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INVARIANCE PRINCIPLES FOR OPERATOR-SCALING GAUSSIAN RANDOM FIELDS

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Recently, Hammond and Sheffield [*Probab. Theory Related Fields* **157** (2013) 691–719] introduced a model of correlated one-dimensional random walks that scale to fractional Brownian motions with long-range dependence. In this paper, we consider a natural generalization of this model to dimension $d \geq 2$. We define a $\mathbb{Z}^d$-indexed random field with dependence relations governed by an underlying random graph with vertices $\mathbb{Z}^d$, and we study the scaling limits of the partial sums of the random field over rectangular sets. An interesting phenomenon appears: depending on how fast the rectangular sets increase along different directions, different random fields arise in the limit. In particular, there is a critical regime where the limit random field is operator-scaling and inherits the full dependence structure of the discrete model, whereas in other regimes the limit random fields have at least one direction that has either invariant or independent increments, no longer reflecting the dependence structure in the discrete model. The limit random fields form a general class of operator-scaling Gaussian random fields. Their increments and path properties are investigated.

1. Introduction. Self-similar processes are important in probability theory because of their connections with limit theorems and their intensive use in modeling; see, for example, [50]. These are processes $(X(t))_{t\in\mathbb{R}}$ that satisfy, for some $H > 0$,

$$(X(\lambda t))_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{=} \lambda^H (X(t))_{t\in\mathbb{R}} \qquad \text{for all } \lambda > 0, \tag{1}$$

where "$\stackrel{\text{f.d.d.}}{=}$" stands for "equal in finite-dimensional distributions." It is well known that the only Gaussian processes that are self-similar and have stationary increments are the fractional Brownian motions. Throughout, we let $(B_H(t))_{t\in\mathbb{R}}$ denote a fractional Brownian motion with Hurst index $H \in (0, 1)$; this is a zero-mean Gaussian process with covariances given by

$$\operatorname{Cov}(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}.$$

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Fractional Brownian motions were first introduced in 1940 by Kolmogorov [25] and their relevance was first recognized by Mandelbrot and Van Ness [32], who gave them their name. Invariance principles for fractional Brownian motions have a long history, since the seminal work of Davydov [15] and Taqqu [49]. As the limiting objects of stochastic models, fractional Brownian motions have appeared in various areas, including random walks in random environment [19], telecommunication processes [35], interacting particle systems [36] and finance [24], just to mention a few.

Recently, Hammond and Sheffield [22] proposed a simple discrete model that scales to fractional Brownian motions with $H > 1/2$. This model, to be described below, can be interpreted as a strongly correlated random walk with ± 1 jumps. As the simple random walk can be viewed as the discrete counterpart of the Brownian motion, the correlated random walks proposed in [22] can be viewed as the discrete counterparts of the fractional Brownian motions for $H > 1/2$. In this regime, the fractional Brownian motion is well known to exhibit long-range dependence [45].

In the present paper, we introduce a discrete random field model that generalizes the Hammond–Sheffield model to any dimension $d \geq 2$ and we study the scaling limits. Based on this model, we establish invariance principles for a new class of operator-scaling Gaussian random fields. The operator-scaling random fields are generalization of self-similar processes (1) to random fields, proposed by Biermé, Meerschaert and Scheffler [8]. Namely, for a matrix E with all eigenvalues having positive real parts, the random field $(X(\boldsymbol{t}))_{\boldsymbol{t}\in\mathbb{R}^d}$ is said to be (E, H)-operator-scaling for some $H > 0$, if

$$(2) \qquad \big(X\big(\lambda^E \boldsymbol{t}\big)\big)_{\boldsymbol{t}\in\mathbb{R}^d} \overset{\text{f.d.d.}}{=} \lambda^H \big(X(\boldsymbol{t})\big)_{\boldsymbol{t}\in\mathbb{R}^d} \qquad \text{for all } \lambda > 0,$$

where $\lambda^E := \sum_{k\geq 0}(\log\lambda)^k E^k/k!$. In this paper, we focus on the case that E is a $d \times d$ diagonal matrix with diagonal entries $\beta_1, \dots, \beta_d$, denoted by $E = \operatorname{diag}(\beta_1, \dots, \beta_d)$. It is worth mentioning that a simple generalization of the self-similarity would be to take E being the identity matrix in (2), and the advantage of taking a general diagonal matrix is to be able to accommodate anisotropic random fields. Examples of operator-scaling Gaussian random fields include fractional Brownian sheets [23] and Lévy Brownian sheets [46]. Here, our results provide a new class to this family with corresponding invariance principles. We also mention that there are other well investigated generalizations of fractional Brownian motions to Gaussian random fields, including distribution-valued ones. See, for example, [6, 30, 47].

We now give a brief description of the Hammond–Sheffield model and its generalization to high dimensions. Let us start with the one-dimensional model. Let μ be a probability distribution with support in $\{1, 2, \dots\}$ that is assumed to be aperiodic (to be defined below). Using the sites of $\mathbb{Z}$ as vertices, one defines a random directed graph $\mathcal{G}_\mu$ by sampling independently one directed edge on each site. The edge starting at the site $i \in \mathbb{Z}$ will point backward to the site $i - Z_i$, where Z_i is a random variable with distribution μ. Here, μ is a probability distribution in the form of

$$\mu(\{n, \dots\}) = n^{-\alpha} L(n), \tag{3}$$

where L is a slowly varying function and $\alpha \in (0, 1/2)$. This choice of α guarantees that the graph $\mathcal{G}_\mu$ has a.s. infinitely many components, each being a tree with infinite vertices. Conditioning on $\mathcal{G}_\mu$, one then defines $(X_j)_{j\in\mathbb{Z}}$ such that:

- $X_j = X_i$ if j and i are in the same component of the graph,
- X_j and X_i are independent otherwise, and
- marginally each X_i has the distribution $(1 - p)\delta_{-1} + p\delta_1$ for some $p \in (0, 1)$.

The partial-sum process $S_n = \sum_{i=1}^{n} X_i, n \geq 1$, can be interpreted as a correlated random walk. Hammond and Sheffield [22], Theorem 1.1, proved that

$$\left(\frac{S_{\lfloor nt\rfloor} - \mathbb{E}S_{\lfloor nt\rfloor}}{n^{\alpha+1/2}L(n)}\right)_{t\in[0,1]} \quad \Rightarrow \quad \sigma\big(B_{\alpha+1/2}(t)\big)_{t\in[0,1]} \tag{4}$$

as $n \to \infty$ in $D([0, 1])$, with the constant σ explicitly given. Here and in the sequel, we let "$\Rightarrow$" denote convergence in distribution [10]. Hammond and Sheffield [22] actually established a strong invariance principle for the convergence (4).

To generalize the Hammond–Sheffield model to high dimensions, we start by constructing a random graph $\mathcal{G}_\mu$ with vertices $\mathbb{Z}^d$. Similarly, at each vertex $\boldsymbol{i} \in \mathbb{Z}^d$ we first sample independently a random edge of length $\boldsymbol{Z_i}$, according to a probability distribution μ, and connect $\boldsymbol{i}$ to $\boldsymbol{i} - \boldsymbol{Z_i}$. The distribution μ has support within $\{1, 2, \dots\}^d$, intuitively meaning that all the edges are directed toward the southwest when $d = 2$. Throughout, we assume that the additive group generated by the support of μ is all $\mathbb{Z}^d$, and in short we say that μ is aperiodic. Most importantly, the distribution μ is assumed to be in the strict domain of normal attraction of (E, ν), denoted by $\mu \in \mathcal{D}(E, \nu)$, for a matrix $E = \operatorname{diag}(1/\alpha_1, \dots, 1/\alpha_d)$ with $\alpha_i \in (0, 1), i = 1, \dots, d$, and an infinitely divisible probability measure ν on $\mathbb{R}_+^d$. That is, if $(\boldsymbol{\xi}_i)_{i\geq1}$ are i.i.d. copies with distribution μ, then

$$n^{-E} \sum_{i=1}^{n} \boldsymbol{\xi}_i \quad \Rightarrow \quad \nu. \tag{5}$$

This assumption is a natural generalization of (3) to high dimensions. We again focus on the case that $\mathcal{G}_\mu$ has infinitely many components, which turns out to be exactly the case that $q(E) := \operatorname{trace}(E) > 2$, and given $\mathcal{G}_\mu$ we define $(X_j)_{j\in\mathbb{Z}^d}$ similarly as in dimension one. Remark that $q(E) > 2$ is trivially satisfied for $d \geq 2$, due to the restriction on $\alpha_i \in (0, 1)$. Remark also that when $d \geq 2$, sometimes it is more practical to express (5) in terms of nonstandard multivariate regular variation, and in this case nothing needs to be assumed in terms of the spectral measure of ν. See Section 2 for detailed descriptions of the measure μ, the random graph $\mathcal{G}_\mu$ and the model.

The key feature of our model is that the underlying random graph induces a partial order of $\mathbb{Z}^d$. Models with such a feature have been considered in literature. In particular, the so-called partially ordered models have been recently introduced by Deveaux and Fernández [18]. Applications of such models include notably image and texture analysis [14]. Our model may be formulated alternatively as a partially ordered model. However, we do not pursue this direction here, as the current setup serves our purpose better.

In this paper, we will investigate the scaling limits of partial sums over increasing rectangles of the random fields described above. For this purpose, we introduce

$$S_n(\boldsymbol{t}) := \sum_{\boldsymbol{j} \in R(\boldsymbol{n},\boldsymbol{t})} X_j, \qquad \boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{N}^d, \boldsymbol{t} = (t_1, \dots, t_d) \in [0,1]^d$$

with $R(\boldsymbol{n},\boldsymbol{t}) = \prod_{k=1}^d [0, n_k t_k - 1] \cap \mathbb{Z}^d$. Surprisingly, the limit theorems are much more complicated in high dimensions. In order to obtain an invariance principle for $S_{\boldsymbol{n}}(\boldsymbol{t})$, one cannot simply require $\min_{i=1,\dots,d} n_i \to \infty$ as most of the limit theorems for random fields do (see, e.g., [5, 16, 28]). Instead, one needs to investigate

$$S_n^{E'}(\boldsymbol{t}) := \sum_{\boldsymbol{j} \in R(n^{E'}\mathbf{1},\boldsymbol{t})} X_{\boldsymbol{j}} \tag{6}$$

with a diagonal matrix $E' = \operatorname{diag}(\beta_1, \dots, \beta_d)$.

The contribution of our main result, Theorem 5, is twofold. First, we establish invariance principles to operator-scaling Gaussian random fields. Such limit theorems, rare in the literature, justify the usage of such Gaussian random fields in various applications, including particularly texture analysis [4, 9, 44] and hydrology [2]. Second, unexpectedly, Theorem 5 reveals the following surprising phenomenon: for different E', the limiting random field may not be the same. Moreover, in the special case with $E' = cE$ for some $c > 0$, the dependence structure of the limiting random field is determined by the measure ν. This case is referred to as the critical regime. For the noncritical regime, one can still obtain invariance principles under different normalizations depending on both E and E', although the limiting random field has degenerate dependence structure (either invariant, i.e., completely dependent, or independent increments) along at least one direction. To the best of our knowledge, the existence of such a critical regime has been rarely seen in the literature, except for the recent results by Puplinskaitė and Surgailis [38, 39]. They investigated a different model in dimension 2, and referred to the same phenomenon as the scaling-transition phenomenon.

Below we briefly summarize the phenomenon of critical regime.

Critical regime: Here, we refer to the case of taking $E' = E$ in (6).

THEOREM 1. *Assume that $\mu \in \mathcal{D}(E, \nu)$ for some diagonal matrix $E = \operatorname{diag}(1/\alpha_1, \dots, 1/\alpha_d)$ with $\alpha_i \in (0,1), i = 1, \dots, d$, and a probability measure ν*

on $\mathbb{R}_+^d$. Assume $\alpha_1 < 1/2$ if $d = 1$. Let ψ be the characteristic function of ν. Then

$$\left(\frac{S_n^E(\boldsymbol{t}) - \mathbb{E}S_n^E(\boldsymbol{t})}{n^{1+q(E)/2}}\right)_{\boldsymbol{t}\in[0,1]^d} \quad \Rightarrow \quad \big(W(\boldsymbol{t})\big)_{\boldsymbol{t}\in[0,1]^d},$$

in the space $D([0,1]^d)$, where the limit Gaussian random field $(W(\boldsymbol{t}))_{\boldsymbol{t}\in\mathbb{R}^d}$ has zero-mean and covariance function

$$\begin{aligned}&\operatorname{Cov}\big(W(\boldsymbol{t}), W(\boldsymbol{s})\big)\\ &\quad = \sigma_X^2 \int_{\mathbb{R}^d} \prod_{k=1}^d \frac{(e^{it_k y_k}-1)\overline{(e^{is_k y_k}-1)}}{2\pi|y_k|^2} \big|\log\psi(\boldsymbol{y})\big|^{-2}\,d\boldsymbol{y}, \qquad \boldsymbol{t},\boldsymbol{s}\in\mathbb{R}^d,\end{aligned}$$

where an explicit expression of σ_X^2 is given in (21) *below.*

The limit Gaussian random field is easily seen to be (E, H)-operator-scaling with $H = 1 + q(E)/2$. For this new class of random fields, we study its increments and the Hölder regularity of the sample paths in Section 5.

Noncritical regime: For the case E' in (6) is not a multiple of E, the situation becomes much more subtle. One can still obtain invariance principles with appropriate normalization depending on both E and E'. However, in the noncritical regime the limiting random fields no longer reflects fully the long-range dependence inherited from $\mathcal{G}_\mu$. In particular, the covariance function of the limiting random field becomes degenerate in certain directions: along these directions, the covariance function becomes the one of a fractional Brownian motion with either $H = 1/2$ (the standard Brownian motion, which is memoryless) or $H = 1$ [the case of complete dependence with $W(t) = tZ$, $t \geq 0$ for a common standard Gaussian random variable Z]. Accordingly, along these directions the increments of the Gaussian random fields are independent or translation invariant, respectively. A general invariance principle is established in Section 4, and properties of the limiting random fields are investigated in Section 5. Here, we only state the invariance principle for $d = 2$. In the noncritical regime, the limit Gaussian random field is a fractional Brownian sheet with Hurst indices H_1 and H_2. However, we do not see a fractional Brownian sheet in the limit in high dimensions most of the time: a complete characterization of when it appears is given in Proposition 6 below.

THEOREM 2. *Assume $d = 2$. Let $\mu \in \mathcal{D}(E, \nu)$ with $E = \operatorname{diag}(1/\alpha_1, 1/\alpha_2)$ and set $E' = \operatorname{diag}(1/\alpha_1, 1/\alpha_2')$ with $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_2 \neq \alpha_2'$. Then, depending on the relation between α_1, α_2 and α_2', the following weak convergence holds*:

$$\left(\frac{S_n^{E'}(\boldsymbol{t}) - \mathbb{E}S_n^{E'}(\boldsymbol{t})}{n^\beta}\right)_{\boldsymbol{t}\in[0,1]^2} \quad \Rightarrow \quad \big(W(\boldsymbol{t})\big)_{\boldsymbol{t}\in[0,1]^2},$$

in the space $D([0,1]^2)$, where the limit Gaussian random field $(W(\boldsymbol{t}))_{\boldsymbol{t}\in\mathbb{R}^2}$ has zero-mean and covariance function in the form of

$$\operatorname{Cov}\big(W(\boldsymbol{t}), W(\boldsymbol{s})\big) = \sigma_X^2\sigma^2 \operatorname{Cov}\big(B_{H_1}(t_1), B_{H_1}(s_1)\big)\operatorname{Cov}\big(B_{H_2}(t_2), B_{H_2}(s_2)\big).$$

Here, $\beta, \sigma^2, H_1, H_2$ *and hence* $\{W(\boldsymbol{t})\}_{\boldsymbol{t}\in[0,1]^2}$ *all depend on* α_1, α_2 *and* α_2'. *In particular, there are four different possibilities as follows*:

(i) $\alpha_2' > \alpha_2, \alpha_2 \in (0, 1/2)$: $\beta = \frac{\alpha_2}{\alpha_2'} + \frac{1}{2}(\frac{1}{\alpha_1} + \frac{1}{\alpha_2'})$, $H_1 = \frac{1}{2}$, $H_2 = \frac{1}{2} + \alpha_2$.

(ii) $\alpha_2' > \alpha_2, \alpha_2 \in (1/2, 1)$: $\beta = 1 + \frac{1}{2\alpha_1} + \frac{1}{\alpha_2'} - \frac{1}{2\alpha_2}$, $H_1 = \frac{1}{2} + \alpha_1(1 - \frac{1}{2\alpha_2})$, $H_2 = 1$.

(iii) $\alpha_2' < \alpha_2, \alpha_1 \in (0, 1/2)$: $\beta = 1 + \frac{1}{2}(\frac{1}{\alpha_1} + \frac{1}{\alpha_2'})$, $H_1 = \frac{1}{2} + \alpha_1$, $H_2 = \frac{1}{2}$.

