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ON ERGODICITY OF SOME MARKOV PROCESSES

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To the memory of Andrzej Lasota (1932–2006)

We formulate a criterion for the existence and uniqueness of an invariant measure for a Markov process taking values in a Polish phase space. In addition, weak-* ergodicity, that is, the weak convergence of the ergodic averages of the laws of the process starting from any initial distribution, is established. The principal assumptions are the existence of a lower bound for the ergodic averages of the transition probability function and its local uniform continuity. The latter is called the *e-property*. The general result is applied to solutions of some stochastic evolution equations in Hilbert spaces. As an example, we consider an evolution equation whose solution describes the Lagrangian observations of the velocity field in the passive tracer model. The weak-* mean ergodicity of the corresponding invariant measure is used to derive the law of large numbers for the trajectory of a tracer.

1. Introduction. The lower bound technique is a useful tool in the ergodic theory of Markov processes. It has been used by Doeblin (see [4]) to show mixing of a Markov chain whose transition probabilities possess a uniform lower bound. A somewhat different approach, relying on the analysis of the operator dual to the transition probability, has been applied by Lasota and Yorke, see, for instance, [15, 17]. For example, in [17], they show that the existence of a lower bound for the iterates of the Frobenius–Perron operator (that corresponds to a piecewise monotonic transformation on the unit interval) implies the existence of a stationary distribution for the deterministic Markov chain describing the iterates of the transformation. In fact, the invariant measure is then unique in the class of measures that are absolutely continuous with respect to one-dimensional Lebesgue measure. It is also statistically stable, that is, the law of the chain, starting from any initial distribution that is absolutely continuous, converges to the invariant measure in the total

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variation metric. This technique has been extended to more general Markov chains, including those which correspond to iterated function systems; see, for example, [18]. However, most of the existing results are formulated for Markov chains taking values in finite-dimensional spaces; see, for example, [29] for a review of the topic.

Generally speaking, the lower bound technique which we have in mind involves deriving ergodic properties of the Markov process from the fact that there exists a "small" set in the state space. For instance, it could be compact, such that the time averages of the mass of the process are concentrated over that set for all sufficiently large times. If this set is compact, then one can deduce the existence of an invariant probability measure without much difficulty.

The question of extending the lower bound technique to Markov processes taking values in Polish spaces that are not locally compact is quite a delicate matter. This situation typically occurs for processes that are solutions of stochastic partial differential equations (SPDEs). The value of the process is then usually an element of an infinite-dimensional Hilbert or Banach space. We stress here that to prove the existence of a stationary measure, it is not enough only to ensure the lower bound on the transition probability over some "thin" set. One can show (see the counterexample provided in [26]) that even if the mass of the process contained in any neighborhood of a given point is separated from zero for all times, an invariant measure may fail to exist. In fact, some general results concerning the existence of an invariant measure and its statistical stability for a discrete-time Markov chain have been formulated in [26]; see Theorems 3.1–3.3.

In the present paper, we are concerned with the question of finding a criterion for the existence of a unique, invariant, ergodic probability measure for a continuous-time Feller Markov process $(Z(t))_{t\geq 0}$ taking values in a Polish space $\mathcal{X}$; see Theorems 1 and 2 below. Suppose that $(P_t)_{t\geq 0}$ is its transition probability semigroup. In our first result (see Theorem 1), we show that there exists a unique, invariant probability measure for the process, provided that for any Lipschitz, bounded function ψ, the family of functions $(P_t\psi)_{t\geq 0}$ is uniformly continuous at any point of $\mathcal{X}$ (we call this the *e-property* of the semigroup) and there exists $z \in \mathcal{X}$ such that for any $\delta > 0$,

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T P_t \mathbf{1}_{B(z,\delta)}(x)\,dt > 0 \qquad \forall x \in \mathcal{X}. \tag{1.1}$$

Here, $B(z,\delta)$ denotes the ball in $\mathcal{X}$ centered at z with radius δ. Observe that, in contrast to the Doeblin condition, we do not require that the lower bound in (1.1) is uniform in the state variable x. If some conditions on uniformity over bounded sets are added [see (2.8) and (2.9) below], then one can also deduce the stability of the ergodic averages corresponding to $(Z(t))_{t\geq 0}$; see Theorem 2. We call this, after [29], *weak-* mean ergodicity.*

This general result is applied to solutions of stochastic evolution equations in Hilbert spaces. In Theorem 3, we show the uniqueness and ergodicity of an invariant measure, provided that the transition semigroup has the e-property and the (deterministic) semi-dynamical system corresponding to the equation without the noise has an attractor which admits a unique invariant measure. This is a natural generalization of the results known for so-called *dissipative systems*; see, for example, [3].

A different approach to proving the uniqueness of an invariant measure for a stochastic evolution equation is based on the strong Feller property of the transition semigroup (see [3, 6, 10] and [22]) or, in a more refined form, on *the asymptotic strong Feller property* (see [11, 12, 19]). In our Theorem 3, we do not require either of these properties of the corresponding semigroup. Roughly speaking, we assume: (1) the existence of a global compact attractor for the system without the noise [hypothesis (i)]; (2) the existence of a Lyapunov function [hypothesis (ii)]; (3) some form of stochastic stability of the system after the noise is added [hypothesis (iii)]; (4) the e-property (see Section 2). This allows us to show lower bounds for the transition probabilities and then use Theorems 1 and 2.

As an application of Theorem 3, we consider, in Sections 5 and 6, the Lagrangian observation process corresponding to the passive tracer model $\dot{x}(t) = V(t, x(t))$, where $V(t, x)$ is a time–space stationary random, Gaussian and Markovian velocity field. One can show that when the field is sufficiently regular [see (2.16)], the process $\mathcal{Z}(t) := V(t, x(t) + \cdot)$ is a solution of a certain evolution equation in a Hilbert space; see (5.5) below. With the help of the technique developed by Hairer and Mattingly [11] (see also [5] and [14]), we verify the assumptions of Theorem 3 when $V(t, x)$ is periodic in the x variable and satisfies a mixing hypothesis in the temporal variable; see (2.17). The latter reflects, physically, quite a natural assumption that the mixing time for the velocity field gets shorter on smaller spatial scales. As a consequence of the statistical stability property of the ergodic invariant measure for the Lagrangian velocity $(\mathcal{Z}(t))_{t\geq 0}$, we obtain the weak law of large numbers for the passive tracer model in a compressible environment; see Theorem 4. It generalizes the corresponding result that holds in the incompressible case, which can be easily deduced due to the fact that the invariant measure is known explicitly in that situation; see [25].

2. Main results. Let $(\mathcal{X}, \rho)$ be a Polish metric space. Let $\mathcal{B}(\mathcal{X})$ be the space of all Borel subsets of $\mathcal{X}$ and let $B_b(\mathcal{X})$ [resp., $C_b(\mathcal{X})$] be the Banach space of all bounded, measurable (resp., continuous) functions on $\mathcal{X}$ equipped with the supremum norm $\|\cdot\|_\infty$. We denote by $\mathrm{Lip}_b(\mathcal{X})$ the space of all bounded Lipschitz continuous functions on $\mathcal{X}$. Denote by

$$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}$$

the smallest Lipschitz constant of f.

Let $(P_t)_{t\geq 0}$ be the transition semigroup of a Markov family $Z = ((Z^x(t))_{t\geq 0}, x \in \mathcal{X})$ taking values in $\mathcal{X}$. Throughout this paper, we shall assume that the semigroup $(P_t)_{t\geq 0}$ is *Feller*, that is, $P_t(C_b(\mathcal{X})) \subset C_b(\mathcal{X})$. We shall also assume that the

Markov family is stochastically continuous, which implies that $\lim_{t\to 0+} P_t\psi(x) = \psi(x)$ for all $x \in \mathcal{X}$ and $\psi \in C_b(\mathcal{X})$.

DEFINITION 2.1. We say that a transition semigroup $(P_t)_{t\geq 0}$ has *the e-property* if the family of functions $(P_t\psi)_{t\geq 0}$ is equicontinuous at every point x of $\mathcal{X}$ for any bounded and Lipschitz continuous function ψ, that is, if

$$\forall \psi \in \mathrm{Lip}_b(\mathcal{X}), x \in \mathcal{X}, \varepsilon > 0, \exists \delta > 0$$

such that

$$\forall z \in B(x,\delta), t \geq 0, \qquad |P_t\psi(x) - P_t\psi(z)| < \varepsilon.$$

$(Z^x(t))_{t\geq 0}$ is then called an *e-process*.

An e-process is an extension to continuous time of the notion of an e-chain introduced in Section 6.4 of [20].

Given $B \in \mathcal{B}(\mathcal{X})$, we denote by $\mathcal{M}_1(B)$ the space of all probability Borel measures on B. For brevity, we write $\mathcal{M}_1$ instead of $\mathcal{M}_1(\mathcal{X})$. Let $(P_t^*)_{t\geq 0}$ be the dual semigroup defined on $\mathcal{M}_1$ by the formula $P_t^*\mu(B) := \int_{\mathcal{X}} P_t \mathbf{1}_B \, d\mu$ for $B \in \mathcal{B}(\mathcal{X})$. Recall that $\mu_* \in \mathcal{M}_1$ is *invariant* for the semigroup $(P_t)_{t\geq 0}$ [or the Markov family $(Z^x(t))_{t\geq 0}$] if $P_t^*\mu_* = \mu_*$ for all $t \geq 0$.

For a given $T > 0$ and $\mu \in \mathcal{M}_1$, define $Q^T\mu := T^{-1}\int_0^T P_s^*\mu \, ds$. We write $Q^T(x,\cdot)$ in the particular case when $\mu = \delta_x$. Let

$$\mathcal{T} := \{x \in \mathcal{X} : \text{the family of measures } (Q^T(x))_{T\geq 0} \text{ is tight}\}. \tag{2.1}$$

Denote by $B(z,\delta)$ the ball in $\mathcal{X}$ with center at z and radius δ, and by "w-lim" the limit in the sense of weak convergence of measures. The proof of the following result is given in Section 3.2.

THEOREM 1. *Assume that $(P_t)_{t\geq 0}$ has the e-property and that there exists $z \in \mathcal{X}$ such that for every $\delta > 0$ and $x \in \mathcal{X}$,*

$$\liminf_{T\uparrow\infty} Q^T(x, B(z,\delta)) > 0. \tag{2.2}$$

The semigroup then admits a unique, invariant probability measure μ_*. Moreover,

$$\operatorname*{w-lim}_{T\uparrow\infty} Q^T\nu = \mu_* \tag{2.3}$$

for any $\nu \in \mathcal{M}_1$ that is supported in $\mathcal{T}$.

REMARK 1. We remark here that the set $\mathcal{T}$ may not be the entire space $\mathcal{X}$. This issue is investigated more closely in [27]. Among other results, it is shown there that if the semigroup $(P_t)_{t\geq 0}$ satisfies the assumptions of Theorem 1, then the set $\mathcal{T}$ is closed. Below, we present an elementary example of a semigroup satisfying the assumptions of the above theorem, for which $\mathcal{T} \neq \mathcal{X}$. Let $\mathcal{X} = (-\infty, -1] \cup [1, +\infty)$, $T(x) := -(x+1)/2 - 1$ for $x \in \mathcal{X}$ and let $P : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \to [0, 1]$ be the transition function defined by the formula

$$P(x, \cdot) = \begin{cases} (1 - \exp(-1/x^2))\delta_{-x}(\cdot) + \exp(-1/x^2)\delta_{x+1}(\cdot), & \text{for } x \geq 1, \\ \delta_{T(x)}(\cdot), & \text{for } x \leq -1. \end{cases}$$

Define the Markov operator $P : B_b(\mathcal{X}) \to B_b(\mathcal{X})$ corresponding to $P(\cdot, \cdot)$, that is,

$$Pf(x) = \int_{\mathcal{X}} f(y) P(x, dy) \qquad \text{for } f \in B_b(\mathcal{X}).$$

Finally, let $(P_t)_{t \geq 0}$ be the semigroup given by the formula

$$P_t f = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} P^n f \qquad \text{for } t \geq 0. \tag{2.4}$$

It is obvious that the semigroup is Feller.

We check that $(P_t)_{t \geq 0}$ satisfies the assumptions of Theorem 1 and that $\mathcal{T} = (-\infty, -1]$. Let $z := -1$. Since, for every $x \in \mathcal{X}$ and $\delta > 0$,

$$\liminf_{t \to +\infty} P_t^* \delta_x (B(z, \delta)) \geq 1 - \exp(-1/x^2),$$

condition (2.2) is satisfied.

To prove the e-property, it is enough to show that for any $f \in \mathrm{Lip}_b(\mathcal{X})$,

$$\lim_{y \to x} \sup_{n \geq 1} |P^n f(x) - P^n f(y)| = 0 \qquad \forall x \in \mathcal{X}. \tag{2.5}$$

If $x \leq -1$, then condition (2.5) obviously holds. We may therefore assume that $x \geq 1$. Observe that

$$P^n f(x) = \sum_{k=0}^{n-1} f\big(T^{n-1-k}(-x-k)\big) G_k(x) + H_n(x) f(x+n), \qquad n \geq 1,$$

where $H_n(x) := \prod_{j=0}^{n-1} \exp(-(x+j)^{-2})$ and $G_k(x) := [1 - \exp(-(x+k)^{-2})] \times H_k(x)$. Here, we interpret $\prod_{j=0}^{-1}$ as equal to 1. After straightforward calculations, we obtain that for $1 \leq x \leq y$, we have

$$\begin{aligned} &|P^n f(x) - P^n f(y)| \\ &\quad \leq \mathrm{Lip}(f)(y-x) + \|f\|_\infty \left(\sum_{k=0}^{n-2} \int_x^y |G_k'(\xi)| \, d\xi + \int_x^y |H_n'(\xi)| \, d\xi \right). \end{aligned}$$

Condition (2.5) follows from the fact that $\sum_{k=0}^{n-2} |G_k'(\xi)|$ and $H_n'(\xi)$ are uniformly convergent on $[1, +\infty)$.

Finally, it can be seen from (2.4) that for any $R > 0$ and $x \geq 1$, we have

$$\liminf_{t \to +\infty} P_t^* \delta_x (B^c(0, R)) \geq \lim_{n \to +\infty} H_n(x) > 0,$$

which proves that $x \notin \mathcal{T}$.

Following [29], page 95, we introduce the notion of weak-* mean ergodicity.

DEFINITION 2.2. A semigroup $(P_t)_{t\ge 0}$ is called *weak-* mean ergodic* if there exists a measure $\mu_* \in \mathcal{M}_1$ such that

$$\text{w-}\lim_{T\uparrow\infty} Q^T\nu = \mu_* \qquad \forall \nu \in \mathcal{M}_1. \tag{2.6}$$

REMARK 2. In some important cases, it is easy to show that $\mathcal{T} = \mathcal{X}$. For example, if $(Z^x(t))_{t\ge 0}$ is given by a stochastic evolution equation in a Hilbert space $\mathcal{X}$, then it is enough to show that there exist a compactly embedded space $\mathcal{V} \hookrightarrow \mathcal{X}$ and a locally bounded, measurable function $\Phi : [0, +\infty) \to [0, +\infty)$ that satisfies $\lim_{R\to+\infty} \Phi(R) = +\infty$ such that

$$\forall x \in \mathcal{X}\ \exists T_0 \ge 0 \qquad \sup_{t\ge T_0} \mathbb{E}\Phi(\|Z^x(t)\|_{\mathcal{V}}) < \infty.$$

Clearly, if $\mathcal{T} = \mathcal{X}$, then the assumptions of Theorem 1 guarantee weak-* mean ergodicity. In Theorem 2 below, the weak-* mean ergodicity is deduced from a version of (2.2) that holds uniformly on bounded sets.

REMARK 3. Of course, (2.6) implies uniqueness of invariant measure for $(P_t)_{t\ge 0}$. Moreover, for any stochastically continuous Feller semigroup $(P_t)_{t\ge 0}$, its weak-* mean ergodicity also implies ergodicity of μ_*, that is, that any Borel set B which satisfies $P_t\mathbf{1}_B = \mathbf{1}_B$, μ_*-a.s. for all $t \ge 0$, must be μ_*-trivial. This can be seen from, for instance, part (iv) of Theorem 3.2.4 of [3].

REMARK 4. Note that condition (2.6) is equivalent to *every point of $\mathcal{X}$ being generic*, in the sense of [8], that is,

$$\text{w-}\lim_{T\uparrow\infty} Q^T(x, \cdot) = \mu_* \qquad \forall x \in \mathcal{X}. \tag{2.7}$$

Indeed, (2.6) obviously implies (2.7) since it suffices to take $\nu = \delta_x$, $x \in \mathcal{X}$. Conversely, assuming (2.7), we can write, for any $\nu \in \mathcal{M}_1$ and $\psi \in C_b(\mathcal{X})$,

$$\lim_{T\uparrow\infty} \int_{\mathcal{X}} \psi(x) Q_T\nu(dx) = \lim_{T\uparrow\infty} \int_{\mathcal{X}} \frac{1}{T} \int_0^T P_s\psi(x)\, ds\, \nu(dx) \stackrel{(2.7)}{=} \int_{\mathcal{X}} \psi(x)\mu_*(dx)$$

and (2.6) follows.

