathfrak{E}(\tilde{\nu}_t)^{-1}.
\end{aligned}$$

□

LEMMA 4.8 (Forward Grönwall inequality). *Let $u$ be a function such that, for $\nu$ a nonnegative Stieltjes measure and $\alpha$ a $\bar{\nu}$-integrable process, $u$ is $\nu$-integrable and*

$$u_t \le \alpha_t + \int_{]0,t]} u_s \, d\nu_s,$$

*then*

$$u_t \le \alpha_t + \mathfrak{E}(\nu_t) \int_{]0,t]} \mathfrak{E}(-\bar{\nu}_s)\alpha_s \, d\bar{\nu}_s.$$

*If $\alpha_t = \alpha$ is constant, this simplifies to*

$$u_t \le \alpha \mathfrak{E}(\nu_t).$$

PROOF. This result follows in an almost identical fashion to Lemma 4.7, and the proof is therefore omitted. □

**5. Existence of BSDE solutions: Fundamental results.** In this section we shall establish the existence of solutions to BSDEs when the process $\mu$ satisfies particular properties.

DEFINITION 5.1. Let $\mu$ be a deterministic nondecreasing right-continuous function $\mu : [0, T] \to \mathbb{R}^+$. The measure $d\mu$ will serve in the place of the Lebesgue measure $dt$ in our BSDE.

As $\mu$ is of finite variation, its discontinuities $\Delta\mu$ are bounded. We assume that $\mu$ assigns positive measure to any nonempty open interval in $[0, T]$.

Unless otherwise indicated, all (in-)equalities should be read as "up to evanescence."

DEFINITION 5.2. We denote by $\|\cdot\|$ the standard Euclidean norm on $\mathbb{R}^K$, and note that $\|y\|^2 = y^* y$, where $[\cdot]^*$ denotes vector transposition.

DEFINITION 5.3. For a given $\mu$ and fixed $K \in \mathbb{N}$, we define the stochastic seminorm $\|\cdot\|_{M_t}$ on $\mathbb{R}^{K\times\infty}$ as follows. For each $i \in \mathbb{N}$, consider $\langle M^i \rangle$ as a measure on the predictable $\sigma$-algebra; cf. Remark 2.1. Let $\langle M^i \rangle$ have the Lebesgue decomposition

$$\langle M^i \rangle_t = m_t^{i,1} + m_t^{i,2},$$

where $m_t^{i,1}$ is absolutely continuous with respect to $\mu \times \mathbb{P}$, and $m_t^{i,2}$ is orthogonal to $\mu \times \mathbb{P}$. As they represent bounded measures on the predictable $\sigma$-algebra, both $m_t^{i,1}$ and $m_t^{i,2}$ will be nondecreasing predictable processes.

We define, for $z_t \in \mathbb{R}^{K\times\infty}$,

$$\|z_t\|^2_{M_t} := \sum_i \left[ \|z_t^i\|^2 \frac{dm_t^{i,1}}{d(\mu \times \mathbb{P})} \right],$$

where $z_t^i \in \mathbb{R}^K$ is the $i$th element in $z_t$, considered as a series of values in $\mathbb{R}^K$.

We note that, for any predictable, progressively measurable process $Z$ taking values in $\mathbb{R}^{K\times\infty}$, and, in particular, for processes satisfying (2) in each of their $K$ components, we have the inequality

$$\begin{aligned} E\left[\int_A \|Z_t\|^2_{M_t}\, d\mu\right] &\le E\left[\sum_i \int_A \|Z_t^i\|^2\, d\langle M_t^i\rangle\right] \\ &= E\left[\sum_i \left\|\int_A Z_t^i\, dM_t^i\right\|^2\right] \qquad (4) \\ &= E\left[\left\|\sum_i \int_A Z_t^i\, dM_t^i\right\|^2\right] \end{aligned}$$

for any predictable set $A \subseteq \Omega \times [0, T]$. (Note the latter equalities are simply the standard isometry used in the construction of the stochastic integral, by the orthogonality of the $M^i$.)

DEFINITION 5.4. We define the following spaces of equivalence classes:

$$H_M^2 = \Big\{ Z : \Omega \times [0, T] \to \mathbb{R}^{K\times\infty}, \text{predictable}, E\left[\sum_i \int_{[0,T]} \|Z_t^i\|^2\, d\langle M^i\rangle_t\right] < +\infty \Big\},$$

$$S^2 = \Big\{ Y : \Omega \times [0, T] \to \mathbb{R}^K, \text{adapted}, E\Big[\sup_{t\in[0,T]} \|Y_t\|^2\Big] < +\infty \Big\},$$

$$H_\mu^2 = \Big\{ Y : \Omega \times [0, T] \to \mathbb{R}^K, \text{progressive}, \int_{]0,T]} E[\|Y_t\|^2]\, d\mu_t < +\infty \Big\},$$

where two elements $Z, \bar{Z}$ of $H_M^2$ are deemed equivalent if

$$E\left[\sum_i \int_{[0,T]} \|Z_t^i - \bar{Z}_t^i\|^2\, d\langle M^i\rangle_t\right] = 0,$$

two elements of $S^2$ are deemed equivalent if they are indistinguishable and two elements of $H_\mu^2$ are equivalent if they are equal $\mu \times \mathbb{P}$-a.s. Note that $K$ is here taken as fixed.

REMARK 5.1. We note that $H^2_M$ is itself a complete metric space, with norm given by $Z \mapsto E[\sum_i \int_{[0,T]} \|Z^i_t\|^2 d\langle M^i\rangle_t]$; similarly for $H^2_\mu$. Note also that the martingale representations constructed in Theorem 2.1 are unique in $H^2_M$.

A key assumption in the study of BSDEs is the continuity of the driver function $F$. When the measure $\mu$ is continuous, we shall show that it is sufficient that $F$ is uniformly Lipschitz continuous for the BSDE (3) to have a solution. On the other hand, as is clear in discrete time (cf. [7]), when $\mu$ is not continuous, a stronger condition is needed on $F$. We shall call this a *firm* Lipschitz bound on $F$, as is defined in the following theorem.

THEOREM 5.1. *For $\mu$ as in Definition 5.1, assume $\mu_T \le 1$. Let $F : \Omega \times [0,T] \times \mathbb{R}^K \times \mathbb{R}^{K\times\infty} \to \mathbb{R}^K$ be a predictable, progressively measurable function such that:*

- $E[\int_{]0,T]} \|F(\omega,t,0,0)\|^2 d\mu_t] < +\infty$;
- *there exists a linear firm Lipschitz bound on $F$, that is, a measurable deterministic function $c_t$ uniformly bounded by some $c \in \mathbb{R}$, such that, for any $y_t, y'_t \in \mathbb{R}^K$, $z_t, z'_t \in \mathbb{R}^{K\times\infty}$,*

$$\begin{aligned}&\|F(\omega,t,y_t,z_t) - F(\omega,t,y'_t,z'_t)\|^2\\ &\qquad \le c_t\|y_t - y'_t\|^2 + c\|z_t - z'_t\|^2_{M_t}, \qquad d\mu \times d\mathbb{P}\text{-a.s.}\end{aligned}$$

  *and*

$$c_t \Delta\mu_t < 1.$$

  *Note that the variable bound $c_t$ need only apply to the behavior of $F$ with respect to $y$.*

*A function satisfying these conditions will be called standard. Then for any $Q \in L^2(\mathbb{R}^K; \mathcal{F}_T)$, the BSDE (3) with driver $F$ has a unique solution $(Y, Z) \in S^2 \times H^2_M$. ($S^2$ and $H^2_M$ are defined in Definition 5.4.)*

To prove this theorem, we first establish the following results.

LEMMA 5.1. *If $\mu$ assigns positive measure to every nonempty open interval, then two càdlàg processes in $H^2_\mu$ are indistinguishable if and only if they are equivalent in $H^2_\mu$. Similarly, two càdlàg processes are equivalent in $H^2_\mu$ if and only if their left limits are equivalent in $H^2_\mu$.*

PROOF. Clearly indistinguishability implies equivalence of the processes, and their left limits, in $H^2_\mu$. By right continuity (resp., left continuity), if on some nonnull set $A$, two processes (resp., their left limits) differ at any point, they must differ on some nonempty open interval. As $\mu$ assigns positive measure to such an interval, it follows that the processes will not be equivalent in $H^2_\mu$. □

LEMMA 5.2. *Let* $(Y, Z)$ *be the solution to a BSDE with data* $(F, Q)$. *If* $F$ *is standard,* $Q \in L^2(\mathbb{R}^K; \mathcal{F}_T)$ *and* $Z \in H_M^2$, *then* $Y \in S^2$ *if and only if the left limit process* $Y_{t-} \in H_\mu^2$.