(iv) $\alpha_2' < \alpha_2, \alpha_1 \in (1/2, 1)$: $\beta = \frac{\alpha_2}{\alpha_2'}(1 - \frac{1}{2\alpha_1}) + \frac{1}{\alpha_1} + \frac{1}{2\alpha_2'}$, $H_1 = 1$, $H_2 = \frac{1}{2} + \alpha_2(1 - \frac{1}{2\alpha_1})$.

Explicit expressions of σ^2 in these cases can be found in the proof of Theorem 2 in Section 5.

The main result of the paper, Theorem 5, is a unified version of invariance principles for general $d \in \mathbb{N}$, $E = \operatorname{diag}(1/\alpha_1, \ldots, 1/\alpha_d)$ and arbitrary E', from which both Theorems 1 and 2 follow as immediate corollaries. Theorem 5 also provides a general principle to determine the correct normalization order, the limit covariance function, and hence the directions of degenerate dependence. We have just seen that in dimension 2 there are already 4 different noncritical regimes. For general $d \geq 3$, the situation becomes more complicated.

The core of the proofs is an application of the martingale central limit theorem, thanks to the key observation that the random field of interest can be represented as a linear random field in the form of

$$X_{\boldsymbol{i}} = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} q_{\boldsymbol{j}} X^*_{\boldsymbol{i}-\boldsymbol{j}}, \qquad \boldsymbol{i} \in \mathbb{Z}^d, \tag{7}$$

of which the innovations $(X^*_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ are multiparameter martingale differences. Hammond and Sheffield [22] also made essential use of the martingale central limit theorem, although the representation as a linear process as in (7) was not explicit. This representation plays a key role in our proofs, as from there when verifying conditions in the martingale central limit theorem, thanks to the structure of the linear process, we can deal with the coefficients $q_{\boldsymbol{j}}$ and innovations $X^*_{\boldsymbol{j}}$ separately. This framework, or more generally the martingale approximation method, has been carried out successfully in dimension one to establish invariance principles for fractional Brownian motions for general stationary processes [17]. To extend this framework to high dimensions, a notorious difficulty is to find a convenient multiparameter martingale to work with. It is well known that the martingale approximation method applied to stationary random fields is not as powerful as to stationary sequences, as pointed out a long time ago by Bolthausen [12]. Fortunately, our specific model can be represented exactly as a simple linear random field with martingale-difference innovations as in (7).

Once the representation of linear random fields in (7) is established, the main work lies in the computation of the limit of the covariance functions. This step is heavily based on the analysis of Fourier transforms of the linear coefficients $(q_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$, the asymptotic property of which is essentially determined by ν. Analyzing the Fourier transforms is a standard tool to compute the covariance functions for stationary linear random fields; see, for example, [28, 38, 39]. To complete the invariance principle, the tightness is established. At last, to develop the sample-path properties we apply recent results in Biermé and Lacaux [7].

The rest of the paper is organized as follows. In Section 2, we describe in details the random-field model. Section 3 provides a general central limit theorem that serves our purpose. Section 4 establishes a general invariance principle that applies to both critical and noncritical regimes. Some properties of the limit random fields are provided in Section 5.

Throughout the paper, we use the following usual notation. Let $d \geq 1$ be an integer. On $\mathbb{R}^d$, we consider the partial order (also denoted by $<$) defined by $\boldsymbol{t} < \boldsymbol{s}$ if $t_j < s_j$ for all $j = 1, \dots, d$, where $\boldsymbol{t} = (t_1, \dots, t_d)$ and $\boldsymbol{s} = (s_1, \dots, s_d)$. In the same way, we use the notation $>$, $\leq$, $\geq$. We write $\boldsymbol{t} \not< \boldsymbol{s}$ as soon as $t_j \geq s_j$ for at least one $j = 1, \dots, d$, and in the same way, we use $\not>$, $\not\leq$, $\not\geq$. We denote by $[\boldsymbol{t}, \boldsymbol{s}]$ the set $[t_1, s_1] \times \cdots \times [t_d, s_d]$ and we write $|\boldsymbol{t}|_\infty$ for $\max\{|t_j|, j = 1, \dots, d\}$, and $|\boldsymbol{t}|_1$ for $\sum_{j=1}^d |t_j|$. Furthermore, write $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}_* = \{1, 2, \dots\}$.

2. The model.

In this section, we will give a detailed description of our random field model $\{X_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{Z}^d}$, of which the dependence structure is determined by an underlying random graph $\mathcal{G}_\mu$. The asymptotic properties of the random graph are determined by a probability measure μ on $\{1, 2, \dots\}^d$, which is assumed to be in the strict domain of normal attraction of an E-operator stable measure ν on $\mathbb{R}_+^d$. Some simple properties of the model will be derived. In particular, we show that the random field of interest can be represented as a linear random field, of which the innovations are stationary multiparameter martingale differences.

2.1. *The random graph.*

On $\mathbb{Z}^d$, we consider the random directed graph $\mathcal{G}_\mu$, associated to μ, defined as follows:

- Let $(\boldsymbol{Z}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be i.i.d. random variables with distribution μ.
- For each $\boldsymbol{n} \in \mathbb{Z}^d$, let $e_{\boldsymbol{n}}$ be the outward edge from $\boldsymbol{n}$ to $\boldsymbol{n} - \boldsymbol{Z}_{\boldsymbol{n}}$.
- $\mathcal{G}_\mu$ is the graph with all sites of $\mathbb{Z}^d$ as vertices and random directed edges $\{e_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{Z}^d\}$.

The graph $\mathcal{G}_\mu$ is then composed of (possibly) several disconnected components and each component is a tree. The upcoming Proposition 1 shows that, almost surely, the number of components of $\mathcal{G}_\mu$ is one or is infinite.

We first introduce the following notation. For $\boldsymbol{n} \in \mathbb{Z}^d$, we denote by $A_{\boldsymbol{n}}$ the ancestral line of $\boldsymbol{n}$, that is the set of all elements $\boldsymbol{k} \in \mathbb{Z}^d$ for which there exists a directed connection from $\boldsymbol{n}$ to $\boldsymbol{k}$ (taking the orientations of the edges into account). Note that, in distribution, $A_{\boldsymbol{n}}$ can be described by the range of the random walk

$(\boldsymbol{n}-\boldsymbol{S}_k)_{k\geq 0}$ where $(\boldsymbol{S}_k)_{k\geq 0}$ is the random walk starting at $\mathbf{0}$ with step distribution μ. In particular, since μ is supported by $\mathbb{N}_*^d$, any element $\boldsymbol{k}$ in $A_{\boldsymbol{n}}$ satisfies $\boldsymbol{k} < \boldsymbol{n}$. Observe that the condition that the support of μ generates the group $\mathbb{Z}^d$ is equivalent to the fact that $\mathbb{P}(A_{\boldsymbol{n}} \cap A_{\boldsymbol{m}} \neq \varnothing) > 0$ for all $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^d$.

For $\boldsymbol{n} \in \mathbb{Z}^d$, we set $q_{\boldsymbol{n}} = \mathbb{P}(\mathbf{0} \in A_{\boldsymbol{n}})$. We clearly have $q_{\boldsymbol{n}} = 0$ as soon as $\mathbf{0} \not< \boldsymbol{n}$, except for $q_{\mathbf{0}} = 1$. Further, since each edge is generated independently at each site, for any $\boldsymbol{n}, \boldsymbol{k} \in \mathbb{Z}^d$,

$$\mathbb{P}(\boldsymbol{k} \in A_{\boldsymbol{n}}) = q_{\boldsymbol{n}-\boldsymbol{k}}.$$

PROPOSITION 1. *If $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2$ converges, then $\mathcal{G}_\mu$ has almost surely infinitely many components whereas if $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2$ diverges, then $\mathcal{G}_\mu$ has almost surely only one component.*

We start by proving the following lemma.

LEMMA 1. (i) *If $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2$ converges then for all $\boldsymbol{n} \in \mathbb{Z}^d$,*

$$\mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) = \left(\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2\right)^{-1} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} q_{\boldsymbol{k}} q_{\boldsymbol{k}+\boldsymbol{n}}.$$

(ii) *If $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2$ diverges then $\mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) = 1$ for all $\boldsymbol{n} \in \mathbb{Z}^d$.*

PROOF. The proof follows an idea developed in Hammond and Sheffield [22], Lemma 3.1, for the dimension 1. Let $\mathcal{G}'_\mu$ be an independent copy of $\mathcal{G}_\mu$. We denote by $A'_{\boldsymbol{n}}$ the ancestral line of $\boldsymbol{n}$ with respect to $\mathcal{G}'_\mu$. On one hand, one has

$$\mathbb{E}|A_{\mathbf{0}} \cap A'_{\boldsymbol{n}}| = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathbb{P}(\boldsymbol{k} \in A_{\mathbf{0}})\mathbb{P}(\boldsymbol{k} \in A_{\boldsymbol{n}}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} q_{\boldsymbol{k}} q_{\boldsymbol{k}+\boldsymbol{n}}.$$

On the other hand,

$$\mathbb{E}|A_{\mathbf{0}} \cap A'_{\boldsymbol{n}}| = \mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing)\mathbb{E}|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| = \mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) \sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2$$

and thus (i) follows.

If $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2 = \infty$, then $\mathbb{E}|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| = \infty$. But $\mathbb{E}|A_{\mathbf{0}} \cap A'_{\mathbf{0}}|$ can also be computed as

$$\begin{aligned} \mathbb{E}|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| &= \sum_{k\geq 0} \mathbb{P}(|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| > k) \\ &= \sum_{k\geq 0} \mathbb{P}(A_{\mathbf{0}} \cap A'_{\mathbf{0}} \neq \{\mathbf{0}\})^k = \frac{1}{1-\mathbb{P}(A_{\mathbf{0}} \cap A'_{\mathbf{0}} \neq \{\mathbf{0}\})}. \end{aligned} \tag{8}$$

Thus, $\mathbb{E}|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| = \infty$ if and only if $\mathbb{P}(A_{\mathbf{0}} \cap A'_{\mathbf{0}} \neq \{\mathbf{0}\}) = 1$, and in this situation $|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| = \infty$ almost surely. Now, since the group generated by the support of μ covers $\mathbb{Z}^d$, we know that, for all $\boldsymbol{n} \in \mathbb{Z}^d$, there exists $\boldsymbol{k}_0 \in \mathbb{Z}^d$ such that

$$\mathbb{P}(\boldsymbol{k}_0 \in A_{\mathbf{0}} \text{ and } \boldsymbol{k}_0 - \boldsymbol{n} \in A'_{\mathbf{0}}) = \mathbb{P}(\boldsymbol{k}_0 \in A_{\mathbf{0}} \cap A'_{\boldsymbol{n}}) > 0.$$

But, since $|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| = \infty$ a.s., we infer that $|A_{\boldsymbol{k}_0} \cap A'_{\boldsymbol{k}_0 - \boldsymbol{n}}| = \infty$ also a.s., and thus

$$\mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) = \mathbb{P}(A_{\boldsymbol{k}_0} \cap A_{\boldsymbol{k}_0 - \boldsymbol{n}} \neq \varnothing) \geq \mathbb{P}(|A_{\boldsymbol{k}_0} \cap A'_{\boldsymbol{k}_0 - \boldsymbol{n}}| = \infty) = 1,$$

which proves (ii). $\square$

PROOF OF PROPOSITION 1. If $C := \sum_{\boldsymbol{k} \in \mathbb{N}^d} q_{\boldsymbol{k}}^2 < \infty$, from Lemma 1(i), we get

$$\begin{aligned}
\mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) &= C^{-1} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} q_{\boldsymbol{k}} q_{\boldsymbol{k}+\boldsymbol{n}} \\
&\leq C^{-1} \Bigg(\sum_{\boldsymbol{k} \in \mathbb{Z}^d, \boldsymbol{k}+\boldsymbol{n} \geq \mathbf{0}} q_{\boldsymbol{k}}^2 \Bigg)^{1/2} \Bigg(\sum_{\boldsymbol{k} \in \mathbb{Z}^d, \boldsymbol{k} \geq \mathbf{0}} q_{\boldsymbol{k}+\boldsymbol{n}}^2 \Bigg)^{1/2} \\
&= C^{-1} \Bigg(\sum_{\boldsymbol{k} \in \mathbb{Z}^d, \boldsymbol{k} \geq -\boldsymbol{n}} q_{\boldsymbol{k}}^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\boldsymbol{k} \in \mathbb{Z}^d, \boldsymbol{k} \geq \boldsymbol{n}} q_{\boldsymbol{k}}^2 \Bigg)^{\frac{1}{2}},
\end{aligned}$$

which goes to 0 as $|\boldsymbol{n}|_\infty \to \infty$. Thus, $\mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) \to 0$ as $|\boldsymbol{n}|_\infty \to \infty$, and we can build a sequence $(\boldsymbol{n}_k)_{k \in \mathbb{N}} \subset \mathbb{Z}^d$, iteratively, such that for each $k \in \mathbb{N}$,

$$\mathbb{P}\Bigg(A_{\boldsymbol{n}_k} \cap \Bigg(\bigcup_{j=0}^{k-1} A_{\boldsymbol{n}_j}\Bigg) \neq \varnothing\Bigg) \leq \frac{1}{k^2}.$$

By the Borel–Cantelli lemma, we see that, almost surely, the ancestral lines $A_{\boldsymbol{n}_k}$, for all k large enough, are disjoint from each other. This proves the first part of the proposition.

The second part of the proposition is clear from Lemma 1(ii). $\square$

2.2. *The measure.* From now on, we always consider a probability measure μ on $\mathbb{N}_*^d$ which is aperiodic (the additive group generated by the support of μ is all $\mathbb{Z}^d$) and such that $\mu \in \mathcal{D}(E, \nu)$ for an infinitely divisible full probability measure ν on $\mathbb{R}_+^d$ and a matrix $E = \operatorname{diag}(1/\alpha_1, \dots, 1/\alpha_d)$ with $\alpha_i \in (0, 1)$ for all $i = 1, \dots, d$. Recall that we mean by $\mu \in \mathcal{D}(E, \nu)$ that if $(\boldsymbol{\xi}_i)_{i \geq 1}$ are i.i.d. copies with distribution μ, then

$$n^{-E} \sum_{i=1}^{n} \boldsymbol{\xi}_i \quad \Rightarrow \quad \nu. \tag{9}$$

Note that, since the distribution of each coordinate is in the strict domain of normal attraction of a positive stable law and since positive α-stable laws only exist for $\alpha \in (0,1)$, the condition $\alpha_i \in (0,1)$ for all $i = 1,\dots,d$ is necessary.

Consider the characteristic function $\psi(\boldsymbol{t}) = \int_{\mathbb{R}_+^d} e^{i\boldsymbol{t}\cdot\boldsymbol{x}}\,d\nu(\boldsymbol{x})$ of ν. It follows from (9) that the log-characteristic function $\log\psi$ is then an E-homogeneous function, that is,

$$\text{for all } t > 0 \text{ and } \boldsymbol{x} \in \mathbb{R}^d, \qquad \log\psi\big(t^E \boldsymbol{x}\big) = t\log\psi(\boldsymbol{x}).$$

See (12) below. Further, $\log\psi(\mathbf{0}) = 0$ and for all $\boldsymbol{x} \neq \mathbf{0}$, $|\log\psi(\boldsymbol{x})| > 0$.

One can also describe μ in the framework of multivariate regular variation. Consider the triplet representation of ν as an infinitely divisible distribution [34], equation (3.17). Then [34], Corollary 8.2.11 states that (9) implies that the triplet has the form $(0,0,\phi)$, with ϕ satisfying

$$\lim_{n\to\infty} n\mu\big(n^E A\big) = \phi(A) \tag{10}$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$ bounded away from $\mathbf{0}$ and $\phi(\partial A) = 0$. Conversely, [34], Corollary 8.2.11, also shows that (10) implies (9) with a possibly centering on the left-hand side and ν determined by the triplet $(a, 0, \phi)$ with a possible drift term a. However, under the assumption $\alpha_i \in (0,1)$, it follows from [34], Theorem 8.2.7, that $a = 0$ and the centering can be set as zero.

In view of (10), μ is said to have nonstandard multivariate regular variation with exponent E and exponent measure ϕ. Most of the applications in the literature of multivariate regular variation, however, focus on the case that $\alpha_1 = \cdots = \alpha_d$. In this case, (10) is referred to as multivariate regular variation in the literature. Standard references on (standard) multivariate regular variation include [42, 43]. References on nonstandard multivariate regular variation include [40], [43], Chapter 6. See also some recent development in [41]. Some examples are given at the end of the subsection.