The proof of the following result is given in Section 3.3.

THEOREM 2. *Let $(P_t)_{t\ge 0}$ satisfy the assumptions of Theorem 1. Assume, also, that there exists $z \in X$ such that for every bounded set A and $\delta > 0$, we have*

$$\inf_{x\in A} \liminf_{T\to+\infty} Q^T(x, B(z, \delta)) > 0. \tag{2.8}$$

Suppose, further, that for every $\varepsilon > 0$ and $x \in X$, there exists a bounded Borel set $D \subset X$ such that

$$\liminf_{T\to+\infty} Q^T(x, D) > 1 - \varepsilon. \tag{2.9}$$

Then, besides the existence of a unique invariant measure μ_* for $(P_t)_{t\geq 0}$, the following are true:

(1) *the semigroup $(P_t)_{t\geq 0}$ is weak-* mean ergodic*;

(2) *for any $\psi \in \mathrm{Lip}_b(\mathcal{X})$ and $\mu \in \mathcal{M}_1$, the weak law of large numbers holds, that is*,

$$\mathbb{P}_\mu\text{-}\lim_{T\to+\infty} \frac{1}{T}\int_0^T \psi(Z(t))\,dt = \int_{\mathcal{X}} \psi\,d\mu_*. \tag{2.10}$$

Here, $(Z(t))_{t\geq 0}$ is the Markov process that corresponds to the given semigroup, whose initial distribution is μ and whose path measure is $\mathbb{P}_\mu$. The convergence takes place in $\mathbb{P}_\mu$ probability.

Using Theorems 1 and 2, we establish the weak-* mean ergodicity for the family defined by the stochastic evolution equation

$$dZ(t) = \bigl(AZ(t) + F(Z(t))\bigr)\,dt + R\,dW(t). \tag{2.11}$$

Here, $\mathcal{X}$ is a real, separable Hilbert space, A is the generator of a C_0-semigroup $S = (S(t))_{t\geq 0}$ acting on $\mathcal{X}$, F maps (not necessarily continuously) $D(F) \subset \mathcal{X}$ into $\mathcal{X}$, R is a bounded linear operator from another Hilbert space $\mathcal{H}$ to $\mathcal{X}$ and $W = (W(t))_{t\geq 0}$ is a cylindrical Wiener process on $\mathcal{H}$ defined over a certain filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Let Z_0 be an $\mathcal{F}_0$-measurable random variable. By a solution of (2.11) starting from Z_0, we mean a solution to the stochastic integral equation (the so-called *mild solution*)

$$Z(t) = S(t)Z_0 + \int_0^t S(t-s)F(Z(s))\,ds + \int_0^t S(t-s)R\,dW(s), \qquad t \geq 0$$

(see, e.g., [2]), where the stochastic integral appearing on the right-hand side is understood in the sense of Itô. We suppose that for every $x \in \mathcal{X}$, there is a unique mild solution $Z^x = (Z_t^x)_{t\geq 0}$ of (2.11) starting from x and that (2.11) defines a Markov family in that way. We assume that for any $x \in \mathcal{X}$, the process Z^x is stochastically continuous.

The corresponding transition semigroup is given by $P_t\psi(x) = \mathbb{E}\psi(Z^x(t))$, $\psi \in B_b(\mathcal{X})$, and we assume that it is Feller.

DEFINITION 2.3. $\Phi : \mathcal{X} \to [0, +\infty)$ is called a *Lyapunov function* if it is measurable, bounded on bounded sets and $\lim_{\|x\|_{\mathcal{X}}\uparrow\infty} \Phi(x) = \infty$.

We shall assume that the deterministic equation

$$\frac{dY(t)}{dt} = AY(t) + F(Y(t)), \qquad Y(0) = x, \tag{2.12}$$

defines a continuous semi-dynamical system $(Y^x, x \in \mathcal{X})$, that is, for each $x \in \mathcal{X}$, there exists a unique continuous solution to (2.12) that we denote by $Y^x = (Y^x(t))_{t\geq 0}$ and for a given t, the mapping $x \mapsto Y^x(t)$ is measurable. Furthermore, we have $Y^{Y^x(t)}(s) = Y^x(t+s)$ for all $t, s \geq 0$ and $x \in \mathcal{X}$.

DEFINITION 2.4. A set $\mathcal{K} \subset \mathcal{X}$ is called *a global attractor* for the semi-dynamical system if:

(1) it is invariant under the semi-dynamical system, that is, $Y^x(t) \in \mathcal{K}$ for any $x \in \mathcal{K}$ and $t \geq 0$;
(2) for any $\varepsilon, R > 0$, there exists T such that $Y^x(t) \in \mathcal{K} + \varepsilon B(0, 1)$ for $t \geq T$ and $\|x\|_{\mathcal{X}} \leq R$.

DEFINITION 2.5. The family $(Z^x(t))_{t\geq 0}, x \in \mathcal{X}$, is *stochastically stable* if

$$\forall \varepsilon, R, t > 0 \qquad \inf_{x \in B(0,R)} \mathbb{P}\big(\|Z^x(t) - Y^x(t)\|_{\mathcal{X}} < \varepsilon\big) > 0. \tag{2.13}$$

In Section 4, from Theorems 1 and 2, we derive the following result concerning ergodicity of Z.

THEOREM 3. *Assume that*:

(i) *the semi-dynamical system* $(Y^x, x \in \mathcal{X})$ *defined by* (2.12) *has a compact*, *global attractor* $\mathcal{K}$;
(ii) $(Z^x(t))_{t\geq 0}$ *admits a Lyapunov function* Φ, *that is*,

$$\forall x \in \mathcal{X} \qquad \sup_{t\geq 0} \mathbb{E}\Phi(Z^x(t)) < \infty;$$

(iii) *the family* $(Z^x(t))_{t\geq 0}, x \in \mathcal{X}$, *is stochastically stable and*

$$\bigcap_{x\in\mathcal{K}} \bigcup_{t\geq 0} \Gamma^t(x) \neq \varnothing, \tag{2.14}$$

where $\Gamma^t(x) = \operatorname{supp} P_t^* \delta_x$;
(iv) *its transition semigroup has the e-property*.

$(Z^x(t))_{t\geq 0}, x \in \mathcal{X}$, *then admits a unique*, *invariant measure* μ_* *and is weak-* mean ergodic*. *Moreover*, *for any bounded*, *Lipschitz observable* ψ, *the weak law of large numbers holds*:

$$\mathbb{P}\text{-}\lim_{T\to+\infty} \frac{1}{T} \int_0^T \psi(Z^x(t))\, dt = \int_{\mathcal{X}} \psi\, d\mu_*.$$

REMARK 5. Observe that condition (2.14) in Theorem 3 is trivially satisfied if $\mathcal{K}$ is a singleton. Also, this condition holds if the semi-dynamical system, obtained after removing the noise, admits a global attractor that is contained in the support of the transition probability function of the solutions of (2.11) corresponding to the starting point at the attractor (this situation occurs, e.g., if the noise is nondegenerate).

Another situation when (2.14) can be guaranteed occurs if we assume (2.13) and uniqueness of an invariant probability measure for $(Y^x, x \in \mathcal{X})$. From stochastic stability condition (2.13), it is clear that the support of such a measure is contained in any $\bigcup_{t\geq 0}\Gamma_t(x)$ for $x \in \mathcal{K}$. We do not know, however, whether there exists an example of a semi-dynamical system corresponding to (2.12) with a nonsingle point attractor and such that it admits a unique invariant measure.

REMARK 6. The e-property used in Theorem 3 can be understood as an intermediary between the strong dissipativity property of [3] and asymptotic strong Feller property (see [11]). A trivial example of a transition probability semigroup that is neither dissipative (in the sense of [3]) nor asymptotic strong Feller, but satisfies the e-property, is furnished by the dynamical system on a unit circle $\{z \in \mathbb{C}: |z| = 1\}$ given by $\dot{z} = i\alpha z$, where $\alpha/(2\pi)$ is an irrational real. For more examples of Markov processes that have the e-property, but are neither dissipative nor have the asymptotic strong Feller property, see [16]. A careful analysis of the current proof shows that the e-property could be viewed as a consequence of a certain version of the asymptotic strong Feller property concerning time averages of the transition operators. We shall investigate this point in more detail in a forthcoming paper.

Our last result follows from an application of the above theorem and concerns the weak law of large numbers for the passive tracer in a compressible random flow. The trajectory of a particle is then described by the solution of an ordinary differential equation,

$$\frac{d\mathbf{x}(t)}{dt} = V(t, \mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{2.15}$$

where $V(t, \xi)$, $(t, \xi) \in \mathbb{R}^{d+1}$, is a d-dimensional random vector field. This is a simple model used in statistical hydrodynamics that describes transport of matter in a turbulent flow. We assume that $V(t, \xi)$ is mean zero, stationary, spatially periodic, Gaussian and Markov in a time random field. Its covariance matrix

$$R_{i,j}(t - s, \xi - \eta) := \mathbb{E}[V_i(t, \xi)V_j(s, \eta)]$$

is given by its Fourier coefficients,

$$\begin{aligned}\widehat{R}_{i,j}(h, k) &:= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-ik\xi} R_{i,j}(h, \xi)\, d\xi \\ &= e^{-\gamma(k)|h|}\mathcal{E}_{i,j}(k), \qquad i, j = 1, \dots, d,\end{aligned}$$

$h \in \mathbb{R}, k \in \mathbb{Z}^d$. Here, $\mathbb{T}^d := [0, 2\pi)^d$, the *energy spectrum* $\mathcal{E} := [\mathcal{E}_{i,j}]$ maps $\mathbb{Z}^d$ into the space $S_+(d)$ of all nonnegative definite Hermitian matrices and the *mixing rates* $\gamma : \mathbb{Z}^d \to (0, +\infty)$. Denote by $\operatorname{Tr} A$ the trace of a given $d \times d$ matrix A and by $\mathbb{P}$-lim the limit in probability. In Section 6, we show the following result.

THEOREM 4. *Assume that*

$$\exists m > d/2 + 1, \alpha \in (0, 1) \qquad |||\mathcal{E}|||^2 := \sum_{k \in \mathbb{Z}^d} \gamma^\alpha(k) |k|^{2(m+1)} \operatorname{Tr} \mathcal{E}(k) < \infty, \tag{2.16}$$

$$\int_0^\infty \sup_{k \in \mathbb{Z}^d} e^{-\gamma(k)t} |k| \, dt < \infty. \tag{2.17}$$

There then exists a constant vector v_* such that

$$\mathbb{P}\text{-}\lim_{t \uparrow \infty} \frac{\mathbf{x}(t)}{t} = v_*.$$

REMARK 7. We will show that $v_* = \mathbb{E}_{\mu_*} V(0, 0)$, where the expectation $\mathbb{E}_{\mu_*}$ is calculated with respect to the path measure that corresponds to the Markov process starting with the initial distribution μ_*, which is invariant under Lagrangian observations of the velocity field, that is, the vector field-valued process $V(t, \mathbf{x}(t) + \cdot)$, $t \geq 0$. In the physics literature, v_* is referred to as the *Stokes drift*. Since V is spatially stationary, the Stokes drift does not depend on the initial value $\mathbf{x}_0$.

REMARK 8. Note that condition (2.17) holds if

$$\exists \varepsilon, K_0 > 0 \text{ such that } \forall k \in \mathbb{Z}^d_* \qquad \gamma(k) \geq K_0 |k|^{1+\varepsilon}.$$

Indeed, it is clear that, under this assumption,

$$\int_1^\infty \sup_{k \in \mathbb{Z}^d_*} e^{-\gamma(k)t} |k| \, dt < \infty.$$

On the other hand, for $t \in (0, 1]$, we obtain

$$\sup_{k \in \mathbb{Z}^d_*} e^{-\gamma(k)t} |k| \leq \sup_{k \in \mathbb{Z}^d_*} \exp\{-K_0 |k|^{1+\varepsilon} t + \log |k|\} \leq \frac{C}{t^{1/(1+\varepsilon)}}$$

for some constant $C > 0$. This, of course, implies (2.17).

3. Proofs of Theorems 1 and 2.

3.1. *Some auxiliary results.* For the proof of the following lemma the reader is referred to [16]; see the argument given on pages 517 and 518.

LEMMA 1. *Suppose that $(\nu_n) \subset \mathcal{M}_1$ is not tight. There then exist an $\varepsilon > 0$, a sequence of compact sets (K_i) and an increasing sequence of positive integers*

(n_i) satisfying

$$\nu_{n_i}(K_i) \ge \varepsilon \qquad \forall i, \tag{3.1}$$

and

$$\min\{\rho(x, y) : x \in K_i, y \in K_j\} \ge \varepsilon \qquad \forall i \ne j. \tag{3.2}$$

Recall that $\mathcal{T}$ is defined by (2.1).

PROPOSITION 1. *Suppose that $(P_t)_{t\ge 0}$ has the e-property and admits an invariant probability measure μ_*. Then* $\operatorname{supp}\mu_* \subset \mathcal{T}$.

PROOF. Let μ_* be the invariant measure in question. Assume, contrary to our claim, that $(Q^T(x))_{T\ge 0}$ is not tight for some $x \in \operatorname{supp}\mu_*$. Then, according to Lemma 1, there exist a strictly increasing sequence of positive numbers $T_i \uparrow \infty$, a positive number ε and a sequence of compact sets (K_i) such that

$$Q^{T_i}(x, K_i) \ge \varepsilon \qquad \forall i, \tag{3.3}$$

and (3.2) holds. We will derive the assertion from the claim that there exist sequences $(\tilde{f}_n) \subset \operatorname{Lip}_b(\mathcal{X})$, $(\nu_n) \subset \mathcal{M}_1$ and an increasing sequence of integers (m_n) such that $\operatorname{supp}\nu_n \subset B(x, 1/n)$ for any n, and

$$\mathbf{1}_{K_{m_n}} \le \tilde{f}_n \le \mathbf{1}_{K_{m_n}^{\varepsilon/4}} \quad \text{and} \quad \operatorname{Lip}(\tilde{f}_n) \le 4/\varepsilon \qquad \forall n. \tag{3.4}$$

Here, $A^\varepsilon := \{x \in \mathcal{X} : \operatorname{dist}(x, A) < \varepsilon\}$, with $\varepsilon > 0$, denotes the ε-neighborhood of $A \subset \mathcal{X}$. Moreover,

$$P_t^*\nu_n\left(\bigcup_{i=n}^{\infty} K_{m_i}^{\varepsilon/4}\right) \le \varepsilon/4 \qquad \forall t \ge 0, \tag{3.5}$$

and

$$|P_t f_n(x) - P_t f_n(y)| < \varepsilon/4 \qquad \forall t \ge 0, \forall y \in \operatorname{supp}\nu_n, \tag{3.6}$$

$f_1 := 0$ and $f_n := \sum_{i=1}^{n-1} \tilde{f}_i$, $n \ge 2$. Temporarily admitting the above claim, we show how to complete the proof of the proposition. First, observe that (3.2) and condition (3.4) together imply that the series $f := \sum_{i=1}^{\infty} \tilde{f}_i$ is uniformly convergent and $\|f\|_\infty = 1$. Also, note that for x, y such that $\rho(x, y) < \varepsilon/8$, we have $\tilde{f}_i(x) \ne 0$, or $\tilde{f}_i(y) \ne 0$, for at most one i. Therefore, for such points, $|f(x) - f(y)| < 16\varepsilon^{-1}\rho(x, y)$. This, in particular, implies that $f \in \operatorname{Lip}(\mathcal{X})$. From (3.3) and (3.4)–(3.6), it follows that

$$\begin{aligned}
&\int_{\mathcal{X}} Q^{T_{m_n}}(x, dy) f(y) - \int_{\mathcal{X}} Q^{T_{m_n}}\nu_n(dy) f(y) \\
&\quad \ge Q^{T_{m_n}}(x, K_{m_n}) + \int_{\mathcal{X}} Q^{T_{m_n}}(x, dy) f_n(y) \\
&\qquad - \int_{\mathcal{X}} Q^{T_{m_n}}\nu_n(dy) f_n(y) - Q^{T_{m_n}}\nu_n\left(\bigcup_{i=n}^{\infty} K_{m_i}^{\varepsilon/4}\right).
\end{aligned} \tag{3.7}$$

By virtue of (3.3), the first term on the right-hand side of (3.7) is greater than or equal to ε. Combining the second and the third terms, we obtain that their absolute value equals

$$\left|\frac{1}{T_{m_n}}\int_0^{T_{m_n}}\int_{\mathcal{X}}[P^s f_n(x) - P^s f_n(y)]\nu_n(dy)\,ds\right| \overset{(3.6)}{\leq} \frac{\varepsilon}{4}.$$

The fourth term is less than or equal to $\varepsilon/4$, by virtue of (3.5). Summarizing, we have shown that

$$\begin{aligned}\int_{\mathcal{X}} Q^{T_{m_n}}(x,dy)f(y) &- \int_{\mathcal{X}} Q^{T_{m_n}}\nu_n(dy)f(y)\\ &= \frac{1}{T_{m_n}}\int_0^{T_{m_n}} ds\int_{\mathcal{X}}[P^s f(x) - P^s f(y)]\nu_n(dy) > \frac{\varepsilon}{2}\end{aligned}$$

for every positive integer n. Hence, there must be a sequence (t_n, y_n) such that $t_n \in [0, T_{m_n}]$, $y_n \in \operatorname{supp}\nu_n \subset B(x, 1/n)$, for which $P_{t_n} f(x) - P_{t_n} f(y_n) > \varepsilon/2$, $n \geq 1$. This clearly contradicts equicontinuity of $(P_t f)_{t\geq 0}$ at x.

