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NONEXPLOSION CRITERIA FOR RELATIVISTIC DIFFUSIONS[1]

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Some general Lorentz covariant operators, associated to the so-called Θ (or Ξ)-relativistic diffusions and making sense in any Lorentzian manifold, have been introduced by Franchi and Le Jan [*Comm. Pure Appl. Math.* **60** (2007) 187–251], Franchi and Le Jan [Curvature diffusions in general relativity (2010). Unpublished manuscript]. Only a few examples have been studied so far. We provide in this work some nonexplosion criteria for these diffusions, which can be used in generic cases.

1. Introduction. It is well known that the metric completeness of a Riemannian manifold does not prevent Brownian motion from exploding within a finite time with positive probability. The situation is now well understood, in particular, thanks to the works of Yau [27], Grigor'yan [16], Takeda [25, 26] and very recently Hsu and Qin [20], to cite but a few names. Different lines of approach have been used. Yau and Grigor'yan treated the analytic counterpart of the completeness problem and investigated the well-posedness of the parabolic Cauchy problem; the former using local information on the geometry under the form of curvature bounds; the latter using a global information under the form of an upper bound for the volume of large balls. Takeda used a purely probabilistic method due to Lyons and Zheng in [21], based on reversibility. This approach was recently improved by Hsu and Qin in [20]. Hsu used stochastic analysis in [19], Theorem 3.5.1, to control the radial process, by estimating the Laplacian of the distance function to a fixed point in terms of curvature bounds. All these results are tied down to the metric framework provided by a complete Riemannian manifold.

A natural analog of Brownian motion in a Lorentzian setting was first introduced by Dudley [10] in the special relativistic case, and extended to the general relativistic framework by Franchi and Le Jan [12]. It belongs to a larger class of relativistic processes introduced in [3] and [13], defined in purely geometric terms and collectively refered to as *relativistic diffusions*. Their trajectories represent the random motion in spacetime of a small massive particle, and make sense only in the unit tangent bundle or in the orthonormal frame bundle. Only a few examples

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have been studied in detail up to now: in Minkowski spacetime (the framework of special relativity) [2, 5, 10], in Robertson–Walker spacetimes (models of universe with a big-bang) [1], Gödel spacetime (a causally paradoxical universe) [11] and Schwarzschild spacetime (a model for an isolated star or a black hole) [12].

Apart from the works [3] and [13], no general study of these intrinsic random processes was done. As a first step towards a better understanding of these processes and their interplay with the geometry of the ambient spacetime, we provide in this work some nonexplosion criteria for some generic classes of Lorentz manifolds. In addition to being a natural question, the completeness issue is strongly related to important questions in general relativity. Indeed, dating back to Penrose and Hawking's incompleteness theorems, the appearance of singularities in Einstein's theory of gravitation has been recognized as unavoidable under quite natural assumptions. Although there is no agreement on what should be called a singularity of a spacetime, the existence of incomplete geodesics has been widely used as an indicator of such a singular feature. In so far as the random dynamics considered in this work (Section 2.2) can be seen as intrinsic perturbations of the geodesic flow, their completeness/incompleteness is a distinguishing feature of a spacetime. We refer the reader to [4] for a first approach of stochastic incompleteness.

The paths of the random processes we shall consider are (almost-)all $\mathcal{C}^1$ paths parametrized by their (proper time) arc length. What could possibly make them explode? In a complete Riemannian manifold, any such path would have to be at time s in a closed ball of radius s with center its starting point, so it cannot explode. There are two problems with the Lorentzian setting: a Lorentzian manifold has no metric or finite distance function associated with its structure, and the set of unit tangent vectors at any point is noncompact. As a result, even in Minkowski spacetime, one can construct exploding paths with finite (proper time) arc length.

To start our investigations, we shall take advantage in Section 3 of the bundle structure of the state space of the process, to exhibit a one-dimensional subprocess whose control is possible in the class of globally hyperbolic spacetimes. This structure, indeed, allows us to define some Lyapounov function, and leads to a nonexplosion criterion by using a simple and well-known observation due to Khasminsky.

With a metric missing, the completeness notion used in a crucial way in the Riemannian setting becomes unavailable. Busemann, Hawking and Ellis and Schmidt, Beem and Ehrlich proposed different notions in replacement. Schmidt's idea is to give a Riemannian structure to the orthonormal frame bundle. We consider Schmidt b-completeness notion in Section 4, showing how it leads to a stochastic completeness result for some of the relativistic diffusions.

This result can be significantly improved by adapting Takeda's strategy [26], as improved by Hsu and Qin [20], to the Lorentzian setting. This is, however, far from being straightforward, since we are working in a nonsymmetric, nonelliptic setting, where the main ingredients of Takeda's method (use of symmetry and reflected

Brownian motion on the boundary of large Riemannian balls) have no obvious Lorentzian counterpart. To overcome this difficulty, we use in Section 5 a sub-Riemannian structure well adapted to our setting, and which will somehow play for us the role of the nonexisting Lorentzian distance.

2. Relativistic diffusions.

2.1. *Basic geometrical setting.* Recall Minkowski space is the product $\mathbb{R}^{1,d} \equiv \mathbb{R} \times \mathbb{R}^d$ equipped with the metric

$$g_{\mathrm{M}}(q,q) := t^2 - |x^1|^2 - \cdots - |x^d|^2 \qquad \text{for any } q = (t,x) \in \mathbb{R}^{1,d},$$

where $(t, x^1, \dots, x^d)$ denote the coordinates of q in the canonical basis $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d\}$ of $\mathbb{R}^{1,d}$.

Let $(\mathbb{M}, g)$ be a smooth $(1+d)$-dimensional Lorentzian manifold (with $d \geq 2$), which we shall always suppose to be oriented and time-oriented. (We refer the reader to the books of Hawking–Ellis [18] and O'Neill [23] for the basics on Lorentzian geometry.) Given any point $m \in \mathbb{M}$, it is usual to consider an orthonormal basis $\{\mathbf{e}_0, \dots, \mathbf{e}_d\}$ of the tangent space $T_m\mathbb{M}$ as an isometry $\mathbf{e}$ from $(\mathbb{R}^{1,d}, g_{\mathrm{M}})$ to $(T_m\mathbb{M}, g_m)$; so, strictly speaking, $\mathbf{e}_i = \mathbf{e}(\varepsilon_i)$. The orthonormal frame bundle of $\mathbb{M}$ is just the collection

$$\mathbb{OM} = \{\Phi = (m, \mathbf{e}) | m \in \mathbb{M}, \mathbf{e} \text{ an orthonormal basis of } (T_m\mathbb{M}, g_m)\}.$$

We shall write $\mathbb{O}\mathcal{U} = \{\Phi = (m, \mathbf{e}) | m \in \mathcal{U}, \mathbf{e} \text{ an orthonormal basis of } T_m\mathbb{M}\}$ for any subset $\mathcal{U}$ of $\mathbb{M}$. For a small enough $\mathcal{U}$ and a chart $x : \mathcal{U} \to \mathbb{R}^{1+d}$ on it, we shall write $\mathbf{e}_j = e_j^k \, \partial_{x^k}$ for each vector $\mathbf{e}_j$ of a frame $\mathbf{e}$; this decomposition provides local coordinates (x^i, e_j^k) on $\mathbb{O}\mathcal{U}$.

Each fiber $\mathbb{O}_m\mathbb{M}$ is modeled on the noncompact orthogonal group $O(1,d)$, which has four connected components. We shall be interested in dynamics leaving these components globally fixed. We choose to consider only one of them, specified by the requirement that $\mathbf{e}_0$ should be future-oriented and that the orientation of $\mathbf{e}$ should be direct. We shall still denote the resulting frame bundle by $\mathbb{OM}$, as there will be no risk of confusion. The Lorentz–Möbius group $SO_0(1,d)$, that is, the connected component of the unit in $O(1,d)$, acts properly on $\mathbb{OM}$. This natural action induces the canonical vertical vector fields $(V_{ij})_{0 \leq i < j \leq d}$. The subgroup of elements in $SO_0(1,d)$ that fix ε_0 can be identified with the rotation group $SO(d)$, and generates the vector fields $(V_{ij})_{1 \leq i < j \leq d}$. To shorten notations we shall write V_j for V_{0j}; it generates boosts, that is, hyperbolic rotations in each fiber, and reads, in the above local coordinates,

$$V_j = e_j^k \frac{\partial}{\partial e_0^k} + e_0^k \frac{\partial}{\partial e_j^k}. \tag{2.1}$$

Throughout this work, $T\mathbb{M}$ and $\mathbb{OM}$ will be endowed with the Levi–Civita connection, inherited from the Lorentzian pseudo-metric g. Last, we denote by H_0 the vector field generating the geodesic flow on $\mathbb{OM}$. Denoting by Γ^{ℓ}_{kj} the Christoffel coefficients, we have, in the above local chart on $\mathbb{OM}$,

$$H_0 = e_0^k \,\partial_{x^k} - e_0^k e_i^j \,\Gamma^{\ell}_{kj} \frac{\partial}{\partial e_i^{\ell}}. \tag{2.2}$$

We shall denote by $T^1\mathbb{M}$ the future-oriented unit tangent bundle over $\mathbb{M}$, with generic element $(m, \dot{m})$. In Minkowski spacetime $\mathbb{R}^{1,d}$, it is the product of $\mathbb{R}^{1,d}$ by the hyperboloid $\mathbb{H} = \{q = (t, x) \in \mathbb{R}^{1,d}; g(q, q) = 1, t > 0\}$. The bundle $T^1\mathbb{M}$ is locally modeled on that product. (Consult [18] or [23] for some background.) Denote by π_1 the projection $(m, \mathbf{e}) \mapsto (m, \mathbf{e}_0 \equiv \dot{m})$ from $\mathbb{OM}$ to $T^1\mathbb{M}$, and by π_0 the canonical projection $\mathbb{OM} \to \mathbb{M}$.

2.2. *Relativistic random dynamics*. Relativistic diffusions model the random motion in spacetime of a small massive particle parametrized by its proper time, providing random timelike paths; so, properly speaking, their mathematical counterpart are random trajectories $(m_s, \dot{m}_s)$ in the future unit bundle $T^1\mathbb{M}$ subject to the condition $\frac{d}{ds} m_s = \dot{m}_s$. Yet it happens to be more convenient to define random dynamics in the orthonormal frame bundle $\mathbb{OM}$ as it bears more structure than $T^1\mathbb{M}$; these diffusions on $\mathbb{OM}$ are constructed so as to have a projection on $T^1\mathbb{M}$ which is itself a diffusion. Such a construction is remniscent of Malliavin–Eells–Elworthy's construction of Brownian motion on a Riemannian manifold as the projection of a diffusion on the orthonormal frame bundle.

2.2.1. *Dynamics in* $\mathbb{OM}$. Given any smooth nonnegative function $\Theta : T^1\mathbb{M} \to \mathbb{R}_+$, identified to a $SO(d)$-invariant function on $\mathbb{OM}$ by setting $\Theta(\Phi) := \Theta(\pi_1(\Phi))$, consider the following Stratonovich differential equation on $\mathbb{OM}$:

$$\begin{aligned} \circ\, d\Phi_s = {} & H_0(\Phi_s)\, ds + \frac{1}{4} \sum_{1 \le j \le d} V_j \Theta(\Phi_s) V_j(\Phi_s)\, ds \\ & + \sqrt{\Theta(\Phi_s)} \sum_{1 \le j \le d} V_j(\Phi_s) \circ dw_s^j, \end{aligned} \tag{2.3}$$

where w is a d-dimensional Brownian motion and where we understand a vector field as a first-order differential operator. This equation has a unique maximal strong solution, defined up to its explosion time ζ.

