ON A GENERAL MANY-DIMENSIONAL EXCITED RANDOM WALK

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In this paper we study a substantial generalization of the model of excited random walk introduced in [*Electron. Commun. Probab.* **8** (2003) 86–92] by Benjamini and Wilson. We consider a discrete-time stochastic process $(X_n, n = 0, 1, 2, \ldots)$ taking values on $\mathbb{Z}^d$, $d \geq 2$, described as follows: when the particle visits a site for the first time, it has a uniformly-positive drift in a given direction ℓ; when the particle is at a site which was already visited before, it has zero drift. Assuming uniform ellipticity and that the jumps of the process are uniformly bounded, we prove that the process is ballistic in the direction ℓ so that $\liminf_{n \to \infty} \frac{X_n \cdot \ell}{n} > 0$. A key ingredient in the proof of this result is an estimate on the probability that the process visits less than $n^{1/2+\alpha}$ distinct sites by time n, where α is some positive number depending on the parameters of the model. This approach completely avoids the use of tan points and coupling methods specific to the excited random walk. Furthermore, we apply this technique to prove that the excited random walk in an i.i.d. random environment satisfies a ballistic law of large numbers and a central limit theorem.

1. Introduction and results. Let $p \in (1/2, 1]$. Consider two discrete time simple random walks on the hyper-cubic lattice $\mathbb{Z}^d$, $d \geq 2$: a symmetric random walk $(Y_n, n \geq 0)$ and a random walk $(Z_n, n \geq 0)$ which jumps to the right with probability p/d, to the left with probability $(1 - p)/d$ and to the other nearest-neighbor sites with probability $1/(2d)$. The *excited* or *cookie random walk* with bias parameter p on $\mathbb{Z}^d$ is a self-interacting random walk $(X_n, n \geq 0)$ starting from 0 and defined as follows: if at time n the walk is at a site x which it visited at some time k such that $k < n$, it jumps according to the transition probabilities of the symmetric random walk (Y_n), so that it jumps with probability $1/(2d)$ to the nearest-neighbor sites of x; if at time n the process visits a site x for the first

time, it jumps according to the transition probabilities of the walk (Z_n), so that it jumps to the right with probability p/d, to the left with probability $(1-p)/d$ and to the other nearest-neighbor sites of x with probability $1/(2d)$ (eating one *cookie* at site x). We will call the walks (Y_n) and (Z_n) defining the excited random walk, the *underlying processes* of the excited random walk.

The excited random walk was introduced in 2003 by Benjamini and Wilson in [3]. They proved that for dimensions $d \geq 2$ it is transient to the right, meaning that a.s.

$$(1.1) \qquad \lim_{n \to \infty} X_n \cdot e_1 = \infty,$$

where $\{e_i : 1 \leq i \leq d\}$ denote the canonical generators of the additive group $\mathbb{Z}^d$. Furthermore, they showed that in dimensions $d \geq 4$ the excited random walk is ballistic to the right so that a.s.

$$(1.2) \qquad \liminf_{n \to \infty} \frac{X_n \cdot e_1}{n} > 0.$$

In [9] and [10], Kozma extended the above ballisticity result to dimensions $d = 3$ and $d = 2$. Standard methods based on regeneration times can be used to deduce from (1.2) that a law of large numbers with deterministic speed v is satisfied, so that a.s.

$$\lim_{n \to \infty} \frac{X_n}{n} = v,$$

where $v \cdot e_1 > 0$ and $v \cdot e_j = 0$ for $2 \leq j \leq n$. In [4], Bérard and Ramírez gave an alternative proof of ballisticity and proved that a central limit theorem is satisfied in dimensions $d \geq 2$, so that

$$(1.3) \qquad \varepsilon^{1/2}(X_{\varepsilon^{-1}n} - \varepsilon^{-1}nv)$$

converges in law as $\varepsilon \to 0$ to a Brownian motion with variance $\sigma^2 > 0$. A variant of the excited random walk, called the *multi-excited random walk*, was introduced by Zerner in [14], where the walk has the possibility of consuming more than one cookie per site, and hence the process exhibits a nontrivial behavior even in dimension $d = 1$. Several papers have been written where the transient and ballisticity properties of this model are studied in random and deterministic environments, mainly in dimension $d = 1$ (see, e.g., [2, 7, 8]). Nevertheless, with the exception of [6] and [15], a very natural issue has not been so far addressed: what happens if the underlying processes are no longer nearest-neighbor spatially homogeneous random walks? For example, if (Y_n) and (Z_n) are random walks on $\mathbb{Z}^d$, which are not spatially homogeneous and do not perform nearest-neighbor jumps, (Y_n) has zero drift, and (Z_n) has a drift to the right, is the corresponding excited random walk transient to the right? It is reasonable to wonder under which conditions the corresponding excited random walk would still be transient to the right as in (1.1), ballistic as in (1.2), or satisfy a central limit theorem as in (1.3). In this paper, we

study a generalization of the excited random walk on $\mathbb{Z}^d$, $d \geq 2$, where the underlying processes of the model are not necessarily homogeneous random walks, and not even Markovian.

From our point of view, part of the reason why these issues have not been considered is related to the techniques so far developed to study the many-dimensional excited random walk. Indeed, the proof of transience to the right of [3], the law of large numbers and the central limit theorem of [4] in dimensions $d \geq 2$, rest on the following two key ingredients: (i) the excited random walk can be coupled to the underlying simple symmetric random walk (Y_n), in such a way that $(X_n - Y_n) \cdot e_1$ is nondecreasing in n and $(X_n - Y_n) \cdot e_j = 0$ for every $2 \leq j \leq d$; (ii) it is possible to get a lower bound for the cardinality of the range at time n of the excited random walk in terms of the *tan points* of the coupled simple symmetric random walk, which exploit reversibility properties of the symmetric random walk (see [3] and [5]). A tan point of the random walk (Y_n) in dimension $d = 2$ is defined as any site $x \in \mathbb{Z}^2$ with the property that the ray $\{x + ke_1 : k \geq 0\}$ is visited by the random walk for the first time at site x. As explained in [4], it can be shown using the ideas of [3] that for every ε the number of tan points visited by (Y_m) at time n is larger than $n^{3/4-\varepsilon}$ with a probability that decays faster than any polynomial in n. Notwithstanding, these methods break down when the underlying processes are even slightly modified. On one hand, the coupling between the excited random walk and the simple random walk does not work in general. Furthermore, the estimation of the number of tan points at a given time is a very specific argument which works only for simple symmetric random walks. Hence, to study excited random walks defined in terms of more general underlying processes, more powerful methods have to be developed: a fundamental ingredient of this paper is the introduction of a new technique to estimate the range of general versions of excited random walks, which completely avoids the use of tan points.

We define a *generalized excited random walk* which will correspond to a process driven by an underlying process (Y_n) which is a (d-dimensional) martingale and a process (Z_n) which satisfies minimal requirements, including the presence of a drift. We develop a machinery which avoids the use of tan points and of coupling, proving that this generalized excited random walk is ballistic. Let $\| \cdot \|$ be the L^2-norm in $\mathbb{Z}^d$ or $\mathbb{R}^d$, $d \geq 2$; also, we define $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ to be the unit sphere in $\mathbb{R}^d$. Consider a $\mathbb{Z}^d$-valued stochastic process $X = (X_n, n = 0, 1, 2, \ldots)$ adapted to a filtration $\mathcal{F} = (\mathcal{F}_n, n = 0, 1, 2, \ldots)$. Unless otherwise stated, we suppose that $X_0 = 0$. Denote by $\mathbb{P}$ the law of X and by $\mathbb{E}$ the corresponding expectation. As mentioned before, the processes we are considering are known also as *cookie random walks*. This terminology is also useful to us, because in the sequel we will need to consider situations when the particle gets the first visit push not in all the sites, but only in the sites of some fixed subset of $\mathbb{Z}^d$. In this case, we say that the initial configuration of cookies (or the initial cookie environment) is such that they are only in this set.

Throughout the paper we suppose that the jumps of the process are uniformly bounded, that is, the following condition holds for the process X:

CONDITION B.   There exists a constant $K > 0$ such that $\sup_{n \geq 0} \|X_{n+1} - X_n\| \leq K$ a.s.

Next, consider the following condition:

CONDITION C^+.   Let $\ell \in \mathbb{S}^{d-1}$. We say that Condition C^+ is satisfied with respect to ℓ if there exist a $\lambda > 0$ such that

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0 \qquad \text{on \{there exists } k < n \text{ such that } X_k = X_n\}$$

and

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) \cdot \ell \geq \lambda \qquad \text{on } \{X_k \neq X_n \text{ for all } k < n\}.$$

The meaning of Condition C^+ is that, when the process X visits a site for the first time, it has drift in the direction ℓ, whereas if it comes to an already visited site, it has zero drift behaving like a martingale.