PROOF OF THE CLAIM. We accomplish this by induction on n. Let $n = 1$. Since $x \in \operatorname{supp}\mu_*$, we have $\mu_*(B(x,\delta)) > 0$ for all $\delta > 0$. Define the probability measure ν_1 by the formula

$$\nu_1(B) = \mu_*(B|B(x,1)) := \frac{\mu_*(B\cap B(x,1))}{\mu_*(B(x,1))}, \qquad B \in \mathcal{B}(\mathcal{X}).$$

Since $\nu_1 \leq \mu_*^{-1}(B(x,1))\mu_*$, from the fact that μ_* is invariant, it follows that the family $(P_t^*\nu_1)_{t\geq 0}$ is tight. Thus, there exists a compact set K such that

$$P_t^*\nu_1(K^c) \leq \varepsilon/4 \qquad \forall t \geq 0. \tag{3.8}$$

Note, however, that $K \cap K_i^{\varepsilon/4} \neq \varnothing$ for only finitely many i's. Otherwise, in light of (3.2), one could construct in K an infinite set of points separated from each other at a distance of at least $\varepsilon/2$, which contradicts its compactness. As a result, there exists an integer m_1 such that

$$P_t^*\nu_1\left(\bigcup_{i=m_1}^{\infty} K_i^{\varepsilon/4}\right) \leq \varepsilon/4 \qquad \forall t \geq 0.$$

Let $\tilde{f}_1$ be an arbitrary Lipschitz function satisfying $\mathbf{1}_{K_{m_1}} \leq \tilde{f}_1 \leq \mathbf{1}_{K_{m_1}^{\varepsilon/4}}$ and $\operatorname{Lip}(\tilde{f}_1) \leq 4/\varepsilon$.

Assume, now, that for a given $n \geq 1$, we have already constructed $\tilde{f}_1, \dots, \tilde{f}_n$, $\nu_1, \dots, \nu_n$, $m_1, \dots, m_n$ satisfying (3.4)–(3.6). Since $(P_t f_{n+1})_{t\geq 0}$ is equicontinuous, we can choose $\delta < 1/(n+1)$ such that $|P_t f_{n+1}(x) - P_t f_{n+1}(y)| < \varepsilon/4$ for all $t \geq 0$ and $y \in B(x,\delta)$. Suppose, further, that $\nu_{n+1} := \mu_*(\cdot|B(x,\delta))$. Since the measure is supported in $B(x,\delta)$, condition (3.6) holds for f_{n+1}. Tightness of

$(P_t^* \nu_{n+1})_{t\ge 0}$ can be argued in the same way as in the case $n = 1$. As a consequence, one can find $m_{n+1} > m_n$ such that

$$P_t^* \nu_{n+1}\left(\bigcup_{i=m_{n+1}}^{\infty} K_i^{\varepsilon/4}\right) \le \varepsilon/4 \qquad \forall t \ge 0.$$

Finally, we let $\tilde{f}_{n+1}$ be an arbitrary continuous function satisfying (3.4). □

For given an integer $k \ge 1$, times $t_1, \dots, t_k \ge 0$ and a measure $\mu \in \mathcal{M}_1$, we let $Q^{t_k,\dots,t_1}\mu := Q^{t_k} \cdots Q^{t_1}\mu$. The following simple lemma will be useful in the sequel. In what follows, $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm.

LEMMA 2. *For all $k \ge 1$ and $t_1, \dots, t_k > 0$,*

$$\limsup_{T\to+\infty} \sup_{\mu\in\mathcal{M}_1} \|Q^{T,t_k,\dots,t_1}\mu - Q^T\mu\|_{\mathrm{TV}} = 0. \tag{3.9}$$

PROOF. To simplify the notation, we assume that $k = 1$. The general case can be argued by the induction on the length of the sequence $t_1, \dots, t_k$ and is left to the reader. For any $T > 0$, we have

$$\begin{aligned} Q^{T,t_1}\mu - Q^T\mu &= (Tt_1)^{-1} \int_0^{t_1} dr \left[\int_0^T P_{s+r}^*\mu\, ds - \int_0^T P_s^*\mu\, ds\right] \\ &= (Tt_1)^{-1} \int_0^{t_1} dr \int_0^r (P_{s+T}^*\mu - P_s^*\mu)\, ds. \end{aligned}$$

The total variation norm of $Q^{T,t_1}\mu - Q^T\mu$ can therefore be estimated by t_1/T and (3.9) follows. □

3.2. *Proof of Theorem* 1. The existence of an invariant measure follows from Theorem 3.1 of [16]. We will show that for arbitrary $x_1, x_2 \in \mathcal{T}$ and $\psi \in \mathrm{Lip}_b(X)$,

$$\lim_{T\uparrow\infty} \left| \int_{\mathcal{X}} \psi(y) Q^T(x_1, dy) - \int_{\mathcal{X}} \psi(y) Q^T(x_2, dy) \right| = 0. \tag{3.10}$$

From this, we can easily deduce (2.3) using, for instance, Example 22, page 74 of [24]. Indeed, for any ν, as in the statement of the theorem,

$$\begin{aligned} &\int_{\mathcal{X}} \psi(y) Q^T\nu(dy) - \int_{\mathcal{X}} \psi\, d\mu_* \\ &\quad = \int_{\mathcal{X}} \int_{\mathcal{X}} \nu(dx)\mu_*(dx') \left(\int_{\mathcal{X}} \psi(y) Q^T(x, dy) - \int_{\mathcal{X}} \psi(y) Q^T(x', dy) \right) \end{aligned}$$

and (2.3) follows directly from (3.10) and Proposition 1. The rest of the argument will be devoted to the proof of (3.10).

Fix a sequence (η_n) of positive numbers monotonically decreasing to 0. Also, fix arbitrary $\varepsilon > 0$, $\psi \in \mathrm{Lip}_b(\mathcal{X})$, $x_1, x_2 \in \mathcal{T}$. For these parameters, we define $\Delta \subset \mathbb{R}$ in the following way: $\alpha \in \Delta$ if and only if $\alpha > 0$ and there exist a positive integer N, a sequence of times $(T_{\alpha,n})$ and sequences of measures $(\mu^n_{\alpha,i})$, $(\nu^n_{\alpha,i}) \subset \mathcal{M}_1$, $i = 1, 2$, such that for $n \geq N$,

$$T_{\alpha,n} \geq n, \tag{3.11}$$

$$\| Q^{T_{\alpha,n}}(x_i) - \mu^n_{\alpha,i} \|_{\mathrm{TV}} < \eta_n, \tag{3.12}$$

$$\mu^n_{\alpha,i} \geq \alpha \nu^n_{\alpha,i} \qquad \text{for } i = 1, 2, \tag{3.13}$$

and

$$\limsup_{T\uparrow\infty} \left| \int_{\mathcal{X}} \psi(x) Q^T \nu^n_{\alpha,1}(dx) - \int_{\mathcal{X}} \psi(x) Q^T \nu^n_{\alpha,2}(dx) \right| < \varepsilon. \tag{3.14}$$

Our main tool is contained in the following lemma.

LEMMA 3. *For given* $\varepsilon > 0$, (η_n), $x_1, x_2 \in \mathcal{T}$ *and* $\psi \in \mathrm{Lip}_b(\mathcal{X})$, *the set* $\Delta \neq \varnothing$. *Moreover, we have* $\sup \Delta = 1$.

Accepting the truth of this lemma, we show how to complete the proof of (3.10). To that end, let us choose an arbitrary $\varepsilon > 0$. Then there exists an $\alpha > 1 - \varepsilon$ that belongs to Δ. By virtue of (3.12), we can replace the $Q^T(x_i, \cdot)$ appearing in (3.10) by $\mu^n_{\alpha,i}$ and the resulting error can be estimated for $T \geq T_{\alpha,n}$ as follows:

$$\begin{aligned}
&\left| \int_{\mathcal{X}} \psi(y) Q^T(x_1, dy) - \int_{\mathcal{X}} \psi(y) Q^T(x_2, dy) \right| \\
&\quad \leq \sum_{i=1}^{2} \left| \int_{\mathcal{X}} \psi(y) Q^T(x_i, dy) - \int_{\mathcal{X}} \psi(y) Q^{T,T_{\alpha,n}}(x_i, dy) \right| \\
&\qquad + \left| \int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,1}(dy) - \int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,2}(dy) \right| \\
&\qquad + \sum_{i=1}^{2} \left| \int_{\mathcal{X}} \psi(y) Q^{T,T_{\alpha,n}}(x_i, dy) - \int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,i}(dy) \right| \\
&\quad \leq \sum_{i=1}^{2} \left| \int_{\mathcal{X}} \psi(y) Q^T(x_i, dy) - \int_{\mathcal{X}} \psi(y) Q^{T,T_{\alpha,n}}(x_i, dy) \right| \\
&\qquad + \left| \int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,1}(dy) - \int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,2}(dy) \right| \\
&\qquad + 2\eta_n \|\psi\|_\infty.
\end{aligned} \tag{3.15}$$

To deal with the second term on the last right-hand side of (3.15), we use condition (3.13). We can then replace $\mu^n_{\alpha,i}$ by $\nu^n_{\alpha,i}$ and obtain

$$\begin{aligned}
&\left|\int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,1}(dy) - \int_{\mathcal{X}} \psi(y) Q^T \mu^n_{\alpha,2}(dy)\right| \\
&\quad \overset{(3.13)}{\le} \alpha\left|\int_{\mathcal{X}} \psi(y) Q^T \nu^n_{\alpha,1}(dy) - \int_{\mathcal{X}} \psi(y) Q^T \nu^n_{\alpha,2}(dy)\right| \\
&\qquad + \sum_{i=1}^{2} \|\psi\|_\infty (\mu^n_{\alpha,i} - \alpha\nu^n_{\alpha,i})(\mathcal{X}) \\
&\quad \le \left|\int_{\mathcal{X}} \psi(y) Q^T \nu^n_{\alpha,1}(dy) - \int_{\mathcal{X}} \psi(y) Q^T \nu^n_{\alpha,2}(dy)\right| + 2\varepsilon\|\psi\|_\infty.
\end{aligned} \tag{3.16}$$

In the last inequality, we have used the fact that $1 - \alpha < \varepsilon$. Summarizing, from Lemma 2, (3.15), (3.16) and (3.14), we obtain that

$$\limsup_{T\uparrow\infty}\left|\int_{\mathcal{X}} \psi(y) Q^T(x_1, dy) - \int_{\mathcal{X}} \psi(y) Q^T(x_2, dy)\right| \le 2\eta_n\|\psi\|_\infty + 2\varepsilon\|\psi\|_\infty + \varepsilon.$$

Since $\varepsilon > 0$ and n were arbitrarily chosen, we conclude that (3.10) follows.

PROOF OF LEMMA 3. First, we show that $\Delta \neq \varnothing$. Let $z \in \mathcal{X}$ be such that for every $\delta > 0$ and $x \in \mathcal{X}$, condition (2.2) is satisfied. Equicontinuity of $(P_t\psi)_{t\ge0}$ at $z \in \mathcal{X}$ implies the existence of $\sigma > 0$ such that

$$|P_t\psi(z) - P_t\psi(y)| < \varepsilon/2 \qquad \text{for } y \in B(z,\sigma) \text{ and } t \ge 0. \tag{3.17}$$

By (2.2), there exist $\beta > 0$ and $T_0 > 0$ such that

$$Q^T(x_i, B(z,\sigma)) \ge \beta \qquad \forall T \ge T_0, i = 1, 2. \tag{3.18}$$

Set $\alpha := \beta$ and $T_{\alpha,n} = n + T_0$ for $n \in \mathbb{N}$, $\mu^n_{\alpha,i} := Q^{T_{\alpha,n}}(x_i)$ and $\nu^n_{\alpha,i}(\cdot) := \mu^n_{\alpha,i}(\cdot|B(z,\sigma))$ for $i = 1, 2$ and $n \ge 1$. Note that $\mu^n_{\alpha,i}(B(z,\sigma)) > 0$, thanks to (3.18). The measures $\nu^n_{\alpha,i}$, $i = 1, 2$, are supported in $B(z,\sigma)$ and, therefore, for all $t \ge 0$, we have

$$\begin{aligned}
&\left|\int_{\mathcal{X}} \psi(x) P_t^*\nu^n_{\alpha,1}(dx) - \int_{\mathcal{X}} \psi(x) P_t^*\nu^n_{\alpha,2}(dx)\right| \\
&\quad = \left|\int_{\mathcal{X}} P_t\psi(x)\nu^n_{\alpha,1}(dx) - \int_{\mathcal{X}} P_t\psi(x)\nu^n_{\alpha,2}(dx)\right| \\
&\quad \le \left|\int_{\mathcal{X}} [P_t\psi(x) - P_t\psi(z)]\nu^n_{\alpha,1}(dx)\right| \\
&\qquad + \left|\int_{\mathcal{X}} [P_t\psi(x) - P_t\psi(z)]\nu^n_{\alpha,2}(dx)\right| \overset{(3.17)}{<} \varepsilon.
\end{aligned}$$

Hence, (3.14) follows. Clearly, conditions (3.10)–(3.13) are also satisfied. Thus, $\Delta \neq \varnothing$.

Next, we show that $\sup \Delta = 1$. Suppose, contrary to our claim, that $\alpha_0 := \sup \Delta < 1$. Thanks to the previous step, we have $\alpha_0 > 0$. Let $(\alpha_n) \subset \Delta$ be such that $\lim_{n\to\infty} \alpha_n = \alpha_0$. Set $T_n := T_{\alpha_n,n}$, $\mu_{n,i} := \mu^n_{\alpha_n,i}$ and $\nu_{n,i} := \nu^n_{\alpha_n,i}$ for $n \geq 1$ and $i = 1, 2$. From conditions (3.12), (3.13) and the fact that the family $(Q^T(x_i))$ is tight for $i = 1, 2$, it follows that the sequences $(\mu_{n,i})$, $(\nu_{n,i})$, $i = 1, 2$, are also tight. Indeed, (3.12) clearly implies tightness of $(\mu_{n,i})$, $i = 1, 2$. As a consequence, for any $\varrho > 0$, there exists a compact set $K \subset \mathcal{X}$ such that $\mu_{n,i}(\mathcal{X} \setminus K) < \varrho$ for all $n \geq 1$, $i = 1, 2$. In turn, condition (3.13) implies that for sufficiently large n, we have

$$\nu_{n,i}(\mathcal{X} \setminus K) < \frac{2\mu_{n,i}(\mathcal{X} \setminus K)}{\alpha_0} < \frac{2\varrho}{\alpha_0}$$

and tightness of $(\nu_{n,i})$, $i = 1, 2$, follows. Therefore, without loss of generality, we may assume that the sequences $(\mu_{n,i})$, $(\nu_{n,i})$, $i = 1, 2$, are weakly convergent. The sequences

$$\bar{\mu}_{n,i} := \mu_{n,i} - \alpha_n \nu_{n,i}, \qquad n \geq 1, \tag{3.19}$$

are therefore also weakly convergent for $i = 1, 2$. The assumption that $\alpha_0 < 1$ implies that the respective limits are nonzero measures; we denote them by $\bar{\mu}_i$, $i = 1, 2$, correspondingly. Let $y_i \in \operatorname{supp} \bar{\mu}_i$, $i = 1, 2$. Analogously to the previous step, we may choose $\sigma > 0$ such that (3.17) is satisfied. By (2.2), we choose $T > 0$ and $\gamma > 0$ for which

$$Q^T\big(y_i, B(z, \sigma/2)\big) \geq \gamma \qquad \text{for } i = 1, 2. \tag{3.20}$$

Since the semigroup $(P_t)_{t\geq 0}$ is Feller, we may find $r > 0$ such that

$$Q^T(y, B(z, \sigma)) \geq \gamma/2 \qquad \text{for } y \in B(y_i, r) \text{ and } i = 1, 2. \tag{3.21}$$

Indeed, it suffices to choose $\phi \in \operatorname{Lip}_b(\mathcal{X})$ such that $\mathbf{1}_{B(z,\sigma/2)} \leq \phi \leq \mathbf{1}_{B(z,\sigma)}$. From (3.20), we have $\int_{\mathcal{X}} \phi(x) Q^T(y_i, dx) \geq \gamma$. The Feller property implies that there exists $r > 0$ such that, for $y \in B(y_i, r)$ and $i = 1, 2$, we have

$$Q^T(y, B(z, \sigma)) \geq \int_{\mathcal{X}} \phi(x) Q^T(y, dx) \geq \frac{\gamma}{2}.$$