PROOF. Clearly, if $Y \in S^2$, then as $Y$ is càdlàg and adapted, and hence progressive, $Y \in H_M^2$. For the converse, write

$$\begin{aligned}
\sup_{t\in[0,T]} \|Y_t\|^2 &\leq 2\|Q\|^2 + 4 \sup_{t\in[0,T]} \left\| \sum_i \int_{]t,T]} Z_u^i \, dM_u^i \right\|^2 \\
&\quad + 4 \sup_{t\in[0,T]} \left\{ \int_{]t,T]} \|F(\omega, u, Y_{u-}, Z_u)\|^2 \, d\mu \right\} \\
&\leq 2\|Q\|^2 + 4 \sup_{t\in[0,T]} \left\| \sum_i \int_{]t,T]} Z_u^i \, dM_u^i \right\|^2 \\
&\quad + 8 \int_{]0,T]} \|F(\omega, u, 0, 0)\|^2 \, d\mu_t \\
&\quad + 8 \int_{]0,T]} [c_t \|Y_{u-}\|^2 + c\|Z_u\|_{M_u}^2] \, d\mu_t,
\end{aligned}$$

and by the assumptions of the lemma, as $Z \in H_M^2$, and so $\sum_i \int_{]0,t]} Z_u^i \, dM_u^i$ is a square integrable martingale, by Doob's inequality [16], Theorem 1.43, this quantity is finite in expectation. □

The following lemma provides the key bounds on BSDE solutions, which we shall use to prove existence and uniqueness of solutions.

LEMMA 5.3. *Let* $(Y, Z)$ *and* $(\bar{Y}, \bar{Z})$ *be the solutions to two BSDEs with standard parameters* $(F, Q)$ *and* $(\bar{F}, \bar{Q})$. *Define*

$$\begin{aligned}
\delta Y &:= Y - \bar{Y}, \qquad \delta Z := Z - \bar{Z}, \\
\delta_2 f_t &:= F(\omega, t, \bar{Y}_{t-}, \bar{Z}_t) - \bar{F}(\omega, t, \bar{Y}_{t-}, \bar{Z}_t), \\
\upsilon_t &:= \int_{]0,t]} [(x_s^{-1} - \Delta\mu_s)(1 + w_s)c_s + x_s] \, d\mu_s, \qquad (5)\\
\pi_t &:= \int_{]0,t]} [(x_s^{-1} - \Delta\mu_s)(1 + w_s^{-1})](1 - \Delta\upsilon_s)^{-1} \, d\mu_s, \\
\rho_t^i &:= \int_{]0,t]} [1 - (x_s^{-1} - \Delta\mu_s)(1 + w_t)c](1 - \Delta\upsilon_s)^{-1} \, d\langle M^i \rangle_t,
\end{aligned}$$

*where* $c_s$ *and* $c$ *are the Lipschitz constants of* $F$, *and* $x_t$, $w_t$ *are any nonnegative measurable functions such that* $\Delta\mu_t \leq x_t^{-1}$ *and* $\Delta\upsilon_t < 1$ *for all* $t$, *and the integrands defining* $\upsilon, \pi$ *and* $\rho^i$ *are uniformly bounded.*

*Then*

$$
\begin{aligned}
&E[\|\delta Y_t\|^2]\mathfrak{E}(\tilde{\upsilon}_t) + E\left[\sum_i \int_{]t,T]} \mathfrak{E}(\tilde{\upsilon}_{s-})\|\delta Z_s^i\|^2\, d\rho_s^i\right] \\
&\qquad \le E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}_T) + \int_{]t,T]} E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\, d\pi_s
\end{aligned}
\tag{6}
$$

*and*

$$
\begin{aligned}
&\int_{]0,T]} E[\|\delta Y_{t-}\|^2]\mathfrak{E}(\tilde{\upsilon}_{t-})\, d\mu_t + E\left[\sum_i \int_{]0,T]} \mu_s \mathfrak{E}(\tilde{\upsilon}_{s-})\|\delta Z_s^i\|^2\, d\rho_s^i\right] \\
&\qquad \le \mu_T E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}_T) + \int_{]0,T]} \mu_s E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\, d\pi_s.
\end{aligned}
\tag{7}
$$

PROOF. Let $\delta F = F(\omega, t, Y_{t-}, Z_t) - \bar{F}(\omega, t, \bar{Y}_{t-}, \bar{Z}_t)$. By application of the differentiation rule for stochastic integrals, we have

$$
\begin{aligned}
d[\|\delta Y_t\|^2] = {}& -2(\delta Y_{t-})^*(\delta F_t)\, d\mu_t + 2\sum_i (\delta Y_{t-})^*(\delta Z_t^i)\, dM_t^i \\
& + \sum_{i,j} (\delta Z_t^i)^*(\delta Z_t^j)\, d[M^i, M^j]_t - 2(\delta F_t)(\Delta\mu_t)\sum_i (\delta Z_t^i)\Delta M_t^i \\
& + \|\delta F_t\|^2(\Delta\mu_t)^2.
\end{aligned}
\tag{8}
$$

As $\delta Y \in S^2$, by the BDG inequality it is clear that $\int_{]0,t]} \sum_i (\delta Y_{s-})^*(\delta Z_s^i)\, dM_s^i$ is a martingale. Similarly the process

$$
\sum_{s\in ]0,t]} \left[(\delta F_s)(\Delta\mu_s)\sum_i (\delta Z_s^i)\Delta M_s^i\right]
$$

is a countable sum of integrable martingale differences and so is also a martingale. Also, $\delta Z \in H_M^2$ and so, by orthogonality of the $M^i$,

$$
\sum_{i,j} (\delta Z_t^i)^*(\delta Z_t^j)\, d[M^i, M^j]_t - \sum_i \|\delta Z_t^i\|^2\, d\langle M^i\rangle_t
$$

is a martingale.

For any $A \in \mathcal{B}([0, T])$, integrating on $A$ and taking an expectation through (8) then yields

$$
\begin{aligned}
\int_A dE[\|\delta Y_t\|^2] = {}& -2\int_A E[(\delta Y_{t-})^*(\delta F_t)]\, d\mu_t + E\left[\sum_i \int_A \|\delta Z_t^i\|^2\, d\langle M^i\rangle_t\right] \\
& + \sum_{t\in A} E[\|\delta F_t\|^2](\Delta\mu_t)^2.
\end{aligned}
$$

Using the fact that $(\Delta\mu_t)^2 = (\Delta\mu_t)(d\mu_t)$ and that for any $x \geq 0$, any $a, b \in \mathbb{R}$, $\pm 2ab \leq xa^2 + x^{-1}b^2$, we have, for any measurable function $x_t \geq 0$,

$$\begin{aligned}
\int_A dE[\|\delta Y_t\|^2] &\geq -\int_A x_t E[\|\delta Y_{t-}\|^2]\,d\mu_t - \int_A x_t^{-1} E[\|\delta F_t\|^2]\,d\mu_t \\
&\quad + E\biggl[\sum_i \int_A \|\delta Z_t^i\|^2\,d\langle M^i\rangle_t\biggr] + \int_A E[\|\delta F_t\|^2](\Delta\mu_t)\,d\mu_t \\
&= -\int_A x_t E[\|\delta Y_{t-}\|^2]\,d\mu_t - \int_A (x_t^{-1} - \Delta\mu_t) E[\|\delta F_t\|^2]\,d\mu_t \\
&\quad + E\biggl[\sum_i \int_A \|\delta Z_t^i\|^2\,d\langle M^i\rangle_t\biggr].
\end{aligned} \tag{9}$$

We now note that, for any measurable $w_t \geq 0$, as $(a+b)^2 \leq (1+w)a^2 + (1+w^{-1})b^2$ for all $w \geq 0$,

$$\begin{aligned}
\|\delta F_t\|^2 &\leq (1+w_t)\|F(\omega, t, Y_{t-}, Z_t) - F(\omega, t, \bar{Y}_{t-}, \bar{Z}_t)\|^2 \\
&\quad + (1+w_t^{-1})\|F(\omega, t, \bar{Y}_{t-}, \bar{Z}_t) - \bar{F}(\omega, t, \bar{Y}_{t-}, \bar{Z}_t)\|^2 \\
&\leq (1+w_t)c_t\|\delta Y_{t-}\|^2 + (1+w_t)c\|\delta Z_t\|_{M_t}^2 + (1+w_t^{-1})\|\delta_2 f_t\|^2.
\end{aligned}$$

Hence, as $x_t^{-1} - \Delta\mu_t \geq 0$,

$$\begin{aligned}
&\int_A (x_t^{-1} - \Delta\mu_t) E[\|\delta F_t\|^2]\,d\mu_t \\
&\quad\leq \int_A (x_t^{-1} - \Delta\mu_t)(1+w_t)c_t E[\|\delta Y_{t-}\|^2]\,d\mu_t \\
&\qquad + \int_A (x_t^{-1} - \Delta\mu_t)(1+w_t)c E[\|\delta Z_t\|_{M_t}^2]\,d\mu_t \\
&\qquad + \int_A (x_t^{-1} - \Delta\mu_t)(1+w_t^{-1}) E[\|\delta_2 f_t\|^2]\,d\mu_t \\
&\quad\leq \int_A (x_t^{-1} - \Delta\mu_t)(1+w_t)c_t E[\|\delta Y_{t-}\|^2]\,d\mu_t \\
&\qquad + E\biggl[\sum_i \int_A (x_t^{-1} - \Delta\mu_t)(1+w_t)c\|\delta Z_t^i\|^2\,d\langle M^i\rangle_t\biggr] \\
&\qquad + \int_A (x_t^{-1} - \Delta\mu_t)(1+w_t^{-1}) E[\|\delta_2 f_t\|^2]\,d\mu_t.
\end{aligned} \tag{10}$$

Combining (9) and (10) gives

$$\begin{aligned}
\int_A dE[\|\delta Y_t\|^2] &\geq -\int_A E[\|\delta Y_{t-}\|^2]\,d\upsilon_t + E\biggl[\sum_i \int_A (1 - \Delta\upsilon_t)\|\delta Z_t^i\|^2\,d\rho_t^i\biggr] \\
&\quad - \int_A E[\|\delta_2 f_t\|^2](1 - \Delta\upsilon_t)\,d\pi_t.
\end{aligned}$$

Let $\phi$ be the signed measure on $\mathcal{B}([0,T])$ defined by

$$\phi(A) = E\left[\sum_i \int_A (1-\Delta\upsilon_t)\|\delta Z_t^i\|^2\, d\rho_t^i\right] - \int_A E[\|\delta_2 f_t\|^2](1-\Delta\upsilon_t)\, d\pi_t.$$

As $d\pi/d\mu$ is bounded, $d\rho^i/d\langle M^i\rangle$ is bounded, $\|\delta_2 f\|^2$ is $\mu$-integrable and $\delta Z_t \in H_M^2$, it follows that $\phi(A)$ is bounded. We see then that $\phi$ is a signed Stieltjes measure, and we equate it with its distribution function $\phi_t := \phi([0,t])$.