We denote by P the Fourier transform of the measure μ, that is,

$$P(\boldsymbol{t}) = \sum_{\boldsymbol{k}\in\mathbb{N}^d} \mu(\{\boldsymbol{k}\}) e^{i\boldsymbol{t}\cdot\boldsymbol{k}}, \qquad \boldsymbol{t} \in \mathbb{R}^d.$$

Note that the assumption that the additive group generated by the support of μ is all $\mathbb{Z}^d$ is equivalent to:

$$P(\boldsymbol{t}) = 1 \text{ if and only if the coordinates of } \boldsymbol{t} \text{ belong to } 2\pi\mathbb{Z};$$

see, for example, Spitzer [48], page 76.

Let $\mathcal{G}_\mu$ be the random graph associated to μ as defined in Section 2.1. The asymptotic behavior of $\{q_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^d}$ will play a key role in our analysis. It is essentially determined by the measure $\mu \in \mathcal{D}(E,\nu)$. We denote by Q the Fourier series with coefficients $q_{\boldsymbol{k}} = \mathbb{P}(\mathbf{0} \in A_{\boldsymbol{k}})$, that is,

$$Q(\boldsymbol{t}) = \sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}} e^{i\boldsymbol{t}\cdot\boldsymbol{k}}.$$

Using that $q_{\boldsymbol{k}} = \sum_{\boldsymbol{j}\in\mathbb{N}_*^d} \mu(\{\boldsymbol{j}\})q_{\boldsymbol{k}-\boldsymbol{j}}$ for $\boldsymbol{k} > \mathbf{0}$, we see that both Fourier series are linked by the relation

$$Q(\boldsymbol{t}) = \frac{1}{1-P(\boldsymbol{t})}.$$

From Lemma 1, we see that

$$\mathbb{P}(A_{\mathbf{0}} \cap A_{\boldsymbol{n}} \neq \varnothing) = \frac{c_{\boldsymbol{n}}(|Q|^2)}{c_{\mathbf{0}}(|Q|^2)},$$

where $c_{\boldsymbol{k}}(|Q|^2)$ denotes the Fourier coefficient of index $\boldsymbol{k}$ of $|Q|^2 = Q\overline{Q}$. This relation explains why the Fourier series Q plays a crucial role in the study of the random graph.

The two following lemmas are key results concerning the behavior of Q at $\mathbf{0}$.

LEMMA 2. *Let $\mu \in \mathcal{D}(E,\nu)$ be as described above and ψ the characteristic function of ν. Then*

$$|Q(\boldsymbol{x})| = |1-P(\boldsymbol{x})|^{-1} = \frac{g(\boldsymbol{x})}{|\log\psi(\boldsymbol{x})|}, \qquad \boldsymbol{x} \in [-\pi,\pi]^d,$$

where g is continuous and positive with $g(\mathbf{0}) = 1$.

PROOF. Let us use a change of variables in polar coordinates. As in [34], Chapter 6, we define a new norm on $\mathbb{R}^d$, related to the matrix E, by

$$\|\boldsymbol{x}\|_E = \int_0^1 |r^E\boldsymbol{x}|\frac{1}{r}\,dr, \tag{11}$$

where here $|\cdot|$ denotes the Euclidean norm. The unit ball $S_E = \{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x}\|_E = 1\}$ associated to this norm is a compact set of $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and every vector in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ can be uniquely written as $r^E\boldsymbol{\theta}$ with $r > 0$ and $\boldsymbol{\theta} \in S_E$, since for any $\boldsymbol{x} \neq \mathbf{0}$, the map $t \mapsto \|t^E\boldsymbol{x}\|_E$ is strictly increasing on $(0,\infty)$.

Since $\mu \in \mathcal{D}(E,\nu)$, we have

$$P(n^{-E}\boldsymbol{\theta})^n \to \psi(\boldsymbol{\theta}) \qquad \text{as } n\to\infty, \text{ uniformly in } \boldsymbol{\theta} \in S_E,$$

from which we infer that

$$t\log P(t^{-E}\boldsymbol{\theta}) \to \log\psi(\boldsymbol{\theta}) \qquad \text{as } t\to\infty, \text{ uniformly in } \boldsymbol{\theta} \in S_E; \tag{12}$$

see [31], page 159. Using that $\log(1+x) \sim x$ as $x \to 0$ and that P is continuous at $\mathbf{0}$, we obtain

$$t(P(t^{-E}\boldsymbol{\theta}) - 1) \to \log\psi(\boldsymbol{\theta}) \qquad \text{as } t\to\infty, \text{ uniformly in } \boldsymbol{\theta} \in S_E.$$

Thus, for all $\varepsilon > 0$, there exists $T > 0$ such that for all $t > T$,

$$\left|\frac{|\log\psi(t^{-E}\boldsymbol{\theta})|}{|P(t^{-E}\boldsymbol{\theta}) - 1|} - 1\right| = \left|\frac{|\log\psi(\boldsymbol{\theta})|}{t|P(t^{-E}\boldsymbol{\theta}) - 1|} - 1\right| \le \varepsilon \qquad \text{uniformly in } \boldsymbol{\theta} \in S_E.$$

Now, set $g(\cdot) = |\log \psi(\cdot)(P(\cdot) - 1)^{-1}|$. The function g is clearly continuous and positive on $[-\pi, \pi]^d \setminus \{\mathbf{0}\}$. Set $\delta = \inf_{\boldsymbol{\theta} \in S_E} \|T^{-E}\boldsymbol{\theta}\|_E > 0$. Then for all $\boldsymbol{x}$ such that $\|\boldsymbol{x}\|_E < \delta$, $\boldsymbol{x} = t_0^{-E}\boldsymbol{\theta}_0$ with $\boldsymbol{\theta}_0 \in S_E$ and $t_0 > T$, and thus

$$|g(\boldsymbol{x}) - 1| = |g(t_0^{-E}\boldsymbol{\theta}_0) - 1| \leq \varepsilon.$$

Thus, g is continuous at $\mathbf{0}$ and $g(\mathbf{0}) = 1$. $\square$

We are thus interested by the function $\boldsymbol{x} \mapsto \log \psi(\boldsymbol{x})$, which is a continuous E-homogeneous function that only vanishes at $\mathbf{0}$. Recall that $q(E) = \mathrm{trace}(E)$.

LEMMA 3. *If $\phi : \mathbb{R}^d \to \mathbb{R}$ is a continuous E-homogeneous function that only vanishes at $\mathbf{0}$, then for any $p > 0$, $\boldsymbol{x} \mapsto |\phi(\boldsymbol{x})|^{-p}$ is locally integrable in $\mathbb{R}^d$ if and only if $q(E) > p$.*

PROOF. There exists a unique finite Radon measure σ_E on S_E which allows the change of variable

$$\int_{\mathbb{R}^d} f(\boldsymbol{t})\, d\boldsymbol{t} = \int_0^{+\infty} \int_{S_E} f(r^E\boldsymbol{\theta}) r^{q(E)-1}\, d\sigma_E(\boldsymbol{\theta})\, dr,$$

for all $f \in L^1(\mathbb{R}^d)$ (see [8], Proposition 2.3). Thus, using the E-homogeneity of ϕ, one has

$$\begin{aligned}\int_{\{\|\boldsymbol{x}\|_E \leq 1\}} |\phi(\boldsymbol{x})|^{-p}\, d\boldsymbol{x} &= \int_0^1 \int_{S_E} r^{q(E)-1} |\phi(r^E\boldsymbol{\theta})|^{-p}\, d\sigma_E(\boldsymbol{\theta})\, dr \\ &= \int_0^1 r^{q(E)-1-p}\, dr \int_{S_E} |\phi(\boldsymbol{\theta})|^{-p}\, d\sigma_E(\boldsymbol{\theta}).\end{aligned}$$

The second integral is finite because $|\phi|$ is continuous and positive on the compact set S_E, and the first integral is finite if and only if $q(E) > p$. $\square$

As a first consequence, we get the following proposition.

PROPOSITION 2. *Let $\mu \in \mathcal{D}(E, \nu)$. The random graph $\mathcal{G}_\mu$ has almost surely infinitely many components if and only if $q(E) > 2$.*

Note that, when $d = 1$, the condition $q(E) > 2$ becomes $\alpha_1 < \frac{1}{2}$, which corresponds to the condition assumed in [22]. When $d \geq 2$, since $\alpha_i \in (0, 1)$ for all $i = 1, \ldots, d$, then the condition $q(E) > 2$ is always satisfied.

PROOF OF PROPOSITION 2. As a consequence of Lemma 2, using Parseval identity, we get

$$\sum_{\boldsymbol{k} \in \mathbb{N}^d} q_{\boldsymbol{k}}^2 = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} |Q(\boldsymbol{x})|^2\, d\boldsymbol{x} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} |g(\boldsymbol{x})|^2 |\log \psi(\boldsymbol{x})|^{-2}\, d\boldsymbol{x}.$$

Since g is bounded and bounded away from 0 on any compact set, we see that $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2 < +\infty$ if and only if $\boldsymbol{x} \mapsto |\log\psi(\boldsymbol{x})|^{-2}$ is integrable on $[-\pi,\pi]^d$. The function $\boldsymbol{x} \mapsto \log\psi(\boldsymbol{x})$ being E-homogeneous, by Lemma 3, it is the case if and only if $q(E) > 2$ and the result follows from Proposition 1. □

To conclude the section, we give few examples of possible probability measure $\mu \in \mathcal{D}(E,\nu)$.

EXAMPLE 1 (Product measure). Let μ be the product measure $\mu_1 \otimes \cdots \otimes \mu_d$, where each μ_i is a regularly varying measure on $\mathbb{N}_*$ with index $\alpha_i \in (0,1)$ such that

$$\mu_i([n,\infty)) \sim c_i n^{-\alpha_i},$$

for some $c_i > 0$. Then, each μ_i belongs to the strict domain of normal attraction (with normalization n^{-1/α_i}) of a positive α_i-stable law ν_i; see [11], Theorem 8.3.1. Positive α-stable laws only exist for $\alpha \in (0,1)$, and then their characteristic functions are given by

$$\varphi(t) = \exp\left\{-\gamma|t|^\alpha\left(1 - i\,\mathrm{sgn}(t)\tan\left(\frac{\pi}{2}\alpha\right)\right)\right\},$$

for some $\gamma > 0$. See [11], Theorem 8.3.2. In this situation, the measure μ belongs to the strict domain of normal attraction of the measure $\nu = \nu_1 \otimes \cdots \otimes \nu_d$ which is a full E-operator stable distribution, with $E = \mathrm{diag}(1/\alpha_1,\dots,1/\alpha_d)$. The characteristic function ψ of ν is such that

$$\log\psi(\boldsymbol{x}) = \sum_{j=1}^d \gamma_j|x_j|^{\alpha_j}\left(1 - i\,\mathrm{sgn}(x_j)\tan\left(\frac{\pi}{2}\alpha_j\right)\right),$$

for some $\gamma_j > 0$.

EXAMPLE 2 (Standard multivariate regular variation). For the standard multivariate regular variation, that is, when $\alpha_1 = \cdots = \alpha_d = \alpha$, many examples have been known from the studies of heavy-tailed random vector $\boldsymbol{X} = (X_1,\dots,X_d) \in \mathbb{R}^d$, in the literature of heavy-tailed time series. An extensively investigated condition for multivariate regular variation is

$$\frac{\mathbb{P}(|\boldsymbol{X}| > ux, \boldsymbol{X}/|\boldsymbol{X}| \in \cdot)}{\mathbb{P}(|\boldsymbol{X}| > u)} \quad\Rightarrow\quad x^{-\alpha}\sigma(\cdot) \qquad \text{as } u \to \infty, \text{ for all } x > 0, \tag{13}$$

for $|\cdot|$ a norm on $\mathbb{R}^d$ and σ a probability measure on $\mathcal{B}(S)$ for $S = \{x \in \mathbb{R}^d : |x| = 1\}$. See, for example, [1]. It is known that (10) implies (13) (see, e.g., [29], Theorem 1.15).

The measure σ is often referred to as the spectral measure, which captures the dependence of extremes. For example, the case that σ concentrates on the d-axis with equal mass means that, in view of (13), the extremes of the stationary processes are asymptotically independent. For more theory and examples on spectral measures reflecting asymptotic dependence of the extremes, we refer to [42], Chapter 5.

EXAMPLE 3 (Polar coordinate). A standard procedure to obtain nonstandard regularly varying random vectors is via the representation using polar coordinates. We use the norm $\|\cdot\|_E$ introduced in (11) to identify $\mathbb{R}^d \setminus \{\mathbf{0}\}$ with $(0,\infty) \times S_E$ for the unit ball $S_E = \{\boldsymbol{x} \in \mathbb{R}^d | \|\boldsymbol{x}\|_E = 1\}$ such that every vector in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ can be uniquely written as $r^E\boldsymbol{\theta}$ with $r > 0$ and $\boldsymbol{\theta} \in S_E$. By [34], Theorem 6.1.7, in case of (9) [equivalently (10)], ϕ can be taken to have the polar coordinate representation

$$\phi(A) = \int_0^\infty \int_{S_E} \mathbb{1}_{\{t^E\boldsymbol{\theta} \in A\}} \sigma(d\boldsymbol{\theta}) \frac{dt}{t^2},$$

for some finite Borel measure σ on S_E. In our case, since μ has support contained in $\mathbb{N}_*^d$, ϕ is a measure on $\mathbb{R}_+^d$, and σ is a finite measure on $S_E^+ = S_E \cap \mathbb{R}_+^d$. Identifying $\mathbb{R}_+^d \setminus \{\mathbf{0}\}$ with $(0,\infty) \times S_E^+$, to obtain a multivariate regular varying measure as in (10), it suffices to show

$$\mu\big((r,\infty) \times \Gamma\big) \sim r^{-1}\sigma(\Gamma) \qquad \text{as } r \to \infty, \text{ for all } \Gamma \in \mathcal{B}(S_E^+). \tag{14}$$

This follows from a standard argument showing that $\{(r,\infty) \times \Gamma\}_{r>0,\Gamma\in\mathcal{B}(S_E^+)}$ are a convergence determining class.

A standard procedure to construct a random vector of which the distribution μ satisfies (14) is the following. Let R be a nonnegative random variable with $\mathbb{P}(R > r) \sim \sigma(S_E^+)r^{-1}$ as $r \to \infty$. Let $\boldsymbol{\Theta}$ be a random element in S_E^+ with probability $\sigma/\sigma(S_E^+)$. Assume that R and $\boldsymbol{\Theta}$ are independent. Then $R^E\boldsymbol{\Theta}$ is regularly varying in $\mathbb{R}_+^d$ in the sense of (14). Indeed,

$$\mathbb{P}\big(R^E\boldsymbol{\Theta} \in (r,\infty) \times \Gamma\big) = \mathbb{P}(R > r, \boldsymbol{\Theta} \in \Gamma) \sim r^{-1}\sigma(\Gamma) \qquad \text{as } r \to \infty.$$

The so-obtained distributions can then be modified to become distributions on $\mathbb{N}_*^d$ with the same regular-variation property. We omit the details.

REMARK 1. For our main results to hold, we do not impose any assumption on the spectral measures σ in Examples 2 and 3. The only assumption is the nonstandard multivariate regular variation with indices $\alpha_1, \dots, \alpha_d \in (0,1)$, and $\alpha_1 < 1/2$ when $d = 1$.

2.3. *The random field.* We now associate a random field $(X_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ to the random graph $\mathcal{G}_\mu$. Assume that $\mu \in \mathcal{D}(E,\nu)$ as in the preceding section, with the diagonal matrix E satisfying $q(E) > 2$, and let $p \in (0,1)$. We proceed as follows.

First, generate the random directed graph $\mathcal{G}_\mu$ as described in previous sections, which has almost surely infinitely many connected components in this situation.

Let $\{\mathcal{C}_i | i \geq 1\}$ denote the collection of disjoint components and associate to each component $\mathcal{C}_i$ a random variable ε_i such that $(\varepsilon_i)_{i\geq 1}$ are i.i.d. with distribution given by $\mathbb{P}(\varepsilon_i = 1) = p$ and $\mathbb{P}(\varepsilon_i = -1) = 1 - p$. Finally, for all $\boldsymbol{j} \in \mathbb{Z}^d$, set $X_{\boldsymbol{j}} = \varepsilon_i$ where i is such that $\boldsymbol{j} \in \mathcal{C}_i$. This construction implies that $X_{\boldsymbol{j}} = X_{\boldsymbol{k}}$ as soon as $\boldsymbol{j}$ and $\boldsymbol{k}$ belong to the same component of $\mathcal{G}_\mu$, and they are independent otherwise.

REMARK 2. The one-dimensional Hammond–Sheffield model can also be formulated as an example of the so-called chains with complete connections [22]. This general class of models has a long history with different names, and is very similar to but not the same as the Gibbs measures on $\mathbb{Z}$; see [20, 21] for more references. Recently, Deveaux and Fernández [18] extended chains with complete connections to the so-called partially ordered models. It would be interesting to formulate our model in their framework.