Set $s_0 = \min\{\bar{\mu}_1(B(y_1, r)), \bar{\mu}_2(B(y_2, r))\} > 0$. Using part (iv) of Theorem 2.1, page 16 of [1], we may find $N \geq 1$ such that

$$\bar{\mu}_{n,i}(B(y_i, r)) > \frac{s_0}{2} \quad \text{and} \quad \alpha_n + s_0 \frac{\gamma}{4} > \alpha_0 \tag{3.22}$$

for $n \geq N$. We prove that $\alpha_0' := \alpha_0 + s_0\gamma/8$ also belongs to Δ, which obviously leads to a contradiction with the hypothesis that $\alpha_0 = \sup \Delta$. We construct sequences $(T_{\alpha_0',n})$, $(\mu^n_{\alpha_0',i})$ and $(\nu^n_{\alpha_0',i})$, $i = 1, 2$, that satisfy conditions (3.11)–(3.14) with α replaced by α_0'. Let $\widehat{\mu}_n^i(\cdot) := \bar{\mu}_{n,i}(\cdot|B(y_i, r))$, $i = 1, 2$, be the measure $\bar{\mu}_{n,i}$ conditioned on the respective balls $B(y_i, r)$, $i = 1, 2$. That is, if $\bar{\mu}_{n,i}(B(y_i, r)) \neq 0$, then we let

$$\widehat{\mu}_n^i(\cdot) := \frac{\bar{\mu}_{n,i}(\cdot \cap B(y_i, r))}{\bar{\mu}_{n,i}(B(y_i, r))}, \tag{3.23}$$

while if $\bar{\mu}_{n,i}(B(y_i, r)) = 0$, we just let $\widehat{\mu}_n^i(\cdot) := \delta_{y_i}$. Also, let $\tilde{\mu}_n^i(\cdot) := (Q^T \bar{\mu}_{n,i}) \times (\cdot|B(z, \sigma))$. From the above definition, it follows that

$$Q^T \mu_{n,i} \geq \frac{s_0 \gamma}{4} \tilde{\mu}_n^i + \alpha_n Q^T \nu_{n,i} \tag{3.24}$$

for $n \geq N$ and $i = 1, 2$. Indeed, note that from (3.22) and (3.23), we have

$$\bar{\mu}_{n,i}(B) \geq \frac{s_0}{2} \widehat{\mu}_n^i(B) \qquad \forall B \in \mathcal{B}(\mathcal{X}), \tag{3.25}$$

hence, also,

$$Q^T \bar{\mu}_{n,i}(B) \geq \frac{s_0}{2} Q^T \widehat{\mu}_n^i(B) \qquad \forall B \in \mathcal{B}(\mathcal{X}). \tag{3.26}$$

On the other hand, by Fubini's theorem, we obtain

$$\begin{aligned} Q^T \widehat{\mu}_n^i(B(z, \sigma)) &= T^{-1} \int_0^T \int_{\mathcal{X}} \mathbf{1}_{B(z,\sigma)}(x) P_s^* \widehat{\mu}_n^i(dx)\, ds \\ &= T^{-1} \int_0^T \int_{\mathcal{X}} P_s \mathbf{1}_{B(z,\sigma)}(x) \widehat{\mu}_n^i(dx)\, ds \\ &= \int_{\mathcal{X}} Q^T(x, B(z, \sigma)) \widehat{\mu}_n^i(dx) \\ &\overset{(3.23)}{=} \int_{B(y_i, r)} Q^T(x, B(z, \sigma)) \widehat{\mu}_n^i(dx) \overset{(3.21)}{\geq} \frac{\gamma}{2} \end{aligned}$$

and, consequently, (3.26) implies that

$$Q^T \bar{\mu}_{n,i}(B(z, \sigma)) \geq \frac{s_0 \gamma}{4}. \tag{3.27}$$

Hence, for any $B \in \mathcal{B}(\mathcal{X})$,

$$\begin{aligned} Q^T \mu_{n,i}(B) &\overset{(3.19)}{=} Q^T \bar{\mu}_{n,i}(B) + \alpha_n Q^T \nu_{n,i}(B) \\ &\geq Q^T \bar{\mu}_{n,i}\big(B \cap B(z, \sigma)\big) + \alpha_n Q^T \nu_{n,i}(B) \\ &\overset{(3.27)}{\geq} \frac{s_0 \gamma}{4} \tilde{\mu}_{n,i}(B) + \alpha_n Q^T \nu_{n,i}(B) \end{aligned}$$

and (3.24) follows. At this point, observe that, by virtue of (3.24), measures $Q^T \mu_{n,i}$ and $(s_0\gamma/4 + \alpha_n)^{-1}[(s_0\gamma/4)\tilde{\mu}_{n,i} + \alpha_n Q^T \nu_{n,i}]$ would satisfy (3.13), with α_0' in place of α, admitted them instead of $\mu^n_{\alpha_0',i}$ and $\nu^n_{\alpha_0',i}$, respectively. Condition (3.12) need not, however, hold in such case. To remedy this, we average

$Q^T \mu_{n,i}$ over a long time, using the operator Q^R corresponding to a sufficiently large $R > 0$, and use Lemma 2. More precisely, since $\eta_n > \|Q^{T_n}(x_i) - \mu_{n,i}\|_{\mathrm{TV}}$ [thus, also, $\eta_n > \|Q^{R,T,T_n}(x_i) - Q^{R,T}\mu_{n,i}\|_{\mathrm{TV}}$ for any $R > 0$], by Lemma 2, we can choose $R_n > T_n$ such that

$$\|Q^{R_n,T,T_n}(x_i) - Q^{R_n}(x_i)\|_{\mathrm{TV}} < \eta_n - \|Q^{R_n,T,T_n}(x_i) - Q^{R_n,T}\mu_{n,i}\|_{\mathrm{TV}}. \tag{3.28}$$

Let

$$T_{\alpha_0',n} := R_n, \qquad \mu^n_{\alpha_0',i} := Q^{R_n} Q^T \mu_{n,i} \tag{3.29}$$

and

$$\nu^n_{\alpha_0',i} := \left(\alpha_n + \frac{s_0\gamma}{4}\right)^{-1} Q^{R_n}\left(\alpha_n Q^T \nu_{n,i} + \frac{s_0\gamma}{4}\tilde{\mu}_{n,i}\right) \tag{3.30}$$

for $i = 1, 2$, $n \geq 1$. By virtue of (3.28), we immediately see that

$$\|Q^{T_{\alpha_0',n}}(x_i) - \mu^n_{\alpha_0',i}\|_{\mathrm{TV}} < \eta_n \qquad \forall n \geq 1.$$

Furthermore, from (3.24), positivity of Q^{R_n} and the definitions of α_0' and measures $\mu^n_{\alpha_0',i}$ $\nu^n_{\alpha_0',i}$, we obtain that

$$\mu^n_{\alpha_0',i} \geq \alpha_0' \nu^n_{\alpha_0',i} \qquad \forall n \geq N, i = 1, 2,$$

when N is chosen sufficiently large. To verify (3.14), note that from (3.30), it follows that

$$\begin{aligned}
&\left|\int_{\mathcal{X}} \psi(x) Q^S \nu^n_{\alpha_0',1}(dx) - \int_{\mathcal{X}} \psi(x) Q^S \nu^n_{\alpha_0',2}(dx)\right| \\
&\quad \leq \alpha_n\left(\alpha_n + \frac{s_0\gamma}{4}\right)^{-1} \\
&\qquad \times \left|\int_{\mathcal{X}} \psi(x) Q^{S,R_n,T}\nu_{n,1}(dx) - \int_{\mathcal{X}} \psi(x) Q^{S,R_n,T}\nu_{n,2}(dx)\right| \\
&\qquad + \frac{s_0\gamma}{4}\left(\alpha_n + \frac{s_0\gamma}{4}\right)^{-1}\left|\int_{\mathcal{X}} \psi(x) Q^{S,R_n}\tilde{\mu}_{n,1}(dx)\, ds\right. \\
&\qquad\qquad \left. - \int_{\mathcal{X}} \psi(x) Q^{S,R_n}\tilde{\mu}_{n,2}(dx)\right|
\end{aligned} \tag{3.31}$$

for all $S \geq 0$. Denote the integrals appearing in the first and the second terms on the right-hand side of (3.31) by $I(S)$ and $II(S)$, respectively. Condition (3.14) will follow if we could demonstrate that the upper limits, as $S \uparrow \infty$, of both of these terms are smaller than ε. To estimate $I(S)$, we use Lemma 2 and condition (3.14), which holds for $\nu_{n,i}$, $i = 1, 2$. We then obtain

$$\begin{aligned}\limsup_{S\uparrow\infty} I(S) \le \limsup_{S\uparrow\infty}&\left|\int_{\mathcal{X}} \psi(x) Q^{S,R_n,T}\nu_{n,1}(dx) - \int_{\mathcal{X}} \psi(x) Q^S \nu_{n,1}(dx)\right| \\ &+ \limsup_{S\uparrow\infty}\left|\int_{\mathcal{X}} \psi(x) Q^S \nu_{n,1}(dx) - \int_{\mathcal{X}} \psi(x) Q^S \nu_{n,2}(dx)\right| \\ &+ \limsup_{S\uparrow\infty}\left|\int_{\mathcal{X}} \psi(x) Q^{S,R_n,T}\nu_{n,2}(dx) - \int_{\mathcal{X}} \psi(x) Q^S \nu_{n,2}(dx)\right| < \varepsilon.\end{aligned}$$

On the other hand, since $\operatorname{supp}\tilde{\mu}_n^i \subset B(z,\sigma)$, $i = 1, 2$, we obtain, from equicontinuity condition (3.17),

$$\begin{aligned}II(S) = \frac{1}{SR_n}\bigg|\int_0^S \int_0^{R_n} \int_{\mathcal{X}}\int_{\mathcal{X}} \big(P_{s_1+s_2}\psi(x) - P_{s_1+s_2}\psi(x')\big)\,ds_1\,ds_2& \\ \times \tilde{\mu}_{n,1}(dx)\tilde{\mu}_{n,2}(dx')\bigg| \le \frac{\varepsilon}{2}.&\end{aligned}$$

Hence, (3.14) holds for $\nu^n_{\alpha_0',i}$, $i = 1, 2$, and function ψ. Summarizing, we have shown that $\alpha_0' \in \Delta$. However, we also have $\alpha_0' > \alpha_0 = \sup\Delta$, which is clearly impossible. Therefore, we conclude that $\sup\Delta = 1$. □

3.3. *Proof of Theorem* 2. Taking Theorem 1 into account, the proof of the first part of the theorem will be completed as soon as we can show that $\mathcal{T} = \mathcal{X}$. Note that condition (2.8) implies that $z \in \operatorname{supp}\mu_*$. Indeed, let B be a bounded set such that $\mu_*(B) > 0$. We can then write, for any $\delta > 0$ and $T > 0$,

$$\begin{aligned}\mu_*(B(z,\delta)) &= \int_{\mathcal{X}} Q^T(y, B(z,\delta))\mu_*(dy) \\ &= \liminf_{T\uparrow\infty}\int_{\mathcal{X}} Q^T(y, B(z,\delta))\mu_*(dy) \\ &\overset{\text{Fatou lem.}}{\ge} \int_{\mathcal{X}} \liminf_{T\uparrow\infty} Q^T(y, B(z,\delta))\mu_*(dy) \\ &\overset{(2.8)}{\ge} \inf_{y\in B}\liminf_{T\uparrow\infty} Q^T(y, B(z,\delta))\mu_*(B) > 0.\end{aligned}$$

According to Proposition 1, the above implies that $z \in \mathcal{T}$. Now, fix an arbitrary $x \in \mathcal{X}$. Let $\mathcal{C}_\varepsilon$ be the family of all closed sets $C \subset \mathcal{X}$ which possess a finite ε-net, that is, there exists a finite set, say $\{x_1, \dots, x_n\}$, for which $C \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$. To prove that the family $(Q^T(x))$ is tight, it suffices to show that for every $\varepsilon > 0$, there exists $C_\varepsilon \in \mathcal{C}_\varepsilon$ such that

$$\liminf_{T\uparrow\infty} Q^T(x, C_\varepsilon) > 1 - \varepsilon; \tag{3.32}$$

for more details, see, for example, pages 517 and 518 of [16]. In light of Lemma 2, this condition would follow if we could prove that for given $\varepsilon > 0$, $k \geq 1$ and $t_1, \dots, t_k \geq 0$, one can find $T_\varepsilon > 0$ and $C_\varepsilon \in \mathcal{C}_\varepsilon$ such that

$$Q^{T,t_1,\dots,t_k}(x, C_\varepsilon) > 1 - \varepsilon \qquad \forall T \geq T_\varepsilon. \tag{3.33}$$

Fix an $\varepsilon > 0$. Since $z \in \mathcal{T}$, we can find $C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2}$ such that (3.32) holds with $\varepsilon/2$ in place of ε and $x = z$. Let $\tilde{C} := C_{\varepsilon/2}^{\varepsilon/2}$ be the $\varepsilon/2$-neighborhood of $C_{\varepsilon/2}$.

LEMMA 4. *There exists $\sigma > 0$ such that*

$$\inf_{\nu \in \mathcal{M}_1(B(z,\sigma))} \liminf_{T \uparrow \infty} Q^T \nu(\tilde{C}) > 1 - \frac{3\varepsilon}{4}. \tag{3.34}$$

In addition, if σ is as above, then for any $k \geq 1$ and $t_1, \dots, t_k \geq 0$, we can choose T_* such that

$$\inf_{\nu \in \mathcal{M}_1(B(z,\sigma))} Q^{T,t_1,\dots,t_k} \nu(\tilde{C}) > 1 - \frac{3\varepsilon}{4} \qquad \forall T \geq T_*. \tag{3.35}$$

PROOF. The claim made in (3.34) follows if we can show that there exists $\sigma > 0$ such that

$$\liminf_{T \to +\infty} Q^T(y, \tilde{C}) > 1 - \frac{3\varepsilon}{4} \qquad \forall y \in B(z, \sigma). \tag{3.36}$$

To prove (3.36), suppose that ψ is a Lipschitz function such that $\mathbf{1}_{C_{\varepsilon/2}} \leq \psi \leq \mathbf{1}_{\tilde{C}}$. Since $(P_t\psi)_{t\geq 0}$ is equicontinuous at z, we can find $\sigma > 0$ such that $|P_t\psi(y) - P_t\psi(z)| < \varepsilon/4$ for all $y \in B(z, \sigma)$. We then have

$$Q^T(y, \tilde{C}) \geq \int_{\mathcal{X}} \psi(y') Q^T(y, dy') \geq \int_{\mathcal{X}} \psi(y') Q^T(z, dy') - \frac{\varepsilon}{4}$$

and, using (3.32), we conclude that

$$\liminf_{T \uparrow \infty} Q^T(y, \tilde{C}) \geq \liminf_{T \uparrow \infty} Q^T(z, C_{\varepsilon/2}) - \frac{\varepsilon}{4} > 1 - \frac{3\varepsilon}{4}. \tag{3.37}$$

Estimate (3.35) follows directly from (3.34) and Lemma 2. □

Let us return to the proof of Theorem 2. Let $\sigma > 0$ be as in the above lemma and let $\gamma > 0$ denote the supremum of all sums $\alpha_1 + \cdots + \alpha_k$ such that there exist $\nu_1, \dots, \nu_k \in \mathcal{M}_1(B(z, \sigma))$ and

$$Q^{t_1^0,\dots,t_{m_0}^0}(x) \geq \alpha_1 Q^{t_1^1,\dots,t_{m_1}^1} \nu_1 + \cdots + \alpha_k Q^{t_1^k,\dots,t_{m_k}^k} \nu_k \tag{3.38}$$

for some $t_1^0, \dots, t_{m_0}^0, \dots, t_1^k, \dots, t_{m_k}^k > 0$. In light of Lemma 4, to deduce (3.33), it is enough to show that $\gamma > 1 - \varepsilon/4$. Assume, therefore, that

$$\gamma \leq 1 - \frac{\varepsilon}{4}. \tag{3.39}$$

Let D be a bounded subset of X, let $T_* > 0$ be such that

$$Q^T(x, D) > 1 - \frac{\varepsilon}{8} \qquad \forall T \geq T_*, \tag{3.40}$$

and let

$$\alpha := \inf_{x \in D} \liminf_{T \uparrow \infty} Q^T(x, B(z, \sigma)) > 0. \tag{3.41}$$

Let $\alpha_1, \dots, \alpha_k > 0$, $t_1^0, \dots, t_{m_0}^0, \dots, t_1^k, \dots, t_{m_k}^k > 0$ and $\nu_1, \dots, \nu_k \in \mathcal{M}_1(B(z, \sigma))$ be such that

$$Q^{t_1^0, \dots, t_{m_0}^0}(x) \geq \alpha_1 Q^{t_1^1, \dots, t_{m_1}^1} \nu_1 + \cdots + \alpha_k Q^{t_1^k, \dots, t_{m_k}^k} \nu_k$$

and

$$\gamma - (\alpha_1 + \cdots + \alpha_k) < \frac{\alpha \varepsilon}{64}. \tag{3.42}$$