Also, we formulate:

CONDITION E.   Let $\ell \in \mathbb{S}^{d-1}$. We say that Condition E is satisfied with respect to ℓ if there exist $h, r > 0$ such that for all n

$$(1.4) \qquad \mathbb{P}[(X_{n+1} - X_n) \cdot \ell > r \mid \mathcal{F}_n] \geq h$$

and for all ℓ' with $\|\ell'\| = 1$, on $\{\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0\}$

$$(1.5) \qquad \mathbb{P}[(X_{n+1} - X_n) \cdot \ell' > r \mid \mathcal{F}_n] \geq h.$$

Condition E is a kind of uniform ellipticity assumption which states that the process can always advance in the direction ℓ by a uniformly positive amount with a uniformly positive probability, and also, when the local drift is equal to zero, the process can do so in any direction. In fact, one may easily verify that if the Conditions B and C^+ are satisfied, then automatically (1.4) holds for some positive h, r. However, we still formulate Condition E this way because in the sequel we will need also to consider processes where the first visit push is not necessarily uniformly positive. Now, given $\ell \in \mathbb{S}^{d-1}$, any stochastic process X adapted to a filtration $\mathcal{F}$, which satisfies Condition B and Conditions C^+ and E with respect to ℓ, will be called a *generalized excited random walk* in direction ℓ. Note that the standard excited random walk with its natural filtration is a generalized random walk in direction e_1. The first result of this paper is that any generalized excited random walk is ballistic.

THEOREM 1.1.   *Let $d \geq 2$ and $\ell \in \mathbb{S}^{d-1}$. Assume that X is a generalized excited random walk in direction ℓ. Then, there exists $v = v(d, K, h, r, \lambda) > 0$ such that*

$$(1.6) \qquad \liminf_{n \to \infty} \frac{X_n \cdot \ell}{n} \geq v \qquad a.s.$$

The proof of Theorem 1.1 rests on two key ingredients: a general result which says that a d-dimensional martingale satisfying Condition B and a condition analogous to Condition E should typically visit much more than $t^{1/2}$ distinct sites by time t; then, the same kind of result is also obtained for any generalized excited random walk with an arbitrary initial configuration of cookies. It is worth noting that the approach of this paper could be applied also to models with other rules of assigning the drift to the particle. For instance, one can consider a model mentioned in [15]: the random walk receives a push in the direction ℓ not at the first visit, but at the kth visit to the site, where $k > 1$. To study such a model, one needs to prove that the set of sites *visited at least k times* should be sufficiently large. However, it is not difficult to do so using (some suitable) uniform ellipticity condition: if the particle visits a site, then in the next few instants of time this site will be visited $k - 1$ times more with a uniformly positive probability.

In this paper, we also consider an excited random walk in an i.i.d. random environment in $\mathbb{Z}^d$, $d \geq 2$, proving a law of large numbers and a central limit theorem for it. Let $\mathcal{P}$ be the set of probability measures on $\{\pm e_i, 1 \leq i \leq d\}$. Let $\mathcal{M} = \mathcal{P}^{\mathbb{N}}$ and $\Omega = \mathcal{M}^{\mathbb{Z}^d}$. An element $\omega = \{\omega(x), x \in \mathbb{Z}^d\} \in \Omega$ is called an *environment*. Here, for each $x \in \mathbb{Z}^d$, $\omega(x) = \{\omega_n(x), n \geq 0\} \in \mathcal{M}$ and $\omega_n(x) = \{\omega_n(x, e) \in \mathcal{P}\}$. Let $\mathbf{P}$ be a probability measure defined on Ω under which the random variables $\{\omega(x), x \in \mathbb{Z}^d\}$ are i.i.d. Let us stress that we do not assume any independence of the random variables $\omega_n(x)$, $n \geq 0$, for a fixed x. We assume that $\mathbf{P}$ is *uniformly elliptic* so that there exists a constant $\kappa > 0$ such that for every $n \geq 0$ and $1 \leq i \leq d$ one has that $\mathbf{P}[\omega_n(0, \pm e_i) \geq \kappa] = 1$. Furthermore, we assume that $\mathbf{P}$ is *uniformly excited* in the direction $\ell \in \mathbb{S}^{d-1}$, so that there exists a $\lambda > 0$ such that

$$\mathbf{P}\left[\sum_{i=1}^{d} (e_i \omega_0(0, e_i) - e_i \omega_0(0, -e_i)) \cdot \ell \geq \lambda \right] = 1,$$

and such that for every $j \geq 1$ we have that

$$\mathbf{P}\left[\sum_{i=1}^{d} (e_i \omega_j(0, e_i) - e_i \omega_j(0, -e_i)) = 0 \right] = 1.$$

We now define the *excited random walk in random environment* (ERWRE) as the process defined for each $n \geq 0$, $x \in \mathbb{Z}^d$ and $e \in \mathbb{Z}^d$ with $|e| = 1$, through the transition probabilities,

$$P_\omega\left[X_{k+1} = x + e \mid X_k = x, \sum_{j=0}^{k-1} \mathbf{1}_{\{X_j = x\}} = n \right] = \omega_n(x, e).$$

In other words, whenever the process visits a site x for the first time, it has a mean drift in the direction ℓ which is larger than λ; whenever it visits a site x which it already visited before, its mean drift is 0. For each environment ω and $x \in \mathbb{Z}^d$, we call $P_{\omega,x}$ the law of such a process starting from x. We define the *annealed* or *averaged* law of the excited random walk in random environment as $P_x = \int P_{\omega,x} \, d\mathbf{P}$, in opposition to $P_{\omega,x}$ which is called the *quenched law*. The second result of this paper is the following theorem.

THEOREM 1.2. *Consider the excited random walk in random environment, uniformly excited in the direction $\ell \in \mathbb{S}^{d-1}$. Then, the following are satisfied*:

(i) (Law of large numbers). *There exists v such that $v \cdot \ell > 0$ and*

$$\frac{X_n}{n} \to v, \qquad P_0\text{-}a.s.$$

(ii) (Central limit theorem). *There exists a nondegenerate matrix A such that*

$$\varepsilon^{1/2}\left(X_{\lfloor n\varepsilon^{-1} \rfloor} - n\varepsilon^{-1}v\right)$$

converges as $\varepsilon \to 0$ in P_0-law to a d-dimensional Brownian motion with covariance matrix A.

In the above theorem, we considered the case of nearest-neighbor jumps only for notational convenience; it extends to the case of uniformly bounded jumps without difficulties (one has to assume also that Condition E holds). Observe also that, taking $\mathbf{P}$ concentrated in one point (so that the environment in all sites is the same), we obtain LLN and CLT for a spatially homogeneous generalized excited random walk.

Let us note that, in [15] transience is proved for an excited random walk in random environment using the method of environment viewed from the particle. As explained in [14], this implies using regeneration times, a law of large numbers with a velocity which could be possibly equal to 0. Nevertheless, we do not see how to use the techniques based on coupling with a simple symmetric random walk and tan points, to prove that the expected value of the regeneration times is finite.

This paper is organized in the following way. First, in Section 2 we define a sequence of regeneration times. The key fact about this sequence (which is essential for proving Theorem 1.1) is that the time intervals between the regenerations behave in a nice way; see Proposition 2.1. The proof of this proposition is rather technical and is postponed to Section 4.2 (generally, in this paper we prefer to postpone the proofs of more technical results). In Section 3 we prove Theorem 1.1 and then apply the previously developed machinery to the excited random walk in random environment proving Theorem 1.2. In Section 4.1 we study typical displacement of excited random walk by time n; these estimates are then used in

Section 4.2. It turns our that, to understand what the typical displacement of the excited random walk should be, one has to understand the typical behavior of the number of different sites visited by time n (i.e., the range of the process). The key result concerning the range (Proposition 4.1) is stated in Section 4.1 without proof. In Section 5.1 we use some martingale techniques to obtain several auxiliary facts, which then are used in Section 5.2 to prove Proposition 4.1.

2. The renewal structure.   The proof of Theorems 1.1 and 1.2, uses classical renewal time methods. Let us note that the ERWRE, in a fixed environment, is a generalized excited random walk. Hence, here we will focus on the construction of the renewal structure for a generalized excited random walk, following the standard approach and notation presented in [4] and in the context of random walk in random environment in [12]. Due to the fact that the generalized excited random walk is neither space homogeneous nor Markovian, we will need to introduce a general notation, and deviate slightly from the construction of [4]. Let $\ell \in \mathbb{S}^{d-1}$. We consider a stochastic process $(X_n, n \geq 0)$ satisfying Conditions B, E and C$^+$, with respect to a filtration $\mathcal{F}_n$ and a direction $\ell \in \mathbb{S}^{d-1}$. For each $u > 0$ let

$$T_u = \min\{k \geq 1 : X_k \cdot \ell \geq u\}.$$

Define

$$\bar{D} = \inf\{m \geq 0 : X_m \cdot \ell < X_0 \cdot \ell\}.$$

Furthermore, define two sequences of $\mathcal{F}_n$-stopping times $\{S_n : n \geq 0\}$ and $\{D_n : n \geq 0\}$ as follows. We let $S_0 = 0$, $R_0 = X_0 \cdot \ell$ and $D_0 = 0$. Next, define by induction in $k \geq 0$

$$S_{k+1} = T_{R_k+1},$$
$$D_{k+1} = \bar{D} \circ \theta_{S_{k+1}} + S_{k+1},$$
$$R_{k+1} = \sup\{X_i \cdot \ell : 0 \leq i \leq D_{k+1}\},$$

where θ is the canonical shift on the space of trajectories. Let

$$\kappa = \inf\{n \geq 0 : S_n < \infty, D_n = \infty\}$$

with the convention that $\kappa = \infty$ when $\{n : S_n < \infty, D_n = \infty\} = \varnothing$. We define the first regeneration time as

$$\tau_1 = S_\kappa.$$

We then define by induction on $n \geq 1$, the sequence of regeneration times $\tau_1, \tau_2, \ldots$ as follows:

$$\tau_{n+1} = \tau_n + \tau_1(X_{\tau_n+\cdot})$$

setting $\tau_{n+1} = \infty$ when $\tau_n = \infty$.