Therefore, as $\Delta\upsilon_t < 1$, $\Delta\mu - x^{-1} \le 0$, an application of Lemma 4.6 yields

$$\begin{aligned}\int_A d[E[\|\delta Y_t\|^2]\mathfrak{E}(\tilde{\upsilon}_t)] &\ge \int_A (1-\Delta\upsilon_t)^{-1}\mathfrak{E}(\tilde{\upsilon}_{t-})\, d\phi_t\\ &= E\left[\sum_i \int_A \mathfrak{E}(\tilde{\upsilon}_{t-})\|\delta Z_t^i\|^2\, d\rho_t^i\right]\\ &\quad - \int_A \mathfrak{E}(\tilde{\upsilon}_{t-})E[\|\delta_2 f_t\|^2]\, d\pi_t.\end{aligned}$$

For $A = ]t,T]$, it follows that

$$\begin{aligned}&E[\|\delta Y_t\|^2]\mathfrak{E}(\tilde{\upsilon}_t) + E\left[\sum_i \int_{]t,T]} \mathfrak{E}(\tilde{\upsilon}_{s-})\|\delta Z_s^i\|^2\, d\rho_s^i\right]\\ &\quad\le E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}_T) + \int_{]t,T]} E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\, d\pi_s,\end{aligned}$$

which is the desired inequality (6). Taking a left-limit in $t$ gives, by the dominated convergence theorem,

$$\begin{aligned}&E[\|\delta Y_{t-}\|^2]\mathfrak{E}(\tilde{\upsilon}_{t-}) + E\left[\sum_i \int_{[t,T]} \mathfrak{E}(\tilde{\upsilon}_{s-})\|\delta Z_s^i\|^2\, d\rho_s^i\right]\\ &\quad\le E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}_T) + \int_{[t,T]} E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\, d\pi_s,\end{aligned}$$

and so by integration and Fubini's theorem, we have that

$$\begin{aligned}&\int_{]0,T]} E[\|\delta Y_{t-}\|^2]\mathfrak{E}(\tilde{\upsilon}_{t-})\, d\mu_t + E\left[\sum_i \int_{]0,T]} \mu_s \mathfrak{E}(\tilde{\upsilon}_{s-})\|\delta Z_s^i\|^2\, d\rho_s^i\right]\\ &\quad\le \mu_T E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}_T) + \int_{]0,T]} \mu_s E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\, d\pi_s.\end{aligned}$$

□

LEMMA 5.4. *Let $F:\Omega\times[0,T]\to\mathbb{R}^K$ be a predictable progressively measurable function such that*

$$E\left[\int_{]0,T]} \|F(\omega,t)\|^2\, d\mu\right] < +\infty.$$

*Then the BSDE*

$$Y_t - \int_{]t,T]} F(\omega, u)\, d\mu + \sum_i \int_{]t,T]} Z^i_u \, dM^i_u = Q$$

*has a unique solution in* $S^2 \times H^2_M$ *for any* $Q \in L^2(\mathbb{R}^K; \mathcal{F}_T)$. (*Note here that* $F$ *does not depend on* $Y$ *or* $Z$.)

PROOF. Using Theorem 2.1, we first construct the processes $Z^i$ which give a representation of the square integrable martingale

$$\sum_i \int_{]0,t]} Z^i_u \, dM^i_u = E\biggl[Q + \int_{]0,T]} F(\omega, u)\, d\mu \bigg| \mathcal{F}_t\biggr].$$

This can clearly be done componentwise, and so we obtain a unique process $Z \in H^2_M$, that is, $Z^i_s(\omega) \in \mathbb{R}^K$. It follows that

$$\begin{aligned}\sum_i \int_{]t,T]} Z^i_u \, dM^i_u &= Q + \int_{]0,T]} F(\omega, u)\, d\mu - E\biggl[Q + \int_{]0,T]} F(\omega, u)\, d\mu \bigg| \mathcal{F}_t\biggr] \\ &= Q + \int_{]t,T]} F(\omega, u)\, d\mu - E\biggl[Q + \int_{]t,T]} F(\omega, u)\, d\mu \bigg| \mathcal{F}_t\biggr],\end{aligned}$$

and so there is an adapted process

$$\begin{aligned}Y_t &:= E\biggl[Q + \int_{]t,T]} F(\omega, u)\, d\mu \bigg| \mathcal{F}_t\biggr] \\ &= Q + \int_{]t,T]} F(\omega, u)\, d\mu - \sum_i \int_{]t,T]} Z^i_u \, dM^i_u,\end{aligned} \tag{11}$$

which satisfies the BSDE. By uniqueness of the right-hand side of (11), this process is unique up to indistinguishability and hence in $S^2$. □

LEMMA 5.5. *Let* $\nu : [0, T] \to \mathbb{R}$ *be a nondecreasing càdlàg function of finite variation and* $c_{(\cdot)} : [0, T] \to \mathbb{R}$ *be a nonnegative bounded measurable function. Then* $c_t \Delta\nu_t = \sup_{s \in [0,T]}\{c_s \Delta\nu_s\}$ *for some* $t$; *that is* $c_t \Delta\nu_t$ *attains its maximum. Consequently, if, for some* $k \in \mathbb{R}$, $c_t \Delta\nu_t < k$ *for all* $t$, *then there exists an* $\varepsilon > 0$ *such that* $c_t \Delta\nu_t < k - \varepsilon$ *for all* $t$.

PROOF. If $c_t \Delta\nu_t \equiv 0$, then the result is trivial. Let $c$ be the upper bound of $c_{(\cdot)}$. As $\nu$ is right-continuous, it has at most countably many jumps. Then, as $\nu$ is nondecreasing, $\sum_t c_t \Delta\nu_t \le c(\sum_t \Delta\nu_t) \le c\nu_T < \infty$. Therefore, $c_t \Delta\nu_t$ is a summable sequence, and hence has finitely many values greater than or equal to $\delta$, for any $\delta > 0$. Let $\delta \in ]0, c_t \Delta\nu_t]$ for some $t$, and so $\{c_t \Delta\nu_t : c_t \Delta\nu_t \ge \delta\}$ is a finite nonempty set, and therefore has a maximum.

Now suppose $c_t \Delta\nu_t < k$ for all $t$. Let $t^*$ be the value at which $c_t \Delta\nu_t$ attains its maximum, hence $c_{t^*} \Delta\nu_{t^*} < k$. For any $\varepsilon < k - c_{t^*} \Delta\nu_{t^*}$ the result then holds. □

PROOF OF THEOREM 5.1. We consider constructing a sequence of approximations in the usual way. For a BSDE with driver $F$ and terminal condition $Q$, we fix an initial approximation $(Y^0, Z^{(0)}) \in S^2 \times H^2_M$. (Note that we denote by $Z^{(n)}$ the $n$th approximation of the infinite-dimensional process $Z$, to distinguish it from $Z^i$, the $i$th component of $Z$.) We shall first allow the $Z$ component of the solution to converge, then allow the $Y$ component to do likewise. This two-stage approach is needed due to the difference in the Lipschitz coefficients of $F$ with respect to $Y$ and $Z$. We shall assume, without loss of generality, that the Lipschitz coefficient of $F$ (with respect to $Z$) satisfies $c > 0$ uniformly.

*Step* 1: *BSDEs where the driver has Y fixed.* To construct the $Z$ solutions, we first fix some càdlàg process $\tilde{Y} \in H^2_\mu$. We wish to define a sequence of approximations of solutions to the BSDE with driver $F(\cdot, \cdot, \tilde{Y}_{\cdot-}, \cdot)$.

For any approximation $Z^{(n)}$, we fix the driver $F^n(\omega, t) = F(\omega, t, \tilde{Y}_{t-}, Z^{(n)}_t)$. Using Lemma 5.4, we obtain a new approximation $(Y^{n+1}, Z^{(n+1)})$. We shall show that the induced map $Z^{(n)} \mapsto Z^{(n+1)}$ is a contraction, and hence that a unique limit exists.