It is clear on this equation that the $(\mathbf{e}_1, \dots, \mathbf{e}_d)$-part of Φ_s is irrelevant in defining the dynamics of $(m_s, \mathbf{e}_0(s))$ since $\Theta(\Phi)$ depends only on $\pi_1(\Phi)$; this is the reason why this diffusion on $\mathbb{OM}$ projects down in $T^1\mathbb{M}$ onto a diffusion. Consult [12], Theorem 1, [13], Theorem 3.2.1 or [3], Section 3.2, for the details. The diffusion in $\mathbb{OM}$ has generator

$$\mathcal{G}_\Theta = H_0 + \frac{1}{2} \sum_{1 \le j \le d} V_j(\Theta V_j). \tag{2.4}$$

We shall generically call these relativistic dynamics Θ-*diffusions* (the Ξ-diffusions of [13]). These diffusions are *covariant*, in the sense that any isometry of $(\mathbb{M}, g)$ maps a Θ-diffusion to a Θ-diffusion (with the same Θ: the law is preserved, up to the starting point), and admit the Liouville measure as an invariant measure. The π_0-projections (on the base manifold $\mathbb{M}$) of their trajectories are almost-surely C^1 paths. A Θ-diffusion $(\Phi_s)_{0\le s<\zeta}$ solving equation (2.3) is parametrized by proper time $s \ge 0$. The particular case $\Theta = 0$ gives back the deterministic geodesic flow, and the case of a nonnull constant Θ gives back the relativistic diffusion as defined first in [12], which we shall call the *basic relativistic diffusion*. It is described in simple terms in Minkowski spacetime. Although the metric g_M is nondefinite positive, its restriction to any tangent space of the half sphere $\mathbb{H}$ of unit tangent vectors is definite negative; this turns $\mathbb{H}$ into a Riemannian manifold with constant negative curvature. Dudley's diffusion $(m_s, \mathbf{e}_s) = (m_s, (\mathbf{e}_0(s), \dots, \mathbf{e}_d(s)))$, which is the basic relativistic diffusion in Minkowski spacetime, corresponds to taking $m_s = m_0 + \int_0^s \mathbf{e}_0(r)\,dr$, and for the velocity $\mathbf{e}_0(r)$ a Brownian motion on $\mathbb{H}$. The remainder $\mathbf{e}_1(r), \dots, \mathbf{e}_d(r)$ of the basis is obtained by paralell transport of $\mathbf{e}_1(0), \dots, \mathbf{e}_d(0)$ along the Brownian path $(\mathbf{e}_0(u))_{0\le u\le r}$.

The following elementary lemma, proved in [4], Section 2.2, gives an intuitive picture of the Θ-diffusions, for Θ depending only on $m \in \mathbb{M}$.

LEMMA 1. *Let $\gamma : [0, T] \to \mathbb{M}$ be a $\mathcal{C}^2$ timelike path parametrized by its proper time, and $\Gamma_0 \in \mathbb{OM}$ such that $\pi_1(\Gamma_0) = (\gamma(0), \dot\gamma(0)) \in T^1\mathbb{M}$. Then there exists a unique $\mathcal{C}^2$ path $(\Psi_s)_{0\le s\le T}$ in $\mathbb{OM}$, and some unique $\mathcal{C}^1$ real-valued controls $h^1, \dots, h^d$ defined on $[0, T]$, such that $\Psi_0 = \Gamma_0$, $\pi_1(\Psi_s) = (\gamma(s), \dot\gamma(s))$ and*

$$\dot\Psi_s = H_0(\Psi_s) + \sum_{j=1}^{d} V_j(\Psi_s) h^j(s).$$

So the Θ-diffusion is obtained in that case by replacing the deterministic controls of a typical $\mathcal{C}^2$ timelike path by Brownian controls with position dependent variance $\Theta(m_s)$.

On a manifold with nonpositive scalar curvature R, taking $\Theta(\Phi) = -\varrho^2 R$ (for a nonnull constant ϱ), one gets a dynamic which can be truly random only in nonempty parts of spacetime; it was called R-*diffusion* in [13]. Denote by $\mathbf{T}$ the energy-momentum tensor of the spacetime. Taking $\Theta(\Phi) = \varrho^2 \mathbf{T}(\mathbf{e}_0, \mathbf{e}_0)$, we get what was named the *energy diffusion* in [13]. See [3] for more general models of diffusions.

2.2.2. *Dynamics in $T^1\mathbb{M}$.* Denote by ∇^v the gradient on $T^1_m\mathbb{M}$, identified with the hyperbolic space $\mathbb{H}^d$ by means of the metric g_m, and by $\mathcal{L}_0$ the vector field generating the geodesic flow on $T^1\mathbb{M}$. Note that $T\pi_1(H_0) = \mathcal{L}_0$ and $T\pi_1(V_j) =:$

$\nabla_j^v = e_j^k \partial_{\dot m^k}$ (with Einstein summation convention). The projection on $T^1\mathbb{M}$ of the $\mathbb{OM}$-valued diffusion has the following $SO(d)$-invariant generator:

$$\mathcal{L}_\Theta = \mathcal{L}_0 + \tfrac{1}{2}\nabla^v(\Theta\nabla^v).$$

For a constant Θ the operator $\mathcal{L}_\Theta$ has the following expression in the local coordinates introduced in Section 2.1:

$$\mathcal{L}_0 + \frac{\Theta}{2}\Delta^v = \dot m^k \frac{\partial}{\partial m^k} + \left(\frac{d}{2}\Theta \dot m^k - \dot m^i \dot m^j \Gamma_{ij}^k(m)\right)\frac{\partial}{\partial \dot m^k} + \frac{\Theta}{2}\big(\dot m^k \dot m^\ell - g^{k\ell}(m)\big)\frac{\partial^2}{\partial \dot m^k\,\partial \dot m^\ell},$$

where Δ^v denotes the vertical Laplacian. We have, for a generic Θ,

$$\mathcal{L}_\Theta = \mathcal{L}_0 + \frac{\Theta}{2}\Delta^v + \frac{1}{2}\big(\dot m^k \dot m^\ell - g^{k\ell}(m)\big)\frac{\partial\Theta}{\partial \dot m^k}\frac{\partial}{\partial \dot m^\ell}. \tag{2.5}$$

The purpose of this work is to provide some conditions under which the Θ-diffusions have almost-surely an infinite lifetime ζ. In so far as we are mainly interested in the $T^1\mathbb{M}$-valued Θ-diffusions as models of physical phenomena, while we shall mainly be working with $\mathbb{OM}$-valued diffusions, it is reassuring to have the following fact, which essentially means that the possible explosion of $(\Phi_s)_{0\le s<\zeta}$ is never due to its $(\mathbf{e}_1,\dots,\mathbf{e}_d)$-part.

PROPOSITION 2. *The Θ-diffusion on $\mathbb{OM}$ and its $T^1\mathbb{M}$-projection have the same lifetime.*

PROOF. Write $\Phi_s = (m_s;(\dot m_s, e_1(s),\dots,e_d(s))) \in \mathbb{OM}$ and $\phi_s := \pi_1(\Phi_s) = (m_s,\dot m_s) \in T^1\mathbb{M}$. Using the local coordinates $(x^k, e_j^\ell)_{0\le k,\ell\le d;1\le j\le d}$, equation (2.3) defining the Θ-diffusion reads

$$\begin{aligned} d\dot m_s^k &= dM_s^k - \Gamma_{i\ell}^k(m_s)\dot m_s^i \dot m_s^\ell\,ds + \frac{d}{2}\Theta(\phi_s)\dot m_s^k\,ds \\ &\quad + \frac{1}{2}\big(\dot m_s^k \dot m_s^\ell - g^{k\ell}(m_s)\big)\frac{\partial\Theta}{\partial \dot m^\ell}(\phi_s)\,ds, \\ de_j^k(s) &= \sqrt{\Theta(\phi_s)}\dot m_s^k\,dw_s^j - \Gamma_{i\ell}^k(m_s)e_j^\ell(s)\dot m_s^i\,ds \\ &\quad + \frac{1}{2}\Theta(\phi_s)e_j^k(s)\,ds + \frac{1}{2}V_j\Theta(\phi_s)\dot m_s^k\,ds, \end{aligned}$$

with the martingale term $dM_s^k := \sqrt{\Theta(\phi_s)}e_j^k(s)\,dw_s^j$. (See Section 3.2 of [13] for the computation of the Itô correction.) Setting $e_0 = \dot m$ and $\eta^{in} := \eta_i^n := \mathbf{1}_{i=n=0} - \mathbf{1}_{1\le i=n\le d}$, and noticing that the matrix $(\eta^{in}e_n^k g_{k\ell})_{0\le i,\ell\le d}$ is the inverse of the matrix $(e^i_\ell)_{0\le i,\ell\le d}$, it follows from the above system that we have, for all $0 \le k \le d$, $1 \le j \le d$,

$$\begin{aligned} de^k_j(s) &= \dot m^k_s \eta^n_j e^q_n(s) g_{q\ell}(m_s)\, dM^\ell_s - \Gamma^k_{i\ell}(m_s) e^\ell_j(s) \dot m^i_s\, ds + \frac{1}{2}\Theta(\phi_s) e^k_j(s)\, ds \\ &\qquad + \frac{1}{2} V_j \Theta(\phi_s) \dot m^k_s\, ds \\ &= -e^\ell_j(s)\Gamma^k_{i\ell}(m_s)\dot m^i_s\, ds + \frac{1}{2} e^k_j(s)\Theta(\phi_s)\, ds + \frac{1}{2} V_j\Theta(\phi_s)\dot m^k_s\, ds \\ &\qquad - e^q_j(s)\dot m^k_s g_{q\ell}(m_s)\Big[d\dot m^\ell_s + \Gamma^\ell_{ip}(m_s)\dot m^i_s \dot m^p_s\, ds - \frac{d}{2}\Theta(\phi_s)\dot m^\ell_s\, ds \\ &\qquad\qquad - \frac{1}{2}[\dot m^p_s \dot m^\ell_s - g^{p\ell}(m_s)]\frac{\partial\Theta}{\partial \dot m^p}(\phi_s)\, ds\Big]. \end{aligned}$$

So the matrix $(e^k_j(s))_{0\le s<\zeta}$ and the frame-valued diffusion $(\Phi_s)_{0\le s<\zeta}$ satisfy a linear stochastic differential equation, conditionally on $(\phi_s)_{0\le s<\zeta}$. It is thus well defined up to the explosion time ζ of the $T^1\mathbb{M}$-valued Θ-diffusion. $\square$

This point being clarified, we shall work freely in the sequel with Θ-diffusions on $\mathbb{OM}$.

3. A first nonexplosion criterion. We give in this section a simple nonexplosion criterion, well suited to investigate the behavior of the Θ-diffusions in the largely used class of globally hyperbolic spacetimes. A Lyapounov function is introduced for this purpose, and leads to a nonexplosion criterion of a different nature than the typical Riemannian criteria mentioned in the Introduction.

The idea is roughly the following: if we can find a function $f = f(\Phi)$ which has compact level sets $\{f \le \lambda\}$, and does not increase along the trajectories, then the dynamics cannot explode. This was noted first by Khasminsky in a stochastic context; we state his observation here for the relativistic diffusions.

LEMMA 3 (Khasminsky). *If there exists a nonnegative function f on $\mathbb{OM}$ and a positive constant C such that $\mathcal{G}_\Theta f \le Cf$, and f goes to infinity along any timelike path leaving any compact in a finite time, then the Θ-diffusion has almost-surely an infinite lifetime.*

PROOF. The condition $\mathcal{G}_\Theta f \le Cf$ implies that the real-valued process $(e^{-Cs} f(\Phi_s))_{s<\zeta}$ is a nonnegative supermartingale. Denote by τ_n the (possibly infinite) exit time from the level set $\{f \le n\}$. By optional stopping, we have

$$f(\Phi_0) \ge \mathbb{E}[e^{-C\tau_n} f(\Phi_{\tau_n})] = n\mathbb{E}[e^{-C\tau_n}].$$

This implies that τ_n goes to infinity as n goes to infinity; as $\zeta = \lim_{n\to\infty}\tau_n$, this proves Khasminsky's statement. $\square$

As Θ-diffusions have no a priori reason not to explode, such a Lyapounov function will generally not exist. Yet, it is possible to construct such a function in some classes of spacetimes of interest for cosmology and theoretical physics. We give below two such examples. The construction of the function f uses the same recipe in both cases: if there exists an intrinsic distinguished future-directed timelike C^1 vector field $U \in T^1\mathbb{M}$, we can define

$$f(\Phi) := g_m(U_m, \dot{m}); \tag{3.1}$$

recall that $\pi_1(\Phi) = (m, \dot{m}) \in T^1\mathbb{M}$. For this choice of $f(\Phi)$, which is the hyperbolic angle between U and $\dot{m}$, we have $f \geq 1$, and

$$H_0 f(\Phi) = \nabla_{\dot{m}}(g(U, \dot{m})) = g(\nabla_{\dot{m}} U, \dot{m}). \tag{3.2}$$

The following lemma shows why f is a good choice to apply Khasminsky's criterion.