Next, we define $D_i^{(0)} = D_i$ and $S_i^{(0)} = S_i$ and for each $k \geq 1$ two sequences $D_i^{(k)}$ and $S_i^{(k)}$ corresponding to the regeneration time τ_{k+1}, analogously to the definition of the sequences of stopping times D_i and S_i related to τ_1. For example, $S_0^{(1)} = \tau_1$, $R_0^{(1)} = X_{\tau_1} \cdot \ell$, $D_0^{(1)} = 0$ and define by induction in $i \geq 0$,

$$S_{i+1}^{(1)} = T_{R_i^{(1)}+1},$$

$$D_{i+1}^{(1)} = \bar{D} \circ \theta_{S_{i+1}^{(1)}} + S_{i+1}^{(1)},$$

$$R_{i+1}^{(1)} = \sup\{X_i \cdot \ell : 0 \leq i \leq D_{i+1}^{(1)}\}.$$

As opposed to the situation which occurs for the standard excited random walk (see [4]), here the sequence of regeneration times is not necessarily i.i.d. For each $k \geq 1$ and $j \geq 0$ such that $S_j^{(k)} < \infty$, we need to introduce the σ-algebra of events up to time $S_j^{(k)}$. We define $\mathcal{G}_j^{(k)}$ as the smallest σ-algebra containing all sets of the form $\{\tau_1 \leq n_1\} \cap \cdots \cap \{\tau_k \leq n_k\} \cap A$, where $n_1 < n_2 < \cdots < n_k$ are integers and $A \in \mathcal{F}_{n_k+S_j^{(k)}}$. In Section 4, we will prove the following proposition, which will play a key role in the proof of Theorem 1.1.

PROPOSITION 2.1.   *Consider a generalized random walk excited in the direction ℓ, and let $(\tau_k, k \geq 1)$ be the associated sequence of regeneration times. Then, there exist $C', \alpha' > 0$ such that for every $n \geq 1$,*

$$\sup_{k \geq 1} \mathbb{P}[\tau_{k+1} - \tau_k > n \mid \mathcal{G}_0^{(k)}] \leq C' e^{-n^{\alpha'}} \qquad a.s.$$

In particular, for every $k \geq 1$ we have that $\tau_k < \infty$ a.s.

Now, let us prove the following result which will be useful in the sequel. Throughout, we denote by ϖ an element of the space of trajectories $(\mathbb{Z}^d)^{\mathbb{N}}$. Furthermore, we define for $n \geq 1$,

$$\bar{D}_n := \inf\{m \geq n : X_m \cdot \ell < X_n \cdot \ell\}.$$

PROPOSITION 2.2.   *Let A be a Borel subset of $(\mathbb{Z}^d)^{\mathbb{N}}$. Then, the following statements are satisfied*:

(i) *For every $k \geq 1$,*

$$\mathbb{P}[X_{\tau_k+\cdot} \in A \mid \mathcal{G}_0^{(k)}] = \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_k=n\}}(\varpi) \mathbb{P}[X_{n+\cdot} \in A \mid \bar{D}_n = \infty, \mathcal{F}_n].$$

(ii) *For every $k, j \geq 1$,*

$$\mathbb{P}[X_{S_j^{(k)}+.} \in A \mid \mathcal{G}_j^{(k)}]$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_k=n\}}(\varpi)\mathbf{1}_{\{S_j^{(k)}=n+m\}}(\varpi)\mathbb{P}[X_{n+m+.} \in A \mid \bar{D}_n = \infty, \mathcal{F}_{n+m}].$$

PROOF.   Since the proof of part (i) is simpler than that of part (ii), we will omit it. For part (ii), we only consider the case $k = 1$ and $j = 1$, being the case $k = 1$ and $j > 1$ similar. Using the fact that for every natural n and $k > 1$, the event $\{\bar{D}_n = \infty\} \cap \{n < \tau_k\}$ is $\mathcal{G}_0^{(k)}$ measurable, the cases when $k > 1$ can be proved in a similar way. For each $n, m \geq 1$, define $\mathcal{S}_{n,m}$ as the set of trajectories $\{x_0, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots, x_{n+m}\}$ satisfying the following properties:

(a)
$$x_n \cdot \ell > \sup_{0 \leq l \leq n-1} x_l \cdot \ell;$$

(b) for each l such that $0 \leq l \leq n - 1$ one has
$$\min_{T_{x_l \cdot \ell} < i \leq n-1} x_i \cdot \ell < x_l \cdot \ell;$$

(c) one has that
$$x_{n+m} \cdot \ell \geq x_n \cdot \ell + 1 > \sup_{0 \leq l \leq n+m-1} x_l \cdot \ell.$$

These three conditions define the trajectories in $\mathcal{S}_{n,m}$ as those for which if $\bar{D}_n = \infty$, then $S_1^{(1)} = n + m$ (and $\tau_1 = n$). We will use the notation $s_{n,m}$ for an element of $\mathcal{S}_{n,m}$. Furthermore, given a trajectory $\varpi \in (\mathbb{Z}^d)^{\mathbb{N}}$, we will denote by ϖ_n its projection to the first n coordinates. Now note that $\mathcal{G}_1^{(1)}$ is generated by the disjoint collection of sets of the form

$$\{\varpi_{n+m} = s_{n,m}\} \cap \{\tau_1 = n\},$$

where n and m vary over the naturals and $s_{n+m} \in \mathcal{S}_{n,m}$. Hence

$$\mathbb{P}[X_{S_1^{(1)}+.} \in A \mid \mathcal{G}_1^{(1)}]$$

$$= \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_1=n\}} \sum_{m=1}^{\infty} \sum_{s_{n+m} \in \mathcal{S}_{n,m}} \mathbf{1}_{\{\varpi_{n+m}=s_{n+m}\}}$$

$$\times \mathbb{P}[X_{n+m+.} \in A \mid \tau_1 = n, \varpi_{n+m} = s_{n+m}].$$

On the other hand, for each n and m we have that when $s_{n+m} \in \mathcal{S}_{n,m}$

$$\{\varpi_{n+m} = s_{n+m}\} \cap \{\tau_1 = n\} = \{\varpi_{n+m} = s_{n,m}, \bar{D}_n = \infty\}.$$

Therefore, since $\mathbf{1}_{\{\tau_1=n\}}(\varpi)\mathbf{1}_{\{S_1^{(1)}=n+m\}}(\varpi)\mathbf{1}_{\{\varpi_{n+m}=s_{n+m}\}} = 0$ whenever $s_{n+m} \notin \mathcal{S}_{n,m}$, we see that

$$\mathbb{P}[X_{S_1^{(1)}+\cdot} \in A \mid \mathcal{G}_1^{(1)}]$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s_{n,m}\in\mathcal{S}_{n,m}} \mathbf{1}_{\{\tau_1=n\}}(\varpi)\mathbf{1}_{\{S_1^{(1)}=n+m\}}(\varpi)$$

$$\times \mathbb{P}[X_{n+m+\cdot} \in A \mid \bar{D}_n = \infty, \varpi_{n+m} = s_{n,m}]$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_1=n\}}(\varpi)\mathbf{1}_{\{S_1^{(1)}=n+m\}}(\varpi)$$

$$\times \mathbb{P}[X_{n+m+\cdot} \in A \mid \bar{D}_n = \infty, \mathcal{F}_{n+m}]. \qquad \square$$

3. Proof of the main results. In this section we will prove Theorems 1.1 and 1.2.

3.1. *Proof of Theorem* 1.1. To prove Theorem 1.1, we first prove the following lemma.