For a given $t \geq 0$, we let

$$\mu_t := Q^{t, t_1^0, \dots, t_{m_0}^0}(x) - \alpha_1 Q^{t, t_1^1, \dots, t_{m_1}^1} \nu_1 - \cdots - \alpha_k Q^{t, t_1^k, \dots, t_{m_k}^k} \nu_k.$$

By virtue of Lemma 2, we can choose $T_* > 0$ such that $\| Q^{t, t_1^0, \dots, t_{m_0}^0}(x) - Q^t(x) \|_{\mathrm{TV}} < \varepsilon/16$ for $t \geq T_*$. Thus, from (3.40), we obtain that for such t,

$$\begin{aligned} \mu_t(D) &> Q^t(x, D) - \| Q^{t, t_1^0, \dots, t_{m_0}^0}(x) - Q^t(x) \|_{\mathrm{TV}} - (\alpha_1 + \cdots + \alpha_k) \\ &\geq 1 - \frac{\varepsilon}{8} - \frac{\varepsilon}{16} - \gamma \overset{(3.39)}{\geq} \frac{\varepsilon}{16}. \end{aligned}$$

However, this means that for $t \geq T_*$,

$$\begin{aligned} \liminf_{T \uparrow \infty} Q^T \mu_t(B(z, \sigma)) &\overset{\text{Fatou lem.}}{\geq} \int_{\mathcal{X}} \liminf_{T \uparrow \infty} Q^T(y, B(z, \sigma)) \mu_t(dy) \\ &\geq \int_D \liminf_{T \uparrow \infty} Q^T(y, B(z, \sigma)) \mu_t(dy) \overset{(3.41)}{\geq} \frac{\alpha \varepsilon}{16}. \end{aligned}$$

Choose $T_* > 0$ such that

$$Q^T \mu_t(B(z, \sigma)) > \frac{\alpha \varepsilon}{32} \qquad \forall t, T \geq T_*. \tag{3.43}$$

Let $\nu(\cdot) := (Q^T \mu_t)(\cdot | B(z, \sigma))$. Of course, $\nu \in \mathcal{M}_1(B(z, \sigma))$. From (3.43) and the definitions of ν, μ_t, we obtain, however, that for t, T as above,

$$Q^{T, t, t_1^0, \dots, t_{m_0}^0}(x) \geq \alpha_1 Q^{T, t, t_1^1, \dots, t_{m_1}^1} \nu_1 + \cdots + \alpha_k Q^{T, t, t_1^k, \dots, t_{m_k}^k} \nu_k + \frac{\alpha \varepsilon}{32} \nu.$$

Hence, $\gamma \geq \alpha_1 + \cdots + \alpha_k + \alpha \varepsilon / 32$, which clearly contradicts (3.42).

Proof of the weak law of large numbers. Recall that $\mathbb{P}_\mu$ is the path measure corresponding to μ, the initial distribution of $(Z(t))_{t\geq 0}$. Let $\mathbb{E}_\mu$ be the corresponding expectation and $d_* := \int \psi \, d\mu_*$. It then suffices to show that

$$\lim_{T\to+\infty} \mathbb{E}_\mu\left[\frac{1}{T}\int_0^T \psi(Z(t))\,dt\right] = d_* \tag{3.44}$$

and

$$\lim_{T\to+\infty} \mathbb{E}_\mu\left[\frac{1}{T}\int_0^T \psi(Z(t))\,dt\right]^2 = d_*^2. \tag{3.45}$$

Equality (3.44) is an obvious consequence of weak-* mean ergodicity. To show (3.45), observe that the expression under the limit equals

$$\begin{aligned} &\frac{2}{T^2}\int_0^T\int_0^t\left(\int_{\mathcal{X}} P^s(\psi P_{t-s}\psi)\,d\mu\right)dt\,ds \\ &\qquad = \frac{2}{T^2}\int_0^T (T-s)\left(\int_{\mathcal{X}} P_s(\psi\Psi_{T-s})\,d\mu\right)ds, \end{aligned} \tag{3.46}$$

where

$$\Psi_t(x) := \int_{\mathcal{X}} \psi(y) Q^t(x,dy) = \frac{1}{t}\int_0^t P_s\psi(x)\,ds. \tag{3.47}$$

The following lemma then holds.

LEMMA 5. *For any $\varepsilon > 0$ and a compact set $K \subset X$, there exists $t_0 > 0$ such that*

$$\forall t \geq t_0 \qquad \sup_{x\in K}\left|\Psi_t(x) - \int_{\mathcal{X}} \psi\,d\mu_*\right| < \varepsilon. \tag{3.48}$$

PROOF. It suffices to show equicontinuity of $(\Psi_t)_{t\geq 0}$ on any compact set K. The proof then follows from pointwise convergence of Ψ_t to d_* as $t\to\infty$ and the Arzela–Ascoli theorem. The equicontinuity of the above family of functions is a direct consequence of the e-property and a simple covering argument. $\square$

Now, suppose that $\varepsilon > 0$. One can find a compact set K such that

$$\forall t \geq 0 \qquad Q^t\mu(K^c) < \varepsilon. \tag{3.49}$$

Then

$$\begin{aligned} &\left|\frac{2}{T^2}\int_0^T (T-s)\left(\int_{\mathcal{X}} P_s(\psi\Psi_{T-s})\,d\mu\right)ds - \frac{2d_*}{T^2}\int_0^T (T-s)\left(\int_{\mathcal{X}} P_s\psi\,d\mu\right)ds\right| \\ &\qquad \leq I + II, \end{aligned}$$

where

$$I := \frac{2}{T^2}\int_0^T (T-s)\left(\int_{\mathcal{X}} P_s\big(\psi(\Psi_{T-s}-d_*)\mathbf{1}_K\big)\,d\mu\right)ds$$

and

$$II := \frac{2}{T^2}\int_0^T (T-s)\left(\int_{\mathcal{X}} P_s\big(\psi(\Psi_{T-s}-d_*)\mathbf{1}_{K^c}\big)\,d\mu\right)ds.$$

According to Lemma 5, we can find t_0 such that (3.48) holds with the compact set K and $\varepsilon\|\psi\|_\infty^{-1}$. We then obtain $|I| \le \varepsilon$. Also, note that

$$|II| \le 2\|\psi\|_\infty(\|\psi\|_\infty + |d_*|)Q^T\mu(K^c) \stackrel{(3.49)}{<} 2\varepsilon\|\psi\|_\infty(\|\psi\|_\infty + |d_*|).$$

The limit on the right-hand side of (3.45) therefore equals

$$\begin{aligned}\lim_{T\to+\infty}&\frac{2d_*}{T^2}\int_0^T (T-s)\left(\int_{\mathcal{X}} P_s\psi\,d\mu\right)ds\\ &= \lim_{T\to+\infty}\frac{2d_*}{T^2}\int_0^T ds\int_0^s\left(\int_{\mathcal{X}} Q^{s'}\psi\,d\mu\right)ds' = d_*^2.\end{aligned}$$

4. Proof of Theorem 3. In what follows, we are going to verify the assumptions of Theorem 2. First, observe that (2.9) follows from (ii) and Chebyshev's inequality. The e-property implies equicontinuity of $(P_t\psi, t \ge 0)$ at any point for any bounded, Lipschitz function ψ. What remains to be shown, therefore, is condition (2.8). The rest of the proof is devoted to that objective. It will be given in five steps.

STEP I. We show that we can find a bounded Borel set B and a positive constant r^* such that

$$\liminf_{T\uparrow\infty} Q^T(x,B) > \frac{1}{2} \qquad \forall x \in \mathcal{K} + r^*B(0,1). \tag{4.1}$$

To prove this, observe, by (ii) and Chebyshev's inequality, that for every $y \in \mathcal{K}$, there exists a bounded Borel set B_y^0 such that $\liminf_{T\uparrow\infty} Q^T(y, B_y^0) > 3/4$. Let B_y be a bounded, open set such that $B_y \supset B_y^0$ and let $\psi \in C_b(\mathcal{X})$ be such that $\mathbf{1}_{B_y} \ge \psi \ge \mathbf{1}_{B_y^0}$. Since $(P_t\psi)_{t\ge0}$ is equicontinuous at y, we can find $r_y > 0$ such that $|P_t\psi(x) - P_t\psi(y)| < 1/4$ for all $x \in B(y, r_y)$ and $t \ge 0$. Therefore, we have

$$\begin{aligned}\liminf_{T\uparrow\infty} Q^T(x, B_y) &\ge \liminf_{T\uparrow\infty}\frac{1}{T}\int_0^T P_s\psi(x)\,ds\\ &\ge \liminf_{T\uparrow\infty}\frac{1}{T}\int_0^T P_s\psi(y)\,ds - \frac{1}{4}\\ &\ge \liminf_{T\uparrow\infty} Q^T(y, B_y^0) - \frac{1}{4} > \frac{1}{2}.\end{aligned}$$

Since the attractor is compact, we can find a finite covering $B(y_i, r_{y_i})$, $i = 1, \dots, N$, of $\mathcal{K}$. The claim made in (4.1) therefore holds for $B := \bigcup_{i=1}^N B_{y_i}$ and $r^* > 0$ sufficiently small so that $\mathcal{K} + r^* B(0,1) \subset \bigcup_{i=1}^N B(y_i, r_{y_i})$.

STEP II. Let $B \subset \mathcal{X}$ be as in Step I. We prove that for every bounded Borel set $D \subset \mathcal{X}$, there exists a $\gamma > 0$ such that

$$\liminf_{T\uparrow\infty} Q^T(x, B) > \gamma \qquad \forall x \in D. \tag{4.2}$$

From the fact that $\mathcal{K}$ is a global attractor for (2.12), for any $r > 0$ and a bounded Borel set D, there exists an $L > 0$ such that $Y^x(L) \in \mathcal{K} + \frac{r}{2} B(0,1)$ for all $x \in D$. By (2.13), we have

$$p(r, D) := \inf_{x\in D} \mathbb{P}\big(\|Z^x(L) - Y^x(L)\|_{\mathcal{X}} < r/2\big) > 0.$$

We therefore obtain that

$$P_L \mathbf{1}_{\mathcal{K}+rB(0,1)}(x) \geq p(r, D) \qquad \forall x \in D. \tag{4.3}$$

Let $r^* > 0$ be the constant given in Step I. Then

$$\begin{aligned}
&\liminf_{T\uparrow\infty} Q^T(x, B) \\
&\quad = \liminf_{T\uparrow\infty} \frac{1}{T}\int_0^T P_{s+L}\mathbf{1}_B(x)\, ds \\
&\quad = \liminf_{T\uparrow\infty} \frac{1}{T}\int_0^T P^*_{s+L}\delta_x(B)\, ds \\
&\quad = \liminf_{T\uparrow\infty} \frac{1}{T}\int_0^T \int_{\mathcal{X}} P_s\mathbf{1}_B(z) P^*_L\delta_z(dz)\, ds \\
&\quad \geq \liminf_{T\uparrow\infty} \frac{1}{T}\int_0^T \int_{\mathcal{K}+r^*B(0,1)} P_s\mathbf{1}_B(z) P^*_L\delta_x(dz)\, ds \qquad (4.4)\\
&\quad \overset{\substack{\text{Fubini}\\ \text{and Fatou}}}{\geq} \int_{\mathcal{K}+r^*B(0,1)} \liminf_{T\uparrow\infty} Q^T(z, B) P^*_L\delta_x(dz) \\
&\quad \overset{(4.1)}{\geq} \frac{1}{2}\int_{\mathcal{X}} \mathbf{1}_{\mathcal{K}+r^*B(0,1)}(z) P^*_L\delta_x(dz) \\
&\quad = \frac{1}{2} P_L \mathbf{1}_{\mathcal{K}+r^*B(0,1)}(x) \\
&\quad \overset{(4.3)}{\geq} \gamma := \frac{p(r^*, D)}{2} \qquad \forall x \in D.
\end{aligned}$$

STEP III. We show here that for every bounded Borel set $D \subset \mathcal{X}$ and any radius $r > 0$, there exists a $w > 0$ such that

$$\inf_{x \in D} \liminf_{T \uparrow \infty} Q^T\big(x, \mathcal{K} + rB(0,1)\big) > w. \tag{4.5}$$

We therefore fix $D \subset \mathcal{X}$ and $r > 0$. From Step II, we know that there exist a bounded set $B \subset \mathcal{X}$ and a positive constant $\gamma > 0$ such that (4.2) holds. By (2.13), we have, as in (4.4),

$$\begin{aligned}
&\liminf_{T \uparrow \infty} Q^T\big(x, \mathcal{K} + rB(0,1)\big) \\
&\quad = \liminf_{T \uparrow \infty} \frac{1}{T} \int_0^T \int_{\mathcal{X}} P_L \mathbf{1}_{\mathcal{K}+rB(0,1)}(z) P_s^* \delta_x(dz)\, ds \\
&\quad \overset{\text{Fubini}}{\geq} \liminf_{T \uparrow \infty} \int_B P_L \mathbf{1}_{\mathcal{K}+rB(0,1)}(z) Q^T(x, dz).
\end{aligned} \tag{4.6}$$

Using (4.3), we can further estimate the last right-hand side of (4.6) from below by

$$p(r, D) \liminf_{T \uparrow \infty} Q^T(x, B) \overset{(4.2)}{>} p(r, D)\gamma. \tag{4.7}$$

We therefore obtain (4.5) with $w = \gamma p(r, D)$.

STEP IV. Choose $z \in \bigcap_{y \in \mathcal{K}} \bigcup_{t \geq 0} \Gamma^t(y) \neq \varnothing$. We are going to show that for every $\delta > 0$, there exist a finite set of positive numbers S and a positive constant $\tilde{r}$ satisfying

$$\inf_{x \in \mathcal{K}+\tilde{r}B(0,1)} \max_{s \in S} P_s \mathbf{1}_{B(z,\delta)}(x) > 0. \tag{4.8}$$

Let $t_x > 0$ for $x \in \mathcal{K}$ be such that $z \in \operatorname{supp} P_{t_x}^* \delta_x$. By the Feller property of $(P_t)_{t \geq 0}$, we may find, for any $x \in \mathcal{K}$, a positive constant r_x such that

$$P_{t_x}^* \delta_y(B(z,\delta)) \geq P_{t_x}^* \delta_x(B(z,\delta))/2 \qquad \text{for } y \in B(x, r_x). \tag{4.9}$$

Since $\mathcal{K}$ is compact, we may choose $x_1, \ldots, x_p \in \mathcal{K}$ such that $\mathcal{K} \subset \bigcup_{i=1}^p B_i$, where $B_i = B(x_i, r_{x_i})$ for $i = 1, \ldots, p$. Choose $\tilde{r} > 0$ such that $\mathcal{K} + \tilde{r}B(0,1) \subset \bigcup_{i=1}^p B_i$.

STEP V. Fix a bounded Borel subset $D \subset \mathcal{X}$, $z \in \bigcap_{y \in \mathcal{K}} \bigcup_{t \geq 0} \Gamma^t(y)$ and $\delta > 0$. Let a positive constant $\tilde{r}$ and a finite set S be such that (4.8) holds. Set

$$u := \inf_{x \in \mathcal{K}+\widehat{r}B(0,1))} \max_{s \in S} P_s \mathbf{1}_{B(z,\delta)}(x) > 0. \tag{4.10}$$

From Step III, it follows that there exists $w > 0$ such that (4.5) holds for $r = \tilde{r}$.

Denote by $\#S$ the cardinality of S. We can easily check that

$$\begin{aligned}\liminf_{T\uparrow\infty}\sum_{q\in S}\frac{1}{T}\int_0^T P_{q+s}\mathbf{1}_{B(z,\delta)}(x)\,ds \\ = \#S\liminf_{T\uparrow\infty} Q^T(x,B(z,\delta)) \qquad \forall x\in D.\end{aligned}\tag{4.11}$$

On the other hand, we have

$$\begin{aligned}&\sum_{q\in S}\frac{1}{T}\int_0^T P_{q+s}\mathbf{1}_{B(z,\delta)}(x)\,ds\\ &\quad=\int_{\mathcal{X}}\sum_{q\in S}P_q\mathbf{1}_{B(z,\delta)}(y)Q^T(x,dy)\\ &\quad\ge\int_{\mathcal{K}+\tilde{r}B(0,1)}\sum_{q\in S}P_q\mathbf{1}_{B(z,\delta)}(y)Q^T(x,dy)\\ &\quad\overset{(4.10)}{\ge} uQ^T\bigl(x,\mathcal{K}+\tilde{r}B(0,1)\bigr) \qquad \forall x\in D.\end{aligned}\tag{4.12}$$

Combining (4.5) with (4.12), we obtain

$$\liminf_{T\uparrow\infty}\sum_{q\in S}\frac{1}{T}\int_0^T P_{q+s}\mathbf{1}_{B(z,\delta)}(x)\,ds > uw \qquad \forall x\in D,$$

and, finally, by (4.11),

$$\liminf_{T\uparrow\infty} Q^T(x,B(z,\delta)) > uw/\#S \qquad \forall x\in D.$$

This shows that condition (2.8) is satisfied with $\alpha = uw/\#S$.