For all $\boldsymbol{n} \in \mathbb{N}^d$, we introduce the partial sum:

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{j}\in[\boldsymbol{0},\boldsymbol{n}-\boldsymbol{1}]} X_{\boldsymbol{j}}.$$

Our aim is to establish a functional central limit theorem (invariance principle) for the partial sums $S_{\boldsymbol{n}}$ (with centering and appropriate normalization) when $\boldsymbol{n}$ goes to infinity with a specific relative speed in each direction. We will distinguish different regimes. We first show, in this section, that $(X_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ can be seen as a linear random field with martingale differences innovations, and thus, $S_{\boldsymbol{n}}$ is a partial sum of a linear random field.

For all $\boldsymbol{j} \in \mathbb{Z}^d$, we define the σ-fields $\sigma_{\boldsymbol{j}} = \sigma\{X_{\boldsymbol{k}} | \boldsymbol{k} < \boldsymbol{j}\}$ and $\overline{\sigma}_{\boldsymbol{j}} = \sigma\{X_{\boldsymbol{k}} | \boldsymbol{k} \ngeq \boldsymbol{j}\}$. Note that, for $\boldsymbol{j} < \boldsymbol{n}$, the value of $X_{\boldsymbol{n}}$ conditioned on $\sigma_{\boldsymbol{j}}$ is obtained by sampling the ancestral line $A_{\boldsymbol{n}}$ and taking the value of $X_{\boldsymbol{k}}$ where $\boldsymbol{k}$ is the first site of the ancestral line $A_{\boldsymbol{n}}$ which is strictly smaller than $\boldsymbol{j}$. We denote

$$X^*_{\boldsymbol{j}} = X_{\boldsymbol{j}} - \mathbb{E}(X_{\boldsymbol{j}} | \sigma_{\boldsymbol{j}}) = X_{\boldsymbol{j}} - \mathbb{E}(X_{\boldsymbol{j}} | \overline{\sigma}_{\boldsymbol{j}}). \tag{15}$$

The equality $\mathbb{E}(X_{\boldsymbol{j}} | \overline{\sigma}_{\boldsymbol{j}}) = \mathbb{E}(X_{\boldsymbol{j}} | \sigma_{\boldsymbol{j}})$ comes from the fact that starting from $\boldsymbol{j}$, the next site in the ancestral line $A_{\boldsymbol{j}}$ is necessarily strictly smaller than $\boldsymbol{j}$. Then for all $\boldsymbol{j} \in \mathbb{Z}^d$, $\mathbb{E}(X^*_{\boldsymbol{j}} | \overline{\sigma}_{\boldsymbol{j}}) = 0$ and $X^*_{\boldsymbol{j}}$ is measurable with respect to $\overline{\sigma}_{\boldsymbol{j}+\boldsymbol{e}_q}$ for all $q = 1, \ldots, d$, where $\boldsymbol{e}_q$ is the qth canonical unit vector of $\mathbb{R}^d$. In particular, the random variables $X^*_{\boldsymbol{j}}$ are orthogonal to each other, that is, $\mathbb{E}(X^*_{\boldsymbol{j}} X^*_{\boldsymbol{k}}) = 0$ as soon as $\boldsymbol{j} \neq \boldsymbol{k}$.

LEMMA 4. *In the above setting*,

$$\operatorname{Var}(X^*_{\boldsymbol{0}}) = \Bigl(\sum_{\boldsymbol{k}\in\mathbb{N}^d} q^2_{\boldsymbol{k}}\Bigr)^{-1} \operatorname{Var}(X_{\boldsymbol{0}}).$$

PROOF. Let $\mathbf{Z_0}$ be the random variable with distribution μ that gives the first ancestor of $\mathbf{0}$. We have $X_{\mathbf{0}} = \sum_{\boldsymbol{k}>\mathbf{0}} \mathbb{1}_{\{\mathbf{Z_0}=\boldsymbol{k}\}} X_{-\boldsymbol{k}}$ and $\mathbb{E}(X_{\mathbf{0}}|\sigma_{\mathbf{0}}) = \sum_{\boldsymbol{k}>\mathbf{0}} p_{\boldsymbol{k}} X_{-\boldsymbol{k}}$, where $p_{\boldsymbol{k}} = \mu(\{\boldsymbol{k}\})$ for all $\boldsymbol{k} > \mathbf{0}$. Therefore,

$$\begin{aligned}\mathbb{E}\big(X_{\mathbf{0}}^{*2}\big) &= \mathbb{E}\bigg(\bigg(\sum_{\boldsymbol{k}>\mathbf{0}}(\mathbb{1}_{\{\mathbf{Z_0}=\boldsymbol{k}\}} - p_{\boldsymbol{k}})X_{-\boldsymbol{k}}\bigg)^2\bigg)\\ &= \sum_{\boldsymbol{k}>\mathbf{0}}\sum_{\boldsymbol{\ell}>\mathbf{0}} \mathbb{E}\big((\mathbb{1}_{\{\mathbf{Z_0}=\boldsymbol{k}\}} - p_{\boldsymbol{k}})(\mathbb{1}_{\{\mathbf{Z_0}=\boldsymbol{\ell}\}} - p_{\boldsymbol{\ell}})\big)\mathbb{E}(X_{-\boldsymbol{k}}X_{-\boldsymbol{\ell}}).\end{aligned} \tag{16}$$

But

$$\mathbb{E}(X_{-\boldsymbol{k}}X_{-\boldsymbol{\ell}}) = \mathbb{P}(A_{-\boldsymbol{k}} \cap A_{-\boldsymbol{\ell}} \neq \varnothing)\mathbb{E}\big(X_{\mathbf{0}}^2\big) + \mathbb{P}(A_{-\boldsymbol{k}} \cap A_{-\boldsymbol{\ell}} = \varnothing)\mathbb{E}(X_{\mathbf{0}})^2 \tag{17}$$

and

$$\mathbb{E}\big((\mathbb{1}_{\{\mathbf{Z_0}=\boldsymbol{k}\}} - p_{\boldsymbol{k}})(\mathbb{1}_{\{\mathbf{Z_0}=\boldsymbol{\ell}\}} - p_{\boldsymbol{\ell}})\big) = \mathbb{1}_{\{\boldsymbol{k}=\boldsymbol{\ell}\}}p_{\boldsymbol{k}} - p_{\boldsymbol{k}}p_{\boldsymbol{\ell}}. \tag{18}$$

Combining (16), (17) and (18), we get

$$\begin{aligned}\mathbb{E}\big(X_{\mathbf{0}}^{*2}\big) &= \mathbb{E}\big(X_{\mathbf{0}}^2\big)\bigg(1 - \sum_{\boldsymbol{k}>\mathbf{0}}\sum_{\boldsymbol{\ell}>\mathbf{0}} p_{\boldsymbol{k}}p_{\boldsymbol{\ell}}\mathbb{P}(A_{-\boldsymbol{k}} \cap A_{-\boldsymbol{\ell}} \neq \varnothing)\bigg)\\ &\quad - \sum_{\boldsymbol{k}>\mathbf{0}}\sum_{\boldsymbol{\ell}>\mathbf{0}} p_{\boldsymbol{k}}p_{\boldsymbol{\ell}}\mathbb{P}(A_{-\boldsymbol{k}} \cap A_{-\boldsymbol{\ell}} = \varnothing)\mathbb{E}(X_{\mathbf{0}})^2\\ &= \big(\mathbb{E}\big(X_{\mathbf{0}}^2\big) - \mathbb{E}(X_{\mathbf{0}})^2\big)\sum_{\boldsymbol{k}>\mathbf{0}}\sum_{\boldsymbol{\ell}>\mathbf{0}} p_{\boldsymbol{k}}p_{\boldsymbol{\ell}}\mathbb{P}(A_{-\boldsymbol{k}} \cap A_{-\boldsymbol{\ell}} = \varnothing)\\ &= \operatorname{Var}(X_{\mathbf{0}})\mathbb{P}\big(A_{\mathbf{0}} \cap A'_{\mathbf{0}} = \{\mathbf{0}\}\big),\end{aligned}$$

where $A'_{\mathbf{0}}$ is an independent copy of $A_{\mathbf{0}}$. Finally, as we saw in (8) in the proof of Lemma 1, $\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2 = \mathbb{E}|A_{\mathbf{0}} \cap A'_{\mathbf{0}}| = \mathbb{P}(A_{\mathbf{0}} \cap A'_{\mathbf{0}} = \{\mathbf{0}\})^{-1}$ and the proof is complete. □

Now, for all $\boldsymbol{j} \in \mathbb{Z}^d$, we introduce

$$\Delta_{\boldsymbol{j}}(X) = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^d} (-1)^{d-|\boldsymbol{\varepsilon}|_1}\mathbb{E}(X|\sigma_{\boldsymbol{j}+\boldsymbol{\varepsilon}}),$$

where $|\boldsymbol{\varepsilon}|_1 = \varepsilon_1 + \cdots + \varepsilon_d$.

Remark that, since $\mathbb{E}(X_{\boldsymbol{j}}|\sigma_{\boldsymbol{j}+\boldsymbol{\varepsilon}}) = \mathbb{E}(X_{\boldsymbol{j}}|\sigma_{\boldsymbol{j}})$ for all $\boldsymbol{\varepsilon} \in \{0, 1\}^d$ with the exception of $\boldsymbol{\varepsilon} = \mathbf{1}$ for which $\mathbb{E}(X_{\boldsymbol{j}}|\sigma_{\boldsymbol{j}+\mathbf{1}}) = X_{\boldsymbol{j}}$, we have

$$\Delta_{\boldsymbol{j}}(X_{\boldsymbol{j}}) = X_{\boldsymbol{j}} - \mathbb{E}(X_{\boldsymbol{j}}|\sigma_{\boldsymbol{j}}) = X_{\boldsymbol{j}}^*. \tag{19}$$

More generally, we have the following lemma.

LEMMA 5. *For all* $\boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^d$,

$$\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}) = q_{\boldsymbol{k}-\boldsymbol{j}}X_{\boldsymbol{j}}^*,$$

which vanishes when $\boldsymbol{k} \not> \boldsymbol{j}$.

PROOF. The result is clear when $\boldsymbol{k}=\boldsymbol{j}$ [see (19)]. In the case $\boldsymbol{k}\le\boldsymbol{j}$, $\boldsymbol{k}\ne\boldsymbol{j}$, we easily see that $\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}})=0$.

Now, assume $\boldsymbol{k}\not\le\boldsymbol{j}$. By linearity, we have

$$\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}})=\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{j}\in A_{\boldsymbol{k}}\}})+\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{j}\notin A_{\boldsymbol{k}}\}}).$$

Using first that $X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{j}\in A_{\boldsymbol{k}}\}}=X_{\boldsymbol{j}}\mathbb{1}_{\{\boldsymbol{j}\in A_{\boldsymbol{k}}\}}$, and then that $\{\boldsymbol{j}\in A_{\boldsymbol{k}}\}$ is independent of $\sigma_{\boldsymbol{j}+\boldsymbol{1}}$, we obtain

$$\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{j}\in A_{\boldsymbol{k}}\}})=\Delta_{\boldsymbol{j}}(X_{\boldsymbol{j}})\mathbb{P}(\boldsymbol{j}\in A_{\boldsymbol{k}})=q_{\boldsymbol{k}-\boldsymbol{j}}X^*_{\boldsymbol{j}}.$$

Denote by $\boldsymbol{a}(\boldsymbol{k},\boldsymbol{j})$ the first element of the ancestral line $A_{\boldsymbol{k}}$ that is $\le\boldsymbol{j}$ and remark that $\boldsymbol{a}(\boldsymbol{k},\boldsymbol{j})$ is independent of $\sigma_{\boldsymbol{j}+\boldsymbol{1}}$. Then

$$\begin{aligned}&\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{j}\notin A_{\boldsymbol{k}}\}})\\&\quad=\sum_{\boldsymbol{\ell}\le\boldsymbol{j},\boldsymbol{\ell}\ne\boldsymbol{j}}\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{a}(\boldsymbol{k},\boldsymbol{j})=\boldsymbol{\ell}\}})=\sum_{\boldsymbol{\ell}\le\boldsymbol{j},\boldsymbol{\ell}\ne\boldsymbol{j}}\Delta_{\boldsymbol{j}}(X_{\boldsymbol{\ell}})\mathbb{P}\big(\boldsymbol{a}(\boldsymbol{k},\boldsymbol{j})=\boldsymbol{\ell}\big).\end{aligned}$$

But $\Delta_{\boldsymbol{j}}(X_{\boldsymbol{\ell}})=0$ for all $\boldsymbol{\ell}\le\boldsymbol{j}$, $\boldsymbol{\ell}\ne\boldsymbol{j}$, and we finally have

$$\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}\mathbb{1}_{\{\boldsymbol{j}\notin A_{\boldsymbol{k}}\}})=0,$$

which completes the proof. □

LEMMA 6. *For all $\boldsymbol{k}\in\mathbb{Z}^d$, the series $\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}})$ converges in L^2 and*

$$X_{\boldsymbol{k}}-\mathbb{E}(X_{\boldsymbol{k}})=\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\Delta_{\boldsymbol{j}}(X_{\boldsymbol{k}}).$$

PROOF. First, remark that by stationarity we may only consider the case where $\boldsymbol{k}=\boldsymbol{0}$. The sum in the statement can be written as $\sum_{\boldsymbol{j}\in\mathbb{N}^d}\Delta_{-\boldsymbol{j}}(X_{\boldsymbol{0}})$ since the other terms vanish. We denote by $n\boldsymbol{1}$ the vector $(n,\dots,n)$ where $n\in\mathbb{N}$. By Lemma 5, we have

$$\mathbb{E}\bigg(\bigg(\sum_{\boldsymbol{j}\in[\boldsymbol{0},n\boldsymbol{1}]}\Delta_{-\boldsymbol{j}}(X_{\boldsymbol{0}})\bigg)^2\bigg)=\mathbb{E}\big(X_{\boldsymbol{0}}^{*2}\big)\bigg(\sum_{\boldsymbol{j}\in[\boldsymbol{0},n\boldsymbol{1}]}q_{\boldsymbol{j}}^2\bigg)$$

and the right-hand side converges to $\mathrm{Var}(X_{\boldsymbol{0}})$ as $n\to\infty$ thanks to Lemma 4. Now, by construction, the random variables $\sum_{\boldsymbol{j}\in[\boldsymbol{0},n\boldsymbol{1}]}\Delta_{-\boldsymbol{j}}(X_{\boldsymbol{0}})$ and $X_{\boldsymbol{0}}-\sum_{\boldsymbol{j}\in[\boldsymbol{0},n\boldsymbol{1}]}\Delta_{-\boldsymbol{j}}(X_{\boldsymbol{0}})$ are orthogonal. To see this last fact, note that for all $\boldsymbol{l}\le\boldsymbol{0}$ and $\boldsymbol{j}\le\boldsymbol{0}$, $\mathbb{E}(\mathbb{E}(X_{\boldsymbol{0}}|\sigma_{\boldsymbol{l}})|\sigma_{\boldsymbol{j}})=\mathbb{E}(X_{\boldsymbol{0}}|\sigma_{\min\{\boldsymbol{l},\boldsymbol{j}\}})$, where the minimum is taken on each coordinate. Thus, we get

$$\begin{aligned}&\mathbb{E}\bigg(\bigg(X_{\boldsymbol{0}}-\mathbb{E}(X_{\boldsymbol{0}})-\sum_{\boldsymbol{j}\in[\boldsymbol{0},n\boldsymbol{1}]}\Delta_{-\boldsymbol{j}}(X_{\boldsymbol{0}})\bigg)^2\bigg)\\&\qquad=\mathrm{Var}(X_{\boldsymbol{0}})-\mathbb{E}\bigg(\bigg(\sum_{\boldsymbol{j}\in[\boldsymbol{0},n\boldsymbol{1}]}\Delta_{-\boldsymbol{j}}(X_{\boldsymbol{0}})\bigg)^2\bigg)\to 0\end{aligned}$$

as $n\to\infty$. □

From Lemmas 6 and 5, we get that $(X_{\boldsymbol{j}} - \mathbb{E}(X_{\boldsymbol{j}}))_{\boldsymbol{j}\in\mathbb{Z}^d}$ is the linear random field given by the innovations $(X^*_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ and the filter $(q_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$. That is, for all $\boldsymbol{k} \in \mathbb{Z}^d$,

$$X_{\boldsymbol{k}} - \mathbb{E}(X_{\boldsymbol{k}}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} q_{\boldsymbol{k}-\boldsymbol{j}} X^*_{\boldsymbol{j}}.$$

Hence, we proved the following proposition.

PROPOSITION 3. *For all* $\boldsymbol{n} \in \mathbb{N}^d$,

$$S_{\boldsymbol{n}} - \mathbb{E}(S_{\boldsymbol{n}}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} b_{\boldsymbol{n},\boldsymbol{j}} X^*_{\boldsymbol{j}},$$

where $b_{\boldsymbol{n},\boldsymbol{j}} = \sum_{\boldsymbol{k}\in[\boldsymbol{0},\boldsymbol{n}-\boldsymbol{1}]} q_{\boldsymbol{k}-\boldsymbol{j}}$. *Further, for any* $\boldsymbol{n} \in \mathbb{N}^d$, $b_{\boldsymbol{n}} = (b_{\boldsymbol{n},\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ *belongs to* $\ell^2(\mathbb{Z}^d)$, *that is*, $\|b_{\boldsymbol{n}}\|^2 := \sum_{\boldsymbol{j}\in\mathbb{Z}^d} b^2_{\boldsymbol{n},\boldsymbol{j}} < \infty$.