LEMMA 3.1. *Consider a generalized excited random walk in the direction ℓ. Let $(\tau_k, k \geq 1)$ be the associated regeneration times. Then, there is a constant $C > 0$ such that a.s.*

$$\limsup_{n\to\infty} \frac{\tau_n}{n} < C.$$

PROOF. Let $C' = \sup_{k\geq 1}\mathbb{E}(\tau_{k+1} - \tau_k \mid \mathcal{G}_0^{(k)})$; by Proposition 2.1, we know that $C' < \infty$. Let $\tau_0 = 0$. Now, consider the process $M_n = \sum_{k=0}^{n-1}(\tau_{k+1} - \tau_k - \mathbb{E}(\tau_{k+1} - \tau_k \mid \mathcal{G}_0^{(k)}))$, for $n \geq 1$, which is a martingale with respect to the filtration $(\mathcal{G}_0^{(n)}, n \geq 1)$. We then have for $C > C'$ that

$$\mathbb{P}[\tau_n > nC] \leq \mathbb{P}[M_n > n(C - C')] \leq \frac{\mathbb{E}[M_n^4]}{n^4(C - C')^4}.$$

But, using the fact that M_n is a martingale and Proposition 2.1, we see that there is a constant $C_1 > 0$ such that $\mathbb{E}[M_n^4] < C_1 n^2$. Hence,

$$\mathbb{P}[\tau_n > nC] \leq \frac{C_2}{n^2}$$

for some constant $C_2 > 0$, which, by Borel–Cantelli, proves the lemma. $\square$

Let us now see how to deduce Theorem 1.1 from Lemma 3.1. By definition of the regeneration times, note that

(3.1) $$\liminf_{n\to\infty} \frac{X_{\tau_n} \cdot \ell}{n} \geq 1.$$

For each $k \geq 0$, define $n_k = \sup\{n \geq 0 : \tau_n \leq k\}$. Since for each n we have $\tau_n \geq n$, it follows that $n_k < \infty$ a.s. Note also that $\lim_{k\to\infty} n_k = \infty$. Also, by definition of τ_k and n_k we have $X_k \cdot \ell \geq X_{\tau_{n_k}} \cdot \ell$, so

$$\liminf_{k\to\infty} \frac{X_k \cdot \ell}{k} = \liminf_{k\to\infty} \frac{X_{\tau_{n_k}} \cdot \ell}{k} = \liminf_{k\to\infty} \frac{n_k}{k} \frac{X_{\tau_{n_k}} \cdot \ell}{n_k}$$

(3.2)

$$\geq \liminf_{k\to\infty} \frac{n_k}{\tau_{n_k+1}} \frac{X_{\tau_{n_k}} \cdot \ell}{n_k} \geq \frac{1}{C},$$

where C is the constant appearing in Lemma 3.1 and in the last inequality we have used (3.1) and Lemma 3.1. This proves Theorem 1.1.

3.2. *Proof of Theorem* 1.2. As explained in the first paragraph of Section 2, let us note that for each environment ω, under the law $P_{\omega,0}$, the ERWRE has a sequence of regeneration times $(\tau_n, n \geq 1)$ which satisfy Propositions 2.1 and 2.2.

Furthermore (as explained, e.g., in [12]), within the context of random walk in random environment, the following proposition is satisfied (the fact that $P_0[\bar{D}_0 = \infty] > 0$ follows from Proposition 4.3 below and the observation that a.s. the random walk satisfies Conditions B, C^+ and E):

PROPOSITION 3.2. *Let $(\tau_n, n \geq 1)$ be the regeneration times of an ERWRE.*

(a) *Under the annealed law P_0, $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$ are independent and $\tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$ are i.i.d. and with the same law as τ_1 under $P_0[\cdot | \bar{D}_0 = \infty]$.*

(b) *Under the annealed law P_0, $X_{(\cdot \wedge \tau_1)}, X_{((\cdot + \tau_1) \wedge \tau_2)} - X_{\tau_1}, X_{((\cdot + \tau_2) \wedge \tau_3)} - X_{\tau_2}, \ldots$ are independent, and $X_{((\cdot + \tau_1) \wedge \tau_2)} - X_{\tau_1}, X_{((\cdot + \tau_2) \wedge \tau_3)} - X_{\tau_2}, \ldots$ are i.i.d. with the same law as $X_{(\cdot \wedge \tau_1)}$ under $P_0[\cdot | \bar{D}_0 = \infty]$.*

Propositions 2.1 and 3.2 have the following two corollaries.

COROLLARY 3.3. *Let A be a Borel subset of $(\mathbb{Z}^d)^{\mathbb{N}}$. Then for every $n \geq 1$,*

$$P_0[X_{\tau_n+\cdot} - X_{\tau_n} \in A \mid \mathcal{G}^{(n)}] = P_0[X_\cdot \in A \mid \bar{D}_0 = \infty].$$

COROLLARY 3.4. *Let $(\tau_n, n \geq 1)$ be the regeneration times of the ERWRE. Then the following are satisfied*:

(a) $E_0[\tau_1^2] < \infty$ *and* $E_0[\tau_1^2 \mid \bar{D}_0 = \infty] < \infty$;
(b) $E_0[X_{\tau_1}^2] < \infty$ *and* $E_0[X_{\tau_1}^2 \mid \bar{D}_0 = \infty] < \infty$.

Now, using standard methods, one can prove part (i) of Theorem 1.2, showing that a.s.

$$v = \lim_{n\to\infty} \frac{X_n}{n} = \frac{E_0[X_{\tau_1} \mid \bar{D}_0 = \infty]}{E_0[\tau_1 \mid \bar{D}_0 = \infty]}.$$

Indeed, by Proposition 3.2 and Corollary 3.4 we have that

$$\lim_{n \to \infty} \frac{\tau_n}{n} = E_0[\tau_1 \mid \bar{D}_0 = \infty] \quad \text{and}$$

(3.3)

$$\lim_{n \to \infty} \frac{X_{\tau_n}}{n} = E_0[X_{\tau_1} \mid \bar{D}_0 = \infty].$$

Then, standard arguments enable us to deduce part (i) of Theorem 1.2 from (3.3). The proof of part (ii) of Theorem 1.2 follows the methods, for example, of [11], using Corollary 3.4, deducing that the covariance matrix of the limiting distribution is given by

$$A = \frac{E_0[(X_{\tau_1} - \tau_1 v)^t (X_{\tau_1} - \tau_1 v) \mid \bar{D}_0 = \infty]}{E_0[\tau_1 \mid \bar{D}_0 = \infty]}.$$

4. Displacement estimates and tails of the regeneration times. In this section we will derive estimates on the displacement of generalized excited random walks with an arbitrary initial cookie configuration. This will be used to prove the tail estimates for the regeneration times in Proposition 2.1. A key ingredient in the proofs will be estimates on the range of the process.

4.1. *Displacement estimates.* For the sake of completeness and for future reference, we introduce also:

CONDITION C. Let $\ell \in \mathbb{S}^{d-1}$. We say that Condition C is satisfied with respect to ℓ if

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0 \qquad \text{on \{there exists } k < n \text{ such that } X_k = X_n\}$$

and

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) \cdot \ell \geq 0 \qquad \text{on } \{X_k \neq X_n \text{ for all } k < n\}.$$

That is, Condition C does not require that on the first visit to a site the particle gets a *uniformly* positive push in the direction ℓ.

Now, we need to consider processes starting at an arbitrary cookie environment.

CONDITIONS C_A and C_A^+. Let $\ell \in \mathbb{S}^{d-1}$ and $A \subset \mathbb{Z}^d$. We say that Condition C_A is satisfied with respect to ℓ, if

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0$$

$$\text{on the event \{there exists } k < n \text{ such that } X_k = X_n \text{ or } X_n \notin A\}$$

and

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) \cdot \ell \geq 0 \qquad \text{on } \{X_k \neq X_n \text{ for all } k < n \text{ and } X_n \in A\}.$$

If (in addition to the first display) there exist $\lambda > 0$ such that

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) \cdot \ell \geq \lambda \qquad \text{on } \{X_k \neq X_n \text{ for all } k < n \text{ and } X_n \in A\},$$

we say that Condition $\mathbf{C}_A^+$ holds.

Note that Condition $\mathbf{C}_A$ ($\mathbf{C}_A^+$) becomes Condition $\mathbf{C}$ ($\mathbf{C}^+$), if $A = \mathbb{Z}^d$. On the other hand, Condition $\mathbf{C}_A$ with $A = \varnothing$ simply means that the process is a d-dimensional martingale. Throughout, given a stochastic process $(\tilde{X}_n, n \geq 0)$ on the lattice $\mathbb{Z}^d$, we denote its range at time n by

$$\mathcal{R}_n^{\tilde{X}} = \{x \in \mathbb{Z}^d : \tilde{X}_k = x \text{ for some } 0 \leq k \leq n\}.$$

Let us use also the notation $|U|$ for the cardinality of U, where $U \subset \mathbb{Z}^d$.

Now, the key fact in this paper is that for *any* process that satisfies Conditions B, E and $\mathbf{C}_A$ with an arbitrary $A \subset \mathbb{Z}^d$, the number of distinct sites visited by time n is typically much larger than $n^{1/2}$:

PROPOSITION 4.1.   *Suppose that Conditions* B, E *and* $\mathbf{C}_A$ (*for some* $A \subset \mathbb{Z}^d$, $d \geq 2$) *hold for a process* $\tilde{X} = (\tilde{X}_n, n = 0, 1, 2, \ldots)$. *Then, there exist positive constants* $\alpha, \gamma_1, \gamma_2$ *which depend only on* d, K, h, r, *such that*

$$(4.1) \qquad \qquad \mathbb{P}[|\mathcal{R}_n^{\tilde{X}}| < n^{1/2+\alpha}] < e^{-\gamma_1 n^{\gamma_2}}$$

for all $n \geq 1$.

The proof of this proposition is postponed to Section 5.