Suppose at the $n$th stage we have two approximations $(Y^{n,1}, Z^{(n,1)})$ and $(Y^{n,2}, Z^{(n,2)})$ of the solution of a BSDE with terminal value $Q$ and driver $F(\cdot, \cdot, \tilde{Y}_{\cdot-}, \cdot)$. We can hence construct new approximations $(Y^{n+1,1}, Z^{(n+1,1)})$ and $(Y^{n+1,2}, Z^{(n+1,2)})$. We consider the difference

$$(\delta Y^{n+1}, \delta Z^{(n+1)}) = (Y^{n+1,1} - Y^{n+1,2}, Z^{(n+1,1)} - Z^{(n+1,2)}).$$

Note that $(Y^{n+1,1}, Z^{(n+1,1)})$ comes from a BSDE with driver $F(\cdot, \cdot, \tilde{Y}_{\cdot-}, Z^{(n,1)}_{\cdot})$ which does not depend on the solutions $(Y^{n+1,1}, Z^{(n+1,1)})$. Hence, for appropriate functions $x_{\cdot}$ and $w_{\cdot}$, the differences $(\delta Y^{n+1}, \delta Z^{(n+1)})$ satisfy our estimate (6), with

$$\delta_2 f_s = F(\omega, s, \tilde{Y}_{s-}, Z^{(n,1)}_s) - F(\omega, s, \tilde{Y}_{s-}, Z^{(n,2)}_s)$$

and $\delta Q = 0$, and when defining $\upsilon$ and $\rho^i$ in (5) we can take $c_s = c = 0$.

We take the values $w_t = 1$, $x_t^{-1} = \frac{1}{4c} + \Delta\mu_t$, and so we see that $\Delta\mu_t - x_t^{-1} \leq 0$,

$$\upsilon_t = \int_{]0,t]} x_s \, d\mu_s = \int_{]0,t]} \frac{4c}{1 + 4c\Delta\mu_s} \, d\mu_s \leq 4c\mu_t$$

is nondecreasing and bounded (and hence of finite variation) and

$$\Delta\upsilon_t = \frac{4c\Delta\mu_t}{1 + 4c\Delta\mu_t} \leq 1 - \frac{1}{1 + 4c} < 1.$$

It follows that the integrands in (5) are bounded, our estimate (6) holds and $\mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}$ is strictly positive and bounded. Hence

$$Z \mapsto E\left[\sum_i \int_{]0,T]} \|Z_s^i\|^2 \mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\langle M^i\rangle_s\right]$$

is an equivalent norm on $H_M^2$.

As we can take $c = 0$ in (5), we have the simplification

$$d\rho_t^i = (1-\Delta\upsilon_t)^{-1}\, d\langle M^i\rangle_t = (1 - x_t\Delta\mu_t)^{-1}\, d\langle M^i\rangle_t.$$

From (6) we obtain

$$\begin{aligned}
E&\left[\sum_i \int_{]t,T]} \|(\delta Z^{(n+1)})_s^i\|^2 \mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\langle M^i\rangle_t\right]\\
&\le \int_{]t,T]} E[\|\delta_2 f_s\|^2][(x_s^{-1} - \Delta\mu_s)(1+w_s^{-1})]\mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\mu_s\\
&= \int_{]t,T]} E[\|\delta_2 f_s\|^2]\left[\frac{1}{2c}\right]\mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\mu_s.
\end{aligned}$$

By the Lipschitz continuity of the original driver, we have

$$E[\|\delta_2 f_s\|^2] \le cE\big[\|\delta Z_s^{(n)}\|_{M_s}^2\big],$$

and so, for our chosen values of $w_t$ and $x_t$, using inequality (4),

$$\begin{aligned}
E&\left[\sum_i \int_{]0,T]} \|(\delta Z^{(n+1)})_s^i\|^2 \mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\langle M^i\rangle_t\right]\\
&\le \frac{1}{2}\int_{]0,T]} E\big[\|\delta Z_s^{(n)}\|_{M_s}^2\big]\mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\mu_s\\
&\le \frac{1}{2}E\left[\sum_i \int_{]0,T]} \|(\delta Z^{(n)})_s^i\|^2 \mathfrak{E}(\tilde{\upsilon}_{s-})(1-\Delta\upsilon_s)^{-1}\, d\langle M^i\rangle_t\right].
\end{aligned}$$

By completeness, the contraction mapping principle gives the existence of a unique limit $Z \in H_M^2$ solving the BSDE with driver $F(\cdot,\cdot,\tilde{Y}_{\cdot-},\cdot)$ and terminal value $Q$. (The solution $Y$ process can, of course, be found using Lemma 5.4, fixing the $Z$ process at the constructed limit.)

*Step* 2: *BSDEs with general drivers*. We now construct a convergent sequence of approximations in $Y$ for a general driver. Consider the Lipschitz bounds of the original driver $F$. Without loss of generality, we assume that $c > 0$ uniformly. As

$c_s \Delta\mu_s < 1$, $\mu$ is nondecreasing and of finite variation, and $c_s$ is bounded, Lemma 5.5 yields a fixed $\varepsilon > 0$ such that $c_s \Delta\mu_s < 1 - \varepsilon$.

Let

$$x_t^{-1} = \frac{1}{c(1+2\varepsilon^{-1})} + \Delta\mu_t,$$
$$w_t^{-1} = \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8-4\varepsilon}.$$

As

$$x_t^{-1} - \Delta\mu_t = \frac{1}{c(1+2\varepsilon^{-1})} < \frac{1}{c(1+w_t)},$$

it is clear that

$$\frac{d\rho_t^i}{d\langle M^i\rangle_t} = [1 - (x_t^{-1} - \Delta\mu_t)(1+w_t)c](1 - x_t\Delta\mu_t)^{-1} > 0,$$

so $\rho^i$ is a nonnegative measure for each $i$.

For any terminal value $Q$, consider an approximation $Y^n \in S^2$. We can then construct a solution $(Y^{n+1}, Z^{(n+1)})$ to the BSDE with driver $F^n(\omega, t, z) = F(\omega, t, Y^n_{t-}, z)$, using the above result. Again we shall show that $Y^n \mapsto Y^{n+1}$ is a contraction, and hence that a unique limit exists.

As above, we consider the sequence of differences $(\delta Y^n, \delta Z^{(n)})$ from two initial approximations. As $Y^{n+1,1}$ is defined using the driver $F^n = F(\omega, t, Y^{n,1}_{t-}, z)$, which does not depend on $Y^{n+1,1}$, we can take $c_s = 0$ when defining $\rho^i$ in (5).

Hence, for our chosen values of $x_t$ and $w_t$, we can again easily verify that the integrands in (5) are bounded, and the resulting $\upsilon$ is nonnegative, bounded and $\Delta\upsilon < 1$. It follows that $\mathfrak{E}(\tilde{\upsilon}_s)$ is strictly positive and bounded.

Considering the difference of any two approximations $\delta Y^n$, by the Lipschitz continuity of the original driver, we have

$$E[\|\delta_2 f_s\|^2] \le c_s E[\|\delta Y^n_{s-}\|^2],$$

so, as $\rho^i$ is a family of nonnegative measures, our estimate (7) gives

$$\begin{aligned}
&\int_{]0,T]} E[\|\delta Y^{n+1}_{t-}\|^2]\mathfrak{E}(\tilde{\upsilon}_{t-})\,d\mu_t \\
&\quad\le \int_{]0,T]} \mu_s E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\,d\pi_s \\
&\quad\le \int_{]0,T]} E[\|\delta Y^n_{s-}\|^2]\mathfrak{E}(\tilde{\upsilon}_{s-})\mu_s c_s[(x_s^{-1} - \Delta\mu_s)(1 + w_s^{-1})] \\
&\qquad\qquad \times (1 - x_s\Delta\mu_s)^{-1}\,d\mu_s.
\end{aligned}$$

By construction we have

$$\begin{aligned}
&\mu_s c_s[(x_s^{-1} - \Delta\mu_s)(1 + w_s^{-1})](1 - x_s\Delta\mu_s)^{-1} \\
&\quad = \mu_s c_s x_s^{-1}(1 + w_s^{-1}) \\
&\quad = \mu_s c_s\left(\frac{1}{c(1+2\varepsilon^{-1})} + \Delta\mu_s\right)(1 + w_s^{-1}) \\
&\quad \le \mu_s\left(\frac{c_s}{c(1+2\varepsilon^{-1})} + 1 - \varepsilon\right)\left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8-4\varepsilon}\right) \\
&\quad \le \left(1 - \frac{\varepsilon}{2}\right)\left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8-4\varepsilon}\right) \\
&\quad = 1 - \frac{\varepsilon^2}{8},
\end{aligned}$$

where the fifth line is because $\mu_s \le \mu_T \le 1$ and

$$\frac{c_s}{c} \le 1 < 1 + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}(1 + 2\varepsilon^{-1}).$$

We then have

$$\int_{]0,T]} E[\|\delta Y_{t-}^{n+1}\|^2]\mathfrak{E}(\tilde{v}_{t-})\,d\mu_t \le \left(1 - \frac{\varepsilon^2}{8}\right)\int_{]0,T]} E[\|\delta Y_{s-}^{n}\|^2]\mathfrak{E}(\tilde{v}_{s-})\,d\mu_s.$$

As $\mathfrak{E}(\tilde{v}_{s-})$ is strictly positive and bounded, $\int_{]0,T]} E[\|\cdot\|^2]\mathfrak{E}(\tilde{v}_{s-})\,d\mu_s$ is an equivalent norm on $H_\mu^2$. By completeness, the contraction mapping principle gives the existence of a limit $Y_t^\infty = \lim_{n\to\infty} Y_{t-}^n$, which is unique in $H_\mu^2$. We also have the existence of a limit $Z$, as $d\rho_t^i/d\langle M^i\rangle_t$ is strictly positive, and from (7),

$$\begin{aligned}
&\lim_{n\to\infty} E\left[\sum_i \int_{]0,T]} \mu_s \mathfrak{E}(\tilde{v}_{s-})\left\|(\delta Z^{(n)})_s^i\right\|^2 d\rho_s^i\right] \\
&\qquad \le \lim_{n\to\infty}\int_{]0,T]} \mu_s E[\|\delta_2 f_s^n\|^2]\mathfrak{E}(\tilde{v}_{s-})\,d\pi_s = 0;
\end{aligned}$$

that is, $\delta Z^{(n)}$ also converges to zero in $H_M^2$.