LEMMA 4. *We have on* $\mathbb{OM}$: $\frac{1}{2}\sum_{j=1}^d V_j(\Theta V_j f) = \frac{d}{2}\Theta f + \frac{1}{2}(f\dot{m}^k - U^k)\frac{\partial\Theta}{\partial\dot{m}^k}$.

PROOF. Choose local coordinates for which $U = \partial_{x^0}$, so $f(\Phi) = \dot{m}^0 = e_0^0$. Using (2.1), we have thus locally:

$$V_j f = \left(e_j^k \frac{\partial}{\partial e_0^k} + e_0^k \frac{\partial}{\partial e_j^k}\right) e_0^0 = e_j^0, \qquad V_j^2 f = e_0^0 = f$$

and

$$\sum_{j=1}^d (V_j\Theta)(V_j f) = \sum_{j=1}^d e_j^0 e_j^k \frac{\partial\Theta}{\partial\dot{m}^k} = (\dot{m}^0\dot{m}^k - g^{0k})\frac{\partial\Theta}{\partial\dot{m}^k} = (f\dot{m}^k - U^k)\frac{\partial\Theta}{\partial\dot{m}^k}.$$

□

It follows from (2.4) and (3.2) that

$$\mathcal{G}_\Theta f = g(\nabla_{\dot{m}} U, \dot{m}) + \frac{d}{2}\Theta f + \frac{1}{2}(f\dot{m}^k - U^k)\frac{\partial\Theta}{\partial\dot{m}^k}.$$

Khasminsky's criterion will thus guarantee the nonexplosion of the Θ-diffusion provided f explodes along exploding trajectories, and there exists a positive constant C such that

$$g(\nabla_{\dot{m}} U, \dot{m}) + \frac{1}{2}(f\dot{m}^k - U^k)\frac{\partial\Theta}{\partial\dot{m}^k} \leq \left(C - \frac{d}{2}\Theta\right) g(U, \dot{m}). \tag{3.3}$$

In order to turn this criterion into an effective tool, we first restrict ourselves to the following general class of spacetimes. This inequality become s particularly simple when Θ depends only on the base point $m \in \mathbb{M}$.

3.1. *Globally hyperbolic spacetimes.* This class of cosmological models is characterized by the existence of a global time function (i.e., a function $\tau : \mathbb{M} \to \mathbb{R}$, with timelike gradient) such that it has connected spacelike level sets $\{\tau = t\}$ of τ, and each integral curve of the vector field $\nabla\tau$ meets each level set of τ in exactly one point. Thus $\mathbb{M}$ is diffeomorphic to the product $I \times S$ of an interval I and a d-dimensional manifold S. Without loss of generality, we can suppose the interval I unbonded from above. With the example of Minkowski spacetime in mind, we see that a given spacetime may have an infinity of time functions; they are not intrinsically associated with the geometry.

Yet, we can take for vector field U in this setting the gradient of the time function $\tau : m = (t, x) \in I \times S \mapsto t$, so

$$f(\Phi) = g(U, \dot{m}) = \nabla_{\dot{m}}\tau = \dot{m}^0 = \dot{t} > 0.$$

There is no hope, though, to prove inequality (3.3) without specifying further the model, as the time function is not intrinsically defined. To proceed further, we shall look at the sub-class of *generalized warped product spacetimes*, in which the time function is supplied by the model and can be seen as an absolute time. These universes are globally hyperbolic spacetimes $\mathbb{M} = I \times S$ whose metric tensor has the form

$$g_m(\dot{m}, \dot{m}) = a_m^2 |\dot{m}^0|^2 - h_m(\dot{m}^S, \dot{m}^S), \tag{3.4}$$

where $\dot{m}^0$ is the image of $\dot{m} \in T_m^1\mathbb{M}$ by the differential of the first projection $I \times S \to I$ and $\dot{m}^S$ the image of $\dot{m}$ by the differential of the second projection $I \times S \to S$. Write $m = (t, x) \in I \times S$. The function a is a positive $\mathcal{C}^1$ function on $\mathbb{M}$, assumed to be bounded on any subset $I' \times S$ where I' is bounded from above and h_m is a positive-definite scalar product on T_xS, depending on m in a $\mathcal{C}^1$ way. This class of spacetimes contains all Robertson–Walker spacetimes (hence in particular de Sitter and Einstein–de Sitter spacetimes and the universal covering of the anti-de Sitter spacetime).

THEOREM 5. *Let* $(\mathbb{M}, g)$ *be a generalized warped product spacetime. If the function*

$$T^1\mathbb{M} \ni (m, \dot{m}) \longmapsto \nabla_{\dot{m}} \log a - \frac{d}{4}\Theta(m, \dot{m}) - \frac{1}{4}\left(\dot{m}^k \frac{\partial\Theta}{\partial\dot{m}^k} - \frac{1}{a^2(m)\dot{m}^0}\frac{\partial\Theta}{\partial\dot{m}^0}\right)$$

is bounded below, then the Θ*-diffusion almost-surely does not explode.*

PROOF. • We first check that if the Θ-diffusion has a finite lifetime ζ then $f(\Phi_s)$ explodes at time ζ^-. To that end, consider a timelike trajectory $\gamma = (m_s, \dot{m}_s)_{0 \le s < T} = ((t_s, x_s), \dot{m}_s)_{0 \le s < T}$ in $T^1\mathbb{M}$, defined on some semi-open interval $[0, T)$, and such that $\frac{d}{ds} m_s = \dot{m}_s$ and $f(\gamma_s) = \dot{t}_s$ is bounded above by some positive constant C. It follows that $t_0 \le t_s \le t_0 + CT$, and $h_{m_s}(\dot{x}_s, \dot{x}_s) \le C^2 a_{m_s}^2$ is bounded above by a constant since a is bounded above on $(\inf I, t_0 + CT] \times S$. This entails that $(x_s)_{0 \le s < T}$ cannot exit a bounded region of S, and so that γ must be trapped in a finite union of sets of the form $\overline{J^+(m_0)} \cap \overline{J^-(q_j)}$, for some $q_j \in \mathbb{M}$. Such a union of sets is compact in a hyperbolic spacetime (see, e.g., [18], Section 6.6), γ is trapped in a compact set. Would γ explode, it would have a cluster point at which the strong causality would fail, leading to a contradiction as globally hyperbolic spacetimes are strongly causal ([18], Section 6.6).

• The condition of the theorem is a rephrasing of the local condition (3.3). To see that, let us work in a neighborhood $\mathcal{V} = [t_1, t_2] \times V$ of a given point m_0, and choose coordinates x^j on V; this provides coordinates (t, x^i) on $\mathcal{V}$, which induce coordinates on $T^1\mathcal{V}$: for $m \in \mathcal{V}$ and $\dot{m} \in T^1_m\mathbb{M}$, write $\dot{m} = \dot{m}^0\,\partial_t + \sum_{1 \le j \le d} \dot{m}^j\,\partial_{x^j}$.

Note first that since $U = a^{-2}\partial_t$, we have

$$\nabla_{\dot{m}} U = \nabla_{\dot{m}}(a^{-2})\,\partial_t + a^{-2}\nabla_{\dot{m}}\,\partial_t.$$

Using Christoffel's symbols Γ^i_{jk} we have

$$(\nabla_{\dot{m}}\,\partial_t)^\alpha = \nabla_{\dot{m}}(a^{-2})\delta^\alpha_0 + a^{-2}\dot{m}^c\Gamma^\alpha_{c0},$$

for $\alpha \in \{0, \dots, d\}$ and a summation over c in $\{0, \dots, d\}$; so

$$H_0 f = g(\nabla_{\dot{m}} U, \dot{m}) = \nabla_{\dot{m}}(\log a^{-2})\dot{m}^0 + a^{-2}\dot{m}^c\Gamma^\alpha_{c0} g_{\alpha\beta}\dot{m}^\beta.$$

The explicit formulas for the Christoffel symbols, in terms of the metric, are

$$\Gamma^0_{00} = \partial_t(\log a), \qquad \Gamma^0_{k0} = \partial_{x^k}(\log a),$$
$$\Gamma^i_{00} = \tfrac{1}{2}h^{i\ell}\,\partial_{x^\ell}(a^2), \qquad \Gamma^i_{k0} = \tfrac{1}{2}h^{i\ell}\,\partial_t h_{\ell k},$$

for $i, k \in \{1, \dots, d\}$ and a sommation over $1 \le \ell \le d$. We thus have, after simplifications,

$$\begin{aligned} H_0 f &= -2\nabla_{\dot{m}}(\log a)\dot{m}^0 + |\dot{m}^0|^2\,\partial_t(\log a) - \frac{a^{-2}}{2}\dot{m}^k\,\partial_t(h_{\ell k})\dot{m}^\ell \\ &= -|\dot{m}^0|^2\,\partial_t \log a - 2\dot{m}^0\dot{m}^k\,\partial_{x^k}\log a - \frac{a^{-2}}{2}\dot{m}^k\,\partial_t(h_{\ell k})\dot{m}^\ell. \end{aligned}$$

Using the unit pseudo-norm relation $a_m^2|\dot{m}^0|^2 - h_{\ell k}(m)\dot{m}^k\dot{m}^\ell = 1$, the above equality becomes

$$H_0 f = -|\dot{m}^0|^2\,\partial_t \log a - 2\dot{m}^0\dot{m}^k\,\partial_{x^k}\log a - \frac{a^{-2}}{2}|\dot{m}^0|^2\,\partial_t(a^2),$$

that is, $H_0 f = -2\dot{m}^0\nabla_{\dot{m}}\log a$. The statement of the theorem follows from (3.3). □

This result takes a particularly simple form in the case where Θ depends only on the base point m, as is the case of the R-diffusion.

COROLLARY 6. *Let $\mathbb{M} = I \times S$ denote a generalized warped product spacetime and Θ be a bounded nonnegative function on $\mathbb{M}$. Then the Θ-diffusion does not explode if ∇a is everywhere nonspacelike and future-directed.*

PROOF. The condition of Theorem 5 reads, in that case, "$T^1\mathbb{M} \ni (m, \dot{m}) \mapsto \nabla_{\dot{m}} \log a$ *is bounded below*." To rephrase this condition into the more synthetic condition of the statement, let us work in local coordinates, (t, x) and $(\dot{t}, \dot{x})$ for m and $\dot{m}$, respectively.

We have $\dot{t} = a^{-1} \operatorname{ch} r$ and $\dot{x} = (\operatorname{sh} r)\sigma$, for some $r \in \mathbb{R}$ and $\sigma \in T_x S$ with $|\sigma|_{h(m)} = 1$.

Define $u := \partial_t \log a$ and $v := \partial_x \log a \in T_x S \equiv \mathbb{R}^d$. Then the condition of Theorem 5 reads, "$ua^{-1} \operatorname{ch} r - (v_i \sigma^i) \operatorname{sh} r \geq C$," for any r and σ. Letting $r \to \pm\infty$, gives $a^{-1}u \geq |v_i \sigma^i| \geq 0$. As the constant C can be taken negative without loss of generality, the reciprocal is clear. Now, since $\max_{|\sigma|_{h(m)}=1} |v_i \sigma^i| = |v|_{h^{-1}(m)}$, the condition reads, "$a^{-1}u \geq |v|_{h^{-1}(m)}$." Finally, as $\nabla = (a^{-2}\,\partial_t, -h^{ij}\,\partial_{x^j})$, the vector $\nabla \log a = (a^{-2}u, -h^{ij} v_j)$ has pseudo-norm $g(\nabla \log a, \nabla \log a) = a^{-2}u^2 - |v|^2_{h^{-1}(m)} \geq 0$. $\square$

This criterion applies in particular to Θ-diffusions in Robertson–Walker spacetimes, recovering the results of Angst [1], who proceeded by direct analysis of the stochastic differential equations of the dynamics.