3. A central limit theorem. We still assume $\mu \in \mathcal{D}(E, \nu)$, where ν is a full E-operator stable law on $\mathbb{R}^d_+$ with $E = \operatorname{diag}(1/\alpha_1, \dots, 1/\alpha_d)$, with $\alpha_i \in (0, 1)$ and $\alpha_1 \in (0, 1/2)$ if $d = 1$. The random field $(X_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ is the random field defined in Section 2.3. In view of Proposition 3, we want to establish central limit theorems for the sequences of L^2 random variables

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d} c_{n,\boldsymbol{j}} X^*_{\boldsymbol{j}}, \qquad n \ge 1$$

with general coefficients $c_n = (c_{n,\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. Recall the definition of $X^*_{\boldsymbol{j}}$ in (15). It turns out that a simple assumption on c_n for a central limit theorem is given by

$$\lim_{n\to\infty} \sup_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{|c_{n,\boldsymbol{j}}|}{\|c_n\|} = 0. \tag{20}$$

The aim of this section is to prove the following central limit theorem.

THEOREM 3. *Let* $c_n = (c_{n,\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ *be a sequence in* $\ell^2(\mathbb{Z}^d)$ *satisfying* (20). *Then*

$$\frac{1}{\|c_n\|} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} c_{n,\boldsymbol{j}} X^*_{\boldsymbol{j}} \quad \Rightarrow \quad \mathcal{N}(0, \sigma^2_X) \qquad \textit{as } n \to \infty,$$

where

$$\sigma^2_X := \operatorname{Var}(X^*_{\boldsymbol{0}}) = \frac{\operatorname{Var}(X_{\boldsymbol{0}})}{\sum_{\boldsymbol{k}\in\mathbb{N}^d} q^2_{\boldsymbol{k}}}. \tag{21}$$

As a preparation, we prove the following theorem which is an adaptation of a theorem of McLeish [33]. McLeish's result applies to triangular arrays of $\mathbb{Z}$-indexed martingale differences, and here we need a version for $\mathbb{Z}^d$-indexed martingale differences in the lexicographical order. Recall that in the lexicographical order, for $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d, \boldsymbol{i} \neq \boldsymbol{j}$, we write $\boldsymbol{i} \prec \boldsymbol{j}$ if, with $m := \min\{q = 1, \dots, d : i_q \neq j_q\}$, $i_m < j_m$. A collection of σ-fields $\{\mathcal{F}_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{Z}^d}$ is called a filtration in the lexicographical order, if $\mathcal{F}_{\boldsymbol{i}} \subset \mathcal{F}_{\boldsymbol{j}}$ for all $\boldsymbol{i} \prec \boldsymbol{j}$. In this case, we say that integrable random variables $(\xi_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ are martingale differences with respect to $\{\mathcal{F}_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{Z}^d}$ if

$$\xi_{\boldsymbol{i}} \in \mathcal{F}_{\boldsymbol{i}+\boldsymbol{e}_d} \quad \text{and} \quad \mathbb{E}(\xi_{\boldsymbol{i}} | \mathcal{F}_{\boldsymbol{i}}) = 0 \qquad \text{for all } \boldsymbol{i} \in \mathbb{Z}^d,$$

where $\boldsymbol{e}_d = (0, \dots, 0, 1) \in \mathbb{Z}^d$.

THEOREM 4 (McLeish [33]). *Let $(\xi_{n,\boldsymbol{j}})_{n \in \mathbb{N}, \boldsymbol{j} \in \mathbb{Z}^d}$ be a collection of random variables satisfying $\sum_{\boldsymbol{j} \in \mathbb{Z}^d} \xi_{n,\boldsymbol{j}} \in L^2$ for all $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$, $(\xi_{n,\boldsymbol{j}})_{\boldsymbol{j} \in \mathbb{Z}^d}$ are martingale differences with respect to a filtration $\{\mathcal{F}_{n,\boldsymbol{j}}\}_{\boldsymbol{j} \in \mathbb{Z}^d}$ in the lexicographical order. If*:

(i) $\lim_{n \to \infty} \max_{\boldsymbol{j} \in \mathbb{Z}^d} |\xi_{n,\boldsymbol{j}}| = 0$ *in probability*,
(ii) $\sup_{n \in \mathbb{N}} \mathbb{E}(\max_{\boldsymbol{j} \in \mathbb{Z}^d} \xi_{n,\boldsymbol{j}}^2) < \infty$,
(iii) $\lim_{n \to \infty} \sum_{\boldsymbol{j} \in \mathbb{Z}^d} \xi_{n,\boldsymbol{j}}^2 = \sigma^2 > 0$ *in probability*,

then

$$\sum_{\boldsymbol{j} \in \mathbb{Z}^d} \xi_{n,\boldsymbol{j}} \quad \Rightarrow \quad \mathcal{N}(0, \sigma^2) \qquad \text{as } n \to \infty.$$

PROOF. Let us explain how one can derive this theorem from Theorem 2.3 in [33] which is stated for finite sets of random variables at each n. First, since $\sum_{\boldsymbol{j} \in \mathbb{Z}^d} \xi_{n,\boldsymbol{j}} \in L^2$, one can find a sequence of finite rectangles Γ_n in $\mathbb{Z}^d$ such that $\sum_{\boldsymbol{j} \in \mathbb{Z}^d \setminus \Gamma_n} \xi_{n,\boldsymbol{j}}$ converges to 0 in L^2 as $n \to \infty$. Thus, the conclusion of Theorem 4 holds as soon as

$$\sum_{\boldsymbol{j} \in \Gamma_n} \xi_{n,\boldsymbol{j}} \quad \Rightarrow \quad \mathcal{N}(0, \sigma^2) \qquad \text{as } n \to \infty.$$

Furthermore, for each n, using the lexicographical order on the finite set Γ_n, one can re-index the random variables $(\xi_{n,\boldsymbol{j}})_{\boldsymbol{j} \in \Gamma_n}$ and the σ-fields $\{\mathcal{F}_{n,\boldsymbol{j}}\}_{\boldsymbol{j} \in \Gamma_n}$ in order to fit with the statement of [33], Theorem 2.3. Now, it suffices to observe that conditions (i), (ii) and (iii) imply those of [33], Theorem 2.3. □

We also need the following lemma.

LEMMA 7. *Let $c_n = (c_{n,\boldsymbol{j}})_{\boldsymbol{j} \in \mathbb{Z}^d}$ be a sequence in $\ell^2(\mathbb{Z}^d)$ such that* (20) *holds. Then*

$$\lim_{n \to \infty} \frac{1}{\|c_n\|^2} \sum_{\boldsymbol{j} \in \mathbb{Z}^d} c_{n,\boldsymbol{j}}^2 X_{\boldsymbol{j}}^{*2} = \mathbb{E}(X_{\boldsymbol{0}}^{*2}) \qquad \text{in } L^2.$$

PROOF. We start by showing that

$$\operatorname{Cov}(X_{\boldsymbol{i}}^{*2}, X_{\boldsymbol{j}}^{*2}) \to 0 \qquad \text{as } |\boldsymbol{i}-\boldsymbol{j}|_\infty \to \infty. \tag{22}$$

Observe that $X_{\boldsymbol{j}}^* = X_{\boldsymbol{j}} - \sum_{\boldsymbol{\ell}>\mathbf{0}} p_{\boldsymbol{\ell}} X_{\boldsymbol{j}-\boldsymbol{\ell}}$. Let $X_{\boldsymbol{j},k}^* = X_{\boldsymbol{j}} - \sum_{\boldsymbol{\ell}\in\{1,\dots,k\}^d} p_{\boldsymbol{\ell}} X_{\boldsymbol{j}-\boldsymbol{\ell}}$. For any $\boldsymbol{j}\in\mathbb{Z}^d$, using that $|X_{\boldsymbol{j}}^*|\le 2$, we get

$$|X_{\boldsymbol{j}}^{*2} - X_{\boldsymbol{j},k}^{*2}| \le 4|X_{\boldsymbol{j}}^* - X_{\boldsymbol{j},k}^*| = 4\left|\sum_{\boldsymbol{\ell}\in[1,\infty)^d\setminus[1,k]^d} p_{\boldsymbol{\ell}} X_{\boldsymbol{j}-\boldsymbol{\ell}}\right|.$$

Thus, since $|X_{\boldsymbol{j}}| = 1$ for all $\boldsymbol{j}\in\mathbb{Z}^d$,

$$\sup_{\boldsymbol{j}\in\mathbb{Z}^d} |X_{\boldsymbol{j}}^{*2} - X_{\boldsymbol{j},k}^{*2}| \le 4\mu([1,\infty)^d \setminus [1,k]^d) \qquad \text{a.s., for all } k>0. \tag{23}$$

Now, introduce

$$R_{\boldsymbol{i},\boldsymbol{j},k} = \left\{\left(\bigcup_{\boldsymbol{\ell}\in\boldsymbol{i}-[0,k]^d} A_{\boldsymbol{\ell}}\right) \cap \left(\bigcup_{\boldsymbol{m}\in\boldsymbol{j}-[0,k]^d} A_{\boldsymbol{m}}\right) = \varnothing\right\}.$$

We have

$$\mathbb{P}(R_{\boldsymbol{i},\boldsymbol{j},k}^c) \le \sum_{\boldsymbol{\ell}\in\boldsymbol{i}-[0,k]^d} \sum_{\boldsymbol{m}\in\boldsymbol{j}-[0,k]^d} \mathbb{P}(A_{\boldsymbol{\ell}}\cap A_{\boldsymbol{m}} \ne \varnothing).$$

But, from Lemma 1(i), we see that $\mathbb{P}(A_{\boldsymbol{\ell}}\cap A_{\boldsymbol{m}} \ne \varnothing) \to 0$ as $|\boldsymbol{\ell}-\boldsymbol{m}|_\infty \to \infty$ and thus, for any $k\ge 1$,

$$\mathbb{P}(R_{\boldsymbol{i},\boldsymbol{j},k}^c) \to 0 \qquad \text{as } |\boldsymbol{i}-\boldsymbol{j}|_\infty \to \infty. \tag{24}$$

Fix $\varepsilon>0$ and, using (23), let $k\in\mathbb{N}$ be such that $\sup_{\boldsymbol{j}\in\mathbb{Z}^d} |X_{\boldsymbol{j}}^{*2} - X_{\boldsymbol{j},k}^{*2}| < \varepsilon$. From (24), for $|\boldsymbol{i}-\boldsymbol{j}|_\infty$ large enough, we have $\mathbb{P}(R_{\boldsymbol{i},\boldsymbol{j},k}^c) < \varepsilon$ and we obtain

$$\begin{aligned}
\mathbb{E}(X_{\boldsymbol{i}}^{*2} X_{\boldsymbol{j}}^{*2}) &= \mathbb{E}(X_{\boldsymbol{i},k}^{*2} X_{\boldsymbol{j},k}^{*2}) + O(\varepsilon) = \mathbb{E}(X_{\boldsymbol{i},k}^{*2} X_{\boldsymbol{j},k}^{*2} | R_{\boldsymbol{i},\boldsymbol{j},k}) + O(\varepsilon) \\
&= \mathbb{E}(X_{\boldsymbol{i},k}^{*2} | R_{\boldsymbol{i},\boldsymbol{j},k}) \mathbb{E}(X_{\boldsymbol{j},k}^{*2} | R_{\boldsymbol{i},\boldsymbol{j},k}) + O(\varepsilon) \\
&= \mathbb{E}(X_{\boldsymbol{i},k}^{*2}) \mathbb{E}(X_{\boldsymbol{j},k}^{*2}) + O(\varepsilon) = \mathbb{E}(X_{\boldsymbol{i}}^{*2}) \mathbb{E}(X_{\boldsymbol{j}}^{*2}) + O(\varepsilon).
\end{aligned}$$

This proves (22).

To prove the lemma, fix $\varepsilon>0$ and let K be such that $|\operatorname{Cov}(X_{\boldsymbol{j}}^{*2}, X_{\boldsymbol{i}}^{*2})| \le \varepsilon$ as soon as $|\boldsymbol{i}-\boldsymbol{j}|_\infty > K$. One has

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{\|c_n\|^2} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} c_{n,\boldsymbol{j}}^2 X_{\boldsymbol{j}}^{*2} - \mathbb{E}(X_{\mathbf{0}}^{*2})\right)^2 \\
&\qquad = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{c_{n,\boldsymbol{j}}^2}{\|c_n\|^2} \sum_{\boldsymbol{i}\in\mathbb{Z}^d} \frac{c_{n,\boldsymbol{i}}^2}{\|c_n\|^2} \operatorname{Cov}(X_{\boldsymbol{j}}^{*2}, X_{\boldsymbol{i}}^{*2})
\end{aligned}$$

$$\leq \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{c_{n,\boldsymbol{j}}^2}{\|c_n\|^2} \sum_{|\boldsymbol{i}-\boldsymbol{j}|_\infty \leq K} \frac{c_{n,\boldsymbol{i}}^2}{\|c_n\|^2} \big|\operatorname{Cov}\big(X_{\boldsymbol{j}}^{*2}, X_{\boldsymbol{i}}^{*2}\big)\big|$$

$$+ \varepsilon \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{c_{n,\boldsymbol{j}}^2}{\|c_n\|^2} \sum_{|\boldsymbol{i}-\boldsymbol{j}|_\infty > K} \frac{c_{n,\boldsymbol{i}}^2}{\|c_n\|^2}$$

$$\leq \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{c_{n,\boldsymbol{k}}^2}{\|c_n\|^2} \sum_{|\boldsymbol{i}-\boldsymbol{0}|_\infty \leq K} \big|\operatorname{Cov}\big(X_{\boldsymbol{0}}^{*2}, X_{\boldsymbol{i}}^{*2}\big)\big| + \varepsilon,$$

and the first term of the right-hand side goes to 0 as $n \to \infty$ because $|\operatorname{Cov}(X_{\boldsymbol{0}}^{*2}, X_{\boldsymbol{i}}^{*2})|$ is bounded and $\sup_{\boldsymbol{k}\in\mathbb{Z}^d} c_{n,\boldsymbol{k}}^2 = o(\|c_n\|^2)$ by assumption. □

PROOF OF THEOREM 3. Recall that we write $\sigma_{\boldsymbol{j}} = \sigma\{X_{\boldsymbol{k}} | \boldsymbol{k} < \boldsymbol{j}\}$ and $\overline{\sigma}_{\boldsymbol{j}} = \sigma\{X_{\boldsymbol{k}} | \boldsymbol{k} \ngeq \boldsymbol{j}\}$, and we already have seen for all $\boldsymbol{j} \in \mathbb{Z}^d$,

$$\mathbb{E}(X_{\boldsymbol{j}} | \sigma_{\boldsymbol{j}}) = \mathbb{E}(X_{\boldsymbol{j}} | \overline{\sigma}_{\boldsymbol{j}}).$$

We now consider the σ-fields $\mathcal{F}_{\boldsymbol{j}} = \sigma\{X_{\boldsymbol{k}} | \boldsymbol{k} \prec \boldsymbol{j}\}$. We have $\sigma_{\boldsymbol{j}} \subset \mathcal{F}_{\boldsymbol{j}} \subset \overline{\sigma}_{\boldsymbol{j}}$ for all $\boldsymbol{j} \in \mathbb{Z}^d$, and thus, for all $\boldsymbol{j} \in \mathbb{Z}^d$, we also have

$$\mathbb{E}(X_{\boldsymbol{j}} | \mathcal{F}_{\boldsymbol{j}}) = \mathbb{E}(X_{\boldsymbol{j}} | \sigma_{\boldsymbol{j}}).$$

Thus, by definition [see (15)], the random field $(X_{\boldsymbol{j}}^*)_{\boldsymbol{j}\in\mathbb{Z}^d}$ is composed of martingale differences with respect to the filtration $\{\mathcal{F}_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbb{Z}^d}$ defined above in the lexicographical order. Now in order to establish Theorem 3, we apply Theorem 4 to

$$\xi_{n,\boldsymbol{j}} := \frac{c_{n,\boldsymbol{j}}}{\|c_n\|} X_{\boldsymbol{j}}^* \quad \text{and} \quad \mathcal{F}_{n,\boldsymbol{j}} := \mathcal{F}_{\boldsymbol{j}} = \sigma\{X_{\boldsymbol{k}} | \boldsymbol{k} \prec \boldsymbol{j}\}.$$

Note that $|X_{\boldsymbol{j}}^*| \leq 2$, and by Lemma 4 the conditions (i), (ii) and (iii) are satisfied with $\sigma_X^2 = \sigma^2 = \mathbb{E}(X_{\boldsymbol{0}}^{*2}) = \operatorname{Var}(X_{\boldsymbol{0}}^*) = (\sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}^2)^{-1} \operatorname{Var}(X_{\boldsymbol{0}})$. The proof is thus complete. □

The following lemma gives another useful condition on the coefficients $(c_{n,\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ for Theorem 3.