5. Ergodicity of the Lagrangian observation process. This section is in preparation for the proof of Theorem 4. Given an $r\ge 0$, we denote by $\mathcal{X}^r$ the Sobolev space which is the completion of

$$\left\{x\in C^\infty(\mathbb{T}^d;\mathbb{R}^d):\int_{\mathbb{T}^d}x(\xi)\,d\xi=0, \widehat{x}(k)\in \operatorname{Im}\mathcal{E}(k), \forall k\in\mathbb{Z}^d_*\right\}$$

with respect to the norm

$$\|x\|^2_{\mathcal{X}^r} := \sum_{k\in\mathbb{Z}^d_*}|k|^{2r}|\widehat{x}(k)|^2,$$

where

$$\widehat{x}(k) := (2\pi)^{-d}\int_{\mathbb{T}^d}x(\xi)e^{-i\xi\cdot k}\,d\xi, \qquad k\in\mathbb{Z}^d,$$

are the Fourier coefficients of x. Note that $\mathcal{X}^u\subset\mathcal{X}^r$ if $u>r$.

Let A_r be an operator on $\mathcal{X}^r$ defined by

$$\widehat{A_r x}(k) := -\gamma(k)\widehat{x}(k), \qquad k\in\mathbb{Z}^d_*,\tag{5.1}$$

with the domain

$$D(A_r) := \left\{ x \in \mathcal{X}^r : \sum_{k \in \mathbb{Z}^d_*} |\gamma(k)|^2 |k|^{2r} |\widehat{x}(k)|^2 < \infty \right\}. \tag{5.2}$$

Since the operator is self-adjoint, it generates a C_0-semigroup $(S_r(t))_{t \geq 0}$ on $\mathcal{X}^r$. Moreover, for $u > r$, A_u is the restriction of A_r and S_u is the restriction of S_r. From now on, we will omit the subscript r when it causes no confusion, writing A and S instead of A_r and S_r, respectively.

Let Q be a symmetric positive definite bounded linear operator on

$$\left\{ x \in L^2(\mathbb{T}^d, d\xi; \mathbb{R}^d) : \int_{\mathbb{T}^d} x(\xi)\, d\xi = 0 \right\}$$

given by

$$\widehat{Qx}(k) := \gamma(k)\mathcal{E}(k)\widehat{x}(k), \qquad k \in \mathbb{Z}^d_*.$$

Let m be the constant appearing in (2.16) and let $\mathcal{X} := \mathcal{X}^m$ and $\mathcal{V} := \mathcal{X}^{m+1}$. Note that, by Sobolev embedding (see, e.g., Theorem 7.10, page 155 of [9]), $\mathcal{X} \hookrightarrow C^1(\mathbb{T}^d, \mathbb{R}^d)$ and hence there exists a constant $C > 0$ such that

$$\|x\|_{C^1(\mathbb{T}^d; \mathbb{R}^d)} \leq C\|x\|_{\mathcal{X}} \qquad \forall x \in \mathcal{X}. \tag{5.3}$$

For any $t > 0$, the operator $S(t)$ is bounded from any $\mathcal{X}^r$ to $\mathcal{X}^{r+1}$. Its norm can be easily estimated by

$$\|S(t)\|_{L(\mathcal{X}^r, \mathcal{X}^{r+1})} \leq \sup_{k \in \mathbb{Z}^d_*} |k| e^{-\gamma(k)t}.$$

Let $e_k(x) := e^{ik\cdot x}$, $k \in \mathbb{Z}^d$. The Hilbert–Schmidt norm of the operator $S(t)Q^{1/2}$ (see Appendix C of [2]) is given by

$$\begin{aligned} \|S(t)Q^{1/2}\|^2_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{V})} &:= \sum_{k \in \mathbb{Z}^d} \|S(t)Q^{1/2}e_k\|^2_{\mathcal{V}} \\ &= \sum_{k \in \mathbb{Z}^d} |k|^{2(m+1)} \gamma(k) e^{-2\gamma(k)t} \operatorname{Tr} \mathcal{E}(k). \end{aligned}$$

Taking into account assumptions (2.16) and (2.17), we easily obtain the following lemma.

LEMMA 6. (i) *For each $t > 0$, the operator $Q^{1/2}S(t)$ is Hilbert–Schmidt from $\mathcal{X}$ to $\mathcal{V}$ and there exists $\beta \in (0, 1)$ such that*

$$\int_0^\infty t^{-\beta} \|S(t)Q^{1/2}\|^2_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{V})}\, dt < \infty.$$

(ii) *For any $r \geq 0$ and $t > 0$, the operator $S(t)$ is bounded from $\mathcal{X}^r$ into $\mathcal{X}^{r+1}$ and*

$$\int_0^\infty \|S(t)\|_{L(\mathcal{X}^r, \mathcal{X}^{r+1})}\, dt < \infty.$$

Let $W = (W(t))_{t \ge 0}$ be a cylindrical Wiener process in $\mathcal{X}$ defined on a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. By Lemma 6(i) and Theorem 5.9, page 127 of [2], for any $x \in \mathcal{X}$, there exists a unique, continuous in t, $\mathcal{X}$-valued process V^x solving, in the mild sense, the Ornstein–Uhlenbeck equation

$$dV^x(t) = AV^x(t)\,dt + Q^{1/2}\,dW(t), \qquad V^x(0) = x. \tag{5.4}$$

Moreover, (5.4) defines a Markov family on $\mathcal{X}$ (see Section 9.2 of [2]) and the law $\mathcal{L}(V(0,\cdot))$ of $V(0,\cdot)$ on $\mathcal{X}$ is its unique invariant probability measure (see Theorem 11.7 of [2]). Note that, since $m > d/2 + 1$, for any fixed t, the realization of $V^x(t,\xi)$ is Lipschitz in the ξ variable. If the filtered probability space $\mathfrak{A}$ is sufficiently rich, that is, if there exists an $\mathcal{F}_0$-measurable random variable with law $\mathcal{L}(V(0,\cdot))$, then the stationary solution to (5.4) can be found as a stochastic process over $\mathfrak{A}$. Its law on the space of trajectories $C([0,\infty) \times \mathbb{T}^d; \mathbb{R}^d)$ coincides with the law of $(V(t,\cdot))_{t\ge 0}$.

5.1. *An evolution equation describing the environment process*. Since the realizations of $V^x(t,\cdot)$ are Lipschitz in the spatial variable, equation (2.15), with $V^x(t,\xi)$ in place of $V(t,\xi)$, has a unique solution $\mathbf{x}_x(t)$, $t \ge 0$, for given initial data $\mathbf{x}_0$. In fact, with no loss of generality, we may, and shall, assume that $\mathbf{x}_0 = 0$. In what follows, we will also denote by $\mathbf{x}$ the solution of (2.15) corresponding to the stationary right-hand side V. Let $\mathcal{Z}(s,\xi) := V(s, \xi + \mathbf{x}(s))$ be the *Lagrangian observation of the environment process* or, in short, *the observation process*. It is known (see [7] and [13]) that $\mathcal{Z}(s,\cdot)$ solves the equations

$$\begin{aligned} &d\mathcal{Z}(t) = [A\mathcal{Z}(t) + B(\mathcal{Z}(t), \mathcal{Z}(t))]\,dt + Q^{1/2}\,d\tilde{W}(t), \\ &\mathcal{Z}(0,\cdot) = V\big(0, \mathbf{x}(0) + \cdot\big), \end{aligned} \tag{5.5}$$

where $\tilde{W}$ is a certain cylindrical Wiener process on the original probability space $\mathfrak{A}$ and

$$\begin{aligned} &B(\psi,\phi)(\xi) := \left(\sum_{j=1}^d \psi_j(0)\,\frac{\partial \phi_1}{\partial \xi_j}(\xi), \ldots, \sum_{j=1}^d \psi_j(0)\,\frac{\partial \phi_d}{\partial \xi_j}(\xi)\right), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \psi, \phi \in \mathcal{X}, \xi \in \mathbb{T}^d. \end{aligned} \tag{5.6}$$

By (5.3), $B(\cdot,\cdot)$ is a continuous bilinear form mapping from $\mathcal{X} \times \mathcal{X}$ into $\mathcal{X}^{m-1}$.

For a given an $\mathcal{F}_0$-measurable random variable Z_0 which is square-integrable in $\mathcal{X}$ and a cylindrical Wiener process W in $\mathcal{X}$, consider the SPDE

$$dZ(t) = [AZ(t) + B(Z(t), Z(t))]\,dt + Q^{1/2}\,dW(t), \qquad Z(0) = Z_0. \tag{5.7}$$

Taking into account Lemma 6(ii), the local existence and uniqueness of a mild solution follow by a standard Banach fixed point argument. For a different type of argument, based on the Euler approximation scheme, see Section 4.2 of [7]. Global existence also follows; see the proof of the moment estimates in Section 5.1.2 below.

Given $x \in \mathcal{X}$, let $Z^x(t)$ denote the value at $t \geq 0$ of a solution to (5.7) satisfying $Z^x(0,\xi) = x(\xi)$, $\xi \in \mathbb{T}^d$. Since the existence of a solution follows from the Banach fixed point argument, $Z = (Z^x, x \in \mathcal{X})$ is a stochastically continuous Markov family and its transition semigroup $(P_t)_{t\geq 0}$ is Feller; for details see, for example, [2] or [23]. Note that

$$P_t\psi(x) := \mathbb{E}\psi\big(V^x\big(t, \mathbf{x}_x(t) + \cdot\big)\big).$$

The following result on ergodicity of the observation process, besides being of independent interest, will be crucial for the proof of Theorem 4.

THEOREM 5. *Under assumptions* (2.16) *and* (2.17), *the transition semigroup* $(P_t)_{t\geq 0}$ *for the family* $Z = (Z^x, x \in \mathcal{X})$ *is weak-* mean ergodic.*

To prove the above theorem, we verify the hypotheses of Theorem 3.

5.1.1. *Existence of a global attractor.* Note that $Y^0(t) \equiv 0$ is the global attractor for the semi-dynamical system $Y = (Y^x, x \in \mathcal{X})$ defined by the deterministic problem

$$\frac{dY^x(t)}{dt} = AY^x(t) + B(Y^x(t), Y^x(t)), \qquad Y^x(0) = x. \tag{5.8}$$

Clearly, this guarantees the uniqueness of an invariant measure ν_* for the corresponding semi-dynamical system; see Definition 2.4. Our claim follows from the exponential stability of Y^0, namely,

$$\forall x \in \mathcal{X}, t > 0 \qquad \|Y^x(t)\|_{\mathcal{X}} \leq e^{-\gamma_* t}\|x\|_{\mathcal{X}}, \tag{5.9}$$

where

$$\gamma_* = \inf_{k \in \mathbb{Z}^d_*} \gamma(k) \tag{5.10}$$

is strictly positive by (2.17). Indeed, differentiating $\|Y^x(t)\|^2_{\mathcal{X}}$ over t, we obtain

$$\frac{d}{dt}\|Y^x(t)\|^2_{\mathcal{X}} = 2\langle AY^x(t), Y^x(t)\rangle_{\mathcal{X}} + 2\sum_{j=1}^d Y^x(t,0)\left\langle \frac{\partial Y^x(t)}{\partial \xi_j}, Y^x(t)\right\rangle_{\mathcal{X}}.$$

The last term on the right-hand side vanishes, while the first one can be estimated from above by $-2\gamma_*\|Y^x(t)\|^2_{\mathcal{X}}$. Combining these observations with Gronwall's inequality, we obtain (5.9).

5.1.2. *Moment estimates.* Let $B(0, R)$ be the ball in $\mathcal{X}$ with center at 0 and radius R. We will show that for any $R > 0$ and any integer $n \geq 1$,

$$\sup_{x \in B(0,R)} \sup_{t \geq 0} \mathbb{E}\|Z^x(t)\|_{\mathcal{X}}^{2n} < \infty. \tag{5.11}$$

Recall that V^x is the solution to (5.4) satisfying $V^x(0) = x$. Let $\mathbf{x}_x = (\mathbf{x}_x(t), t \geq 0)$ solve the problem

$$\frac{d\mathbf{x}_x}{dt}(t) = V^x(t, \mathbf{x}_x(t)), \qquad \mathbf{x}_x(0) = 0. \tag{5.12}$$

We then obtain

$$\|Z^x(t)\|_{\mathcal{X}}^2 \stackrel{d}{=} \int_{\mathbb{T}^d} \big|\nabla^m V^x\big(t, \mathbf{x}_x(t) + \xi\big)\big|^2 \, d\xi = \|V^x(t)\|_{\mathcal{X}}^2, \tag{5.13}$$

where the first equality means equality in law. Since V^x is Gaussian, there is a constant $C_1 > 0$ such that

$$\mathbb{E}\|V^x(t)\|_{\mathcal{X}}^{2n} \leq C_1(\mathbb{E}\|V^x(t)\|_{\mathcal{X}}^2)^n.$$

Hence, there is a constant $C_2 > 0$ such that for $\|x\|_{\mathcal{X}} \leq R$,

$$\begin{aligned} \mathbb{E}\|V^x(t)\|_{\mathcal{X}}^{2n} &\leq C_2(1 + R^{2n})\left(\int_0^t \|S(t-s)Q^{1/2}\|_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{X})}^2 \, ds\right)^n \\ &\leq C_2(1 + R^{2n})\left(\int_0^\infty \|S(s)Q^{1/2}\|_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{X})}^2 \, ds\right)^n. \end{aligned}$$

Note that there is a constant C_3 such that

$$\int_0^\infty \|S(s)Q^{1/2}\|_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{X})}^2 \, ds \leq C_3 |||\mathcal{E}|||^2 < \infty,$$

where $|||\mathcal{E}|||^2$ appears in (2.16) and (5.11) indeed follows.

5.1.3. *Stochastic stability.* Define $\tilde{Z}^x(t) := V^x(t, \mathbf{x}_x(t) + \cdot)$. This satisfies equation (5.5) and so the laws of $(\tilde{Z}^x(t))_{t \geq 0}$ and $(Z^x(t))_{t \geq 0}$ are identical. On the other hand, for $\tilde{V}^x(t) := S(t)x$ and

$$\frac{d\mathbf{y}_x}{dt} = \tilde{V}^x(t, \mathbf{y}_x(t)), \qquad \mathbf{y}(0) = 0,$$

we have that $Y^x(t, \cdot) := \tilde{V}^x(t, \mathbf{y}_x(t) + \cdot)$ satisfies (5.8). To show stochastic stability, it suffices to prove that

$$\forall \varepsilon, R, T > 0 \qquad \inf_{\|x\|_{\mathcal{X}} \leq R} \mathbb{P}\big(\|\tilde{Z}^x(T) - Y^x(T)\|_{\mathcal{X}} < \varepsilon\big) > 0. \tag{5.14}$$

Let $M = (M(t))_{t \geq 0}$ be the stochastic convolution process

$$M(t) := \int_0^t S(t-s)Q^{1/2} \, dW(s), \qquad t \geq 0. \tag{5.15}$$

It is a centered, Gaussian, random element in the Banach space $C([0,T],\mathcal{X})$ whose norm we denote by $\|\cdot\|_\infty$. We will use the same notation for the norm on $C[0,T]$. Note that $V^x(t)=\tilde V^x(t)+M(t)$. Since M is a centered, Gaussian, random element in the Banach space $C([0,T],\mathcal{X})$, its topological support is a closed linear subspace; see, for example, [28], Theorem 1, page 61. Thus, in particular, 0 belongs to the support of its law and

$$\forall\delta>0\qquad q:=\mathbb{P}(F_\delta)>0, \tag{5.16}$$

where $F_\delta:=[\|M\|_\infty<\delta]$. Since $\|V^x-\tilde V^x\|_\infty<\delta$ on F_δ, we can choose δ sufficiently small so that $\|\mathbf{x}_x-\mathbf{y}_x\|_\infty<\rho$, where ρ is chosen in such a way that

$$\begin{aligned}\|\tilde Z^x(T)&-Y^x(T)\|_{\mathcal{X}}\\ &\le\big\|V^x\big(T,\mathbf{x}_x(T)+\cdot\big)-\tilde V^x\big(T,\mathbf{x}_x(T)+\cdot\big)\big\|_{\mathcal{X}}\\ &\quad+\big\|\tilde V^x\big(T,\mathbf{x}_x(T)+\cdot\big)-\tilde V^x\big(T,\mathbf{y}_x(T)+\cdot\big)\big\|_{\mathcal{X}}<\varepsilon\qquad\forall x\in B(0,R)\end{aligned}$$

on F_δ. Hence, (5.14) follows.