3.2. *Perfect fluids.* Our second class of examples, where to apply Lyapounov's method to prove nonexplosion, will be the set of spacetimes with normal matter whose energy-momentum tensor $\mathbf{T}$ is that of a perfect fluid. They are characterized by the datum of a timelike vector field U, the four velocity of the fluid and two functions ρ and p on $\mathbb{M}$, respectively, the energy density and pressure of the fluid. See [7, 18]. We have then $\mathbf{T} = \rho U \otimes U + p(g + U \otimes U)$, or in local coordinates,

$$\mathbf{T}_{ij} = (\rho + p) U_i U_j + p g_{ij}.$$

Such a spacetime is said to be of *perfect fluid type*. Notice that contrarily to the globally hyperbolic spacetimes, no topological assumption is made on a perfect fluid type spacetime.

Gödel's universe is such a spacetime. This is the manifold $\mathbb{R}^4$ with the metric $ds^2 = dt^2 - dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 - dz^2 - 2e^{\sqrt{2}\omega x}\, dt\, dy$, where $\omega > 0$ is a constant. It is a solution to Einstein's equation with cosmological constant ω^2 and represents a pressure-free perfect fluid. It has energy-momentum tensor $\mathbf{T} = U \otimes U$, where $(U_j) = (\sqrt{2}\omega, 0, \sqrt{2}\omega e^{\sqrt{2}\omega x}, 0)$ represents the four-velocity covector of the matter, and ω is the vorticity of this field. This spacetime has constant scalar curvature $2\omega^2$. See Section 2.4 in [11]. As above, the function f is defined by formula (3.1) and can be used as a Lyapounov function under some conditions. The computations made in Section 3.1 work equally well in that setting and lead to the following results.

PROPOSITION 7. *Let* $(\mathbb{M}, g)$ *be a Lorentzian manifold of perfect fluid type and* f *be defined by formula* (3.1). *Suppose* f *goes almost-surely to infinity along any exploding timelike path. If there exists a constant* C *such that*

$$H_0 f + \frac{d}{2}\Theta f + \frac{1}{2}(f\dot{m}^k - U^k)\frac{\partial\Theta}{\partial\dot{m}^k} \leq Cf,$$

then the Θ-*diffusion has almost-surely an infinite lifetime*.

In the particular case of Gödel's universe, the gradient ∇U of the velocity vanishes (since $U^i = \delta_0^i$), so that $H_0 f = 0$, by formula (3.2); and f is the square root of the energy.

COROLLARY 8. *Let us work in Gödel's universe and suppose that* $3\Theta + (\dot{m}^k \frac{\partial\Theta}{\partial\dot{m}^k} - \frac{1}{f}\frac{\partial\Theta}{\partial\dot{m}^0})$ *is bounded above in* $T^1\mathbb{M}$. *Then the* Θ-*diffusion has almost-surely an infinite lifetime. This condition holds in particular if* $\Theta(\Phi) = \Theta(m)$ *depends only on the base point and is bounded, as this is the case for the basic relativistic diffusion and the* R-*diffusion in Gödel's universe*.

Note that this criterion does not apply to the energy diffusion in Gödel's universe. Indeed one can see in that case (see Section 2.4 of [11]) that the above quantity is equal to $5\Theta - 4\omega^2$ and that the energy Θ is unbounded along the trajectories of the energy diffusion.

REMARK 9. In Einstein–de Sitter spacetime the energy diffusion explodes with positive probability, as proved in Proposition 5.4.2 of [13]. (This Robertson–Walker universe is both a warped product and a perfect fluid type spacetime.) Consult [4] for a first study of stochastic incompleteness for relativistic diffusions.

4. b-completeness. The study of dynamics in the orthonormal frame bundle is not new in general relativity, and essentially dates back to Cartan's moving frame method. However, Schmidt [24] was the first to notice that the geometry of $\mathbb{OM}$ itself may be used to provide a conceptual framework in which studying the nature of spacetime singularities. For that purpose, he introduced on the parallelizable manifold $\mathbb{OM}$ a Riemannian metric, turning $\{H_0, \dots, H_d, (V_{ij})_{0\leq i<j\leq d}\}$ into a Riemannian orthonormal basis, and called it the bundle metric, or *b-metric*. The completeness of this metric structure on $\mathbb{OM}$ can essentially be phrased in terms of $\mathbb{M}$-valued paths. To state that fact, recall that one can associate to any $\mathbb{M}$-valued $\mathcal{C}^1$ path $\gamma : [0, T[\to \mathbb{M}$ and $\mathbf{e} \in \mathbb{O}_{\gamma_0}\mathbb{M}$ a unique horizontal lift $\gamma^\uparrow : [0, T) \to \mathbb{OM}$ of γ, starting from $(\gamma_0, \mathbf{e})$, and charactarized by the properties

$$\frac{d}{ds}\gamma_s^\uparrow \in \operatorname{span}(H_0, \dots, H_d) \quad \text{and} \quad \pi_0(\gamma_s^\uparrow) = \gamma_s \qquad \text{for all } s \in [0, T).$$

The $S_\mathbf{e}$-*length* of γ is defined as the Riemannian length of its horizontal lift $\gamma^\uparrow$; it depends on $\mathbf{e} \in \mathbb{O}_{\gamma_0}\mathbb{M}$. In other words, given $\mathbf{e} \in \mathbb{O}_{\gamma_0}\mathbb{M}$, seen as orthonormal in the Euclidean sense, the $S_{\mathbf{e}}$-length of the $\mathbb{M}$-valued $\mathcal{C}^1$ path γ is the Euclidean length of its anti-development in $(T_{\gamma_0}\mathbb{M}, \mathbf{e})$. Although this length depends on $\mathbf{e}$, its finiteness is independent of it; we can thus talk of *finite S-length* of a $\mathcal{C}^1$ path without mentioning the frame $\mathbf{e}$. Note that in a Riemannian setting the $S_{\mathbf{e}}$-lenth of a $\mathcal{C}^1$ path is its usual Riemannian length.

THEOREM 10 (Schmidt [24]). *$\mathbb{OM}$ is complete for the above b-metric if and only if any $\mathcal{C}^1$ path $\gamma : [0, T) \to \mathbb{M}$ with a bounded S-length converges in $\mathbb{M}$ at time T^-.*

The above completeness hypothesis is usually called *b-completeness*. The Riemannian version of this statement is trivial as the orthonormal frame bundle with its b-metric is complete iff the Riemannian manifold is complete. The Lorentzian situation is more involved as there exists (timelike, spacelike and lightlike) complete Lorentzian manifolds $\mathbb{M}$ which have an incomplete path of bounded acceleration, so $\mathbb{OM}$ is not b-complete (see, e.g., [14] and [6]). The noncompactness of $SO_0(1, d)$ lies at the core of this phenomenon.

However, the Riemannian view of a Lorentzian manifold provided by Schmidt's metric offers a bridge to investigate some features of the latter using the tools of Riemannian geometry, as the following proposition shows.

PROPOSITION 11. *Let Θ be a bounded function on $\mathbb{M}$. Then the Θ-diffusion does not explode if $\mathbb{OM}$ is b-complete.*

One should not be confused about that statement. It does not mean that the Riemannian completeness of $\mathbb{OM}$ implies the completeness of its Brownian trajectories, which is false. One cannot assign an $S_{\mathbf{e}}$-length to a Brownian path in $\mathbb{OM}$ as it is not regular enough.

PROOF OF PROPOSITION 11. • Given a horizontal $\mathcal{C}^1$-path $(\rho_s)_{0\le s<T}$ in $\mathbb{OM}$, write γ for its projection $\pi_0 \circ \rho$ in $\mathbb{M}$, so $\rho = \gamma^{\uparrow}$. For $0 \le s < T$, denote by $\tau^{\gamma}_{0\to s}$ the parallel transport operator along the curve $(\gamma_r)_{0\le r\le s}$, with inverse $\tau^{\gamma}_{0\leftarrow s}$. Also, denote by $(p_s)_{0\le s<T}$ the anti-development of γ: this $T_{\gamma_0}\mathbb{M}$-valued $\mathcal{C}^1$-path is defined for all $s \in [0, T[$ by the formula $p_s = \int_0^s \tau^{\gamma}_{0\leftarrow r}\dot{\gamma}_r \, dr$. Last, we shall denote by $\dot{p}^j_r$ the coordinates of $\dot{p}_r$ in the frame ρ_0, and by $\|.\|_{\rho_s}$ the Euclidean norm in $(T_{\gamma_s}\mathbb{M}, \rho_s)$. We have by construction $d\rho_s = \sum_{0\le j\le d} H_j(\rho_s)\dot{p}^j_s \, ds$ and $\dot{\gamma}_s = \tau^{\gamma}_{0\to s}\dot{p}_s$, as well as the identity $\|\dot{\gamma}_s\|^2_{\rho_s} = \|\dot{p}_s\|^2_{\rho_0} = \sum_{0\le j\le d}(\dot{p}^j_s)^2$. The b-completeness assumption means that γ has a limit γ_T in $\mathbb{M}$ at time T if

$$\int_0^{T-} \|\dot{p}_s\|_{\rho_0} \, ds < \infty. \tag{4.1}$$

• The basic relativistic diffusion $(m_s, \mathbf{e}_s)_{0\leq s<\zeta}$ is by construction the development in $\mathbb{M}$ of the relativistic Dudley diffusion in Minkowski spacetime, identified with $T_{m_0}\mathbb{M}$ (see Theorem 3.2 in [12]). As trajectories of the latter over a time a bounded time interval have almost-surely a finite length in the Eulidean norm associated with any frame of $\mathbb{R}^{1,d}$, the b-completeness of $\mathbb{OM}$ ensures the nonexplosion of the basic relativistic diffusion.

• For a generic Θ-diffusion, formula (2.5) implies the existence for each $s \in [0, \zeta[$ of an orthonormal basis $(\varphi_1(s), \dots, \varphi_d(s))$ of $\dot{p}_s^{\perp}$ in $\mathbb{R}^{1,d}$ such that one has

$$d\dot{p}_s^k = \sum_{j=1}^{d} \sqrt{\Theta(m_s)}\varphi_j^k(s)\, dw_s^j + \frac{d}{2}\Theta(m_s)\dot{p}_s^k\, ds$$

for some d-dimensional Brownian motion w. We have used the fact that Θ depends only on m to simplify the general expression. The path $(p_s, \dot{p}_s)_{0\leq s<\zeta}$ appears then as a time change of Dudley's diffusion, by means of the map $s \mapsto \inf\{u \mid \int_0^u \Theta(m_r)\, dr > s\}$. The result follows for a bounded function Θ. □

This result can be improved in two ways: by relaxing the boundedness hypothesis on Θ and by relaxing the geometric completeness assumption. The next section explains how this can be done in a sub-Riemannian framework by using ideas from the theory of reversible Markov processes.

5. A volume growth nonexplosion criterion. We prove in this section a nonexplosion criterion involving only the volume growth of some sub-Riemannian boxes in $\mathbb{OM}$ and the function Θ, as described in Theorem 13 below. This result is proved in Section 5.4 following Takeda's method, as improved recently by Hsu and Qin in [20]. Yet, there is a real difficulty in doing this, as we are working with a nonsymmetric, hypoelliptic diffusion, and on a principal bundle with noncompact fibers. To overcome these difficulties, we introduce a sub-Riemannian structure on $\mathbb{OM}$, well adapted to our setting, and which will somehow play for us the role of the missing Lorentzian distance.

5.1. *Sub-Riemannian framework and main results.*

5.1.1. *Sub-Riemannian distance function.* We have seen in Section 4 that the completeness of the natural Riemannian metric of the parallelizable manifold $\mathbb{OM}$ implies the stochastic completeness of all the Θ-diffusions with a bounded Θ. One can significantly improve that conclusion by working with the sub-Riemannian structure on $\mathbb{OM}$ induced by the field of $(d+1)$-planes generated by the vector fields $H_0, V_1, \dots, V_d$. In that setting, one can assign a length only to $\mathcal{C}^1$ paths $\rho : [0, T] \to \mathbb{OM}$ whose tangent vector belong at any time s to the vector space spanned by $H_0, V_1, \dots, V_d$ in $T_{\rho_s}\mathbb{OM}$, say $\dot{\rho}_s = \dot{\rho}_s^0 H_0(\rho_s) + \dot{\rho}_s^1 V_1(\rho_s) + \dots + \dot{\rho}_s^d V_d(\rho_s)$. Such a path is said to be *admissible*; its length is then defined as

$\int_0^T (\sum_{i=0}^d (\dot\rho_s^i)^2)^{1/2}\,ds$. The sub-Riemannian distance between two points of $\mathbb{OM}$ is defined as the infimum of the length of the admissible paths joining these two points, with the convention $\inf\emptyset = +\infty$. Chow's theorem [8] ensures that the sub-Riemannian distance function $\mathcal{D}(\cdot,\cdot)$ is finite and continuous in its two arguments if (see, e.g., [22]) the Lie algebra generated by $H_0, V_1, \dots, V_d$ has full dimension, which holds here. Fix a reference point $\Phi_{\mathrm{ref}} \in \mathbb{OM}$.