We take the right limits of a left-continuous version of the process $Y^\infty$, namely

$$\begin{aligned}
Y_t &= E\left[Q + \int_{]t,T]} F(\omega, s, Y_s^\infty, Z_s)\,d\mu_s \middle| \mathcal{F}_t\right] \\
&= E\left[Q + \int_{]t,T]} F(\omega, s, Y_{s-}, Z_s)\,d\mu_s \middle| \mathcal{F}_t\right].
\end{aligned}$$

By Lemma 5.2, $Y \in S^2$ and by Lemma 5.1 it is unique in $S^2$. This pair $(Y, Z)$ will solve the BSDE with driver $F$ and terminal value $Q$. This limit is unique for $Z \in H_M^2$, as can by seen by fixing $Y$ and using our earlier result. □

REMARK 5.2. In discrete time, we have shown in [6] that a necessary and sufficient condition for the existence of a solution to the discrete BSDE is that $F$ is invariant with respect to equivalent $Z$ in $\|\cdot\|_{M_t}$ norm, and that $y \to y - F(\omega, t, y, z)$ is a bijection in $y$ for all $z, t$ and almost all $\omega$. The requirement that $F$ is firmly Lipschitz is sufficient, but not necessary, to guarantee that these conditions hold.

**6. Existence of BSDE solutions: General results.** We now wish to extend our above solution to allow $\mu$ to be any Stieltjes measure, by relaxing the condition that $\mu_T \le 1$. In so doing, we shall also weaken slightly the firm Lipschitz requirement.

LEMMA 6.1. *Let $\nu$ be a nonnegative Stieltjes measure with $\Delta\nu < 1$. Then there exists an $\eta > 0$ and a finite sequence $\{0 = t_0 < t_1 < \cdots < t_B = T\}$ such that $\nu(]t_i, t_{i+1}]) \le 1 - \eta$ for all $i$.*

PROOF. By Lemma 5.5 with $c_t \equiv 1$, there exists an $\eta > 0$ with $\Delta\nu < 1-\eta$. Let $t_K = T$ for some large $K$. Define recursively for integers $j < K$, $t_j = \sup\{t : \nu_t < \nu_{t_{j+1}} - 1 + \eta\} \vee 0$. By right continuity, $\nu(]t_j, t_{j+1}]) = \nu_{t_{j+1}} - \nu_{t_j} \le 1 - \eta$. For any $j$, it is also easy to show that $\nu_{t_{j+2}} - \nu_{t_j} > 1 - \eta$. Hence, as $\nu_T$ is finite, the sequence $t_j$ has only finitely many nonzero terms. Let $k = \max\{j : t_k = 0\}$, let $B = K - k$ and rescale the index of our sequence accordingly. We then have a sequence with the desired properties. □

THEOREM 6.1. *Let $\mu$ be any deterministic Stieltjes measure assigning positive measure to every open interval. (Note $\|\cdot\|_M$ is still well defined in relation to $\mu$.) Let $F : \Omega \times [0, T] \times \mathbb{R}^K \times \mathbb{R}^{K\times\infty} \to \mathbb{R}^K$ be a predictable, progressively measurable function such that:*

- $E[\int_{]0,T]} \|F(\omega, t, 0, 0)\|^2 \, d\mu_t] < +\infty$.
- *There exists a quadratic firm Lipschitz bound on $F$, that is, a measurable deterministic function $c_t$ uniformly bounded by some $c \in \mathbb{R}$, such that, for any $y_t, y_t' \in \mathbb{R}^K$, $z_t, z_t' \in \mathbb{R}^{K\times\infty}$,*

$$\|F(\omega, t, y_t, z_t) - F(\omega, t, y_t', z_t')\|^2 \le c_t \|y_t - y_t'\|^2 + c\|z_t - z_t'\|^2_{M_t}, \qquad d\mu \times d\mathbb{P}\text{-a.s.}$$

*and*

$$c_t (\Delta\mu_t)^2 < 1.$$

*Note that the variable bound $c_t$ need only apply to the behavior of $F$ with respect to $y$.*

*A function satisfying these conditions will be called standard. Then for any* $Q \in L^2(\mathbb{R}^K; \mathcal{F}_T)$, *the BSDE* (3) *with driver* $F$ *has a unique solution* $(Y, Z) \in S^2 \times H^2_M$. ($S^2$ *and* $H^2_M$ *are defined in Definition* 5.4.)

PROOF. We assume, without loss of generality, that $c \geq 1$. By Lemma 5.5, as $(\mu_t)^2$ is a nondecreasing càdlàg function of finite variation and $c_t(\Delta\mu_t)^2 < 1$ for all $t$, there exists an $\varepsilon > 0$ such that $c_t(\Delta\mu_t)^2 \leq 1 - \varepsilon$. Let

$$\nu_t = \int_{]0,t]} \frac{2(1+\varepsilon^{-1})c}{\varepsilon + 2(1+\varepsilon^{-1})c\Delta\mu_t}\, d\mu_t =: \int_{]0,t]} \lambda_t^{-1}\, d\mu_t.$$

Then $\nu \sim \mu$, and $\Delta\nu_t = \lambda_t^{-1}\Delta\mu_t < 1$. As $\nu_t$ is right continuous, deterministic and has no jumps of size equal to or greater than one, by Lemma 6.1 there exists a finite sequence $\{t_0 = 0 < t_1 < \cdots < t_B = T\}$ such that $\nu(]t_j, t_{j+1}]) \leq 1$ for all $j$.

We now note that, omitting the $\omega$ and $t$ arguments, our BSDE (3) can be written

$$Q = Y_t - \int_{]t,T]} \lambda_u F(Y_{u-}, Z_u)\, d\nu_u + \sum_{i=1}^{\infty} \int_{]t,T]} Z_u^i\, dM_u^i, \tag{12}$$

which is a BSDE in $\nu$ with Lipschitz property

$$\|\lambda_t F(y_t, z_t) - \lambda_t F(y_t', z_t')\|^2 \leq \lambda_t^2 c_t \|y_t - y_t'\|^2 + \lambda_t^2 c \|z_t - z_t'\|^2_{M_t}, \qquad d\nu \times d\mathbb{P}\text{-a.s.}$$

We write

$$\bar{c} = \sup_t \{\lambda_t^2 c\} \leq \left(\frac{\varepsilon}{2(1+\varepsilon^{-1})c} + \mu_T\right)^2 c < \infty$$

and $\bar{c}_t = \lambda_t^2 c_t$. Note that as $\varepsilon < 1$, $c_t/c < 1$,

$$\begin{aligned}
\bar{c}_t \Delta\nu_t &= \left(\frac{\varepsilon + 2(1+\varepsilon^{-1})c\Delta\mu_t}{2(1+\varepsilon^{-1})c}\right)^2 c_t \Delta\nu_t \\
&\leq \left(\frac{\varepsilon}{2(1+\varepsilon^{-1})c} + \Delta\mu_t\right)^2 c_t \\
&\leq (1+\varepsilon^{-1}) \frac{\varepsilon^2 c_t}{4(1+\varepsilon^{-1})^2 c^2} + (1+\varepsilon) c_t (\Delta\mu_t)^2 \\
&\leq \frac{\varepsilon^2}{4} + (1+\varepsilon)(1-\varepsilon) \\
&\leq 1 - \frac{3\varepsilon^2}{4} \\
&< 1.
\end{aligned} \tag{13}$$

Finally, we define the measures

$$\nu_t^k = \int_{]0, t\wedge t_{k+1}]} \left(\frac{\eta}{\nu_{t_k}} + \left(1 - \frac{\eta}{\nu_{t_k}}\right) I_{t > t_k}\right) d\nu_t.$$

It is easy then to show that $\nu^k_{t_{k+1}} \leq 1$ for all $k$. Furthermore, as

$$\frac{d\nu^k}{d\nu} = \frac{\eta}{\nu_{t_k}} I_{t<t_k} + I_{t\in[t_k,t_{k+1}]} > 0,$$

we see $\nu^k$ assigns positive measure to every interval in $]0, t_{k+1}]$. Hence, on $]0, t_{k+1}]$, $\nu^k$ is a measure of the type considered in Theorem 5.1. Also, $\Delta\nu^k_t \leq \Delta\nu_t < 1$, and $\nu^k$ agrees with $\nu$ for all subsets of $]t_k, t_{k+1}]$.