5.1.4. *e-property of the transition semigroup.* It suffices to show that for any $\psi\in C_b^1(\mathcal{X})$ and $R>0$, there exists a positive constant C such that

$$\sup_{t\ge0}\sup_{\|x\|_{\mathcal{X}}\le R}\|DP_t\psi(x)\|_{\mathcal{X}}\le C\|\psi\|_{C_b^1(\mathcal{X})}. \tag{5.17}$$

Here, $D\phi$ denotes the Fréchet derivative of a given function $\phi\in C_b^1(\mathcal{X})$. Indeed, let $\rho_n\in C_0^2(\mathbb{R}^n)$ be supported in the ball of radius $1/n$, centered at 0 and such that $\int_{\mathbb{R}^n}\rho_n(\xi)\,d\xi=1$. Suppose that (e_n) is an orthonormal base in $\mathcal{X}$ and Q_n is the orthonormal projection onto $\operatorname{span}\{e_1,\dots,e_n\}$. Define

$$\psi_n(x):=\int_{\mathbb{R}^n}\rho_n(Q_nx-\xi)\psi\left(\sum_{i=1}^n\xi_ie_i\right)d\xi,\qquad x\in\mathcal{X}.$$

One can deduce (see part 2 of the proof of Theorem 1.2, pages 164 and 165 in [22]) that for any $\psi\in\operatorname{Lip}(\mathcal{X})$, the sequence (ψ_n) satisfies $(\psi_n)\subset C_b^1(\mathcal{X})$ and $\lim_{n\to\infty}\psi_n(x)=\psi(x)$ pointwise. In addition, $\|\psi_n\|_{L^\infty}\le\|\psi\|_{L^\infty}$ and $\sup_z\|D\psi_n(z)\|_{\mathcal{X}}\le\operatorname{Lip}(\psi)$. Let $R>0$ be arbitrary and $x,y\in B(0,R)$. We can write

$$\begin{aligned}|P_t\psi(x)-P_t\psi(y)|&=\lim_{n\to\infty}|P_t\psi_n(x)-P_t\psi_n(y)|\\ &\le\sup_{\|z\|_{\mathcal{X}}\le R}\|DP_t\psi_n(z)\|_{\mathcal{X}}\|x-y\|_{\mathcal{X}}\\ &\stackrel{(5.17)}{\le}C\|\psi_n\|_{C_b^1(\mathcal{X})}\|x-y\|_{\mathcal{X}}\\ &\le C[\|\psi\|_\infty+\operatorname{Lip}(\psi)]\|x-y\|_{\mathcal{X}}.\end{aligned}$$

This shows equicontinuity of $(P_t\psi)_{t\ge 0}$ for an arbitrary Lipschitz function ψ in the neighborhood of any x and the e-property follows.

To prove (5.17), we adopt the method from [11]. First, note that $DP_t\psi(x)[v]$, the value of $DP_t\psi(x)$ at $v\in\mathcal{X}$, is equal to $\mathbb{E}\{D\psi(Z^x(t))[U(t)]\}$, where $U(t):=\partial Z^x(t)[v]$ and

$$\partial Z^x(t)[v]:=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\big(Z^{x+\varepsilon v}(t)-Z^x(t)\big),$$

the limit here taken in $L^2(\Omega,\mathcal{F},\mathbb{P};\mathcal{X})$. The process $U=(U(t),t\ge 0)_{t\ge 0}$ satisfies the linear evolution equation

$$\begin{aligned}\frac{dU(t)}{dt}&=AU(t)+B(Z^x(t),U(t))+B(U(t),Z^x(t)),\\ U(0)&=v.\end{aligned}\tag{5.18}$$

Suppose that H is a certain Hilbert space and $\Phi:\mathcal{X}\to H$ a Borel measurable function. Given an $(\mathcal{F}_t)_{t\ge 0}$-adapted process $g:[0,\infty)\times\Omega\to\mathcal{X}$ satisfying $\mathbb{E}\int_0^t\|g_s\|^2_{\mathcal{X}}\,ds<\infty$ for each $t\ge 0$, we denote by $\mathcal{D}_g\Phi(Z^x(t))$ the Malliavin derivative of $\Phi(Z^x(t))$ in the direction of g. That is, the $L^2(\Omega,\mathcal{F},\mathbb{P};H)$-limit, if exists, of

$$\mathcal{D}_g\Phi(Z^x(t)):=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}[\Phi(Z^x_{\varepsilon g}(t))-\Phi(Z^x(t))],$$

where $Z^x_g(t)$, $t\ge 0$, solves the equation

$$\begin{aligned}dZ^x_g(t)&=[AZ^x_g(t)+B(Z^x_g(t),Z^x_g(t))]\,dt+Q^{1/2}\big(dW(t)+g_t\,dt\big),\\ Z^x_g(0)&=x.\end{aligned}$$

In particular, one can easily show that when $H=\mathcal{X}$ and $\Phi=I$, where I is the identity operator, the Malliavin derivative of $Z^x(t)$ exists and the process $D(t):=\mathcal{D}_gZ^x(t)$, $t\ge 0$, solves the linear equation

$$\begin{aligned}\frac{dD}{dt}(t)&=AD(t)+B(Z^x(t),D(t))+B(D(t),Z^x(t))+Q^{1/2}g(t),\\ D(0)&=0.\end{aligned}\tag{5.19}$$

The following two facts about the Malliavin derivative will be crucial for us in the sequel. Directly from the definition of the Malliavin derivative, we derive *the chain rule*: if we suppose that $\Phi\in C^1_b(\mathcal{X};H)$, then

$$\mathcal{D}_g\Phi(Z^x(t))=D\Phi(Z^x(t))[D(t)].\tag{5.20}$$

In addition, the *integration by parts formula* holds; see Lemma 1.2.1, page 25 of [21]. If we suppose that $\Phi\in C^1_b(\mathcal{X})$, then

$$\mathbb{E}[\mathcal{D}_g\Phi(Z^x(t))]=\mathbb{E}\bigg[\Phi(Z^x(t))\int_0^t\langle g(s),Q^{1/2}\,dW(s)\rangle_{\mathcal{X}}\bigg].\tag{5.21}$$

We also have the following proposition.

PROPOSITION 2. *For any given $v, x \in \mathcal{X}$ such that $\|v\|_{\mathcal{X}} \le 1$, $\|x\|_{\mathcal{X}} \le R$, one can find an $(\mathcal{F}_t)$-adapted $\mathcal{X}$-valued process $g_t = g_t(v, x)$ that satisfies*

$$\sup_{\|v\|_{\mathcal{X}}\le 1} \sup_{\|x\|_{\mathcal{X}}\le R} \int_0^\infty \mathbb{E}\|Q^{1/2} g_s\|_{\mathcal{X}}^2\, ds < \infty, \tag{5.22}$$

$$\sup_{\|v\|_{\mathcal{X}}\le 1} \sup_{\|x\|_{\mathcal{X}}\le R} \sup_{t\ge 0} \mathbb{E}\|DZ^x(t)[v] - \mathcal{D}_g Z^x(t)\|_{\mathcal{X}} < \infty. \tag{5.23}$$

We prove this proposition shortly. First, however, let us demonstrate how it can be used to complete the argument for the e-property. Let $\omega_t(x) := \mathcal{D}_g Z^x(t)$ and $\rho_t(v, x) := DZ^x(t)[v] - \mathcal{D}_g Z^x(t)$. Then

$$\begin{aligned} DP_t\psi(x)[v] &= \mathbb{E}\{D\psi(Z^x(t))[\omega_t(x)]\} + \mathbb{E}\{D\psi(Z^x(t))[\rho_t(v,x)]\} \\ &= \mathbb{E}\{\mathcal{D}_g\psi(Z^x(t))\} + \mathbb{E}\{D\psi(Z^x(t))[\rho_t(v,x)]\} \\ &\overset{(5.21)}{=} \mathbb{E}\left\{\psi(Z^\xi(t)) \int_0^t \langle g(s), Q^{1/2}\, dW(s)\rangle_{\mathcal{X}}\right\} \\ &\qquad + \mathbb{E}\{D\psi(Z^x(t))[\rho_t(v,x)]\}. \end{aligned}$$

We have

$$\left|\mathbb{E}\left\{\psi(Z^x(t)) \int_0^t \langle g(s), Q^{1/2}\, dW(s)\rangle_{\mathcal{X}}\right\}\right| \le \|\psi\|_{L^\infty}\left(\mathbb{E}\int_0^\infty \|Q^{1/2} g(s)\|_{\mathcal{X}}^2\, ds\right)^{1/2}$$

and

$$|\mathbb{E}\{D\psi(Z^x(t))[\rho_t(v,x)]\}| \le \|\psi\|_{C_b^1(\mathcal{X})} \mathbb{E}\|\rho_t(v,x)\|_{\mathcal{X}}.$$

Hence, by (5.22) and (5.23), we derive the desired estimate (5.17) with

$$\begin{aligned} C = {} & \left(\mathbb{E}\int_0^\infty \|Q^{1/2} g(s)\|_{\mathcal{X}}^2\, ds\right)^{1/2} \\ & + \sup_{\|v\|_{\mathcal{X}}\le 1} \sup_{\|x\|_{\mathcal{X}}\le R} \sup_{t\ge 0} \mathbb{E}\|DZ^x(t)[v] - \mathcal{D}_g Z^x(t)\|_{\mathcal{X}}. \end{aligned}$$

Therefore, the e-process property would be shown if we could prove Proposition 2.

5.1.5. *Proof of Proposition* 2. Let us denote by $\Pi_{\ge N}$ the orthogonal projection onto $\mathrm{span}\{ze^{ik\xi} : |k| \ge N, z \in \mathrm{Im}\,\mathcal{E}(k)\}$ and let $\Pi_{<N} := I - \Pi_{\ge N} = \Pi_{\ge N}^{\perp}$. Write

$$A_N := \Pi_{\ge N} A, \qquad Q_N := \Pi_{\ge N} Q, \qquad A_N^{\perp} := \Pi_{<N} A, \qquad Q_N^{\perp} := \Pi_{<N} Q.$$

Given an integer N, let $\zeta^N(v, x)(t)$ be the solution of the problem

$$\begin{aligned} \frac{d\zeta^N}{dt}(t) &= A_N\zeta^N(t) + \Pi_{\ge N}\big(B(Z^x(t), \zeta^N(t)) + B(\zeta^N(t), Z^x(t))\big) \\ &\quad - \frac{1}{2}\Pi_{<N}\zeta^N(t)\|\Pi_{<N}\zeta^N(t)\|_{\mathcal{X}}^{-1}, \\ \zeta^N(0) &= v. \end{aligned} \tag{5.24}$$

We adopt the convention that

$$\Pi_{<N}\zeta^N\|\Pi_{<N}\zeta^N\|_{\mathcal{X}}^{-1} := 0 \qquad \text{if } \Pi_{<N}\zeta^N = 0. \tag{5.25}$$

Let

$$g := Q^{-1/2} f, \tag{5.26}$$

where

$$\begin{aligned} f(t) :=\ & A_N^{\perp}\zeta^N(v,x)(t) \\ & + \Pi_{<N}[B(Z^x(t),\zeta^N(v,x)(t)) + B(\zeta^N(v,x)(t), Z^x(t))] \\ & + \tfrac{1}{2}\Pi_{<N}\zeta^N(v,x)(t)\|\Pi_{<N}\zeta^N(v,x)(t)\|_{\mathcal{X}}^{-1} \end{aligned} \tag{5.27}$$

and where N will be specified later. Note that f takes values in a finite-dimensional space, where Q is invertible, by the definition of the space $\mathcal{X}$. Recall that $\rho_t(v,x) := DZ^x(t)[v] - \mathcal{D}_g Z^x(t)$. We have divided the proof into a sequence of lemmas.

LEMMA 7. *We have*

$$\rho_t(v,x) = \zeta^N(v,x)(t) \qquad \forall t \geq 0. \tag{5.28}$$

PROOF. Adding $f(t)$ to both sides of (5.24), we obtain

$$\begin{aligned} & \frac{d\zeta^N(v,x)}{dt}(t) + f(t) \\ & \qquad = A\zeta^N(v,x)(t) + B(Z^x(t),\zeta^N(v,x)(t)) \\ & \qquad\qquad + B(\zeta^N(v,x)(t), Z^x(t)), \\ & \zeta^N(v,x)(0) = v. \end{aligned} \tag{5.29}$$

Recall that $DZ^x(t)[v]$ and $\mathcal{D}_g Z^x(t)$ obey equations (5.18) and (5.19), respectively. Hence, $\rho_t := \rho_t(v,x)$ satisfies

$$\begin{aligned} \frac{d\rho_t}{dt} &= A\rho_t + B(Z^x(t),\rho_t) + B(\rho_t, Z^x(t)) - Q^{1/2}g(t), \\ \rho_0 &= v. \end{aligned}$$

Since $f(t) = Q^{1/2}g_t$, we conclude that ρ_t and $\zeta^N(v,x)(t)$ solve the same linear evolution equation with the same initial value. Thus, the assertion of the lemma follows. □

LEMMA 8. *For each $N \geq 1$, we have $\Pi_{<N}\zeta^N(v,x)(t) = 0$ for all $t \geq 2$.*

PROOF. Applying $\Pi_{<N}$ to both sides of (5.24), we obtain

$$\begin{aligned} &\frac{d}{dt}\Pi_{<N}\zeta^N(v,x)(t) = -\frac{1}{2}\|\Pi_{<N}\zeta^N(v,x)(t)\|_{\mathcal{X}}^{-1}\Pi_{<N}\zeta^N(v,x)(t), \\ &\zeta^N(v,x)(0) = v. \end{aligned} \tag{5.30}$$

Multiplying both sides of (5.30) by $\Pi_{<N}\zeta^N(v,x)(t)$, we obtain that $z(t) := \|\Pi_{<N}\zeta^N(v,x)(t)\|_{\mathcal{X}}^2$ satisfies

$$\frac{dz}{dt}(t) = -\frac{1}{2}\sqrt{z(t)}. \tag{5.31}$$

Since $\|v\|_{\mathcal{X}} \le 1$, $z(0) \in (0,1]$ and the desired conclusion holds from elementary properties of the solution of the ordinary differential equation (5.31). □

LEMMA 9. *For any $R > 0$, the following hold*:
(i) *for any N*,

$$\sup_{\|v\|_{\mathcal{X}}\le 1}\ \sup_{\|x\|_{\mathcal{X}}\le R}\ \sup_{t\in[0,2]} \mathbb{E}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^4 < \infty; \tag{5.32}$$

(ii) *there exists an $N_0 \in \mathbb{N}$ such that for any $N \ge N_0$*,

$$\sup_{\|v\|_{\mathcal{X}}\le 1}\ \sup_{\|x\|_{\mathcal{X}}\le R} \int_0^\infty (\mathbb{E}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^4)^{1/2}\,dt < \infty \tag{5.33}$$

and

$$\sup_{\|v\|_{\mathcal{X}}\le 1}\ \sup_{\|x\|_{\mathcal{X}}\le R}\ \sup_{t\ge 0} \mathbb{E}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^4 < \infty. \tag{5.34}$$

Since the proof of the lemma is quite lengthy and technical, we postpone its presentation until the next section. However, we can now complete the proof of Proposition 2.

First, we assume that f is given by (5.27) with an arbitrary $N \ge N_0$, where N_0 appears in the formulation of Lemma 9. By Lemma 7, $\rho_t(v,x) = \zeta^N(v,x)(t)$. Of course, (5.34) implies (5.23). We show (5.22). As a consequence of Lemma 8, we have $\Pi_N\zeta^N(v,x)(t) = 0$ for $t \ge 2$. The definition of the form $B(\cdot,\cdot)$ [see (5.6)] and the fact that the partial derivatives commute with the projection operator $\Pi_{<N}$ together imply that

$$\Pi_{<N}B(Z^x(t),\zeta^N(v,x)(t)) = B(Z^x(t),\Pi_{<N}\zeta^N(v,x)(t)).$$

As a consequence of Lemma 8 and convention (5.25), we conclude from (5.27) that

$$f(t) = \Pi_{<N}B(\zeta^N(v,x)(t), Z^x(t)) = B(\zeta^N(v,x)(t), \Pi_{<N}Z^x(t)) \qquad \forall t \ge 2.$$

By (5.3), for $t \ge 2$, we have

$$\begin{aligned}\|f(t)\|_{\mathcal{X}} &\le C\|\zeta^N(v,x)(t)\|_{\mathcal{X}}\|\Pi_{<N}Z^x(t)\|_{\mathcal{X}^{m+1}}\\ &\le CN\|\zeta^N(v,x)(t)\|_{\mathcal{X}}\|\Pi_{<N}Z^x(t)\|_{\mathcal{X}}\\ &\le CN\|\zeta^N(v,x)(t)\|_{\mathcal{X}}\|Z^x(t)\|_{\mathcal{X}}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\mathbb{E}\int_2^\infty \|Q^{1/2}g_t(v,x)\|_{\mathcal{X}}^2\,dt\\ &\quad=\mathbb{E}\int_2^\infty \|f(t)\|_{\mathcal{X}}^2\,dt\\ &\quad\le C^2N^2\sup_{t\ge 2}(\mathbb{E}\|Z^x(t)\|_{\mathcal{X}}^4)^{1/2}\int_2^\infty(\mathbb{E}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^4)^{1/2}\,dt.\end{aligned}$$

Hence, by (5.11) and (5.33), we obtain

$$\sup_{\|x\|_{\mathcal{X}}\le R,\|v\|_{\mathcal{X}}\le 1}\mathbb{E}\int_2^\infty \|Q^{1/2}g_t(v,x)\|_{\mathcal{X}}^2\,dt<\infty.$$

Clearly, by Lemma 9(i) and (5.11), we have

$$\sup_{\|x\|_{\mathcal{X}}\le R,\|v\|_{\mathcal{X}}\le 1}\mathbb{E}\int_0^2 \|Q^{1/2}g_t(v,x)\|_{\mathcal{X}}^2\,dt<\infty$$

and the proof of (5.22) is completed.