(H) COMPLETENESS HYPOTHESIS. *The closed boxes $B_\lambda := \{\mathcal{D}(\Phi_{\mathrm{ref}},\cdot) \le \lambda\}$ are compact for any $\lambda > 0$.*

This completeness hypothesis rules out the pathological examples of Geroch [14] and Beem [6]; it does not depend on the arbitrary choice of Φ_{ref}. Unlike its Riemannian analog, the sub-Riemannian distance function $\mathcal{D}(\Phi_{\mathrm{ref}},\cdot)$ is not smooth in any neighborhood of Φ_{ref}, [22]; however, it is a viscosity solution of the equation

$$|H_0\mathcal{D}|^2 + |V_1\mathcal{D}|^2 + \cdots + |V_d\mathcal{D}|^2 = 1$$

on $\mathbb{OM} \setminus \{\Phi_{\mathrm{ref}}\}$ (see, e.g., Theorem 2 in [9]; we do not use that fact in the sequel). We shall use that quantitative information in Section 5.4 under the classical form given in the following proposition.

PROPOSITION 12. *Fix $\lambda > 0$. One can associate to any positive constant η a smooth function $F : \mathbb{OM} \to \mathbb{R}_+$ such that*

$$\max_{\Phi \in B_\lambda} |F(\Phi) - \mathcal{D}(\Phi_{\mathrm{ref}}, \Phi)| \le \eta$$

and we have on B_λ

$$|H_0F|^2 + |V_1F|^2 + \cdots + |V_dF|^2 \le 2.$$

PROOF. Let us introduce the Riemannian metric g_ε on $\mathbb{OM}$ for which $H_0, H_1, \dots, H_d$ and the $(V_{ij})_{0\le i<j\le d}$ are *orthogonal*, with H_0 and the $V_{0j} (= V_j)$ of norm 1 and the other vectors of norm ε^{-1}. Denote by $\mathcal{D}_\varepsilon(\cdot) = \mathcal{D}_\varepsilon(\Phi_{\mathrm{ref}},\cdot)$ the distance function associated with g_ε. It is a 1-Lipschitz-continuous function (with respect to the distance function $\mathcal{D}_\varepsilon$) which is differentiable almost-everywhere, by Rademacher's theorem, and has a gradient of norm 1 almost-everywhere.

$$\begin{aligned}(5.1)\qquad &|H_0\mathcal{D}_\varepsilon|^2 + |V_1\mathcal{D}_\varepsilon|^2 + \cdots + |V_d\mathcal{D}_\varepsilon|^2 \\ &\quad + \varepsilon^{-2}\left(\sum_{i=1}^d |H_i\mathcal{D}_\varepsilon|^2 + \sum_{1\le i<j\le d} |V_{ij}\mathcal{D}_\varepsilon|^2\right) = 1.\end{aligned}$$

(Indeed, the set of conjugate points to Φ_0 in B_λ is closed and has null measure. In the complementary, relatively open, set the distance is attained along a unique geodesic whose unit tangent vector at the final point is the gradient of the distance function to Φ_0.) The function $\mathcal{D}_\varepsilon$ is easily seen to converge uniformly to $\mathcal{D}(\Phi_{\mathrm{ref}}, \cdot)$ on the compact box B_λ (this is where we need these boxes to be compact); see, for example, Sections 0.8.A and 1.4.D of Gromov's article [17]. As we have almost-everywhere

$$|H_0\mathcal{D}_\varepsilon|^2 + |V_1\mathcal{D}_\varepsilon|^2 + \cdots + |V_d\mathcal{D}_\varepsilon|^2 \le 1,$$

by (5.1), a standard regularization procedure yields the conclusion. □

5.1.2. *Main results.* We use the natural volume measure on $\mathbb{OM}$ associated with the Lorentzian structure. It is defined by the formula

$$\mathrm{Vol}(d\Phi) = \mathrm{Vol}_{\mathbb{M}}(dm) \otimes \mathrm{Vol}_m(d\mathbf{e}), \qquad \Phi = (m, \mathbf{e}),$$

where $\mathrm{Vol}_{\mathbb{M}}(dm)$ is the Lorentzian volume measure and $\mathrm{Vol}_m(d\mathbf{e})$ is the image of a given Haar measure on $SO_0(1,d)$ by the identification of the fiber $\pi_0^{-1}(m)$ with $SO_0(1,d)$; see, for example, [18], Section 2.8, for the Lorentzian volume measure. The volume measure Vol on $\mathbb{OM}$ is uniquely defined up to a multiplicative constant. In order to avoid some unpleasant pathologies, we shall make the following rather mild assumption on the causal structure of spacetime.

HYPOTHESIS. $(\mathbb{M}, g)$ *is strongly causal.*

It means that any point of $\mathbb{M}$ has arbitrarily small neighborhoods which no nonspacelike path intersects more than once; see [18], page 192, or [7].

THEOREM 13. *Let* $(\mathbb{M}, g)$ *be a strongly causal Lorentzian manifold satisfying the Completeness hypothesis* (H). *Set* $\Theta_r := \sup_{\Phi\in B_r}\Theta(\Phi)$, *for any* $r > 0$, *and suppose*

$$\int^\infty \frac{r\,dr}{\Theta_r \log(\Theta_r \mathrm{Vol}(B_r))} = \infty. \tag{5.2}$$

Then the Θ*-diffusion has almost-surely an infinite lifetime, from any starting point.*

Condition (5.2) has the form of the classical nonexplosion condition for Brownian motion, $\int^\infty \frac{r\,dr}{\log \mathrm{Vol}(B_r)} = \infty$, first proved by Grigor'yan [16] and has precisely that form for Θ bounded. Note that no topological assumption on $\mathbb{M}$ is needed, contrary to the results of Section 3.1. One can give a quantitative version of the above theorem by providing an upper rate function.

COROLLARY 14. *Let* $\mathbb{M}$ *be a strongly causal Lorentzian manifold satisfying the Completeness hypothesis* (H). *Set* $h(\rho) \equiv \rho$ *if* $\Theta \equiv 0$; *otherwise, pick a constant* R_0 *such that* $\Theta_{R_0} > 0$ *and set for* $\rho > 0$

$$h(\rho) := \inf\left\{R > R_0 \,\middle|\, \int_{R_0}^R \frac{r\,dr}{\Theta_r \log[\Theta_r \mathrm{Vol}(B_r)]} > \rho\right\}.$$

Then, given any $\Phi_0 \in \mathbb{OM}$, *there exist* $R_0 > 0$ *and a positive constant* C *such that we have* $\mathbb{P}_{\Phi_0}$*-almost-surely*

$$\mathcal{D}(\Phi_0, \Phi_s) \le Ch(Cs).$$

We prove Theorem 13 following Takeda's method, explained in the next section. To adapt it to our setting, we shall introduce in Section 5.3 a modified Θ-diffusion on some compact space; it is used crucially in the proof of Theorem 13 given in Section 5.4.

5.2. *Takeda's method.*

5.2.1. *The main ingredients.* Using an idea of Lyons and Zheng [21], Takeda devised [25, 26] a remarkably simple and sharp nonexplosion criterion for Brownian motion on a Riemannian manifold $\mathbb{V}$. Loosely speaking, his reasoning works as follows. Suppose we have a diffusion $(x_s)_{s\ge 0}$ on $\mathbb{V}$ which is symmetric (with respect to the Riemannian volume measure VOL, say) and conservative; denote by L its generator, and let f be a sufficiently smooth function. Denote by $\mathbb{P}_{\mathrm{VOL}}$ the measure $\int \mathbb{P}_x \mathrm{VOL}(dx)$ on the path space, where $\mathbb{P}_x$ is the law of the diffusion started from x. Fix a time $T > 0$. As the reversed process $(x_{T-s})_{0\le s\le T}$ is an L-diffusion under $\mathbb{P}_{\mathrm{VOL}}$, applying Itô's formula to both $f(x_s)$ and $f(x_{T-s})$ provides two martingales M and $\widetilde{M}$ [with respect to the two different filtrations $\sigma(x_s; 0 \le s \le T)$ and $\sigma(x_{T-s}; 0 \le s \le T)$, resp.] such that

$$f(x_s) = f(x_0) + M_s + \int_0^s Lf(x_r)\,dr,$$

$$f(x_s) = f\bigl(x_{T-(T-s)}\bigr) = f(x_T) + \widetilde{M}_{T-s} + \int_0^{T-s} Lf(x_{T-r})\,dr.$$

It follows that $f(x_s) = \frac{f(x_0)+f(x_T)}{2} + \frac{M_s+\widetilde{M}_{T-s}}{2} + \int_0^T Lf(x_s)\,dr$, and consequently,

$$f(x_T) - f(x_0) = \tfrac{1}{2}(M_T - \widetilde{M}_T).$$

If $\frac{d\langle M\rangle_s}{ds}$ and $\frac{d\langle \widetilde{M}\rangle_s}{ds}$ are bounded above, by 1 say, the previous identity provides a control of $(f(x_T) - f(x_0))$ by the supremum of the absolute value of a Brownian motion over the time interval $[0, T]$.

Back to the nonexplosion problem for Brownian motion on $\mathbb{V}$, fix a point $m \in \mathbb{V}$ and a radius $R > 1$, and consider the Brownian motion $(x_s)_{s\ge 0}$ reflected on the boundary of the Riemannian ball $B(m; R)$, started under its invariant measure $\mathbf{1}_{B(m;R)}\mathrm{VOL}$. It is a symmetric conservative diffusion; denote by $\overline{\mathbb{P}}_{B(m;R)}$ its law. Using the Dirichlet forms approach to symmetric diffusions one can apply the above reasoning to the (nonsmooth, but 1-Lipschitz) Riemannian distance function $d(m, \cdot)$, which gives the estimate

$$\overline{\mathbb{P}}_{B(m;R)}\Bigl(x_0 \in B(m; 1), \sup_{s\le T} d(m, x_s) = R\Bigr) \le \mathrm{VOL}(B(m; R)) \times 2\mathbb{P}\Bigl(\sup_{s\le T} |B_s| > R\Bigr).$$

But as the Brownian motion on $\mathbb{V}$ behaves in the ball $B(m; R)$ as the Brownian motion reflected on the boundary of $B(m; R)$, the above inequality also gives an upper bound for the probability that the Brownian motion on $\mathbb{V}$, started uniformly from $B(m; 1)$, exits the ball $B(m; R)$ before time T. Combining this estimate with the Borel–Cantelli lemma, Takeda proved that the Brownian motion on $\mathbb{V}$ is conservative provided

$$\liminf_{R\to\infty} R^{-2} \log \mathrm{VOL}(B(m; R)) < \infty,$$

re-proving in a simple way a criterion due to Karp and Li. Takeda's method has been refined by several authors, culminating with Hsu and Qin's recent work [20], in which they give an elegant and simple proof of a sharp nonexplosion criterion, due to Grigor'yan [16], for Brownian motion on a Riemannian manifold in terms of volume growth, as well as an escape rate function. We shall follow their method to deal with relativistic diffusions.

5.2.2. *The difficulties.* The main difficulty in implementing this approach is in finding what can play the role of the pair "*Riemannian distance function–reflected Brownian motion*" in our Lorentzian, hypoelliptic framework. We describe in the remainder of this section a nonstandard reflection mechanism for a Brownian motion in a Riemannian manifold which will serve us as a guide in the construction of the Θ-diffusion reflected on the boundary of the sub-Riemannian boxes, as described in Section 5.3.