LEMMA 8. *If $(c_{n,\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ is a sequence in $\ell^2(\mathbb{Z}^d)$ that satisfies, for all $q = 1, \dots, d$,*

$$\lim_{n\to\infty} \frac{1}{\|c_n\|^2} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} |c_{n,\boldsymbol{j}}^2 - c_{n,\boldsymbol{j}+\boldsymbol{e}_q}^2| = 0, \tag{25}$$

where $\boldsymbol{e}_q$ is the qth vector of the canonical basis of $\mathbb{R}^d$, then (20) holds.

PROOF. We use an idea of [37]. Assume that (20) does not hold. Then there exist $\varepsilon > 0$, a sequence $(n_k)_{k\geq 1}$ such that $n_k \to \infty$ as $k \to \infty$, and a sequence $(\boldsymbol{j}_k)_{k\geq 1}$ such that $c_{n_k,\boldsymbol{j}_k} > \varepsilon \|c_{n_k}\|$ for all $k \in \mathbb{N}$. Choose $M > 0$ such that $M^d \varepsilon^2 > 1$. One has, for all $k \in \mathbb{N}$,

$$\|c_{n_k}\|^2 \geq \sum_{\boldsymbol{j}\in[0,M-1]^d} c^2_{n_k,\boldsymbol{j}_k+\boldsymbol{j}} \geq M^d c^2_{n_k,\boldsymbol{j}_k} - \sum_{\boldsymbol{j}\in[0,M-1]^d} |c^2_{n_k,\boldsymbol{j}_k+\boldsymbol{j}} - c^2_{n_k,\boldsymbol{j}_k}|.$$

Hence,

$$(M^d\varepsilon^2 - 1)\|c_{n_k}\|^2 \leq \sum_{\boldsymbol{j}\in[0,M-1]^d} |c^2_{n_k,\boldsymbol{j}_k+\boldsymbol{j}} - c^2_{n_k,\boldsymbol{j}_k}|. \tag{26}$$

But, if $\boldsymbol{j} \in [0, M-1]^d$, then

$$|c^2_{n_k,\boldsymbol{j}_k} - c^2_{n_k,\boldsymbol{j}_k+\boldsymbol{j}}| \leq \sum_{i=1}^{\ell(\boldsymbol{\lambda})} |c^2_{n_k,\boldsymbol{\lambda}_i} - c^2_{n_k,\boldsymbol{\lambda}_{i+1}}|,$$

where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_\ell)$ is any path from $\boldsymbol{\lambda}_0 = \boldsymbol{j}_k$ to $\boldsymbol{\lambda}_\ell = \boldsymbol{j}_k + \boldsymbol{j}$, with $|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_{i+1}|_1 = 1$. Since $\boldsymbol{j} \in [0, M-1]^d$, we can always choose the path $\boldsymbol{\lambda}$ of length $\ell = \ell(\boldsymbol{\lambda})$ smaller than dM. Thus, we get

$$\begin{aligned}|c^2_{n_k,\boldsymbol{j}_k} - c^2_{n_k,\boldsymbol{j}_k+\boldsymbol{j}}| &\leq dM \sup_{q=1,\ldots,d} \sup_{\boldsymbol{k}\in\mathbb{Z}^d} |c^2_{n_k,\boldsymbol{k}} - c^2_{n_k,\boldsymbol{k}+\boldsymbol{e}_q}| \\ &\leq dM \sum_{q=1}^{d} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |c^2_{n_k,\boldsymbol{k}} - c^2_{n_k,\boldsymbol{k}+\boldsymbol{e}_q}|.\end{aligned}$$

Together with (26), this contradicts (25). □

REMARK 3. Using the Cauchy–Schwarz inequality, we also see that the condition

$$\lim_{n\to\infty} \frac{1}{\|c_n\|^2} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} (c_{n,\boldsymbol{j}} - c_{n,\boldsymbol{j}+\boldsymbol{e}_q})^2 = 0 \qquad \text{for all } q = 1, \ldots, d, \tag{27}$$

implies (25), and thus by Lemma 8, implies (20). This last observation leads to an improvement in Theorem 3.1 in Biermé and Durieu [5]. The conditions (i) and (ii) of this theorem are equivalent to our conditions (20) and (27), respectively. Thus, the condition (i) in [5], Theorem 3.1, is unnecessary.

4. An invariance principle. The aim of the section is to establish a general invariance principle for partial sums of the random field $(X_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ defined in Section 2.

4.1. *Main result.* Recall that $(X_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ are associated to the random graph $\mathcal{G}_\mu$, with $\mu \in \mathcal{D}(E, \nu)$. We consider partial sums on finite rectangular subsets of $\mathbb{Z}^d$. As we will see, the growth of the rectangles will be determinant in the invariance principle and different limit random fields appear at different regimes. For the general case, consider a matrix $E' = \operatorname{diag}(1/\alpha'_1, \dots, 1/\alpha'_d)$ with $\alpha'_i > 0, i = 1, \dots, d$ and the partial-sum process

$$S_n^{E'}(\boldsymbol{t}) = \sum_{\boldsymbol{j}\in[\boldsymbol{0}, n^{E'}\boldsymbol{t}-\boldsymbol{1}]} X_{\boldsymbol{j}} \qquad n \geq 1 \text{ and } \boldsymbol{t} = (t_1, \dots, t_d) \in [0,1]^d.$$

The result will depend on both E' and E.

We introduce several parameters. First, for all $k = 1, \dots, d$, set $\rho_k := \alpha_k/\alpha'_k$, and consider

$$\gamma_0 = \gamma_0(E, E') := \min\left\{\gamma \in \{\rho_1, \dots, \rho_d\} \Big| \sum_{k:\gamma\geq\rho_k} \frac{1}{\alpha_k} > 2, \sum_{k:\gamma>\rho_k} \frac{1}{\alpha_k} \leq 2\right\}. \tag{28}$$

Note that γ_0 is well defined by the assumption $q(E) > 2$, and is completely determined by E and E'. Given $\gamma_0 > 0$, define the sets

$$I_< := \{k \in \{1, \dots, d\} | \gamma_0 < \rho_k\},$$
$$I_= := \{k \in \{1, \dots, d\} | \gamma_0 = \rho_k\},$$
$$I_> := \{k \in \{1, \dots, d\} | \gamma_0 > \rho_k\}.$$

This gives a partition of $\{1, \dots, d\}$. We also write $I_\leq := I_< \cup I_=$ and $I_\geq := I_= \cup I_>$. The sets $I_>$ and $I_<$ consist of the directions in which the limit random field exhibit degenerate dependence structure. Remark that, by construction,

$$|I_=| \geq 1 \quad \text{and} \quad |I_>| \leq 1.$$

According to these subsets of $\{1, \dots, d\}$, we consider subspaces of $\mathbb{R}^d$ given by

$$\mathcal{H}_< := \{x \in \mathbb{R}^d | x_k = 0 \text{ for } k \notin I_<\},$$

and similarly $\mathcal{H}_=, \mathcal{H}_>, \mathcal{H}_\leq, \mathcal{H}_\geq$. Let $\pi_<$, $\pi_=$, $\pi_>$, $\pi_\leq$, and $\pi_\geq$ denote orthogonal projections to the corresponding subspaces, and let $\lambda_<$, $\lambda_=$, $\lambda_>$, $\lambda_\leq$, and $\lambda_\geq$ denote the Lebesgue measures on the corresponding subspaces. For π of any proceeding projection, πE is a linear operator on $\mathbb{R}^d$; accordingly there is a corresponding diagonal matrix, which we also denote by πE with a slight abuse of notation.

Next, we define another diagonal matrix E'' (that only depends on E and E') by

$$E'' := \operatorname{diag}(\gamma_1/\alpha'_1, \dots, \gamma_d/\alpha'_d) \qquad \text{with } \gamma_k := \frac{\gamma_0}{\rho_k} \vee 1, k = 1, \dots, d.$$

By the definition of E'', one has

$$\pi_\leq E'' = \pi_\leq E' \quad \text{and} \quad \pi_\geq E'' = \gamma_0 \pi_\geq E. \tag{29}$$

Further, $E'' - \gamma_0 E$ is strictly positive on $\mathcal{H}_<$. We can now state our main result.

THEOREM 5. *Assume $\mu \in \mathcal{D}(E, \nu)$ with $E = \operatorname{diag}(1/\alpha_1, \dots, 1/\alpha_d)$ with $\alpha_i \in (0, 1), i = 1, \dots, d$, and $\alpha_1 \in (0, 1/2)$ if $d = 1$. Let $E' = \operatorname{diag}(1/\alpha'_1, \dots, 1/\alpha'_d)$, with $\alpha'_i > 0, i = 1, \dots, d$, and γ_0 defined as in (28). If $q(\pi_> E) < 2$, then*

$$\left(\frac{S_n^{E'}(\boldsymbol{t}) - \mathbb{E}(S_n^{E'}(\boldsymbol{t}))}{n^{\gamma_0 + q(E') - q(E'')/2}}\right)_{\boldsymbol{t} \in [0,1]^d} \quad \Rightarrow \quad \big(W(\boldsymbol{t})\big)_{\boldsymbol{t} \in [0,1]^d},$$

as $n \to \infty$, in the Skorohod space $D([0, 1]^d)$, where $(W(\boldsymbol{t}))_{\boldsymbol{t} \in \mathbb{R}^d}$ is a zero-mean Gaussian process with covariances given by

$$\begin{aligned}
&\operatorname{Cov}\big(W(\boldsymbol{t}), W(\boldsymbol{s})\big)\\
&\quad = \sigma_X^2 \left(\prod_{k \in I_<} \operatorname{Cov}\big(B_{1/2}(t_k), B_{1/2}(s_k)\big)\right)\\
&\qquad \times \left(\prod_{k \in I_>} \frac{t_k s_k}{2\pi}\right) \int_{\mathcal{H}_\geq} \left|\log \psi(\boldsymbol{y})\right|^{-2} \prod_{k \in I_=} \frac{(e^{it_k y_k} - 1)\overline{(e^{is_k y_k} - 1)}}{2\pi |y_k|^2} \, d\lambda_\geq(\boldsymbol{y}),
\end{aligned}$$

with $B_{1/2}$ a standard Brownian motion on $\mathbb{R}$, ψ is the characteristic function of ν, and σ_X^2 is given in (21).

In the expression of covariance above and in the sequel, it is understood that when $I_<$ or $I_>$ is empty, the corresponding product equals 1.

This theorem reveals that taking different summing rectangles may lead to different limits, under different normalizations. To the best of our knowledge, such a phenomenon has not been noticed in the literature until very recently [38, 39] for a different model. We elaborate on this phenomenon of scaling transition in Section 5.

REMARK 4. Observe that one can write $\operatorname{Cov}(W(\boldsymbol{t}), W(\boldsymbol{s})) = \frac{\sigma_X^2}{(2\pi)^d} \overline{C}(\boldsymbol{t}, \boldsymbol{s})$ with

$$\overline{C}(\boldsymbol{t}, \boldsymbol{s}) := \left(\prod_{k \in I_>} t_k s_k\right) \int_{\mathbb{R}^d} \left|\log \psi(\pi_\geq \boldsymbol{y})\right|^{-2} \left(\prod_{k \in I_\leq} \frac{(e^{it_k y_k} - 1)\overline{(e^{is_k y_k} - 1)}}{|y_k|^2}\right) d\boldsymbol{y}, \tag{30}$$

because of the identity ([46], Proposition 7.2.8)

$$\int_{\mathbb{R}} \frac{(e^{ity} - 1)\overline{(e^{isy} - 1)}}{2\pi |y|^{1+2H}} \, dy = C_H \operatorname{Cov}\big(B_H(t), B_H(s)\big), \qquad t, s \in \mathbb{R}, H \in (0, 1) \tag{31}$$

with

$$C_H = \frac{\pi}{H \Gamma(2H) \sin(H\pi)}.$$

Both Theorems 1 and 2 follow directly from Theorem 5.

PROOF OF THEOREM 1. In the critical regime corresponding to Theorem 1, $E = E'$ and $I_< = I_> = \varnothing$. In addition to $I_< = I_> = \varnothing$, it also follows that $\mathcal{H}_\geq = \mathbb{R}^d$ and $\lambda_\geq$ is the Lebesgue measure on $\mathbb{R}^d$. Theorem 1 now follows immediately. □

The proof of Theorem 2 is slightly more computational. As it corresponds to a special case that the limit W is a fractional Brownian sheet, which will be discussed in Section 5.2, the proof is postponed there.

REMARK 5. By definition (28), $q(\pi_> E) \leq 2$. The condition $q(\pi_> E) < 2$ in Theorem 5 cannot be dropped. It is easy to construct an example with $q(\pi_> E) = 2$, and as can be seen at the end of the proof of Lemma 9, in this case the asymptotic estimate of the covariance no longer holds. Moreover, in view of results for $d = 2$ (Theorem 2), this corresponds to the cases $\alpha_2' > \alpha_2, \alpha_2 = 1/2$ or $\alpha_2' < \alpha_2, \alpha_1 = 1/2$. In these cases, we conjecture that the limiting random fields are still fractional Brownian sheets with $(H_1, H_2) = (1/2, 1)$ and $(H_1, H_2) = (1, 1/2)$, respectively, that is, the dependence is degenerate in both directions. So $q(\pi_> E) = 2$ may be viewed as another *critical regime in the noncritical regime*. Since the paper is already quite involved, and this regime seems to preserve the least dependence, we do not pursue the investigation of this case here.

REMARK 6. As we will see below in the proof, essentially we establish an invariance principle for linear random field $(X_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ with

$$X_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathbb{Z}^d} q_{\boldsymbol{j}-\boldsymbol{i}} X^*_{\boldsymbol{i}}, \qquad \boldsymbol{j} \in \mathbb{Z}^d,$$

where $(X^*_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ are stationary martingale-difference innovations and $(q_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ are real Fourier coefficients of certain function $Q(\boldsymbol{t})$. This is a standard framework to obtain linear random fields in the literature, and we comment briefly on connections between our results and others:

(i) First, the same invariance principle should hold if the innovations are replaced by other weakly dependent random fields (weakly dependent in the sense of, e.g., [5, 28, 51]). These results can be viewed as generalizations of the seminal work of Davydov [15] on invariance principles for linear processes.

(ii) Second, from the modeling point of view, the specific choices of $Q(t)$ [in terms of $\mu \in \mathcal{D}(E, \nu)$], and hence $(q_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ are new. However, although our assumption on $Q(\boldsymbol{t})$ is very general, not all possible operator-scaling Gaussian random fields can show up in the limit; in particular, the Hammond–Sheffield model in high dimensions does not scale to fractional Brownian sheets except for a few cases in terms of Hurst indices shown in Proposition 6. The aforementioned results [5, 28, 51] all include linear random-field models scaling to fractional Brownian sheets, for flexible choices of Hurst indices.

(iii) At last, when the innovation random fields exhibit strong dependence, the limiting object could be more complicated ([28]).

4.2. *Proof of the main result.* The rest of the section is devoted to the proof of Theorem 5. Using Proposition 3, we get

$$S_n^{E'}(\boldsymbol{t}) - \mathbb{E}\big(S_n^{E'}(\boldsymbol{t})\big) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} b_{n,\boldsymbol{j}}(\boldsymbol{t}) X^*_{\boldsymbol{j}}, \tag{32}$$

with $b_n(\boldsymbol{t}) = (b_{n,\boldsymbol{j}}(\boldsymbol{t}))_{\boldsymbol{j}\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ and

$$b_{n,\boldsymbol{j}}(\boldsymbol{t}) = \sum_{\boldsymbol{k}\in[0,n^{E'}\boldsymbol{t}-\mathbf{1}]} q_{\boldsymbol{k}-\boldsymbol{j}}. \tag{33}$$

Recall that $(X^*_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ are stationary martingale differences.

The proof of Theorem 5 is now divided into three steps. The key step is to compute the covariance, which is done in Section 4.2.1. Then we proceed with the standard argument to show the weak convergence by first establishing finite-dimensional convergence in Section 4.2.2 and then the tightness in Section 4.2.3. The matrices E and E', and thus γ_0 and E'', are fixed as in the assumptions of the theorem.

4.2.1. *Covariances.* From (32), we obtain for $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^d$,

$$\operatorname{Cov}\big(S_n^{E'}(\boldsymbol{t}), S_n^{E'}(\boldsymbol{s})\big) = \sigma_X^2 \langle b_n(\boldsymbol{t}), b_n(\boldsymbol{s})\rangle,$$

where, $\langle b_n(\boldsymbol{t}), b_n(\boldsymbol{s})\rangle := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} b_{n,\boldsymbol{k}}(\boldsymbol{t}) b_{n,\boldsymbol{k}}(\boldsymbol{s})$. The asymptotic behavior of the covariances are given in the following lemma where $u_n \underset{n\to\infty}{\sim} v_n$ stands for $\lim_{n\to\infty} u_n/v_n = 1$.