We now consider the sequence of BSDEs

$$Y^{k+1}_{t_{k+1}} = Y^k_t - \int_{]t,t_{k+1}]} \lambda_t F(Y^k_{u-}, Z^k_u)\, d\nu^k_u + \sum_{i=1}^{\infty} \int_{]t,t_{k+1}]} (Z^k)^i_u \, dM^i_u \tag{14}$$

with $Y^B_T = Q$. For each $k$, (14) is a standard BSDE with a driver $\lambda_t F$, which has Lipshitz coefficients of $\bar{c}_t$ and $\bar{c}$, and hence is (linearly) firmly Lipschitz by (13). Hence, the existence of a unique solution for each $k$ is guaranteed by Theorem 5.1.

For $k = B - 1$, (14) agrees with (12), and hence with the original BSDE (3), for all $t \in [t_k, t_{k+1}]$. It follows that the solution $Y^{B-1}_t$ is a solution to our original BSDE on the interval $[t_{B-1}, t_B]$. Similarly, for $k = B - 2$, this argument then implies that $Y^{B-2}_t$ is a solution to our original BSDE on the interval $[t_{B-2}, t_{B-1}]$, etc.

We now piece together these solutions to define $Y_t = Y^k_t$ where $t \in ]t_k, t_{k+1}]$, and similarly for $Z$. By an inductive argument, we can see that this will solve the desired BSDE. Furthermore, this solution will be unique, as the solution is unique on each subsection $]t_k, t_{k+1}]$. □

REMARK 6.1. We note that, even when $\mu_T \leq 1$, the conditions of Theorem 6.1 are strictly weaker than those of Theorem 5.1. In this case, the jumps of $\mu$ satisfy $\Delta\mu \leq 1$, and it follows that a quadratic firm Lipschitz bound is weaker than a linear firm Lipschitz bound.

REMARK 6.2. Clearly if $\Delta\mu = 0$, then the requirement that $F$ is firmly Lipschitz degenerates into the classical requirement that $F$ is uniformly Lipschitz. It is to be expected that many of the generalisations of the Lipschitz conditions which are known in the case where our filtration is generated by a Brownian motion, that is, to drivers with a stochastic Lipschitz bound, to drivers with quadratic growth, to drivers with linear growth and a monotonicity condition, etc., will also be possible in this situation. There is, however, considerable difficulty involved in obtaining these results in the simple continuous case, and it is to be expected that this difficulty will be increased by the discontinuities present here.

REMARK 6.3. The situation where $F$ has stochastic Lipschitz bounds is of particular interest here, as it would then be possible to consider replacing $\mu$ with a general predictable process of finite variation, and consequently, with any square integrable special semimartingale. Such a general situation is arguably as general as can be expected within the context of stochastic integration.

**7. A comparison theorem.** Given we have now established the existence of solutions to these equations, we now wish to prove a comparison theorem for them. This is based on the theorem in [8], for BSDEs of the type of (1).

THEOREM 7.1 (Comparison theorem). *Suppose we have two BSDEs corresponding to standard coefficients and terminal values* $(F, Q)$ *and* $(\bar{F}, \bar{Q})$. *Let* $(Y, Z)$ *and* $(\bar{Y}, \bar{Z})$ *be the associated solutions. Suppose that for some* $s$, *the following conditions hold*:

(i) $Q \geq \bar{Q}$ $\mathbb{P}$-*a.s.*;

(ii) $\mu \times \mathbb{P}$-*a.s. on* $[s, T] \times \Omega$,

$$F(\omega, u, \bar{Y}_{u-}, \bar{Z}_u) \geq \bar{F}(\omega, u, \bar{Y}_{u-}, \bar{Z}_u);$$

(iii) *for each* $j$, *there exists a measure* $\tilde{\mathbb{P}}_j$ *equivalent to* $\mathbb{P}$ *such that the* $j$*th component of* $X$, *as defined for* $r \geq s$ *by*

$$\begin{aligned} e_j^* X_r := &- \int_{]s,r]} e_j^*[F(\omega, u, \bar{Y}_{u-}, Z_u) - F(\omega, u, \bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \\ &+ \sum_i \int_{]s,r]} e_j^*[Z_u^i - \bar{Z}_u^i]\, dM_u^i \end{aligned}$$

*is a* $\tilde{\mathbb{P}}_j$ *supermartingale on* $[s, T]$;

(iv) *if, for all* $r \in [s, T]$,

$$\begin{aligned} e_i^* Y_r &- E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F(\omega, u, Y_{u-}, Z_u)\, d\mu_u \Big| \mathcal{F}_r\right] \\ &\geq e_i^* \bar{Y}_r - E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,t]} e_i^* F(\omega, u, \bar{Y}_{u-}, Z_u)\, d\mu_u \Big| \mathcal{F}_r\right] \end{aligned}$$

*for all* $i$, *then* $Y_r \geq \bar{Y}_r$ *for all* $r \in [s, t]$ *componentwise.*

*It is then true that* $Y \geq \bar{Y}$ *on* $[s, T]$, *except possibly on some evanescent set.*

PROOF. We omit the $\omega$ and $t$ arguments of $F$ for clarity.

Then, for $r \in [s, T]$

$$\begin{aligned} &Y_r - \bar{Y}_r - \int_{]r,T]} [F(Y_{u-}, Z_u) - \bar{F}(\bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \\ &\quad + \sum_i \int_{]r,T]} [Z_u^i - \bar{Z}_u^i]\, dM_u^i \\ &\quad = Y_\tau - \bar{Y}_\tau \geq 0. \end{aligned} \tag{15}$$

This can be rearranged to give

$$
\begin{aligned}
Y_r - \bar{Y}_r - \int_{]r,T]} [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \\
\geq \int_{]r,T]} [F(\bar{Y}_{u-}, \bar{Z}_u) - \bar{F}(\bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \\
+ \int_{]r,T]} [F(\bar{Y}_{u-}, Z_u) - F(\bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \\
- \sum_i \int_{]r,T]} [Z_u^i - \bar{Z}_u^i]\, dM_u^i.
\end{aligned}
\tag{16}
$$

We have that

$$
\int_{]r,T]} [F(\bar{Y}_{u-}, \bar{Z}_u) - \bar{F}(\bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \geq 0
$$

by assumption (ii). As $e_j^* X_r$ is a $\tilde{\mathbb{P}}_j$ supermartingale, we know that the process given by

$$
\begin{aligned}
e_j^* \tilde{X}_r &:= e_j^* X_r - E_{\tilde{\mathbb{P}}_i}[e_j^* X_T | \mathcal{F}_r] \\
&= E_{\tilde{\mathbb{P}}_j}\biggl[\int_{]r,T]} e_j^* [F(\bar{Y}_{u-}, Z_u) - F(\bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \\
&\qquad - \sum_i \int_{]r,T]} e_j^* [Z_u^i - \bar{Z}_u^i]\, dM_u^i \Big| \mathcal{F}_r\biggr]
\end{aligned}
\tag{17}
$$

is also a $\tilde{\mathbb{P}}_j$-supermartingale, with $e_j^* \tilde{X}_T = 0$ $\tilde{\mathbb{P}}_j$-a.s. Hence $e_j^* \tilde{X}_r \geq 0$.

For each $j$, taking a $\tilde{\mathbb{P}}_j | \mathcal{F}_r$ conditional expectation throughout (16) and premultiplying by $e_j^*$ gives

$$
e_j^* Y_r - e_j^* \bar{Y}_r - E_{\tilde{\mathbb{P}}_j}\biggl[\int_{]r,T]} e_j^* [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \Big| \mathcal{F}_r\biggr] \geq 0.
$$

This must hold for all $r \in [s, T]$ and almost all $\omega$. By assumption (iv), for almost all $\omega$, it follows that the comparison $Y_r \geq \bar{Y}_r$ must hold for all $r \in [s, T]$.

As $Y - \bar{Y}$ is càdlàg, we have that $Y - Y$ is indistinguishable from a nonnegative process and, therefore, the inequality holds up to evanescence. □

REMARK 7.1. Assumption (iv) is clearly trivial whenever $F$ does not depend on $Y$.

REMARK 7.2. Assumption (iii) is very closely related to the *Fundamental theorem of asset pricing* (see [11]), as it relates an inequality in current values to the existence of an equivalent (super-)martingale measure.

COROLLARY 7.1.1. *If assumption* (iv) *holds for any* $T$ *whenever* $s \geq T - \varepsilon$ *for some fixed* $\varepsilon$, *then the comparison also holds.*

PROOF. In this case, we can show that the comparison holds on $[T-\varepsilon, T]$. We can then replace $T$ with $T-\varepsilon$ throughout the theorem, replacing $Q$ and $\bar{Q}$ with $Y_{T-\varepsilon}$ and $\bar{Y}_{T-\varepsilon}$ in assumption (i). It is clear that assumptions (ii) and (iii) will continue to hold, with the same choice of measures $\tilde{\mathbb{P}}_i$. By the statement of the corollary, assumption (iv) will then hold on the interval $[T-2\varepsilon, T-\varepsilon]$. By induction, it follows that the comparison holds on $[T-n\varepsilon, T]$ for all $n \in \mathbb{N}$. For $n$ sufficiently large, this implies the comparison holds on $[s, T]$ as desired. □

DEFINITION 7.1. A standard driver $F$ such that assumptions (iii) and (iv) of Theorem 7.1 hold on $[0, T]$ for all $Y, \bar{Y} \in S^2$ and $Z, \bar{Z} \in H_M^2$ will be called *balanced*.