Brownian motion reflected on the boundary of a ball $B(m; R)$ is the simplest diffusion process which coincides with Brownian motion on the ball $B(m; R)$ and has a state space with finite volume. One cannot take a smaller state space if the former property is to be satisfied. Yet, one can make different choices if one is ready to loose the minimality property. To explain that fact, let us suppose that $(\mathbb{V}, g)$ is a Cartan–Hadamard manifold. Given a point $m \in \mathbb{V}$ let us use the exponential map $\exp_m$ at m as a global chart on $\mathbb{V}$; this identifies the geodesic ball $B(m; R)$ on M to the (Euclidean-shaped) ball $B'(0; R)$ in $T_m\mathbb{V}$. Given $\varepsilon > 0$, let us modify the metric on $B'(0; R + \varepsilon) \setminus B'(0; R)$ so as to interpolate smoothly between $\exp_m^* g$ on $B'(0; R)$ and the constant metric g_m outside $B'(0; R + \varepsilon)$ (primed balls refer to the pull-back metric $\exp_m^* g$). Denote by $\widetilde{g}$ the restriction to $B'(0; R + 2\varepsilon)$ of this modified metric, and define the compact space $\mathbf{K}$ as the quotient of the closed ball $\overline{B}'(0; R + 2\varepsilon)$ by the identification of $m' \in \partial\overline{B}'(0; R + 2\varepsilon)$ and $-m'$. Then the $\widetilde{g}$-Brownian motion on $\mathbf{K}$ coincides with the $\exp_m^* g$-Brownian motion on $B'(0; R)$ and has a state space with finite $\widetilde{g}$-volume $\mathrm{VOL}_{\widetilde{g}}(\mathbf{K}) = (1 + o(\varepsilon))\mathrm{VOL}_g(B(m; R))$. The construction of a modified Θ-diffusion given in Section 5.3 will be reminiscent of the preceding nonstandard reflected Brownian motion.

5.3. *A modified process.* We start our construction of the "reflected" Θ-diffusion by constructing the compact space on which it is going to live. Fix for that purpose a reference point $\Phi_{\mathrm{ref}} \in \mathbb{OM}$, the center of the boxes B_λ, and set $\mathcal{D}(\Phi) = \mathcal{D}(\Phi_{\mathrm{ref}}, \Phi)$ for all $\Phi \in \mathbb{OM}$. Fix also two positive constants λ and ε and consider the relatively compact open region

$$\mathcal{U} := \{\lambda < \mathcal{D} < \lambda + \varepsilon\} = B_{\lambda+\varepsilon} \setminus \overline{B_\lambda}.$$

LEMMA 15. *There exists in $\mathcal{U}$ a smooth hypersurface V of $\mathbb{OM}$ separating ∂B_λ from $\partial B_{\lambda+\varepsilon}$ such that the subset $V_0 := \{\Phi \in V | H_0(\Phi) \in T_\Phi V\}$ is a smooth hypersurface of V.*

The separation property means that $\partial B_\lambda \cup \partial B_{\lambda+\varepsilon}$ does not intersect V but any continuous path from ∂B_λ to $\partial B_{\lambda+\varepsilon}$ hits V. We thank A. Oancea and P. Pansu for their help in proving this statement.

PROOF OF LEMMA 15. Let us use the function F of Proposition 12, with $\eta < \varepsilon/4$ and $R > \lambda + \varepsilon$, and fix some constants $\eta < \varepsilon_1 < \varepsilon_2 < \varepsilon/2 - \eta$ such that $B_\lambda \subset \{\varepsilon_1 \le F - \lambda \le \varepsilon_2\} \subset B_{\lambda+\varepsilon/2}$. The set of regular values of $(F - \lambda)$ is dense in the interval $(\varepsilon_1, \varepsilon_2)$, by Sard's theorem. Fix a regular value $c \in (\varepsilon_1, \varepsilon_2)$, so the level set $S := \{F = c\}$ is a smooth hypersurface separating ∂B_λ from $\partial B_{\lambda+\varepsilon/2}$.

We shall now be working in $\mathcal{U}' \equiv S \times [0, \frac{\varepsilon}{2})$, where we are going to construct the separating hypersurface V as the graph of some function $f : S \to [0, \frac{\varepsilon}{2})$, resorting to the transversality lemma. Denote by $\mathrm{Gr}(T\mathcal{U}')$ the Grassmannian bundle over $\mathcal{U}'$ made up of all the hyperplanes of $T\mathcal{U}'$, and associate to any function $f : S \to (0, \frac{\varepsilon}{2})$ the function $G_f : S \to \mathrm{Gr}(T\mathcal{U}')$ defined by $G_f(m) := \{(\sigma, df_m(\sigma)) | \sigma \in T_m S\}$. Let $\mathcal{H}$ denote the smooth hypersurface of $\mathrm{Gr}(T\mathcal{U}')$, made up of all hyperplanes containing H_0. Then $G_f^{-1}(\mathcal{H})$ is a smooth hypersurface of $\mathrm{Graph}(f)$ as soon as G_f is transverse to $\mathcal{H}$. Therefore the statement reduces to finding a function f such that G_f be transverse to $\mathcal{H}$.

Consider for that purpose a smooth partition of unity: $\mathbf{1}_S = \sum_{j=1}^k \alpha_j$, with $\{\alpha_j > 0\} = \psi_j(\mathcal{B}^\nu)$ diffeomorphic under ψ_j to the unit ball $\mathcal{B}^\nu \subset \mathbb{R}^\nu$ [with $\nu = \dim(\mathbb{OM}) - 1 = (d+3)d/2$]. Denoting by $\mathcal{A}$ the space of (the restictions to $\mathcal{B}^\nu$ of) affine functions on $\mathbb{R}^\nu$, consider the map $F : \mathcal{A}^n \times S \to \mathrm{Gr}(T\mathcal{U}')$ defined by the formula

$$G(\varphi_1, \dots, \varphi_k, m) := G_f(m),$$

where $f = \sum_{j=1}^k \alpha_j \varphi_j \circ \psi_j^{-1}$. This is easily seen to be a submersion. It follows from the transversality lemma that such a G_f is transversal to $\mathcal{H}$ for almost-every $(\varphi_1, \dots, \varphi_k) \in \mathcal{A}^n$. The graph of the function f corresponding to a small multiple of such a k-tuple has the properties of the statement. $\square$

Let O be the set of points of the box $B_{\lambda+\varepsilon}$ of the form $\gamma(1)$ for some continuous path $\gamma : [0, 1] \to B_{\lambda+\varepsilon}$ starting from a point of B_λ and not hitting V; this is an open set with V as a boundary. Denote also by W another smooth hypersurface, separating V from ∂B_λ and transverse to H_0 except on a relative hypersurface. Let now denote by $\mathbb{O}'\mathbb{M}$ a disjoint copy of the set of past-directed frames

$$\{(m,\mathbf{e}) \in GL\mathbb{M} | \mathbf{e} = (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d) \text{ such that } (m, (-\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in \mathbb{O}\mathbb{M}\},$$

and let O', V', V'_0 and W' be the subsets of $\mathbb{O}'\mathbb{M}$ corresponding to O, V, V_0 and W. The equivalence relation

$$(m, (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in V \sim (m, (-\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in V'$$

defines a manifold structure on the quotient space $(O \cup V) \sqcup (O' \cup V')/\sim$, which we denote by $\mathcal{E}$. Note that $\mathcal{E}$ is compact and that its volume is in between $2\mathrm{VOL}(B_\lambda)$ and $2\mathrm{VOL}(B_{\lambda+\varepsilon})$. Write $\mathcal{V}$ for the image in $\mathcal{E}$ of V, and $\mathcal{V}_0$ for the image in $\mathcal{E}$ of V_0; define the primed sets $\mathcal{V}'$ and $\mathcal{V}'_0$ accordingly.

REMARK 16. The geodesic flow is naturally well defined on $\mathcal{E} \setminus \mathcal{V}_0$, getting instantly from O to O' or from O' to O at its crossings of $\mathcal{V} \setminus \mathcal{V}_0$. Indeed by the above definition, for any $\Phi \in \mathcal{V} \setminus \mathcal{V}_0$, either $H_0(\Phi)$ points outwards seen from O and inwards seen from O', or $H_0(\Phi)$ points inwards seen from O and outwards seen from O'. There is, however, no a priori convenient way to extend the geodesic flow on $\mathcal{V}_0$. This is the reason why we need to take care of this exceptional set.

We define the *modified relativistic diffusion* on the compact manifold $\mathcal{E}$ as follows. Let $a : B_{\lambda+\varepsilon} \to [0,1]$ be a smooth function equal to 1 on B_λ, and whose vanishing set is exactly the closed part $\mathfrak{C}$ of $\mathcal{U}$ in between W and V [this means that $\mathfrak{C}$ is the union of the trajectories $(\gamma_s)_{s\in(0,1)} \subset \mathcal{U}$ of continuous paths γ such that $\gamma_0 \in W$, $\gamma_1 \in V$, and $(\gamma_s)_{s\in(0,1)}$ does not intersect the oriented hypersurface $W \cup V$]. We extend to $\mathcal{E}$ the restiction of a to $O \cup V$, by setting $a(\mathbf{e}') = a(\mathbf{e})$ for $\mathbf{e}' = (m, (-\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in \mathbb{O}'\mathbb{M}$ and $\mathbf{e} = (m, (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_d)) \in \mathbb{O}\mathbb{M}$. We define the generator of the modified diffusion to be the following variant of $\mathcal{G}_\Theta$:

$$\mathcal{G} := H_0 + \frac{1}{2}\sum_{j=1}^{d} V_j(a\Theta V_j). \tag{5.3}$$

Denote by $\mathrm{VOL}_{\mathcal{E}}$ (resp., VOL_V, VOL_W) the natural volume element on $\mathcal{E}$ (resp., V, W).

LEMMA 17. *For $\mathrm{VOL}_{\mathcal{E}}$-almost all starting point $\Phi_0 \in \mathcal{E}$, the modified relativistic diffusion is a well-defined $\mathcal{E}$-valued process having an almost-surely infinite lifetime.*

PROOF. This modified diffusion has generator $\mathcal{G}_\Theta$ in B_λ and in its mirror copy B'_λ, and reduces to the geodesic flow in the region $\{a = 0\}$ in between W and W'. After Remark 16, we need first make sure that the set $\mathcal{V}_0 \cup \mathcal{V}'_0$ of bad points is polar.

Let $\mathcal{N}$ and $\mathcal{N}'$ be the orbits in the region $\{a=0\}$ of $\mathcal{V}_0$ and $\mathcal{V}_0'$ by the geodesic flow. They have, as a consequence of Lemma 15, null $\text{Vol}_{\mathcal{E}}$-measure. But as the modified diffusion started from any $\Phi_0 \in \{a>0\}$ is hypoelliptic, its hitting distribution of $W \cup W'$ has a density with respect to $\text{Vol}_{W\cup W'}$. It follows that the modified diffusion, started from any point of $\Phi_0 \setminus (\mathcal{N} \cup \mathcal{N}')$, will almost surely never hit $\mathcal{N} \cup \mathcal{N}'$, proving that this $\mathcal{E}$-valued process is well defined.

It can behave in two ways as it approaches its lifetime: either crossing infinitely many times $\mathcal{V}$, or remaining eventually in a compact subset of O or O'. In the latter case, its projection on $\mathbb{M}$ is a (future or past-directed) timelike path confined in a compact subset of O. As such it has a cluster point at which the strong causality condition cannot hold, preventing $\mathbb{M}$ from being strongly causal, a contradiction.

In the former case, either the path eventually remains in the region $\{a=0\}$, or it performs before some finite proper time an infinite number of crossings from $W \cup W'$ to $\mathcal{V}$. Since the geodesic flow does not explode in $\{a=0\}$, we are left with the latter possibility. It cannot lead to explosion either, since the geodesic flow needs a traveling time bounded away from 0 to travel from $W \cup W'$ to $\mathcal{V}$. □

Note that the volume measure $\text{Vol}_{\mathcal{E}}$ of the compact manifold $\mathcal{E}$ is an invariant finite measure for the modified diffusion.

5.4. *Crossing times and escape rate of Θ-diffusions.* Fix a reference point $\Phi_{\text{ref}} \in \mathbb{OM}$, and set $\mathcal{D}(\cdot) = \mathcal{D}(\Phi_{\text{ref}}, \cdot)$. Let us emphasize that $\mathcal{D}$ is a two-points function, so it is easy to pass from $\mathcal{D}(\Phi_{\text{ref}}, \Phi)$ to $\mathcal{D}(\Phi_0, \Phi)$, or the other way round, using the triangle inequality, for any $\Phi_0 \in \mathbb{OM}$.