Now, let us recall Azuma's inequalities. Let $a > 0$. If $\{Z_n\}_{n \in \mathbb{N}}$ is a martingale with respect to some filtration, and such that $|Z_k - Z_{k-1}| < c$ a.s., then (cf., e.g., Theorem 7.2.1 of [1])

$$(4.2) \qquad \qquad \mathbb{P}[|Z_n - Z_0| \geq a] \leq 2 \exp\left(-\frac{a^2}{2nc^2}\right).$$

If $\{\tilde{Z}_n\}_{n \in \mathbb{N}}$ is a super-martingale, $|\tilde{Z}_k - \tilde{Z}_{k-1}| < c$ a.s., then (see, e.g., Lemma 1 of [13])

$$(4.3) \qquad \qquad \mathbb{P}[\tilde{Z}_n - \tilde{Z}_0 \geq a] \leq \exp\left(-\frac{a^2}{2nc^2}\right).$$

Let $H(a, b) \subset \mathbb{Z}^d$ be defined as

$$H(a, b) = \{x \in \mathbb{Z}^d : x \cdot \ell \in [a, b]\}.$$

Now we obtain an important consequence of Proposition 4.1: if the cookies' configuration A is such that there are enough cookies around the starting point, then the process is likely to advance in the direction ℓ.

PROPOSITION 4.2.   *Suppose that the process X satisfies Conditions* B, E, C_A^+, *and there exists* $\delta_0 > 0$ *such that for some* $n \geq 1$

$$(4.4) \qquad \left| (\mathbb{Z}^d \setminus A) \cap H\left(-n^{1/2+\delta_0}, \tfrac{2}{3}\lambda n^{1/2+\alpha}\right) \right| \leq \tfrac{1}{3} n^{1/2+\alpha},$$

where α *is from Proposition* 4.1. *Then, for some positive constants* γ_3, γ_4 *that depend only on* $d, K, h, r, \lambda, \delta_0$, *we have*

$$(4.5) \qquad \mathbb{P}\left[X_n \cdot \ell < \tfrac{1}{3} \lambda n^{1/2+\alpha} \right] < e^{-\gamma_3 n^{\gamma_4}}.$$

PROOF.   First, note that, by (4.3)

$$\mathbb{P}\left[\max_{k \leq n} X_k \cdot \ell > \frac{2}{3}\lambda n^{1/2+\alpha}, X_n \cdot \ell < \frac{1}{3}\lambda n^{1/2+\alpha} \right] \leq C_1 n e^{-C_2 n^{2\alpha}}$$

for some $C_1, C_2 > 0$. Observe also that, again by (4.3),

$$\mathbb{P}\left[\min_{k \leq n} X_k \cdot \ell < -n^{1/2+\delta_0} \right] \leq C_3 n e^{-C_4 n^{2\delta_0}}$$

for some $C_3, C_4 > 0$. Next, let

$$\mathcal{D}_k = \mathbb{E}(X_{k+1} - X_k \mid \mathcal{F}_k)$$

be the (conditional) drift at time k. Then, the process

$$Y_n = X_n - \sum_{k=0}^{n-1} \mathcal{D}_k$$

is a martingale with bounded increments. By (4.4) and Condition C_A, on the event

$$\{ |\mathcal{R}_n^X| \geq n^{1/2+\alpha} \} \cap \{ X_k \in H\left(-n^{1/2+\delta_0}, \tfrac{2}{3}\lambda n^{1/2+\alpha}\right) \text{ for all } k \leq n \},$$

we have

$$\left(\sum_{k=0}^{n-1} \mathcal{D}_k \right) \cdot \ell > \frac{2}{3}\lambda n^{1/2+\alpha}.$$

Hence, using (4.2) and Proposition 4.1, we conclude the proof of Proposition 4.2. □

REMARK 4.1.   Condition (4.4) is enough for our needs in this paper, but, in fact, from the proof of Proposition 4.2 one can see that it can be relaxed in the following way: if in the strip of width $O(n^{1/2+\hat{\alpha}})$ (where $0 < \hat{\alpha} < \alpha$) there are not more than $O(n^{1/2+\alpha})$ eaten cookies, then the process is likely to advance by at least $O(n^{1/2+\hat{\alpha}})$ by time n.

For each $\ell \in \mathbb{S}^{d-1}$, define the half-space $\mathbb{M}_\ell = \{ x \in \mathbb{Z}^d : x \cdot \ell > 0 \}$. Next, we prove that if for some $\ell \in \mathbb{S}^{d-1}$ we know that all the cookies in the half-space $\mathbb{M}_\ell$ are present and that Condition C_A is satisfied with respect to ℓ, then with uniformly positive probability the process never goes below its initial location:

PROPOSITION 4.3. *Assume Conditions* B, E, C_A^+, *and suppose that A is such that* $\mathbb{M}_\ell \subset A$. *Then there exists* $\psi = \psi(d, K, h, r, \lambda) > 0$ *such that*

$$(4.6) \qquad \mathbb{P}[\bar{D}_0 = \infty] \geq \mathbb{P}[X_n \cdot \ell > 0 \text{ for all } n \geq 1] \geq \psi.$$

PROOF.   Clearly, since on $\{X_n \cdot \ell > 0 \text{ for all } n \geq 1\}$ the process does not see the cookies' configuration on $\mathbb{Z}^d \setminus \mathbb{M}_\ell$, it is enough to prove this proposition for the case $A = \mathbb{Z}^d$. For a (large enough) integer m, consider the event

$$U_0 = \{(X_{k+1} - X_k) \cdot \ell \geq r \text{ for all } k = 0, \ldots, \lceil r^{-1} \rceil m - 1\};$$

observe that $X_{\lceil r^{-1} \rceil m} \cdot \ell \geq m$ on U_0. By (1.4) we have for any fixed $m > 0$

$$(4.7) \qquad \mathbb{P}[U_0] \geq h^{\lceil r^{-1} \rceil m}.$$

Let $A' = \mathbb{Z}^d \setminus \{X_0, \ldots, X_{\lceil r^{-1} \rceil m - 1}\}$, $y_0 = X_{\lceil r^{-1} \rceil m}$, and abbreviate $W_k = X_{\lceil r^{-1} \rceil m + k}$, $k \geq 0$, so that $W_0 = y_0$. Observe that, if m is large enough, then the process $W - y_0$ satisfies Conditions B and E, and $A' - y_0$ satisfies (4.4) for all $n \geq m^{2-\varepsilon}$ for some small enough δ_0. Now, suppose that the event U_0 occurs and let us fix ε such that $(2 - \varepsilon)(\frac{1}{2} + \alpha) > 1$. Denote $m_0 = 0, m_1 = m$ and $m_{k+1} = \frac{\lambda}{3} m_k^{(2-\varepsilon)(1/2+\alpha)}$ for $k \geq 1$. Consider the events

$$G_k = \left\{ \min_{j \leq m_k^{2-\varepsilon}} (W_j - W_0) \cdot \ell > -m_k \right\},$$

$$U_k = \left\{ W_{\lfloor m_k^{2-\varepsilon} \rfloor} \geq m_{k+1} \right\}, \qquad k \geq 1.$$

Observe that

$$(4.8) \qquad \{X_n \cdot \ell > 0 \text{ for all } n \geq 1\} \supset \left(\bigcap_{k=1}^{\infty} (G_k \cap U_k) \right) \cap U_0$$

(indeed, on $G_k \cap U_{k-1}$ we have $X_n \cdot \ell > 0$ for all $n \in (m_{k-1}^{2-\varepsilon}, m_k^{2-\varepsilon}]$). Also, by (4.3),

$$(4.9) \qquad \mathbb{P}[G_k \mid U_0] \geq 1 - m_k^{2-\varepsilon} e^{-2K^2 m_k^{\varepsilon}}$$

and, by Proposition 4.2,

$$(4.10) \qquad \mathbb{P}[U_k \mid U_0] \geq 1 - e^{-\gamma_3 m_k^{(2-\varepsilon)\gamma_4}}.$$

Since $\mathbb{P}[\bigcap_{k=1}^{\infty} (G_k \cap U_k) \mid U_0] \geq 1 - \sum_{k=1}^{\infty} (\mathbb{P}[G_k^c \mid U_0] + \mathbb{P}[U_k^c \mid U_0])$, Proposition 4.3 now follows from (4.7) and (4.8). $\square$

4.2. *Proof of Proposition* 2.1. Here we will prove Proposition 2.1, closely following [4]. We will need the following result:

PROPOSITION 4.4. *Consider a generalized excited random walk with respect to a filtration $(\mathcal{F}_k, k \geq 0)$ and in the direction ℓ. Let $(\tau_k, k \geq 1)$ be the corresponding regeneration times. Then*:

(i) *there exists a positive constant ψ depending only on d, K, h, r, λ, such that*

$$(4.11) \qquad \sup_{j,k \geq 1} \mathbb{P}[D_j^{(k)} < \infty \mid \mathcal{G}_j^{(k)}] < 1 - \psi;$$

(ii) *there exist positive constants γ_3, γ_4 depending only on d, K, h, r, λ, such that*

$$(4.12) \qquad \sup_{k \geq 1} \mathbb{P}\left[(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell < \frac{1}{3}\lambda n^{1/2+\alpha} \,\Big|\, \mathcal{G}_0^{(k)}\right] < e^{-\gamma_3 n^{\gamma_4}};$$