THEOREM 7.2. *In the scalar* ($K = 1$) *case*, *assumption* (iv) *of Theorem* 7.1 *holds for any standard* $F$.

PROOF. As we are in the scalar case, we can omit the $e_i$ from the statement of the assumption. Hence, we wish to show that, given for all $r \in [s, T]$

$$\begin{aligned} & Y_r - E_{\tilde{\mathbb{P}}}\left[\int_{]r,T]} F(\omega, u, Y_{u-}, Z_u)\, d\mu_u \middle| \mathcal{F}_r\right] \\ & \quad \geq \bar{Y}_r - E_{\tilde{\mathbb{P}}}\left[\int_{]r,T]} F(\omega, u, \bar{Y}_{u-}, Z_u)\, d\mu_u \middle| \mathcal{F}_r\right], \end{aligned}$$

we must have $Y_r \geq \bar{Y}_r$. For simplicity, let $\delta Y := Y - \bar{Y}$.

It is clear from the problem and the recursivity of BSDE solutions that we can replace $T$ with any stopping time $\tau \leq T$ such that $\delta Y_\tau \geq 0$. By applying Lemma 6.1, we can also assume that $s$ is such that $\int_{]s,T]} c_u\, d\mu_u < 1$, and simply piece together the result for general $s$.

Suppose on some nonnull set $A \in \mathcal{F}$, $\delta Y_u < 0$ for some $u \in [s, T]$. As $\delta Y$ is adapted and right continuous, this implies that there are stopping times $\sigma, \tau$ such that $\delta Y_u < 0$ for all $u \in [\sigma, \tau[$, and $s \leq \sigma < \tau$ on $A$. Without loss of generality, let $\tau$ be the largest such upper bound. Then, as $\delta Y_T \geq 0$ and $\tau \leq T$, it follows that $\delta Y_\tau \geq 0$. Replacing $T$ with $\tau$ in the above inequality, we know that

$$\begin{aligned} & E_{\tilde{\mathbb{P}}}[I_{r\in[\sigma,\tau[}|\delta Y_r|] \\ & \quad = E_{\tilde{\mathbb{P}}}[-I_{r\in[\sigma,\tau[}\delta Y_r] \\ & \quad \leq E_{\tilde{\mathbb{P}}}\left[-I_{r\in[\sigma,\tau[}\int_{]r,\tau]} F^1(\omega, u, Y^1_{u-}, Z^1_u) - F^1(\omega, u, Y^2_{u-}, Z^1_u)\, d\mu_u\right] \\ & \quad \leq E_{\tilde{\mathbb{P}}}\left[I_{r\in[\sigma,\tau[}\int_{]r,\tau]} c_u|\delta Y_{u-}|\, d\mu_u\right] \\ & \quad \leq \int_{]r,T]} E_{\tilde{\mathbb{P}}}[I_{u\in[\sigma,\tau[}|\delta Y_{u-}|]c_u\, d\mu_u. \end{aligned}$$

Taking a left limit in $r$, we see

$$E_{\tilde{\mathbb{P}}}[I_{r\in[\sigma,\tau[}|\delta Y_{r-}|] \le \int_{[r,T]} E_{\tilde{\mathbb{P}}}[I_{u\in[\sigma,\tau[}|\delta Y_{u-}|]c_u\, d\mu_u.$$

By assumption, this quantity is strictly positive. Integration on $]t, T]$ and Fubini's theorem gives, for $t > s$,

$$\begin{aligned}\int_{]t,T]} E_{\tilde{\mathbb{P}}}[I_{r\in[\sigma,\tau[}|\delta Y_{r-}|]c_r\, d\mu_r &\le \int_{]t,T]}\left(\int_{[r,T]} E_{\tilde{\mathbb{P}}}[I_{u\in[\sigma,\tau[}|\delta Y_{u-}|]c_u\, d\mu_u\right)c_r\, d\mu_r \\ &= \int_{]t,T]}\left(\int_{]t,u]} c_r\, d\mu_r\right)E_{\tilde{\mathbb{P}}}[I_{u\in[\sigma,\tau[}|\delta Y_{u-}|]c_u\, d\mu_u \\ &< \int_{]t,T]} E_{\tilde{\mathbb{P}}}[I_{u\in[\sigma,\tau[}|\delta Y_{u-}|]c_u\, d\mu_u,\end{aligned}$$

where the last line is due to our assumption that $\int_{]s,T]} c_t\, d\mu_t < 1$. This contradicts our assumption that this quantity is strictly positive. Therefore, $A$ is a null set, that is, $\delta Y_u \ge 0$ for all $u \in [s, t]$. □

DEFINITION 7.2. The comparison between $Y$ and $\bar{Y}$ will be called *strict* on $[s, T]$ if the conditions of Theorem 7.1 hold, and, for any $A \in \mathcal{F}_s$ such that $Y_s = \bar{Y}_s$ $\mathbb{P}$-a.s. on $A$, we have $Y_u = \bar{Y}_u$ on $[s, T] \times A$, up to evanescence.

LEMMA 7.1. *If the comparison is strict on* $[s, T]$, *then for any* $A \in \mathcal{F}_s$ *such that* $Y_s = \bar{Y}_s$ $\mathbb{P}$*-a.s. on* $A$, *it follows that*:

- $Q = \bar{Q}$ $\mathbb{P}$*-a.s. on* $A$;
- $F(\omega, u, \bar{Y}_{u-}, \bar{Z}_u) = \bar{F}(\omega, u, \bar{Y}_{u-}, \bar{Z}_u)$ $\mu \times \mathbb{P}$*-a.s. on* $[s, T] \times A$;
- $Z \equiv \bar{Z}$ *in* $H^2_M$ *on* $[s, T] \times A$.

PROOF. We omit the $\omega$ and $t$ arguments of $F$ and $\bar{F}$ for clarity. Let $\tilde{X}$ be as in (17), and let $S$ be the process defined by

$$\begin{aligned} e_j^* S_r &:= e_j^* E_{\tilde{\mathbb{P}}_i}[Q - \bar{Q}|\mathcal{F}_r] \\ &\quad + e_j^* E_{\tilde{\mathbb{P}}_i}\left[\int_{]r,T]}[F(\bar{Y}_{u-}, \bar{Z}_u) - \bar{F}(\bar{Y}_{u-}, \bar{Z}_u)]\, d\mu_u \middle| \mathcal{F}_r\right] + e_j^* \tilde{X}_r. \end{aligned} \tag{18}$$

Then $e_j^* S$ is a $\tilde{\mathbb{P}}_j$-supermartingale, as the first term is a $\tilde{\mathbb{P}}_j$-martingale, the second is nonincreasing in $r$ by assumption (ii) of Theorem 7.1 and the third is a $\tilde{\mathbb{P}}_j$-supermartingale by assumption (iii) of Theorem 7.1. Furthermore, each of these terms is nonnegative.

Taking a $\tilde{\mathbb{P}}_j|\mathcal{F}_r$ conditional expectation through (3), we have that, for all $r \in [s, T]$,

$$e_j^*(Y_r - \bar{Y}_r) = e_j^* S_r + E_{\tilde{\mathbb{P}}}\left[\int_{]r,t]} e_j^*[F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \middle| \mathcal{F}_r\right]. \tag{19}$$

If $Y_r = \bar{Y}_r$ on $[s, T] \times A$ up to evanescence, then it is clear from (19) that $S_r = 0$ $\mathbb{P}$-a.s. on $[s, T] \times A$. Hence, by nonnegativity, each of the terms on the right-hand side of (18) must be zero. The first two points of the lemma immediately follow.

Consider the BSDE (3) satisfied by $\bar{Y}$. As $F(\bar{Y}_{u-}, Z_u) = \bar{F}(Y_{u-}, Z_u)$ $\mu \times \mathbb{P}$-a.s. on $[s, T] \times A$ and $Q = \bar{Q}$ $\mathbb{P}$-a.s. on $A$, we know that

$$\bar{Y}_r - \int_{]r,T]} \bar{F}(\bar{Y}_{u-}, \bar{Z}_u)\, d\mu_u + \sum_i \int_{]r,T]} \bar{Z}_u^i \, dM_u^i = \bar{Q}$$

is $\mathbb{P}$-a.s. equal to

$$\bar{Y}_r - \int_{]r,T]} F(\bar{Y}_{u-}, \bar{Z}_u)\, d\mu_u + \sum_i \int_{]r,T]} \bar{Z}_u^i \, dM_u^i = Q.$$

Hence, in $A$, $(\bar{Y}, \bar{Z})$ is a solution at time $r$ to the BSDE defining $(Y, Z)$.

As the solution to this BSDE is unique, it follows that, on $[s, T] \times A$, $\bar{Z} \equiv Z$ in $H^2_{M_t}$. $\square$

THEOREM 7.3 (Strict comparison). *Consider the scalar* ($K = 1$) *case, where* $F$ *is balanced. Then the comparison is strict on* $[s, T]$ *for all* $s$.