Given an increasing sequence $(R_n)_{n\ge 1}$ of positive reals, set $\tau_0 = 0$ and associate to each R_n

$$\text{the exit time } \tau_n \text{ from the box } B^{(n)} := \{\mathcal{D} \le R_n\}.$$

It takes the diffusion an amount of proper time $(\tau_n - \tau_{n-1})$ to go from the box $B^{(n-1)}$ to the box $B^{(n)}$. The strategy in [20] is to estimate $\mathbb{P}_\Phi(\tau_n - \tau_{n-1} \le t_n)$ for a suitably chosen deterministic sequence $\{t_n\}_{n\ge 0}$ of increments of time. Set for $n \ge 1$

$$T_n := \sum_{k=1}^{n} t_k \quad \text{and} \quad r_n := R_n - R_{n-1}.$$

If one can show that

$$\sum_{n\ge 1} \mathbb{P}_\Phi(\tau_n - \tau_{n-1} \le t_n) < \infty \tag{5.4}$$

for a convenient choice of the sequences $(R_n)_{n\ge 1}$ and $(T_n)_{n\ge 1}$, then the Borel–Cantelli lemma tells us that the diffusion does not exit $B^{(n)}$ before time T_n, for n

large enough, preventing explosion. Following [20], we are going to consider the events

$$E_n := \{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\},$$

so as to be able to use our modified process run backwards from the *fixed* time T_n, when estimating the probability that the process crosses from $B^{(n-1)}$ to $B^{(n)}$ not too fast. Lemma 2.1 of [20] (an application of the Borel–Cantelli lemma) justifies that considering these events leads to the same nonexplosion conclusion as (5.4). We recall it here for the reader's convenience.

LEMMA 18 ([20]). *Fix $\Phi \in \mathbb{OM}$. If $\sum_{n\geq 1} \mathbb{P}_\Phi(E_n) < \infty$, then there exists $\mathbb{P}_\Phi$-almost-surely δ such that $\tau_n \geq T_n - \delta$, for all $n \geq 1$.*

We shall use the results of Sections 5.1.1 and 5.3 to prove the fundamental estimate of Proposition 19 below. Given any compact subset B of $\mathbb{OM}$, denote by $\mathbb{P}_B$ the law of the relativistic diffusion in $\mathbb{OM}$ started under the uniform probability in B

$$\mathbb{P}_B(\cdot) = \frac{1}{\mathrm{VOL}(B)} \int_B \mathbb{P}_\Phi(\cdot)\mathrm{VOL}(d\Phi).$$

Similarly, and given any compact subset A of $\mathcal{E}$, write $\mathbb{Q}_A$ for the law of the modified Θ-diffusion in $\mathcal{E}$ started under the uniform probability in A.

PROPOSITION 19. *There exists a constant C such that we have, for any $n \geq 1$,*

$$\begin{aligned}
&\mathbb{P}_{B^{(1)}}(\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n)\\
&\qquad \leq C \frac{\mathrm{VOL}(B^{(n)})}{\mathrm{VOL}(B^{(1)})} \frac{T_n\sqrt{\widehat{\Theta}_n/t_n}}{(r_n - 1 - 4t_n)} \exp\left[-\frac{(r_n - 1 - 4t_n)^2}{32\widehat{\Theta}_n t_n}\right],
\end{aligned}$$

where $\widehat{\Theta}_n$ denotes the supremum of Θ over the box $\{\mathcal{D} \leq R_n + 1\}$.

The proof mimics Takeda's original proof, as adapted by Hsu and Qin in [20], with the noticeable difference that we are working with a nonsymmetric, nonelliptic diffusion.

PROOF OF PROPOSITION 19. We start by embedding the box $B^{(n)}$ into the set $\mathcal{E}^{(n)}$ constructed in Section 5.3, with $\lambda = R_n$ and $\varepsilon = \frac{1}{2}$, say. From now on we work on the path space over $\mathcal{E}^{(n)}$ and use the coordinate process X, whose filtration is denoted by $(\mathcal{F}_s)_{s\geq 0}$. We still denote by τ_n the exit time from (the image in $\mathcal{E}^{(n)}$ of) $B^{(n)}$; the event

$$E_n := \{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\}$$

belongs to $\mathcal{F}_{\tau_n}$. As explained above in Section 5.2, the proof has two main ingredients, the first of which is inequality (5.5) below, where $\mathbb{Q}_{\mathcal{E}^{(n)}}$ denotes the distribution of the modified Θ-diffusion in $\mathcal{E}^{(n)}$, with generator $\mathcal{G}$ given in (5.3).

As the Θ-diffusion and the modified Θ-diffusion have the same law before the stopping time τ_n, we have $\mathbb{P}_{B^{(n)}}(E_n) = \mathbb{Q}_{B^{(n)}}(E_n) \le 2\mathbb{Q}_{\mathcal{E}^{(n)}}(E_n)$, and so

$$\mathbb{P}_{B^{(1)}}(E_n) \le 2\frac{\mathrm{Vol}(B^{(n)})}{\mathrm{Vol}(B^{(1)})}\mathbb{Q}_{\mathcal{E}^{(n)}}(E_n), \tag{5.5}$$

by the obvious inequality $\mathbb{P}_{B^{(1)}}(E_n) \le \frac{\mathrm{Vol}(B^{(n)})}{\mathrm{Vol}(B^{(1)})}\mathbb{P}_{B^{(n)}}(E_n)$. The second ingredient involves the Lyons–Zheng decomposition of $\mathcal{D}(X_s)$ under $\mathbb{Q}_{\mathcal{E}^{(n)}}$. As $\mathcal{D}$ is not a priori sufficiently regular to use Itô's formula, we apply it to its smooth approximation F constructed in Proposition 12 (with $R = R_n$ and $\eta = \frac{1}{2}$). As the process $(X_{T_n - s})_{0\le s\le T_n}$ is under $\mathbb{Q}_{\mathcal{E}^{(n)}}$ a homogeneous diffusion process with generator $\mathcal{G}^* = -H_0 + \frac{1}{2}\sum_{j=1}^d V_j(a\Theta V_j)$, it follows from Itô's formula that there exist two martingales $(M_s)_{0\le s\le T_n}$ and $(\widetilde{M}_s)_{0\le s\le T_n}$, with respect to the forward and backward filtrations of the process, respectively, such that

$$F(X_s) = F(X_0) + M_s + \int_0^s \mathcal{G}F(X_r)\,dr,$$

$$F(X_s) = F\big(X_{T_n-(T_n-s)}\big) = F(X_{T_n}) + \widetilde{M}_{T_n-s} + \int_s^{T_n} \mathcal{G}^* F(X_r)\,dr,$$

with

$$\begin{aligned}\langle M\rangle_s &= \sum_{j=1}^d \int_0^s a(X_r)\Theta(X_r)|V_jF|^2(X_r)\,dr \le 4\widehat{\Theta}_n s,\\ \langle \widetilde{M}\rangle_s &= \sum_{j=1}^d \int_0^s a(X_{T_n-r})\Theta(X_{T_n-r})|V_jF|^2(X_{T_n-r})\,dr \le 4\widehat{\Theta}_n s.\end{aligned} \tag{5.6}$$

Setting $M'_s := \widetilde{M}_{T_n-s}$ and noting that $\mathcal{G} - \mathcal{G}^* = 2H_0$, we thus have

$$d(F(X_s)) = d\left(\frac{M_s + M'_s}{2}\right) + H_0F(X_s)\,ds \tag{5.7}$$

with a controlled drift term $|H_0F| \le 2$, by Proposition 12. By construction, we have

$$\sup_{0\le s\le t_n} |F(X_{\tau_{n-1}+s}) - F(X_{\tau_{n-1}})| \ge r_n - 1$$

on the event E_n, where X hits the set $\{F \ge R_n - \frac{1}{2}\}$ in the time interval $[\tau_{n-1}, \tau_{n-1} + t_n]$. To control the $\mathbb{Q}_{\mathcal{E}^{(n)}}$-probability of E_n, we use Hsu and Qin's trick. Cut the interval $[0, T_n] = \bigcup_{k=1}^{\ell_n}[(k-1)t_n, kt_n]$ into $\ell_n := T_n/t_n$ sub-intervals of length t_n (to lighten the notations, we shall neglect the fact that ℓ_n may not be an integer; this fact causes no trouble but notational), and write on each event $\{(k-1)t_n \le \tau_{n-1} \le kt_n\}$

$$F(X_{\tau_{n-1}+s}) - F(X_{\tau_{n-1}}) = F(X_{\tau_{n-1}+s}) - F(X_{kt_n}) + F(X_{kt_n}) - F(X_{\tau_{n-1}}).$$

This simple remark shows that the event $\{\sup_{0\le s\le t_n} |F(X_{\tau_{n-1}+s}) - F(X_{\tau_{n-1}})| \ge r_n - 1\}$ is included in one of the ℓ_n events $\{\sup_{0\le |s|\le t_n} |F(X_{kt_n+s}) - F(X_{kt_n})| \ge \frac{r_n-1}{2}\}$, where $1 \le k \le \ell_n$. By (5.7) and the inequality $|H_0 F| \le 2$, the kth of these events is included in the union $A_k \cup \widetilde{A}_k$, where

$$A_k := \left\{ \sup_{0\le |s|\le t_n} |M_{kt_n+s} - M_{kt_n}| \ge \frac{r_n - 1}{2} - 2t_n \right\}$$

and

$$\widetilde{A}_k := \left\{ \sup_{0\le |s|\le t_n} |\widetilde{M}'_{kt_n+s} - \widetilde{M}'_{kt_n}| \ge \frac{r_n - 1}{2} - 2t_n \right\}.$$

Let W be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By (5.6) we have

$$\begin{aligned} \mathbb{Q}_{\mathcal{E}^{(n)}}(A_k) &\le 2\mathbb{P}\left(\sup_{0\le s\le t_n} |W_s| \ge \frac{r_n - 1 - 4t_n}{4\sqrt{\widehat{\Theta}_n}} \right) \\ &\le \frac{C\sqrt{\widehat{\Theta}_n/t_n}}{r_n - 1 - 4t_n} \exp\left(-\frac{(r_n - 1 - 4t_n)^2}{32\widehat{\Theta}_n t_n} \right) \end{aligned}$$

for some positive constant C; the same identity holds for $\widetilde{A}_k$, using (5.6). Summing over k and using inequality (5.5) yields the statement of the proposition since $E_n \subset \bigcup_{k=1}^{\ell_n} (A_k \cup \widetilde{A}_k)$. $\square$

This key proposition being proved, it becomes easy to prove Theorem 13.

PROOF OF THEOREM 13. Taking $R_n = 2^{n+5}$ and $t_n \le 2^{n+1}$ in Proposition 19, so that $T_n \le 2^{n+2}$, we get for any $n \ge 1$

$$\begin{aligned} \mathbb{P}_{B^{(1)}}(E_n) &= \mathbb{P}_{B^{(1)}}(\tau_n - \tau_{n-1} \le t_n, \tau_n \le T_n) \\ &\le C \frac{\mathrm{VOL}(B^{(n)})}{\mathrm{VOL}(B^{(1)})} \sqrt{\frac{\widehat{\Theta}_n}{t_n}} \exp\left[-\frac{4^n}{\widehat{\Theta}_n t_n} \right]. \end{aligned} \tag{5.8}$$

Specifying the choice of t_n by setting

$$t_n := \min\left\{ 2^{n+1}, \frac{4^{n-1}}{(1 + \log^+[\widehat{\Theta}_n \mathrm{VOL}(B^{(n)})])\widehat{\Theta}_n} \right\},$$

the right-hand side of (5.8) is seen to be bounded above by a constant multiple of 2^{-n}, ensuring as a consequence the convergence of the series $\sum_{n\geq 1}\mathbb{P}_{B^{(1)}}(E_n)$. Indeed, we get from (5.8), with the above t_n,

$$\mathbb{P}_{B^{(1)}}(E_n)\leq C'\mathrm{Vol}\big(B^{(n)}\big)\sqrt{\frac{\widehat{\Theta}_n^2\log[\widehat{\Theta}_n\mathrm{Vol}(B^{(n)})]}{4^n}}e^{-4\log[\widehat{\Theta}_n\mathrm{Vol}(B^{(n)})]}\leq C''/2^n.$$

[Ignoring the trivial case $\Theta\equiv 0$, we can suppose without loss of generality that we have $\widehat{\Theta}_n\mathrm{Vol}(B^{(n)})\geq 3$ for n large enough.] Note that the above choice of time increments t_n is simpler than Hsu and Qin's choice in [20]; there is in particular no need to introduce their auxiliary function $h(R)\equiv\log\log R$, to get Grigor'yan's criterion, if the second upper bound of their Section 3 is not used.