LEMMA 9. *For all* $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^d$,

$$\sigma_X^2 \langle b_n(\boldsymbol{t}), b_n(\boldsymbol{s})\rangle \underset{n\to\infty}{\sim} n^{2\gamma_0 + 2q(E') - q(E'')} \operatorname{Cov}\big(W(\boldsymbol{t}), W(\boldsymbol{s})\big).$$

PROOF. Define for $m \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$D_m(x) = \sum_{l=0}^{m} e^{ilx} = \frac{e^{i(m+1)x} - 1}{e^{ix} - 1},$$

and for $\boldsymbol{x} \in \mathbb{R}^d$, the trigonometric polynomial

$$K_n(\boldsymbol{t}, \boldsymbol{x}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \mathbb{1}_{\boldsymbol{j}\in[0,n^{E'}\boldsymbol{t}-\mathbf{1}]} e^{i\boldsymbol{j}\cdot\boldsymbol{x}} = \prod_{k=1}^{d} D_{\lfloor n^{1/\alpha'_k} t_k - 1\rfloor}(x_k),$$

where $\lfloor\cdot\rfloor$ stands for the integer part. Recall that since

$$Q(\boldsymbol{x}) = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} q_{\boldsymbol{j}} e^{i\boldsymbol{j}\cdot\boldsymbol{x}},$$

the sequence $b_n(\boldsymbol{t})$ [defined in (33)] is obtained by the convolution product of the Fourier coefficients of $K_n(\boldsymbol{t},\cdot)$ and $\overline{Q}$ with $\overline{Q}(\boldsymbol{x}) = Q(-\boldsymbol{x})$ since $(q_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}^d}$ is a real sequence. It follows that $b_{n,\boldsymbol{k}}(\boldsymbol{t})$ is the $\boldsymbol{k}$th Fourier coefficient of $\overline{Q}K_n(\boldsymbol{t},\cdot)$. Therefore, using Bessel–Parseval identity, we get

$$\begin{aligned}\langle b_n(\boldsymbol{t}), b_n(\boldsymbol{s})\rangle &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \overline{Q}(\boldsymbol{x})K_n(\boldsymbol{t},\boldsymbol{x})\overline{\overline{Q}(\boldsymbol{x})K_n(\boldsymbol{s},\boldsymbol{x})}\,d\boldsymbol{x}\\ &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} |Q(\boldsymbol{x})|^2\prod_{k=1}^d D_{\lfloor n^{1/\alpha'_k}t_k-1\rfloor}(x_k)\overline{D_{\lfloor n^{1/\alpha'_k}s_k-1\rfloor}(x_k)}\,d\boldsymbol{x} \qquad (34)\\ &= \frac{n^{-q(E'')}}{(2\pi)^d}\int_{n^{E''}[-\pi,\pi]^d}\Phi_n(\boldsymbol{y},\boldsymbol{t},\boldsymbol{s})\,d\boldsymbol{y},\end{aligned}$$

where

$$\begin{aligned}&\Phi_n(\boldsymbol{y},\boldsymbol{t},\boldsymbol{s})\\ &\quad = |Q(n^{-E''}\boldsymbol{y})|^2\prod_{k=1}^d D_{\lfloor n^{1/\alpha'_k}t_k-1\rfloor}(n^{-\gamma_k/\alpha'_k}y_k)\overline{D_{\lfloor n^{1/\alpha'_k}s_k-1\rfloor}(n^{-\gamma_k/\alpha'_k}y_k)}.\end{aligned}$$

According to Lemma 2 and the E-homogeneity of $\log\psi$, one has

$$\begin{aligned}n^{-2\gamma_0}|Q(n^{-E''}\boldsymbol{y})|^2 &= n^{-2\gamma_0}|g(n^{-E''}\boldsymbol{y})|^2|\log\psi(n^{-\gamma_0E}n^{-(E''-\gamma_0E)}\boldsymbol{y})|^{-2}\\ &= |g(n^{-E''}\boldsymbol{y})|^2|\log\psi(n^{-(E''-\gamma_0E)}\boldsymbol{y})|^{-2}.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} n^{-2\gamma_0}|Q(n^{-E''}\boldsymbol{y})|^2 = |\log\psi(\pi_{\geq}\boldsymbol{y})|^{-2},$$

because $E''-\gamma_0E$ is null on $\mathcal{H}_{\geq}$ and strictly positive on $\mathcal{H}_{<}$ and $g(\boldsymbol{0}) = 1$. Further, for all $n\in\mathbb{N}_*$, $\boldsymbol{y}\in n^{E''}[-\pi,\pi]^d$,

$$n^{-2\gamma_0}|Q(n^{-E''}\boldsymbol{y})|^2 \leq \max_{\boldsymbol{x}\in[-\pi,\pi]^d}|g(\boldsymbol{x})|^2\sup_{z\in\mathcal{H}_<}|\log\psi(\boldsymbol{z}+\pi_{\geq}\boldsymbol{y})|^{-2}. \qquad (35)$$

Now, remark that for all $t\in[0,1]$ and $y\in\mathbb{R}$,

$$\lim_{n\to\infty} n^{-1}D_{\lfloor nt-1\rfloor}(n^{-\gamma}y) = \begin{cases}\dfrac{e^{ity}-1}{iy}, & \text{if } \gamma = 1,\\ t, & \text{if } \gamma > 1,\end{cases}$$

and if $|n^{-\gamma}y|\leq\pi$, then

$$|n^{-1}D_{\lfloor nt-1\rfloor}(n^{-\gamma}y)| = \left|\frac{\sin(\lfloor nt\rfloor n^{-\gamma}y/2)}{n\sin(n^{-\gamma}y/2)}\right| \leq \begin{cases}\pi\min\left\{1,\dfrac{1}{|y|}\right\}, & \text{if } \gamma = 1,\\ \dfrac{\pi}{2}, & \text{if } \gamma > 1,\end{cases}$$

where we have used that $\frac{2}{\pi}|x| \le |\sin(x)| \le |x| \wedge 1$ for $x \in [-\pi/2, \pi/2]$ and that $|t| \le 1$. Since $\gamma_k > 1$ if and only if $k \in I_>$, we infer

$$\begin{aligned}\Phi_n(\boldsymbol{y}, \boldsymbol{t}, \boldsymbol{s}) &\sim n^{2\gamma_0+2q(E')} \bigl|\log \psi(\pi_{\ge} \boldsymbol{y})\bigr|^{-2} \\ &\quad \times \Bigl(\prod_{k\in I_>} t_k s_k\Bigr)\Bigl(\prod_{k\in I_\le} \frac{(e^{it_k y_k}-1)\overline{(e^{is_k y_k}-1)}}{|y_k|^2}\Bigr)\end{aligned} \tag{36}$$

as $n \to \infty$ and for all $\boldsymbol{t}, \boldsymbol{s} \in [0, 1]^d$,

$$n^{-2\gamma_0-2q(E')}\bigl|\Phi_n(\boldsymbol{y}, \boldsymbol{t}, \boldsymbol{s})\bigr| \le \pi^{2d} \max_{\boldsymbol{x}\in[-\pi,\pi]^d} \bigl|g(\boldsymbol{x})\bigr|^2 h(\boldsymbol{y}), \tag{37}$$

with

$$h(\boldsymbol{y}) := \sup_{\boldsymbol{x}\in\mathcal{H}_<} \bigl|\log \psi(\boldsymbol{x} + \pi_{\ge}\boldsymbol{y})\bigr|^{-2} \prod_{k\in I_\le} \min\Bigl\{1, \frac{1}{|y_k|^2}\Bigr\}. \tag{38}$$

Applying the dominated convergence theorem to (34), (36) and (37) and using (30), to show the desired result it remains to prove that h is integrable on $\mathbb{R}^d$.

Formally, write

$$\begin{aligned}\int_{\mathbb{R}^d} h(\boldsymbol{y})\, d\boldsymbol{y} &= \int_{\mathcal{H}_<}\int_{\mathcal{H}_\ge} h(\boldsymbol{y})\, d\lambda_< \otimes \lambda_\ge(\boldsymbol{y}) \\ &= \int_{\mathcal{H}_<} \prod_{k\in I_<} \min\Bigl\{1, \frac{1}{|y_k|^2}\Bigr\} d\lambda_<(\boldsymbol{y}) \\ &\quad \times \int_{\mathcal{H}_\ge} \sup_{\boldsymbol{x}\in\mathcal{H}_<} \bigl|\log \psi(\boldsymbol{x}+\boldsymbol{y})\bigr|^{-2} \prod_{k\in I_=} \min\Bigl\{1, \frac{1}{|y_k|^2}\Bigr\} d\lambda_\ge(\boldsymbol{y}),\end{aligned}$$

where the first integral in the right-hand side is understood to be 1 if $\mathcal{H}_< = \{\boldsymbol{0}\}$ (i.e. $I_< = \varnothing$). By Fubini's theorem, h is integrable over $\mathbb{R}^d$ if

$$\int_{\mathcal{H}_<} \prod_{k\in I_<} \min\Bigl\{1, \frac{1}{|y_k|^2}\Bigr\} d\lambda_<(\boldsymbol{y}) < \infty \tag{39}$$

and

$$\begin{aligned}&\int_{\mathcal{H}_\ge} h(\boldsymbol{y})\, d\lambda_\ge(\boldsymbol{y}) \\ &\quad = \int_{\mathcal{H}_\ge} \sup_{\boldsymbol{x}\in\mathcal{H}_<} \bigl|\log \psi(\boldsymbol{x}+\boldsymbol{y})\bigr|^{-2} \prod_{k\in I_=} \min\Bigl\{1, \frac{1}{|y_k|^2}\Bigr\} d\lambda_\ge(\boldsymbol{y}) < \infty.\end{aligned} \tag{40}$$

The integrability condition (39) is obvious. For (40), let us remark that the function $\boldsymbol{y} \in \mathcal{H}_\ge \mapsto \inf_{\boldsymbol{x}\in\mathcal{H}_<} |\log \psi(\boldsymbol{x} + \boldsymbol{y})|$ is $(\pi_\ge E)$-homogeneous and since $q(\pi_\ge E) > 2$, by Lemma 3, the function $\boldsymbol{y} \in \mathcal{H}_\ge \mapsto \sup_{\boldsymbol{x}\in\mathcal{H}_<} |\log \psi(\boldsymbol{x} + \boldsymbol{y})|^{-2}$ is locally integrable on $\mathcal{H}_\ge$ with respect to $\lambda_\ge$. Together with the fact that

$\sup_{\boldsymbol{x}\in\mathcal{H}_<} |\log\psi(\boldsymbol{x}+\boldsymbol{y})|^{-2}$ is bounded by 1 for $\pi_=\boldsymbol{y}$ large enough, this shows that (40) holds in the case $\mathcal{H}_> = \{0\}$. For the case $\mathcal{H}_> \neq \{\boldsymbol{0}\}$, the preceding considerations show that

$$\int_{\mathcal{H}_\geq} \mathbb{1}_{\{\|\pi_>\boldsymbol{y}\|_{\pi_> E}\leq 1\}} h(\boldsymbol{y})\, d\lambda_\geq(\boldsymbol{y}) < \infty,$$

with the definition of $\|\cdot\|_{\pi_> E}$ given in (11). Moreover,

$$\begin{aligned}\int_{\mathcal{H}_\geq} &\mathbb{1}_{\{\|\pi_>\boldsymbol{y}\|_{\pi_> E}>1\}} h(\boldsymbol{y})\, d\lambda_\geq(\boldsymbol{y})\\ &\leq \int_{\mathcal{H}_>} \mathbb{1}_{\{\|\boldsymbol{y}\|_{\pi_> E}>1\}} \sup_{\boldsymbol{x}\in\mathcal{H}_\leq} \left|\log\psi(\boldsymbol{x}+\boldsymbol{y})\right|^{-2} d\lambda_>(\boldsymbol{y})\\ &\quad\times \int_{\mathcal{H}_=} \prod_{k\in I_=} \min\left\{1, \frac{1}{|y_k|^2}\right\} d\lambda_=(\boldsymbol{y}).\end{aligned}$$

The second integral is clearly finite. For the first one, since $\boldsymbol{y} \in \mathcal{H}_> \mapsto \inf_{\boldsymbol{x}\in\mathcal{H}_\leq} |\log\psi(\boldsymbol{x}+\boldsymbol{y})|$ is $(\pi_> E)$-homogeneous and $q(\pi_> E) < 2$, one has

$$\begin{aligned}\int_{\mathcal{H}_>} &\mathbb{1}_{\{\|\boldsymbol{y}\|_{\pi_> E}>1\}} \sup_{\boldsymbol{x}\in\mathcal{H}_\leq} \left|\log\psi(\boldsymbol{x}+\boldsymbol{y})\right|^{-2} d\lambda_>(\boldsymbol{y})\\ &= \int_1^{+\infty} r^{q(\pi_> E)-3} \int_{S_{\pi_> E}} \sup_{\boldsymbol{x}\in\mathcal{H}_\leq} \left|\log\psi(\boldsymbol{x}+\boldsymbol{\theta})\right|^{-2} d\sigma_{\pi_> E}(\boldsymbol{\theta}) < \infty,\end{aligned}$$

where $S_{\pi_> E}$ is the unit sphere of $\mathcal{H}_>$ with respect to $\|\cdot\|_{\pi_> E}$ and $\sigma_{\pi_> E}$ is the Radon measure on $S_{\pi_> E}$ such that $d\lambda_> = r^{q(\pi_> E)-1}\, dr\, d\sigma_{\pi_> E}$. This shows that (40) holds, and thus the function h in (38) is integrable over $\mathbb{R}^d$. □

4.2.2. *Finite-dimensional convergence*. We start by showing that the coefficients $b_{n,\boldsymbol{j}}(\boldsymbol{t})$ defined in (33) satisfy the condition (20) of Theorem 3 in the following lemma.

LEMMA 10. *For all $\boldsymbol{t} \in (0,1]^d$ and all $q = 1, \dots, d$,*

$$\lim_{n\to\infty} \frac{1}{\|b_n(\boldsymbol{t})\|^2} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \left|b_{n,\boldsymbol{j}}^2(\boldsymbol{t}) - b_{n,\boldsymbol{j}+\boldsymbol{e}_q}^2(\boldsymbol{t})\right| = 0$$

and (20) *holds*.