(iii) *there exist positive constants γ_5, γ_6 depending only on d, K, h, r, λ, such that*

$$(4.13) \qquad \sup_{j,k \geq 1} \mathbb{P}[n \leq D_j^{(k)} - S_j^{(k)} < \infty \mid \mathcal{G}_0^{(k)}] < e^{-\gamma_5 n^{\gamma_6}}.$$

Before proving Proposition 4.4, let us see how it implies Proposition 2.1. We follow the proof of Proposition 1 in [4]. Let a_1, a_2, a_3 be positive real numbers such that $a_1 < \frac{1}{2} + \alpha$ and $a_2 + a_3 < a_1$. For each integer $n > 0$, let $u_n = \lfloor n^{a_1} \rfloor$, $v_n = \lfloor n^{a_2} \rfloor$ and $w_n = \lfloor n^{a_3} \rfloor$. We now choose n large enough so that $(K + 1)v_n(w_n + 1) + 2 + K \leq u_n$. Let

$$A_n = \{(X_{\tau_k+n} - X_{\tau_k}) \cdot \ell \leq u_n\}, \qquad B_n = \bigcap_{j=0}^{v_n} \{D_j^{(k)} < \infty\}$$

and

$$F_n = \bigcup_{j=0}^{v_n} \{w_n \leq D_j^{(k)} - S_j^{(k)} < \infty\}.$$

We will show that

$$(4.14) \qquad A_n^c \cap B_n^c \cap F_n^c \subset \{\tau_{k+1} - \tau_k < n\}.$$

To this end, for each natural $n \geq \tau_k$ define

$$r_n = \max\{(X_j - X_{\tau_k}) \cdot \ell : \tau_k \leq j \leq n\}.$$

On the event $A_n^c \cap B_n^c \cap C_n^c$, we can define

$$M = \inf\{0 \leq j \leq v_n : D_j^{(k)} = \infty\},$$

and it is true that $\tau_{k+1} = S_M^{(k)}$. Hence, we need to prove that $S_M^{(k)} - \tau_k < n$. Let us set $D_{-1}^{(k)} = \tau_k$. By definition we know that $D_{M-1}^{(k)} < \infty$. We now write

$$r_{D_{M-1}^{(k)}} = \sum_{j=0}^{M-1} \left(r_{D_j^{(k)}} - r_{S_j^{(k)}} \right) + \left(r_{S_j^{(k)}} - r_{D_{j-1}^{(k)}} \right)$$

with the convention $\sum_{j=0}^{-1} = 0$. Since, by Condition B, the range of each jump is at most K, we have that for each $0 \le j \le M-1$, $r_{D_j^{(k)}} - r_{S_j^{(k)}} \le K(D_j^{(k)} - S_j^{(k)})$. And by definition, for each $0 \le j \le M-1$, we have that $r_{S_j^{(k)}} - r_{D_{j-1}^{(k)}} \le K+1$. But on the event F_n^c, since for each $0 \le j \le M-1$ we have $D_j^{(k)} < \infty$, it is true that $D_j^{(k)} - S_j^{(k)} \le w_n$. It follows that

$$r_{D_{M-1}^{(k)}} \le (K+1)v_n(w_n + 1).$$

Since we have chosen n sufficiently large so that $(K+1)v_n(w_n+1) + 2 + K \le u_n$, we have $r_{D_{M-1}^{(k)}} + 2 + K \le u_n$. Now, on A_n^c, we have that $X_{\tau_k+n} \cdot \ell - X_{\tau_k} \cdot \ell > u_n$. Hence, $X_{\tau_k+n} \cdot \ell - X_{\tau_k} \cdot \ell > r_{D_{M-1}^{(k)}} + 2 + K$ and the smallest i such that $X_{\tau_k+i} \cdot \ell - X_{\tau_k} \cdot \ell > r_{D_{M-1}^{(k)}} + 1$ must be smaller than n. It follows that $S_M^{(k)} - \tau_k < n$, and this concludes the proof of (4.14).

Now let us show that (4.14) is enough to prove Proposition 2.1. Indeed, by parts (ii) and (iii) of Proposition 4.4, we have that

$$(4.15) \qquad \mathbb{P}\big[A_n \mid \mathcal{G}_0^{(k)}\big] \le e^{-\gamma_3 n^{\gamma_4}}$$

and

$$(4.16) \qquad \mathbb{P}\big[F_n \mid \mathcal{G}_0^{(k)}\big] \le \frac{1}{\gamma_7} e^{-n^{\gamma_7}}$$

for some γ_7 such that $0 < \gamma_7 < \gamma_6$. Furthermore, by part (i) of Proposition 4.4, we see that

$$(4.17) \qquad \mathbb{P}\big[B_n \mid \mathcal{G}_0^{(k)}\big] \le (1-\psi)^{v_n}.$$

It is clear now that estimates (4.15), (4.16) and (4.17) applied to inclusion (4.14), imply the statement of Proposition 2.1.

PROOF OF PROPOSITION 4.4.
PROOF OF PART (i). Let k and j be fixed positive integers. By part (ii) of Proposition 2.2, we have that

$$\mathbb{P}\big[D_j^{(k)} = \infty \mid \mathcal{G}_j^{(k)}\big]$$

$$(4.18) \qquad = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_k=n\}}(\varpi) \mathbf{1}_{\{S_j^{(k)}=n+m\}}(\varpi)$$

$$\times \mathbb{P}[\bar{D}_{n+m} = \infty \mid \bar{D}_n = \infty, \mathcal{F}_{n+m}].$$

Now, on the paths such that $\inf_{n \le j \le n+m} X_j \cdot \ell \ge X_n \cdot \ell$ (which happens when $\tau_k = n$) we have that

$$\mathbb{P}[\bar{D}_{n+m} = \infty \mid \bar{D}_n = \infty, \mathcal{F}_{n+m}] = \frac{\mathbb{P}[\bar{D}_{n+m} = \infty \mid \mathcal{F}_{n+m}]}{\mathbb{P}[\bar{D}_n = \infty \mid \mathcal{F}_{n+m}]}$$
$$\ge \mathbb{P}[\bar{D}_{n+m} = \infty \mid \mathcal{F}_{n+m}].$$

Hence, using Proposition 4.3 we get that

(4.19) $$\mathbb{P}[\bar{D}_{n+m} = \infty \mid \bar{D}_n = \infty, \mathcal{F}_{n+m}] \ge \psi > 0.$$

Substituting (4.19) into (4.18) we obtain that

$$\mathbb{P}[D_j^{(k)} = \infty \mid \mathcal{G}_j^{(k)}] \ge \psi \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_k = n\}}(\varpi) \mathbf{1}_{\{S_j^{(k)} = n+m\}}(\varpi) \ge \psi > 0.$$

PROOF OF PART (ii). Note that

$$\mathbb{P}\left[(X_{\tau_k + n} - X_{\tau_k}) \cdot \ell < \lambda \frac{1}{3} n^{1/2 + \alpha} \;\Big|\; \mathcal{G}_0^{(k)} \right]$$
$$= \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_k = m\}}(\varpi) \mathbb{P}\left[(X_{m+n} - X_m) \cdot \ell < \lambda \frac{1}{3} n^{1/2 + \alpha} \;\Big|\; \bar{D}_m = \infty, \mathcal{F}_m \right]$$
$$\le \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_k = m\}}(\varpi) \frac{1}{\mathbb{P}[\bar{D}_m = \infty \mid \mathcal{F}_m]} \mathbb{P}\left[(X_{m+n} - X_m) \cdot \ell < \lambda \frac{1}{3} n^{1/2 + \alpha} \;\Big|\; \mathcal{F}_m \right]$$
$$\le \sum_{m=1}^{\infty} \mathbf{1}_{\{\tau_k = m\}}(\varpi) \frac{1}{\mathbb{P}[\bar{D}_m = \infty \mid \mathcal{F}_m]} e^{-\gamma_3 n^{\gamma_4}}$$
$$\le \frac{1}{\psi} e^{-\gamma_3 n^{\gamma_4}},$$

where in the first equality we have used part (i) of Proposition 2.2, in the second to last inequality Proposition 4.2 and in the last step Proposition 4.3.

PROOF OF PART (iii). This follows from part (ii) in analogy with the proof of Lemma 9 of [4]. $\square$

5. On the number of distinct sites visited by the cookie process. In this section, after obtaining some auxiliary results, we prove Proposition 4.1.

5.1. *Some preliminary facts.* Denote by

$$L_n(m) = \sum_{j=0}^{n} \mathbf{1}_{\{X_j \cdot \ell \in [m, m+1)\}}$$

the local time on the corresponding strip. Let also

$$\tau_U^X = \min\{n \geq 0 : X_n \in U\} \tag{5.1}$$

be the entrance time to set U for the process X.

In the sequel we will use the following simple facts: if τ is a finite stopping time, and X is a process that satisfies Conditions B, E and C_A, then the process $X \circ \theta_\tau$ also satisfies Conditions B, E and C_A (but maybe with different A, which is random and $\mathcal{F}_\tau$-measurable). If Y satisfies Conditions B, E and C_A with $A = \varnothing$, then $Y \circ \theta_\tau$ also satisfies Conditions B, E and C_A with $A = \varnothing$.