5.1.6. *Proof of Lemma* 9. Recall that for any r, A is a self-adjoint operator when considered on the space $\mathcal{X}^r$, and that

$$\langle A\psi,\psi\rangle_{\mathcal{X}^r}\le -\gamma_*\|\psi\|_{\mathcal{X}^r}^2,\qquad \psi\in D(A),\tag{5.35}$$

where $\gamma_*>0$ was defined in (5.10). Recall that V^x is the solution to the Ornstein–Uhlenbeck equation (5.4) starting from x and that $\mathbf{x}_x$ is the corresponding solution to (5.12). The laws of the processes $(Z^x(t))_{t\ge 0}$ and $(V^x(t,\cdot+\mathbf{x}_x(t)))_{t\ge 0}$ are the same. By virtue of this and the fact that $\|V^x(t,\cdot+\mathbf{x}_x(t))\|_{\mathcal{X}}=\|V^x(t)\|_{\mathcal{X}}$, we obtain that for each $N\ge 1$ and $r\ge 0$,

$$\mathcal{L}((\|\Pi_{\ge N}V^x(t)\|_{\mathcal{X}^r})_{t\ge 0})=\mathcal{L}((\|\Pi_{\ge N}Z^x(t)\|_{\mathcal{X}^r})_{t\ge 0}),\tag{5.36}$$

where, as we recall, $\mathcal{L}$ stands for the law of the respective process.

In order to show the first part of the lemma, note that from (5.24), upon scalar multiplication (in $\mathcal{X}$) of both sides by $\zeta^N(v,x)(t)$ and use of (5.35), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2\\ &\quad\le -\gamma_*\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2\\ &\qquad+|\zeta^N(v,x)(t,0)|\|Z^x(t)\|_{\mathcal{V}}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}+\frac{1}{2}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}.\end{aligned}$$

Here, as we recall, $\mathcal{V} = \mathcal{X}^{m+1}$. Taking into account (5.3) and the rough estimate $a/2 \le 1 + a^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 \le (1-\gamma_*)\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 + C\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2\|Z^x(t)\|_{\mathcal{V}} + 1.$$

Using Gronwall's inequality and (5.36), we obtain

$$\begin{aligned}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 &\le (\|v\|_{\mathcal{X}}^2 + t)\exp\left\{2(1-\gamma_*)t + 2C\int_0^t \|Z^x(s)\|_{\mathcal{V}}\,ds\right\}\\ &\le (1+t)\exp\left\{2(1-\gamma_*)t + C\int_0^t \|V^x(s)\|_{\mathcal{V}}\,ds\right\}\\ &\le (1+t)\exp\left\{2(1-\gamma_*)t + C\int_0^t \big(\|S(s)x\|_{\mathcal{V}} + \|M(s)\|_{\mathcal{V}}\big)\,ds\right\},\end{aligned}$$

where $M = V^0$ is given by (5.15). By Lemma 6(ii),

$$\sup_{\|x\|_{\mathcal{X}}\le 1}\int_0^\infty \|S(s)x\|_{\mathcal{V}}\,ds < \infty.$$

Thus, the proof of the first part of the lemma will be completed as soon as we can show that

$$\mathbb{E}\exp\left\{C\int_0^2 \|M(s)\|_{\mathcal{V}}\,ds\right\} < \infty. \tag{5.37}$$

By Lemma 6, M is a Gaussian element in $C([0,2],\mathcal{V})$. Therefore, (5.37) is a direct consequence of the Fernique theorem (see, e.g., [2]).

To prove the second part of the lemma, first observe that for any $N \ge 1$,

$$\begin{aligned}&\langle \Pi_{\ge N} B(Z^x(t), \zeta^N(v,x)(t)), \zeta^N(v,x)(t)\rangle_{\mathcal{X}}\\ &\quad= \langle B(Z^x(t), \Pi_{\ge N}\zeta^N(v,x)(t)), \Pi_{\ge N}\zeta^N(v,x)(t)\rangle_{\mathcal{X}}\\ &\quad= 0.\end{aligned}$$

Multiplying both sides of (5.24) by $\zeta^N(v,x)(t)$ and remembering that $\Pi_{<N}\zeta^N(v,x)(t) = 0$ for $t \ge 2$, we obtain that, for those times,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2\\ &\quad\le -\gamma_*\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 + |\zeta^N(v,x)(t,0)|\,\|\Pi_{\ge N}Z^x(t)\|_{\mathcal{V}}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}\\ &\quad\overset{(5.3)}{\le} -\gamma_*\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 + C\|\Pi_{\ge N}Z^x(t)\|_{\mathcal{V}}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2\\ &\quad\le -\gamma_*\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2\\ &\qquad + C\big(\|\Pi_{\ge N}S(t)x\|_{\mathcal{V}}^2 + \|\Pi_{\ge N}Z^0(t)\|_{\mathcal{V}}\big)\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2.\end{aligned}$$

Define

$$h(z) = \frac{z^2}{\sqrt{1+\gamma_*^{-1}|z|^2}}, \qquad z \ge 0.$$

Note that there exists a constant $\tilde{C}$ such that

$$Cz\zeta^2 \le \frac{\gamma_*}{2}\zeta^2 + \frac{\tilde{C}}{4}h(z)\zeta^2, \qquad z \ge 0, \zeta \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} &C\|\Pi_{\ge N}Z^0(t)\|_{\mathcal{V}}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 \\ &\quad \le \frac{\gamma_*}{2}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 + \frac{\tilde{C}}{4}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 h(\|\Pi_{\ge N}Z^0(t)\|_{\mathcal{V}}). \end{aligned}$$

Using Gronwall's inequality, we obtain, for $t \ge 2$,

$$\begin{aligned} &\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^2 \\ &\quad \le \|\zeta^N(v,x)(2)\|_{\mathcal{X}}^2 \exp\biggl\{-\gamma_*(t-2) + L\|x\|_{\mathcal{X}}^2 \\ &\qquad\qquad + \frac{\tilde{C}}{2}\int_2^t h(\|\Pi_{\ge N}Z^0(s)\|_{\mathcal{V}})\,ds\biggr\}, \end{aligned}$$

where $L := 2C\int_2^\infty \|S(t)\|_{L(\mathcal{X},\mathcal{V})}\,dt$. We have, therefore, by virtue of the Cauchy–Schwarz inequality,

$$\begin{aligned} &\mathbb{E}\|\zeta^N(v,x)(t)\|_{\mathcal{X}}^4 \\ &\quad \le \mathbb{E}\|\zeta^N(v,x)(2)\|_{\mathcal{X}}^4 \mathbb{E}\exp\biggl\{-2\gamma_*(t-2) + 2L\|x\|_{\mathcal{X}}^2 \\ &\qquad\qquad + \tilde{C}\int_2^t h(\|\Pi_{\ge N}Z^0(s)\|_{\mathcal{V}})\,ds\biggr\} \\ &\quad \overset{(5.36)}{=} \mathbb{E}\|\zeta^N(v,x)(2)\|_{\mathcal{X}}^4 \mathbb{E}\exp\biggl\{-2\gamma_*(t-2) + 2L\|x\|_{\mathcal{V}}^2 \\ &\qquad\qquad + \tilde{C}\int_2^t h(\|\Pi_{\ge N}M(s)\|_{\mathcal{V}})\,ds\biggr\}, \end{aligned}$$

where M is given by (5.15). Write

$$\Psi_N(t) := \exp\biggl\{\tilde{C}\int_0^t h(\|\Pi_{\ge N}M(s)\|_{\mathcal{V}})\,ds\biggr\}.$$

The proof of part (ii) of the lemma will be completed as soon as we can show that there exists an N_0 such that, for all $N \ge N_0$,

$$\sup_{t\ge 0} e^{-4\gamma_*(t-2)}\mathbb{E}\Psi_N(t) < \infty \quad \text{and} \quad \int_2^\infty e^{-2\gamma_* t}(\mathbb{E}\Psi_N(t))^{1/2}\,dt < \infty.$$

To do this, it is enough to show that

$$\forall \kappa > 0, \exists N(\kappa) \geq 1 \qquad \text{such that } \sup_{t\geq 0} e^{-\kappa t}\mathbb{E}\Psi_N(t) < \infty \qquad \forall N \geq N(\kappa). \tag{5.38}$$

To do this, note that for any $N_1 > N$, $M_{N,N_1}(t) := \Pi_{<N_1}\Pi_{\geq N} M(t)$ is a strong solution to the equation

$$dM_{N,N_1}(t) = A_N M_{N,N_1}(t)\,dt + \Pi_{<M}\Pi_{\geq N} Q^{1/2} dW(t).$$

Therefore, we can apply the Itô formula to $M_{N,N_1}(t)$ and the function

$$H(x) = \big(1 + \|(-A_N)^{-1/2}x\|_{\mathcal{V}}^2\big)^{1/2}.$$

As a result, we obtain

$$\begin{aligned}
&\big(1 + \|(-A_N)^{-1/2}M_{N,N_1}(t)\|_{\mathcal{V}}^2\big)^{1/2} \\
&\quad = 1 - \int_0^t \|M_{N,N_1}(s)\|_{\mathcal{V}}^2\big(1 + \|(-A_N)^{-1/2}M_{N,N_1}(s)\|_{\mathcal{V}}^2\big)^{-1/2}\,ds \\
&\qquad + \frac{1}{2}\int_0^t \|H''((-A_N)^{-1/2}M_{N,N_1}(s))\Pi_{<N_1}\Pi_{\geq N} Q^{1/2}\|_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{V})}^2\,ds \\
&\qquad + \int_0^t \big(1 + \|(-A_N)^{-1/2}M_{N,N_1}(s)\|_{\mathcal{V}}^2\big)^{-1/2} \\
&\qquad\qquad \times \langle(-A_N)^{-1}M_{N,N_1}(s), \Pi_{<N_1}\Pi_{\geq N} Q^{1/2} dW(s)\rangle_{\mathcal{V}}.
\end{aligned}$$

Taking into account the spectral gap property of A, we obtain

$$\begin{aligned}
&\|M_{N,N_1}(s)\|_{\mathcal{V}}^2\big(1 + \|(-A_N)^{-1/2}M_{N,N_1}(s)\|_{\mathcal{V}}^2\big)^{-1/2} \\
&\quad \geq \|M_{N,N_1}(s)\|_{\mathcal{V}}^2\big(1 + \gamma_*^{-1}\|M_{N,N_1}(s)\|_{\mathcal{V}}^2\big)^{-1/2} \\
&\quad = h(\|M_{N,N_1}(s)\|_{\mathcal{V}}).
\end{aligned}$$

Therefore,

$$\tilde{C}\int_0^t h(\|M_{N,N_1}(s)\|_{\mathcal{V}})\,ds \leq \tilde{C} + \mathcal{M}_{N,N_1}(t) + \mathcal{R}_{N,N_1}(t),$$

where

$$\begin{aligned}
\mathcal{M}_{N,N_1}(t) = \tilde{C}&\int_0^t \big(1 + \|(-A_N)^{-1/2}M_{N,N_1}(s)\|_{\mathcal{V}}^2\big)^{-1/2} \\
&\qquad \times \langle(-A_N)^{-1}M_{N,N_1}(s), \Pi_{<N_1}\Pi_{\geq N} Q^{1/2}\,dW(s)\rangle_{\mathcal{V}} \\
&- \frac{\tilde{C}^2}{2}\int_0^t \big(1 + \|(-A_N)^{-1/2}M_{N,N_1}(s)\|_{\mathcal{V}}^2\big)^{-1} \\
&\qquad \times \|Q^{1/2}(-A_N)^{-1}M_{N,N_1}(s)\|_{\mathcal{V}}^2\,ds
\end{aligned}$$

and

$$\mathcal{R}_{N,N_1}(t) = \frac{\tilde{C}}{2}\int_0^t \|H''((-A_N)^{-1/2}M_{N,N_1}(s))\Pi_{<N_1}\Pi_{\geq N}Q^{1/2}\|^2_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{V})}\,ds$$
$$+ \frac{\tilde{C}^2}{2}\int_0^t \bigl(1+\|(-A_N)^{-1/2}M_{N,N_1}(s)\|^2_{\mathcal{V}}\bigr)^{-1}$$
$$\times \|Q^{1/2}(-A_N)^{-1}M_{N,N_1}(s)\|^2_{\mathcal{V}}\,ds.$$

Since

$$H''(x) = \bigl(1+\|(-A_N)^{-1/2}x\|^2_{\mathcal{V}}\bigr)^{-1/2}(-A_N)^{-1}$$
$$- \bigl(1+\|(-A_N)^{-1/2}x\|^2_{\mathcal{V}}\bigr)^{-3/2}(-A_N)^{-1}x$$
$$\otimes (-A_N)^{-1}x,$$

there exists a constant C_1 such that for all N, N_1 and t,

$$\mathcal{R}_{N,N_1}(t) \leq tC_1\bigl(\|(-A_N)^{-1}\|^4_{L(\mathcal{X},\mathcal{V})}\|Q^{1/2}\|^2_{L(\mathcal{X},\mathcal{V})}$$
$$+ \|(-A_N)^{-1}Q^{1/2}\|^2_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{V})}\bigr)$$

for all $N_1 > N$. Let $\kappa > 0$. We can choose sufficiently large N_0 such that for $N \geq N_0$,

$$C_1\bigl(\|(-A_N)^{-1}\|^4_{L(\mathcal{X},\mathcal{V})}\|Q^{1/2}\|^2_{L(\mathcal{X},\mathcal{V})} + \|(-A_N)^{-1}Q^{1/2}\|^2_{L_{(\mathrm{HS})}(\mathcal{X},\mathcal{V})}\bigr) \leq \kappa.$$

Since $(\exp\{\mathcal{M}_{N_0,N_1}(t)\})$ is a martingale, we have shown, therefore, that for $N \geq N_0$,

$$\mathbb{E}\int_0^t \exp\{h(\|M_{N,N_1}(s)\|_{\mathcal{V}})\,ds\} \leq \exp\{\tilde{C} + \kappa t\}.$$

Letting $N_1 \to \infty$, we obtain (5.38).

6. Proof of Theorem 4. With no loss of generality, we will assume that the initial position of the tracer $\mathbf{x}_0 = 0$. By definition,

$$\mathbf{x}(t) = \int_0^t V(s,\mathbf{x}(s))\,ds = \int_0^t \mathcal{Z}(s,0)\,ds,$$

where $\mathcal{Z}(t,x) = V(t,\mathbf{x}(t)+x)$ is the observation process. Recall that $\mathcal{Z}(t)$ is a stationary solution to (5.5). Obviously, uniqueness and the law of a stationary solution do not depend on the particular choice of the Wiener process. Therefore,

$$\mathcal{L}\biggl(\frac{\mathbf{x}(t)}{t}\biggr) = \mathcal{L}\biggl(\frac{1}{t}\int_0^t \tilde{Z}(s,0)\,ds\biggr)$$

and

$$\mathcal{L}\left(\frac{d\mathbf{x}}{dt}(t)\right) = \mathcal{L}(\tilde{Z}(t,0)),$$

where, as before, $\mathcal{L}(X)$ stands for the law of a random element X and $\tilde{Z}$ is, by Theorem 5, a unique (in law) stationary solution of the equation

$$d\tilde{Z}(t) = [A\tilde{Z}(t) + B(\tilde{Z}(t), \tilde{Z}(t))]\,dt + Q^{1/2}\,dW(t).$$

Let $F : \mathcal{X} \to \mathbb{R}$ be given by $F(x) = x(0)$. The proof of the first part of the theorem will be completed as soon as we can show that the limit (in probability)

$$\mathbb{P}\text{-}\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \tilde{Z}(s,0)\,ds$$

exists and is equal to $\int_{\mathcal{X}} F(x)\mu_*(dx)$, where μ_* is the unique invariant measure for the Markov family Z defined by (5.7). Since the semigroup $(P_t)_{t\geq 0}$ satisfies the e-property and is weak-* mean ergodic, part (2) of Theorem 2 implies that for any bounded Lipschitz continuous function ψ,

$$\mathbb{P}\text{-}\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \psi(\tilde{Z}(s))\,ds = \int_{\mathcal{X}} \psi(x)\mu_*(dx).$$

Since $\mathcal{X}$ is embedded in the space of bounded continuous functions, F is Lipschitz. The theorem then follows by an easy truncation argument.

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