To conclude that the Θ-diffusion does not explode we need to check that $T_n=\sum_{k=1}^n t_k$ increases to infinity. For the above choice of time increments t_n, we have $\mathbb{P}_{B^{(1)}}$-almost-surely, for n larger than some n_0, and for a positive universal constant c,

$$\begin{aligned}T_n&\geq\sum_{k=n_0}^{n}\min\left\{2^{k+1},\frac{4^{k-1}}{\Theta_{2^{k+5}+1}(\log^+[\Theta_{2^{k+5}+1}\mathrm{Vol}(B_{2^{k+5}})]+1)}\right\}\\&\geq c\int_{2^{n_0+1}}^{2^n}\min\left\{8,\frac{r}{\Theta_r\log[\Theta_r\mathrm{Vol}(B_r)]}\right\}dr.\end{aligned}\tag{5.9}$$

Leaving aside the trivial case $\Theta\equiv 0$ and recalling that the map $r\mapsto\Theta_r=\max_{B_r}\Theta$ is nondecreasing, we can suppose without loss of generality that $\Theta_r\geq 3$. The divergence of the sequence (T_n) is then granted by the integral criterion

$$\int^{\infty}\min\left\{8,\frac{r}{\Theta_r\log[\Theta_r\mathrm{Vol}(B_r)]}\right\}dr=\infty.$$

As Θ_r increases, this condition is equivalent to

$$\sum_{n\geq 1}\min\left\{8,\frac{n}{\Theta_n\log[\Theta_n\mathrm{Vol}(B_n)]}\right\}=\infty,$$

that is to

$$\sum_{n\geq 1}\frac{n}{\Theta_n\log[\Theta_n\mathrm{Vol}(B_n)]}=\infty,$$

since the former holds obviously if an infinite number of terms were larger than 8. The previous condition is equivalent to condition (5.2) of Theorem 13.

Using the Borel–Cantelli lemma under the form of Lemma 18, it follows that we have

$$\mathbb{P}_{B^{(1)}}\Big(\sup_{0\leq s\leq T_n-\delta}\mathcal{D}(\Phi_s)\leq 2^{n+5}\text{ for any large enough }n\Big)=1,\tag{5.10}$$

so $\sup_{0\le s\le t}\mathcal{D}(\Phi_s)<\infty$, for all $t>0$, since T_n increases to ∞. Would a realization of the path Φ_s explode by time t, its projection in $\mathbb{M}$ would provide a timelike path with an accumulation point [for it stays in the projection of a compact set by Hypothesis (H)], contradicting the strong causality assumption on $\mathbb{M}$.

To prove that the same happens under any $\mathbb{P}_{\Phi_0}$, notice that since the nonexplosion event E belongs to the invariant σ-algebra, the function $\mathbb{OM}\ni\Phi\mapsto\mathbb{P}_\Phi(E)$ is $\mathcal{G}_\Theta$-harmonic, hence continuous, as $\mathcal{G}_\Theta$ is hypoelliptic. It follows that since

$$\mathbb{P}_{B^{(1)}}(E)=\frac{1}{\mathrm{VOL}(B^{(1)})}\int_{B^{(1)}}\mathbb{P}_\Phi(E)\mathrm{VOL}(d\Phi),$$

the probability $\mathbb{P}_\Phi(E)$ must be equal to 1 for all $\Phi\in B^{(1)}$. But as the ball $B^{(1)}$ was arbitrarily chosen, $\mathbb{P}_\Phi(E)$ is identically equal to 1 everywhere. □

5.5. *Upper rate function.* Using essentially the same reasoning as in Section 4 of [20], the above proof yields almost for free the upper rate function for the Θ-diffusion given in Corollary 14. See also [15] for related results. We keep the preceding notation.

PROOF OF COROLLARY 14. We follow the argument of [20], Section 4, making sure that it works here as well with our choice for t_n, and without their auxiliary function log log. Suppose first Θ nonidentically null and recall inequality (5.9), in which we can forget to take the minimum with 8, by Proposition 20 below. By (5.10), this yields, almost-surely, the inequality

$$\sup_{0\le s\le ch^{-1}(2^n)-\delta}\mathcal{D}(\Phi_s)\le 2^{n+5},$$

that is

$$\sup_{0\le s\le ch^{-1}(R)-\delta}\mathcal{D}(\Phi_s)\le 32R,$$

for large enough R. Letting $R=h((t+\delta)/c)$, this entails $\sup_{0\le s\le t}\mathcal{D}(\Phi_s)\le 32h((t+\delta)/c)$, hence $\sup_{0\le s\le t}\mathcal{D}(\Phi_s)\le 32h(Ct)$, for large enough t. This shows the claim under the probability $\mathbb{P}_{B^{(1)}}$, and then under $\mathbb{P}_{\Phi_0}$ as well, by the same argument already used at the end of the proof of Theorem 13. Finally, in the geodesic case ($\Theta\equiv 0$), the same holds with $T_n\ge c2^n=ch(2^n)$. □

5.6. *Estimates of the volume of the sub-Riemannian boxes and application.* Let us begin with a crude lower estimate of the volume of the boxes B_r based on the vertical expansion in the $SO_0(1,d)$-fiber of $\mathbb{OM}$, without taking into account the horizontal expansion which depends on the curvature of the base Lorentzian manifold $\mathbb{M}$. We used this lower bound in the proof of Corollary 14.

PROPOSITION 20. *We have* $\liminf_{r\to\infty}\frac{\log\mathrm{VOL}(B_r)}{r}\ge d-1$.

PROOF. Fix a relatively compact neighborhood $\mathcal{U}$ of m_0 in $\mathbb{M}$, above which $\mathbb{O}\mathcal{U}$ is trivialized in $\mathcal{U} \times SO_0(1,d)$. Assume without loss of generality that Φ_0 corresponds to $(m_0, \mathbf{1})$. By the ball-box theorem (see, e.g., [22]), the box $B_r = \{\mathcal{D} \leq r\}$ contains a neighborhood $\mathcal{V} \times B(\mathbf{1}, \varepsilon)$ of Φ_0, for some $\varepsilon > 0$ and for r larger than some fixed r_1. Using this argument a finite number of times, together with the triangle inequality for $\mathcal{D}$, we see that the box $\{\mathcal{D} \leq r\}$ contains any neighborhood $\mathcal{U} \times B(\mathbf{1}, \varrho)$ of Φ_0, for any $\varrho > 0$, provided r is large enough, say no less than $r_0 = r_0(\mathcal{U}, \varrho)$. Take ϱ larger than the diameter of $SO(d)$.

We easily see that the boxes $\{\mathcal{D} \leq r\}$ dilate in the vertical directions $V_1, \ldots, V_d$ with speed r, as r increases. So $\{\mathcal{D} \leq r\}$ contains the product of $\mathcal{U}$ by the ball of radius $(r - r_0)$ in $SO_0(1,d)$ for r large enough. This provides a lower bound on $\mathrm{VOL}(\{\mathcal{D} \leq r\})$ by some constant multiple of the volume of the hyperbolic ball of radius $(r - r_0)$, from which it follows that there exists some positive constant c such that $\log \mathrm{VOL}(B_r) \geq (d-1)r + \log c$, for r large enough. $\square$

To close this work, we give a nonexplosion criterion involving only the geometry of $\mathbb{M}$, rather than the geometry of $\mathbb{O}\mathbb{M}$ as it appears in Theorem 13 through the sub-Riemannian boxes B_r.

PROPOSITION 21. *Fix $\Phi_0 = (m_0, \mathbf{e}_0) \in \mathbb{O}\mathbb{M}$, and define the S_{Φ_0}-radius $\rho^S_{\Phi_0}(m)$ of any $m \in \mathbb{M}$ as the infimum of the S_{Φ_0}-length of C^1 paths joining m_0 to m. Define the S_{Φ_0}-ball $B^S_{\Phi_0}(r)$ of radius r as the set $B^S_{\Phi_0}(r) := \{m \in \mathbb{M} | \rho^S_{\Phi_0}(m) \leq r\}$, and set*

$$V^S(r) := \mathrm{VOL}_{\mathbb{M}}(B^S_{\Phi_0}(r)).$$

Then there exists a constant C such that we have for all $r > 0$

$$\log \mathrm{VOL}(B_r) \leq C + (d-1)r + \log V^S(Ce^r).$$

Note that the S_{Φ_0}-balls $B^S_{\Phi_0}(r)$ and their volume depend only on the choice of $\Phi_0 = (m_0, \mathbf{e}_0) \in \mathbb{O}\mathbb{M}$ and on the geometry of $\mathbb{M}$. We noticed indeed in Section 4 that the $S_{\mathbf{e}_0}$-length of a path in $\mathbb{M}$ started from m_0 is the Euclidean length of its anti-development in $(T_{m_0}\mathbb{M}, \mathbf{e}_0)$.

PROOF OF PROPOSITION 21. By the definitions in Sections 4 and 5.1.1, the b-distance of Φ_0 to any $\Phi \in \mathbb{O}\mathbb{M}$ is not larger than $\mathcal{D}_{\Phi_0}(\Phi)$, so $B_r \subset B^b(\Phi_0; r)$, where B^b denotes the ball in $\mathbb{O}\mathbb{M}$ of the b-metric. Vertically, that is to say in the frame $\tau^\gamma_{0\to s}(\Phi_0)$ parallely transported along a minimizing curve γ, the maximal hyperbolic distance reached by the velocity component $\dot{m}_s$ of γ_s is s, which is responsible for a maximal vertical volume $\mathcal{O}(e^{(d-1)r})$.

Having accelerated till reaching a maximal velocity $\mathcal{O}(e^r)$, a minimizing curve in $B^b(\Phi_0; r)$ can perform a maximal horizontal displacement $\mathcal{O}(e^r)$. Hence we have the inclusions

$$B^S_{\Phi_0}(r) \subset \pi_0(B^b(\Phi_0; r)) \subset B^S_{\Phi_0}(\mathcal{O}(e^r)),$$

and so $\mathrm{VOL}(B_r) \le Ce^{(d-1)r} V^S(Ce^r)$. $\square$

Applying Proposition 21 to the integral condition of Theorem 13 yields in the case of a bounded Θ the nonexplosion criterion $\int^{\infty} \frac{r\,dr}{r+\log V^S(e^r)} = \infty$. Using the increasing character of the map $(r \mapsto V^S(e^r))$, discretizing and distinguishing whether or not there are infinitely many n such that $\log V^S(e^n) \le n$, we easily see that this condition is equivalent to the condition $\int^{\infty} \frac{r\,dr}{\log V^S(e^r)} = \infty$.

COROLLARY 22. *Let* $(\mathbb{M}, g)$ *be a strongly causal Lorentz manifold satisfying the Completeness hypothesis* (H) *and the volume growth condition* $\int^{\infty} \frac{r\,dr}{\log V^S(e^r)} = \infty$. *Then all* Θ*-diffusions with a bounded* Θ *are stochastically complete.*

It is easy to see that this volume growth integral criterion does not depend on the choice of $\Phi_0 \in \mathbb{OM}$. Contrary to Proposition 20, it relies on the horizontal expansion and not on the vertical expansion. This criterion does not apply to Gödel's universe, for which $\log V^S(e^r)$ is of order e^r; the nonexplosion criterion of Section 3.2 covers the case of that spacetime. Corollary 22 applies, for example, to Lorentz manifolds which are topologically $\mathbb{R}^{1+d}$ and have a pseudo-metric g such that g, g^{-1} and the first-order derivatives of g with respect to the canonical coordinates are bounded, since then $\log V^S(e^r)$ is of order r, as is the case in Minkowski spacetime.

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