We need the following.

LEMMA 5.1.   *Under Conditions B, E, C_A, for any $\delta > 0$ there exists a constant γ_1' such that for all m we have*

$$\mathbb{P}[L_n(m) \geq n^{1/2+2\delta}] \leq e^{-\gamma_1' n^\delta}.$$

PROOF.   Without restricting generality, suppose that $m = -1$. Denote $\hat{T}_0 = 0$, and

$$\hat{T}_{k+1} = \min\{j > \hat{T}_k : X_j \cdot \ell \in [-1, 0)\}.$$

With this notation,

$$L_n(-1) = \max\{k : \hat{T}_k \leq n\}. \tag{5.2}$$

By Condition E, there exist $C_1 > 0$ and $i_0 \geq 1$ such that for any stopping time τ

$$\mathbb{P}[(X_{\tau+i_0} - X_\tau) \cdot \ell \geq 2 \mid \mathcal{F}_\tau] \geq C_1. \tag{5.3}$$

Using the optional stopping theorem, for any m and for any x such that $x \cdot \ell \geq 1$, we obtain that

$$
\begin{aligned}
1 \leq (n^{1/2+\delta} + K) \\
\times \mathbb{P}\big[\tau_{H(n^{1/2+\delta},+\infty)}^X \circ \theta_{\hat{T}_m+i_0} < \tau_{H(-\infty,0)}^X \circ \theta_{\hat{T}_m+i_0} \mid \mathcal{F}_{\hat{T}_m+i_0}, \\
X_{\hat{T}_m+i_0} \cdot \ell = x\big].
\end{aligned}
\tag{5.4}
$$

Then, by (5.3) and (5.4), for any m it holds that

$$\mathbb{P}\big[X_{\hat{T}_m+k} \cdot \ell > 0 \text{ for all } i_0 \leq k \leq \tau_{H(n^{1/2+\delta},+\infty)}^X \circ \theta_{\hat{T}_m} \mid \mathcal{F}_{\hat{T}_m}\big] \geq C_2 n^{-1/2-\delta}.$$

That is, starting at any y such that $y \cdot \ell \in [-1, 0)$, by (5.3) with a uniformly positive probability the particle advances by at least distance 2 in direction ℓ during the first i_0 steps, so that it comes to $H(1, +\infty)$; then, by (5.4), the particle will visit $H(n^{1/2+\delta}, +\infty)$ before coming back to the "negative" half-space with probability at least $O(n^{-1/2-\delta})$. But, if the particle managed to visit $H(n^{1/2+\delta}, +\infty)$, by (4.3)

it is quite likely that it will take more than n time units to go back to $H(-\infty, 0)$. So, by (4.3), we have

$$\mathbb{P}[\hat{T}_{k+i_0} - \hat{T}_k > n \mid \mathcal{F}_{\hat{T}_k}] \geq C_2 n^{-1/2-\delta}(1 - e^{-n^{2\delta}/(2K^2)}) \geq C_3 n^{-1/2-\delta}.$$

Thus, one can write

$$\mathbb{P}[\text{there exists } k \leq n^{1/2+2\delta} - i_0 \text{ such that } \hat{T}_{k+i_0} - \hat{T}_k > n] \geq 1 - e^{-C_4 n^{\delta}},$$

and so, using (5.2), we prove Lemma 5.1. $\square$

Denote by $B(x, s) = \{y \in \mathbb{Z}^d : \|y - x\| \leq s\}$ the discrete ball centered in x and with radius s.

Consistently with the discussion in Section 1, let $\{Y_n\}_{n \in N}$ be a process which satisfies Conditions B, E, C_A with $A = \varnothing$ (i.e., Y is a d-dimensional martingale with uniformly bounded increments and some uniform ellipticity).

Now we obtain some properties of the process Y needed in the course of the proof of our results.

LEMMA 5.2. *There exist $b \in (0, 1)$ and $\gamma_2' > 0$ (depending only on K, h, r) such that*

$$(5.5) \qquad \mathbb{E}(\|Y_{n+1}\|^b \mid \mathcal{F}_n) \geq \|Y_n\|^b \mathbf{1}_{\{\|Y_n\| > \gamma_2'\}}.$$

PROOF. Observe that, for any $\delta > 0$ there exists $\varepsilon' > 0$ such that for all $u \in \mathbb{R}$ with the property $|u| \leq \varepsilon'$ and all $b \in (\frac{1}{2}, 1)$ we have

$$(5.6) \qquad (1 + u)^{b/2} \geq 1 + \frac{b}{2}u - (1 + \delta)\frac{b}{4}\left(1 - \frac{b}{2}\right)u^2.$$

Now, abbreviate

$$P_{y, y+z} = \mathbb{P}[Y_{n+1} - Y_n = z \mid \mathcal{F}_n, Y_n = y];$$

take y such that $\frac{K}{\|y\|} < \varepsilon'$, and write

$$\mathbb{E}(\|Y_{n+1}\|^b - \|Y_n\|^b \mid \mathcal{F}_n, Y_n = y)$$
$$= \sum_z P_{y, y+z}(\|y + z\|^b - \|y\|^b)$$
$$= \|y\|^b \sum_z P_{y, y+z}\left(\left(\frac{\|y + z\|^2}{\|y\|^2}\right)^{b/2} - 1\right)$$
$$= \|y\|^b \sum_z P_{y, y+z}\left(\left(1 + \frac{2y \cdot z + \|z\|^2}{\|y\|^2}\right)^{b/2} - 1\right).$$

Denote by $\varphi_{y,z}$ the angle between y and z regarded as vectors in $\mathbb{R}^d$. By (5.6), Condition B and the fact that $\sum_z z P_{y,y+z} = 0$, we can write

$$\mathbb{E}(\|Y_{n+1}\|^b - \|Y_n\|^b \mid Y_n = y)$$

$$\geq \|y\|^b \sum_z P_{y,y+z} \left(\frac{by \cdot z}{\|y\|^2} + \frac{b\|z\|^2}{2\|y\|^2} - (1+\delta)b\left(1 - \frac{b}{2}\right)\frac{(y \cdot z)^2}{\|y\|^4} \right.$$

$$- (1+\delta)b\left(1 - \frac{b}{2}\right)\frac{(y \cdot z)\|z\|^2}{\|y\|^4}$$

$$\left. - (1+\delta)\frac{b}{4}\left(1 - \frac{b}{2}\right)\frac{\|z\|^4}{\|y\|^4} \right)$$

$$\geq \frac{b}{\|y\|^{2-b}} \sum_z P_{y,y+z} \left(\frac{b}{2}\|z\|^2 - (1+\delta)\left(1 - \frac{b}{2}\right)\|z\|^2 \cos^2 \varphi_{y,z} \right)$$

$$+ C_1 \frac{K^3}{\|y\|^{3-b}} + C_2 \frac{K^4}{\|y\|^{4-b}},$$

where C_1 and C_2 are some (not necessarily positive) constants. Since $d \geq 2$ and using uniform ellipticity, we obtain that if δ is small enough, and b is close enough to 1, we have for any y

$$\sum_z P_{y,y+z} \left(\frac{b}{2}\|z\|^2 - (1+\delta)\left(1 - \frac{b}{2}\right)\|z\|^2 \cos^2 \varphi_{y,z} \right) > \delta' > 0$$

[use Condition B and (1.5) with some ℓ' such that $\ell' \cdot y = 0$]. Together with the previous computation, this completes the proof of Lemma 5.2. $\square$

Next, we prove the following fact (b is from Lemma 5.2):

LEMMA 5.3. *Assume that a process Y satisfies Conditions B, E, C_A with $A = \varnothing$, and suppose also that $Y_0 = x_0$. Then, for any $\delta > 0$ there exists $\gamma_3' > 0$ such that for all $x_0, y_0 \in \mathbb{Z}^d$ and for all n we have*

$$(5.7) \qquad \mathbb{P}\left[\sum_{j=1}^n \mathbf{1}_{\{Y_j = y_0\}} > n^{b/2+\delta} \right] \leq e^{-\gamma_3' n^\delta}.$$