PROOF. Again, as $K = 1$ we can omit $e_j$ from all equations, and we omit the $\omega$ and $t$ arguments of $F$ and $\bar{F}$ for clarity. Let $S_r$ be as defined in (18), and note that $S$ is a nonnegative $\tilde{\mathbb{P}}$-supermartingale.

Taking a $\tilde{\mathbb{P}}|\mathcal{F}_s$ conditional expectation of (19) gives

$$\begin{aligned} E_{\tilde{\mathbb{P}}}[Y_r - \bar{Y}_r | \mathcal{F}_s] &= E_{\tilde{\mathbb{P}}}\biggl[S_r + \int_{]s,t]} [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \bigg| \mathcal{F}_s\biggr] \\ &\quad - E_{\tilde{\mathbb{P}}}\biggl[\int_{]s,r]} [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \bigg| \mathcal{F}_s\biggr] \\ &\le S_s + E_{\tilde{\mathbb{P}}}\biggl[\int_{]s,t]} [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \bigg| \mathcal{F}_s\biggr] \\ &\quad + \int_{]s,r]} E_{\tilde{\mathbb{P}}}[|F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)| | \mathcal{F}_s]\, du \\ &\le S_s + E_{\tilde{\mathbb{P}}}\biggl[\int_{]s,t]} [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \bigg| \mathcal{F}_s\biggr] \\ &\quad + c \int_{]s,r]} E_{\tilde{\mathbb{P}}}[|Y_{u-} - \bar{Y}_{u-}| | \mathcal{F}_s]\, d\mu_u. \end{aligned} \tag{20}$$

We know from (19) and the assumption $Y_s - \bar{Y}_s = 0$ on $A$ that

$$I_A S_s + I_A E_{\tilde{\mathbb{P}}}\biggl[\int_{]s,t]} [F(Y_{u-}, Z_u) - F(\bar{Y}_{u-}, Z_u)]\, d\mu_u \bigg| \mathcal{F}_s\biggr] = I_A(Y_s - \bar{Y}_s) = 0,$$

and so, as $Y-\bar{Y}$ is nonnegative by Theorem 7.1, premultiplication of (20) by $I_A$ and then taking an expectation gives

$$E_{\tilde{\mathbb{P}}}[I_A(Y_r-\bar{Y}_r)] \le c\int_{]s,r]} E_{\tilde{\mathbb{P}}}[I_A(Y_{u-}-\bar{Y}_{u-})]\,d\mu_u.$$

As all quantities are nonnegative, taking a limit from below yields

$$E_{\tilde{\mathbb{P}}}[I_A(Y_{r-}-\bar{Y}_{r-})] \le c\int_{]s,r]} E_{\tilde{\mathbb{P}}}[I_A(Y_{u-}-\bar{Y}_{u-})]\,d\mu_u,$$

and an application of (the forward version of) Grönwall's lemma implies

$$E_{\tilde{\mathbb{P}}}[I_A(Y_r-\bar{Y}_r)] \le 0.$$

By nonnegativity, it follows that $Y_r=\bar{Y}_r$, $\tilde{\mathbb{P}}$-a.s. on $A$. Again, as $Y-\bar{Y}$ is càdlàg, this shows that $Y=\bar{Y}$ on $[s,t]\times A$, up to evanescence. □

COROLLARY 7.3.1. *If the $i$th component of $F(\omega,t,y,z)$ depends only on the $i$th component of $y$ (as well as on $\omega$, $t$ and $z$), then the comparison is strict.*

PROOF. As the $i$th component of $F$ depends only on the $i$th component of $y$, we can repeat the construction of Theorem 7.3 in each component. The result follows. □

REMARK 7.3. In the scalar case, with a simple Brownian filtration ($M^1=W$, $M^i=0$ for $i\ge 2$) and $d\mu=dt$, we can use Girsanov's transformation to construct the measure required for assumption (iii) of Theorem 7.1. We write

$$\Lambda_t := 1+\int_{]0,t]}\Lambda_{u-}\frac{F(\omega,u,\bar{Y}_{u-},Z_u)-F(\omega,u,\bar{Y}_{u-},\bar{Z}_u)}{Z_u-\bar{Z}_u}\,dW_u,$$

then $d\tilde{\mathbb{P}}/d\mathbb{P}=\Lambda_T$. It is then easy to verify that $X$ is a martingale. In this case, using Theorem 7.2 we can see that any Lipschitz continuous $F$ is balanced.

**8. Nonlinear expectations.** We are now in a position to explicitly construct nonlinear expectations in a general probability space. We shall not here consider the more general theory of nonlinear evaluations. An approach without these restrictions can be seen in [8]. These operators, discussed in [19], are closely related to the theory of dynamic risk measures, as in [1, 3, 21] and others, as each concave nonlinear expectation $\mathcal{E}(\cdot|\mathcal{F}_t)$ corresponds to a dynamic convex risk measure through the relationship

$$\rho_t(Q)=-\mathcal{E}(Q|\mathcal{F}_t).$$

A further discussion of this relationship can be found in [21].

DEFINITION 8.1. A family of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t): L^2(\mathcal{F}_T)\to L^2(\mathcal{F}_t), \qquad 0\le t\le T,$$

is called an $\mathcal{F}_t$-consistent *nonlinear expectation* if $\mathcal{E}(\cdot|\mathcal{F}_t)$ satisfies the following properties:

(1) If $Q \geq \bar{Q}$ $\mathbb{P}$-a.s. componentwise

$$\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(\bar{Q}|\mathcal{F}_t), \qquad \mathbb{P}\text{-a.s. componentwise}$$

with equality iff $Q = \bar{Q}$ $\mathbb{P}$-a.s.

(2) For $Q \in L^2(\mathcal{F}_t)$, $\mathcal{E}(Q|\mathcal{F}_t) = Q$ $\mathbb{P}$-a.s.

(3) For any $s \leq t$,

$$\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s), \qquad \mathbb{P}\text{-a.s.}$$

(4) For any $A \in \mathcal{F}_t$,

$$I_A\mathcal{E}(Q|\mathcal{F}_t) = \mathcal{E}(I_A Q|\mathcal{F}_t), \qquad \mathbb{P}\text{-a.s.}$$

THEOREM 8.1. *Let $F$ be a balanced driver which does not depend on $Y$* (*i.e.*, $c_t \equiv 0$) *and satisfies $F(\omega, t, y, 0) = 0$ $\mu \times \mathbb{P}$-a.s. Then the operator defined by*

$$\mathcal{E}(Q|\mathcal{F}_t) = Y_t,$$

*where $Y$ is the solution to a BSDE* (3) *with driver $F$, is a nonlinear expectation.*

PROOF. (1) As $F$ is balanced, this result follows directly from the comparison theorem (Theorem 7.1). As $F$ does not depend on $Y$, the strict comparison will also hold, by Corollary 7.3.1.

(2) Consider the BSDE (3) on $[t, T]$

$$Y_s - \int_{]s,T]} F(\omega, u, Y_{u-}, Z_u)\, d\mu_u + \sum_i \int_{]s,T]} Z_u^i\, dM_u^i = Q.$$

This has a solution $Y_s = Q$, $Z_s = 0$. As $Q \in L^2(\mathcal{F}_t)$, this solution is adapted and, by Theorem 5.1, unique. Therefore $\mathcal{E}(Q|\mathcal{F}_t) = Y_t = Q$ as desired.

(3) By definition the BSDE with terminal condition $Q$ at time $T$ has solution $Y_t$ at time $t$. Simple manipulation of the BSDE (3) at time $s$ shows that $Y_s$ is also the time $s$ solution to the BSDE with terminal condition $Y_t$ at time $t$. Hence, by property 2, $Y_s$ solves both the BSDE with terminal condition $Y_t = \mathcal{E}(Q|\mathcal{F}_t)$ and the BSDE with terminal condition $Q$.

(4) Consider the BSDE with driver $F$ and terminal condition $Q$. Multiplying by $I_A$, as $I_A F(\omega, t, y, z) = F(\omega, t, I_A y, I_A z)$, we see that $(I_A Y, I_A Z)$ is the solution to the BSDE with driver $F$ and terminal condition $I_A Q$, as desired. □

REMARK 8.1. It is known in discrete time [6], and under some conditions in continuous time [9], that BSDEs describe all nonlinear expectations, subject to some boundedness conditions. It is likely that a similar result will hold in this setting. However, obtaining such a result is beyond the scope of this paper.

**9. Conclusions.** We have constructed BSDEs in a general filtered probability space, using only basic properties of the filtration. We have presented conditions for the existence of unique solutions to these equations, and seen how these are related to the conditions in both the classical setting, and the discrete time setting. We have given a comparison theorem for these solutions, which allows the construction of nonlinear expectations in these spaces.

These results are significantly more general than those previously available, as they make very few assumptions on the underlying probability space. A consequence of this is that a possibly infinite-dimensional martingale representation theorem is required. In full generality, they also make no assumptions regarding the relationship of the integrator of the driver and the quadratic variations of the martingale terms. At the same time, this general setting provides an approach unifying the theory of BSDEs in discrete and continuous time.

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