PROOF. Without restriction of generality, we may assume that $x_0 = y_0 = 0$. Abbreviate $\tilde{\tau} = \tau_{\mathbb{Z}^d \setminus B(0, \gamma_2'+1)}^Y$ and $\tilde{V} = \{Y_m \neq 0 \text{ for all } 1 \leq m \leq \tilde{\tau}\}$, where γ_2' is from Lemma 5.2. By uniform ellipticity, there exists $C_1 > 0$ such that

$$(5.8) \qquad\qquad\qquad \mathbb{P}[\tilde{V}] > C_1.$$

By the optional stopping theorem and Lemma 5.2 we have

$$(\gamma_2' + 1)^b \leq (\gamma_2')^b + C_2(n^{1/2} + K)^b \mathbb{P}[\tau_{\mathbb{Z}^d \setminus B(0, C_2 n^{1/2})}^Y \circ \theta_{\tilde{\tau}} < \tau_{B(0,\gamma_2')}^Y \circ \theta_{\tilde{\tau}} \mid \tilde{V}, \mathcal{F}_{\tilde{\tau}}],$$

where C_2 is a (large) constant to be chosen later. This implies that

$$(5.9) \qquad \mathbb{P}\big[\tau^Y_{\mathbb{Z}^d \setminus B(0, C_2 n^{1/2})} \circ \theta_{\tilde{\tau}} < \tau^Y_{B(0, \gamma'_2)} \circ \theta_{\tilde{\tau}} \mid \tilde{V}, \mathcal{F}_{\tilde{\tau}}\big] \geq \frac{C_3}{n^{b/2}}$$

for some positive C_3 depending on C_2. Next we assume that C_2 is sufficiently large so that (4.2) implies that for any stopping time $\hat{\tau}$

$$(5.10) \quad \begin{aligned} &\mathbb{P}[\tau^Y_0 \circ \theta_{\hat{\tau}} > n \mid \mathcal{F}_{\hat{\tau}}, Y_{\hat{\tau}} = y] \\ &\qquad \geq 1 - 2\exp\left(-\frac{(C_2 n^{1/2})^2}{2nK^2}\right) \geq \frac{1}{2} \qquad \text{for any } y \in \mathbb{Z}^d \setminus B(0, C_2 n^{1/2}) \end{aligned}$$

(to reach 0 from y, the martingale Y has to advance by at least $C_2 n^{1/2}$ units in some fixed direction). Now, (5.8), (5.9) and (5.10) imply that

$$(5.11) \qquad \mathbb{P}[Y_m \neq 0 \text{ for all } m = 1, \ldots, n] \geq \frac{C_1 C_3}{2n^{b/2}}.$$

Then, proceeding as in the proof of Lemma 5.1 [the argument after (5.4) up to the end of the proof], we obtain that (5.11) implies (5.7). $\quad\square$

Next, we prove that the process Y typically hits sets which contain enough points close to the starting place of the process:

LEMMA 5.4. *Assume that a process Y satisfies Conditions B, E, C_A with $A = \varnothing$, and suppose that $Y_0 = x$. Consider an arbitrary $\delta > 0$ and a set U and suppose that $|B(x, m^{1/2}) \setminus U| \leq m^{1-b/2-2\delta}$, for some m. Then, there exists $\gamma'_4 > 0$ such that*

$$\mathbb{P}[\tau^Y_U \geq m^{1-\delta}] \leq e^{-\gamma'_4 m^\delta}.$$

PROOF. First, by (4.2), we have that

$$\mathbb{P}[Y_k \in B(x, m^{1/2}) \text{ for all } k \leq m^{1-\delta}] \geq 1 - e^{-Cm^\delta}.$$

Then, by Lemma 5.3, with probability at least $1 - e^{-Cm^\delta}$ by time $m^{1-\delta}$ every site from $B(x, m^{1/2})$ will be visited less than $m^{b/2+\delta}$ times, so we have

$$|\mathcal{R}^Y_{m^{1-\delta}}| = |\{Y_0, \ldots, Y_{m^{1-\delta}}\}| > \frac{m^{1-\delta}}{m^{b/2+\delta}} = m^{1-b/2-2\delta}$$

with probability at least $1 - e^{-C'm^\delta}$. To complete the proof of Lemma 5.4, it remains to observe that, since $|B(x, m^{1/2}) \setminus U| \leq m^{1-b/2-2\delta}$, on the event

$$\{|B(x, m^{1/2}) \cap \mathcal{R}^Y_{m^{1-\delta}}| > m^{1-b/2-2\delta}\}$$

we have $\{Y_0, \ldots, Y_{m^{1-\delta}}\} \cap U \neq \varnothing$. $\quad\square$

5.2. *Proof of Proposition* 4.1. Fix $a \in (0, \frac{1}{2})$ and $\varepsilon > 0$ in such a way that $(1 - a + \varepsilon) \wedge (\frac{1}{2} + \frac{a}{2}(1 - b) - 4\varepsilon) > \frac{1}{2}$, where b is from Lemma 5.2; also, fix $n \geq 1$. In the rest of this section, we will not explicitly indicate the dependence on a, b, ε, n by sub/superscripts in the notation. Let us denote by

$$H_j = H\big(2(j - 1)n^{a/2}, 2(j + 1)n^{a/2}\big)$$

the corresponding strip of width $4n^{a/2}$. We say that the strip H_j is a *trap* if $|\mathcal{R}_n^X \cap H_j| \geq n^{a(1-b/2)-2\varepsilon}$.

Consider the event

$$G = \big\{|\mathcal{R}_n^X| \geq \tfrac{1}{2}n^{(1-a+\varepsilon)\wedge(1/2+(a/2)(1-b)-4\varepsilon)}\big\}.$$

We are going to prove that

$$(5.12) \qquad\qquad \mathbb{P}[G] \geq 1 - e^{-C_1 n^{\varepsilon/2}},$$

thus establishing Proposition 4.1. Let us introduce the event

$$G_1 = \{L_n(k) \leq n^{1/2+\varepsilon} \text{ for all } k \in [-Kn, Kn]\}.$$

By Lemma 5.1, it holds that

$$(5.13) \qquad\qquad \mathbb{P}[G_1] \geq 1 - (2Kn + 1)e^{-\gamma_1' n^{\varepsilon/2}}.$$

Next, denote $\sigma_0 = 0$, and, inductively (assuming, of course, that $\lfloor n^{a-\varepsilon} \rfloor \geq 1$),

$$(5.14) \qquad \sigma_{k+1} = \min\big\{j \geq \sigma_k + \lfloor n^{a-\varepsilon} \rfloor : |\mathcal{R}_j^X \cap B(X_j, n^{a/2})| \leq n^{a(1-b/2)-2\varepsilon}\big\}$$

(formally, if such j does not exist, we put $\sigma_{k+1} = \infty$). Consider the event (to hit a new point means to visit a previously unvisited site)

$$G_2 = \big\{\text{at least one new point is hit on each}$$
$$\text{of the time intervals } [\sigma_{j-1}, \sigma_j), j = 1, \ldots, \tfrac{1}{2}n^{1-a+\varepsilon}\big\}.$$

Now, the key observation is the following: when the process is walking on previously visited sites, it has zero drift. So, if we only want to assure that at least one new point it visited, this is equivalent to considering the first moment when the process Y (the process without cookies) enters the previously unvisited set. Then, by Lemma 5.4 we have

$$\mathbb{P}\big[\text{at least one new point is hit on each}$$
$$\text{of the time intervals } [\sigma_{j-1}, \sigma_j), j = 1, \ldots, k\big]$$
$$\geq 1 - ke^{-\gamma_4' n^{\varepsilon/a}},$$

so we obtain that

$$(5.15) \qquad\qquad \mathbb{P}[G_2] \geq 1 - \tfrac{1}{2}n^{1-a+\varepsilon}e^{-\gamma_4' n^{\varepsilon/a}}.$$

Next, assuming that n is so large that $8n^{1-\varepsilon} < n/2$, let us show that $(G_1 \cap G_2) \subset G$. Indeed, suppose that both G_1 and G_2 occur, but $|\mathcal{R}_n^X| < n^{1/2+a/2(1-b)-4\varepsilon}$. Denote by

$$\hat{L}_j = \sum_{k=2(j-1)n^{a/2}}^{2(j+1)n^{a/2}-1} L_n(k)$$

the total number of visits to H_j. Then, on $\{|\mathcal{R}_n^X| < n^{1/2+a/2(1-b)-4\varepsilon}\}$ the number of traps is at most $2n^{1/2-a/2-2\varepsilon}$. On the event G_1, we can write

$$\sum_j \hat{L}_j \mathbf{1}_{\{H_j \text{ is a trap}\}} \leq 4n^{a/2} \times 2n^{1/2-a/2-2\varepsilon} \times n^{1/2+\varepsilon} = 8n^{1-\varepsilon}.$$

On the other hand, note that, since for $j \leq n$ we have $\mathcal{R}_j^X \subset \mathcal{R}_n^X$, if $|\mathcal{R}_j^X \cap B(X_j, n^{a/2})| > n^{a(1-b/2)-2\varepsilon}$ then X_j must be in a trap. Since n is such that $8n^{1-\varepsilon} < n/2$, we obtain that, on the event

$$\left\{ \sum_j \hat{L}_j \mathbf{1}_{\{H_j \text{ is a trap}\}} \leq 8n^{1-\varepsilon} \right\}$$

we have that $\sigma_{n^{1-a+\varepsilon}/2} < n$ (indeed, the total time spent in nontraps is at least $n/2$; on the other hand, from the definition (5.14) one can see that up to the moment σ_k we can have at most $kn^{a-\varepsilon}$ instances j such that $|\mathcal{R}_j^X \cap B(X_j, n^{a/2})| \leq n^{a(1-b/2)-2\varepsilon}$). But then, on the event G_2 we have that $|\mathcal{R}_n^X| \geq \frac{1}{2}n^{(1-a+\varepsilon)}$. So, indeed $(G_1 \cap G_2) \subset G$, and (5.12) follows from (5.13) and (5.15). The proof of Proposition 4.1 is finished.

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