PROOF. Fix $\ell \in \{1, \dots, d\}$ and $t \in (0,1]^d$ be fixed. Using the Cauchy–Schwarz inequality,

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \left|b_{n,\boldsymbol{j}}^2(\boldsymbol{t}) - b_{n,\boldsymbol{j}+\boldsymbol{e}_\ell}^2(\boldsymbol{t})\right| \leq \left(\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \left(b_{n,\boldsymbol{j}}(\boldsymbol{t}) - b_{n,\boldsymbol{j}+\boldsymbol{e}_\ell}(\boldsymbol{t})\right)^2\right)^{\frac{1}{2}} 2\|b_n(\boldsymbol{t})\|.$$

So, it is enough to show that

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\big(b_{n,\boldsymbol{j}}(\boldsymbol{t})-b_{n,\boldsymbol{j}+\boldsymbol{e}_\ell}(\boldsymbol{t})\big)^2=o\big(\|b_n(\boldsymbol{t})\|^2\big).$$

But we have

$$b_{n,\boldsymbol{j}}(\boldsymbol{t})-b_{n,\boldsymbol{j}+\boldsymbol{e}_\ell}(\boldsymbol{t})=\sum_{\substack{\boldsymbol{k}\in[\boldsymbol{0},n^{E'}\boldsymbol{t}-\boldsymbol{1}]\\ \text{with } k_\ell=\lfloor n^{1/\alpha'_\ell}t_\ell\rfloor-1}}q_{\boldsymbol{k}-\boldsymbol{j}}-\sum_{\substack{\boldsymbol{k}\in[\boldsymbol{0},n^{E'}\boldsymbol{t}-\boldsymbol{1}]\\ \text{with } k_\ell=0}}q_{\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{e}_\ell}.$$

Thus,

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\big(b_{n,\boldsymbol{j}}(\boldsymbol{t})-b_{n,\boldsymbol{j}+\boldsymbol{e}_\ell}(\boldsymbol{t})\big)^2\le 2\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\Bigg(\sum_{\substack{\boldsymbol{k}\in[\boldsymbol{0},n^{E'}\boldsymbol{t}-\boldsymbol{1}]\\ \text{with } k_\ell=0}}q_{\boldsymbol{k}-\boldsymbol{j}}\Bigg)^2.$$

Let $\varepsilon>0$. Using Lemma 9, we get

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\|b_n(\boldsymbol{t})\|^2}\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\Bigg(\sum_{\substack{\boldsymbol{k}\in[\boldsymbol{0},n^{E'}\boldsymbol{t}-\boldsymbol{1}]\\ \text{with } k_\ell=0}}q_{\boldsymbol{k}-\boldsymbol{j}}\Bigg)^2\\
&\quad\le\limsup_{n\to\infty}\frac{1}{\|b_n(\boldsymbol{t})\|^2}\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\Bigg(\sum_{\substack{\boldsymbol{k}\in[\boldsymbol{0},n^{E'}\boldsymbol{t}-\boldsymbol{1}]\\ \text{with } k_\ell\le\varepsilon n^{1/\alpha'_\ell}t_\ell-1}}q_{\boldsymbol{k}-\boldsymbol{j}}\Bigg)^2\\
&\quad=\limsup_{n\to\infty}\frac{\|b_n(t_1,\dots,t_{\ell-1},\varepsilon t_\ell,t_{\ell+1},\dots,t_d)\|^2}{\|b_n(\boldsymbol{t})\|^2}\\
&\quad=\frac{V(t_1,\dots,t_{\ell-1},\varepsilon t_\ell,t_{\ell+1},\dots,t_d)}{V(\boldsymbol{t})},
\end{aligned}$$

where $V(\boldsymbol{t}):=\overline{C}(\boldsymbol{t},\boldsymbol{t})$ with the covariance function $\overline{C}(\cdot,\cdot)$ defined in (30). We conclude the proof of the lemma using that, for any $\boldsymbol{t}\in(0,1]^d$,

$$V(t_1,\dots,t_{\ell-1},\varepsilon t_\ell,t_{\ell+1},\dots,t_d)\to 0\qquad\text{as }\varepsilon\to 0.$$

The fact that (20) holds is a consequence of Lemma 8. $\square$

To prove the finite-dimensional convergence, we use the Cramèr–Wold device. Let $m\in\mathbb{N}$, $\boldsymbol{t}_1,\dots,\boldsymbol{t}_m\in[0,1]^d$, $\lambda_1,\dots,\lambda_m\in\mathbb{R}$, and consider $S_n^{(m)}=\sum_{k=1}^m\lambda_k S_n^{E'}(\boldsymbol{t}_k)$. One has

$$S_n^{(m)}-\mathbb{E}\big(S_n^{(m)}\big)=\sum_{j\in\mathbb{Z}^d}d_{n,\boldsymbol{j}}X_{\boldsymbol{j}}^*,$$

where $d_{n,\boldsymbol{j}} := \sum_{k=1}^m \lambda_k b_{n,\boldsymbol{j}}(\boldsymbol{t}_k)$ and $\operatorname{Var}(S_n^{(m)}) = \|d_n\|^2 \operatorname{Var}(X_{\boldsymbol{0}}^*)$. Using Lemma 9, we get

$$\begin{aligned}\|d_n\|^2 &= \sum_{k=1}^m \sum_{\ell=1}^m \lambda_k \lambda_\ell \langle b_n(\boldsymbol{t}_k), b_n(\boldsymbol{t}_\ell)\rangle \\ &\underset{n\to\infty}{\sim} \frac{n^{2\gamma_0+2q(E')-q(E'')}}{(2\pi)^d} \sum_{k=1}^m \sum_{\ell=1}^m \lambda_k \lambda_\ell \overline{C}(\boldsymbol{t}_k, \boldsymbol{t}_\ell),\end{aligned}$$

where $\overline{C}$ is defined in (30).

If $\sum_{k=1}^m \sum_{\ell=1}^m \lambda_k \lambda_\ell \overline{C}(\boldsymbol{t}_k, \boldsymbol{t}_\ell) = 0$, then $\frac{1}{n^{\gamma_0+q(E')-q(E'')/2}}(S_n^{(m)} - \mathbb{E}(S_n^{(m)}))$ converges to 0 in L^2. If $\sum_{k=1}^m \sum_{\ell=1}^m \lambda_k \lambda_\ell \overline{C}(\boldsymbol{t}_k, \boldsymbol{t}_\ell) > 0$, we get that for each $k = 1, \dots, m$,

$$\|b_n(\boldsymbol{t}_k)\|^2 \underset{n\to\infty}{\sim} \|d_n\|^2 \frac{\overline{C}(\boldsymbol{t}_k, \boldsymbol{t}_k)}{\sum_{k=1}^m \sum_{\ell=1}^m \lambda_k \lambda_\ell \overline{C}(\boldsymbol{t}_k, \boldsymbol{t}_\ell)}.$$

Thus, since the $b_{n,\boldsymbol{j}}(\boldsymbol{t}_k)$ satisfy (20),

$$\sup_{\boldsymbol{j}} |d_{n,\boldsymbol{j}}| \le \sum_{k=1}^m \lambda_k \sup_{\boldsymbol{j}} |b_{n,\boldsymbol{j}}(\boldsymbol{t}_k)| = \sum_{k=1}^m \lambda_k o(\|b_n(\boldsymbol{t}_k)\|) = o(\|d_n\|).$$

This proves that (20) also holds for the $d_{n,\boldsymbol{j}}$ and Theorem 3 applies to $S_n^{(m)}$. We thus proved the finite-dimensional convergence.

4.2.3. *Tightness*. To prove the tightness, by Bickel and Wichura [3], following [51] and [27], it is enough to show that for some $p > 0$ there exist $\gamma > 1$ and $C > 0$ such that for all $\boldsymbol{t} = (t_1, \dots, t_d) \in [0, 1]^d$,

$$\mathbb{E}\left|\frac{S_n^{E'}(\boldsymbol{t}) - \mathbb{E}(S_n^{E'}(\boldsymbol{t}))}{n^{\gamma_0+q(E')-\frac{q(E'')}{2}}}\right|^p \le C \prod_{j=1}^d t_j^\gamma.$$

Recall from equation (34) that for all $\boldsymbol{t} \in [0, 1]^d$, we have

$$\begin{aligned}\|b_n(\boldsymbol{t})\|^2 &= \frac{n^{-q(E'')}}{(2\pi)^d} \\ &\quad \times \int_{n^{E''}[-\pi,\pi]^d} |Q(n^{-E''}\boldsymbol{y})|^2 \left(\prod_{k=1}^d |D_{\lfloor n^{1/\alpha'_k} t_k - 1\rfloor}(n^{-\gamma_k/\alpha'_k} y_k)|^2\right) d\boldsymbol{y}.\end{aligned}$$

For any $\delta \in (0, 1)$, observe that $|\sin^2(x)| = |\sin^{1-\delta}(x)\|\sin^{1+\delta}(x)| \le \min\{|x|^{1-\delta}, |x|^2\}$ for all x, and $|\sin(x)| \ge \frac{2}{\pi}|x|$ for $x \in [-\pi/2, \pi/2]$. Then, for all n and y such that $|ny| \le \pi$ and all $t \in [0,1]$, one has

$$n^{-2}\big|D_{\lfloor nt-1\rfloor}(n^{-1}y)\big|^2 = \frac{\sin^2(\lfloor nt\rfloor \frac{y}{2n})}{n^2\sin^2(\frac{y}{2n})}$$
$$\le \min\left\{\frac{\pi^2}{2^{1-\delta}}\frac{t^{1-\delta}}{|y|^{1+\delta}}, \frac{\pi^2}{4}t^2\right\} \le \frac{\pi^2}{2^{1-\delta}} t^{1-\delta}\min\left\{\frac{1}{|y|^{1+\delta}}, 1\right\},$$

and thus,

$$n^{-2}\big|D_{\lfloor nt-1\rfloor}(n^{-\gamma}y)\big|^2 \le \begin{cases} \dfrac{\pi^2}{2^{1-\delta}} t^{1-\delta}\min\left\{\dfrac{1}{|y|^{1+\delta}}, 1\right\}, & \text{if } \gamma = 1,\\ \dfrac{\pi^2}{4}t^2, & \text{if } \gamma > 1.\end{cases}$$

Recalling that $\gamma_k/\alpha_k' > 1$ if and only if $k \in I_>$, together with (35), this shows that there exists a constant $C > 0$ such that

$$\begin{aligned} &n^{-2\gamma_0 - 2q(E') + q(E'')}\|b_n(\boldsymbol{t})\|^2 \\ &\quad \le C\left(\prod_{j\in I_>} t_j^2\right)\left(\prod_{j\in I_\le} t_j^{1-\delta}\right) \\ &\qquad \times \int_{\mathbb{R}^d} \sup_{\boldsymbol{x}\in\mathcal{H}_<} \big|\log\psi(\boldsymbol{x} + \pi_\ge \boldsymbol{y})\big|^{-2} \prod_{j\in I_\le} \min\left\{\frac{1}{|y_j|^{1+\delta}}, 1\right\} d\boldsymbol{y}. \end{aligned}$$

One can show that this last integral is finite by proceeding exactly as we did to show the integrability of the function h in (38). The important point is that $1 + \delta > 1$ to guarantee the integrability of $\frac{1}{|y|^{1+\delta}}$ at infinity. Hence, for a new constant $C' > 0$,

$$n^{-2\gamma_0 - 2q(E') + q(E'')}\|b_n(\boldsymbol{t})\|^2 \le C'\left(\prod_{j\in I_>} t_j^2\right)\left(\prod_{j\in I_\le} t_j^{1-\delta}\right) \le C' \prod_{j=1}^d t_j^{1-\delta}.$$

Let $p > 2$. Using Burkholder's inequality and the preceding inequality, there exists a constant $c_p > 0$ such that

$$\begin{aligned} \mathbb{E}\left|\frac{S_n^{E'}(\boldsymbol{t}) - \mathbb{E}(S_n^{E'}(\boldsymbol{t}))}{n^{\gamma_0 + q(E') - q(E'')/2}}\right|^p &\le c_p \mathbb{E}\left(\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \frac{b_{n,\boldsymbol{j}}^2(\boldsymbol{t})}{n^{2\gamma_0 + 2q(E') - q(E'')}} X_{\boldsymbol{j}}^{*2}\right)^{\frac{p}{2}} \\ &\le c_p\left(\frac{\|b_n(\boldsymbol{t})\|^2}{n^{2\gamma_0 + 2q(E') - q(E'')}}\right)^{\frac{p}{2}} \\ &\le c_p C'^{p/2} \prod_{j=1}^d t_j^{(1-\delta)p/2}, \end{aligned}$$

which gives the tightness by choosing $\delta > 1 - \frac{2}{p}$.

5. Properties of the limit field. In this section, we focus on the zero-mean Gaussian random field $(W(\boldsymbol{t}))_{\boldsymbol{t}\in\mathbb{R}^d}$ arising in the limit in Theorem 5. Recall that this random field depends on both E and E'.

5.1. *Increments.* We may consider a harmonizable representation of W, defined on the whole space $\mathbb{R}^d$ by

$$W(\boldsymbol{t}) = \sigma_X\biggl(\prod_{k\in I_>} t_k\biggr)\int_{\mathbb{R}^d}\biggl(\prod_{k\in I_\le}\frac{e^{it_k y_k}-1}{iy_k}\biggr)\bigl|\log\psi(\pi_\ge \boldsymbol{y})\bigr|^{-1}\widetilde{\mathcal{M}}(d\boldsymbol{y}),$$

for all $\boldsymbol{t}\in\mathbb{R}^d$, with σ_X given in (21), and $\widetilde{\mathcal{M}}$ is a centered complex-valued Gaussian measure on $\mathbb{R}^d$ with Lebesgue control measure (see [52]). The harmonizable representation shows that the random field has stationary rectangular increments. In the sequel, we let $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_d)$ denote the canonical basis of $\mathbb{R}^d$. Rectangular increments of W are defined for $\boldsymbol{s}<\boldsymbol{t}$ by

$$\begin{aligned} W([\boldsymbol{s},\boldsymbol{t}]) &= \sum_{\varepsilon\in\{0,1\}^d}(-1)^{d+|\varepsilon|_1}W\bigl(s_1+\varepsilon_1(t_1-s_1),\dots,s_d+\varepsilon_d(t_d-s_d)\bigr)\\ &= \Delta^{(1)}_{t_1-s_1}\Delta^{(2)}_{t_2-s_2}\cdots\Delta^{(d)}_{t_d-s_d}W(\boldsymbol{s}),\end{aligned}$$

where $|\varepsilon|_1=\varepsilon_1+\cdots+\varepsilon_d$ and $\Delta^{(j)}_\delta$ corresponds to the directional increment of step $\delta\in\mathbb{R}$ in direction j for $1\le j\le d$, defined by

$$\Delta^{(j)}_\delta W(\boldsymbol{t}) = W(\boldsymbol{t}+\delta\boldsymbol{e}_j)-W(\boldsymbol{t}).$$

A direct consequence of Theorem 5 are the following properties of the random field W.

PROPOSITION 4. *The random field W satisfies the following properties*:

(i) *stationary rectangular increments*: *for any fixed $\boldsymbol{s}\in\mathbb{R}^d$*,

$$\bigl(W([\boldsymbol{s},\boldsymbol{t}])\bigr)_{\boldsymbol{s}<\boldsymbol{t}} \overset{f.d.d.}{=} \bigl(W([\boldsymbol{0},\boldsymbol{t}-\boldsymbol{s}])\bigr)_{\boldsymbol{s}<\boldsymbol{t}} \equiv \bigl(W(\boldsymbol{t}-\boldsymbol{s})\bigr)_{\boldsymbol{s}<\boldsymbol{t}};$$

(ii) *(E',H)-operator-scaling property*: *for all $\lambda>0$*

$$\bigl(W(\lambda^{E'}\boldsymbol{t})\bigr)_{\boldsymbol{t}\in\mathbb{R}^d} \overset{f.d.d.}{=} \bigl(\lambda^H W(\boldsymbol{t})\bigr)_{\boldsymbol{t}\in\mathbb{R}^d},$$

with $H=\gamma_0+q(E')-\frac{q(E'')}{2}$ and E'' satisfying (29).

PROOF. Property (i) can be proved by observing that $W[\boldsymbol{s},\boldsymbol{t}]$ corresponds to the limit of partial sums of a stationary random field over a rectangle area, after normalization. By stationarity, the distributions of the partial sums, and hence the limit, depend only on $\boldsymbol{t}-\boldsymbol{s}$, up to certain boundary effect which needs to be taken care of. Alternatively, the stationary increment property can also be derived from the covariance function. The proof is omitted.

We only prove (ii) here. Recall the definition of $\overline{C}(\boldsymbol{t},\boldsymbol{s})$ in (30). By the change of variables,

$$y_k \mapsto \begin{cases} \lambda^{1/\alpha_k'} y_k, & k \in I_{\le}, \\ \lambda^{\gamma_0/\alpha_k} y_k, & k \in I_{>}, \end{cases}$$

we have

$$\begin{aligned}
\overline{C}(\lambda^{E'}\boldsymbol{t},\lambda^{E'}\boldsymbol{s}) &= \biggl(\prod_{k\in I_>} \lambda^{2/\alpha_k'} t_k s_k\biggr) \int_{\mathbb{R}^d} \frac{1}{|\log\psi(\lambda^{-\gamma_0 E}\lambda^{\gamma_0 E}\pi_{\ge}\boldsymbol{y})|^2} \\
&\quad \times \biggl(\prod_{k\in I_\le} \frac{(e^{it_k\lambda^{1/\alpha_k'} y_k}-1)\overline{(e^{is_k\lambda^{1/\alpha_k'} y_k}-1)}}{2\pi|y_k|^2}\biggr) d\boldsymbol{y} \\
&= \biggl(\prod_{k\in I_>} \lambda^{2/\alpha_k'}\biggr)\lambda^{2\gamma_0}\biggl(\prod_{k\in I_\le}\lambda^{1/\alpha_k'}\biggr)\biggl(\prod_{k\in I_>}\lambda^{-\gamma_0/\alpha_k}\biggr)\overline{C}(\boldsymbol{t},\boldsymbol{s}) \\
&= \lambda^{2\gamma_0+q(E')+q(\pi_> E')-\gamma_0 q(\pi_> E)}\overline{C}(\boldsymbol{t},\boldsymbol{s}),
\end{aligned}$$

where in the second equality we also used the fact that $\log\psi(\lambda^E\boldsymbol{y}) = \lambda\log\psi(\boldsymbol{y})$. On the other hand, recalling (29), we have

$$\begin{aligned}
q(E') - \frac{1}{2}q(E'') &= q(E') - \frac{1}{2}q(\pi_{\le}E') - \frac{\gamma_0}{2}q(\pi_> E) \\
&= \frac{1}{2}q(E') + \frac{1}{2}q(\pi_> E') - \frac{\gamma_0}{2}q(\pi_> E).
\end{aligned}$$

The desired result thus follows. □

We can say more about the directional increments $\Delta_\delta^{(j)} W(\boldsymbol{t})$. First of all, as a special case of Proposition 4(i), $W(\boldsymbol{t})$ viewed as a process indexed by $t_j \in \mathbb{R}$ has stationary increments. Moreover, simple dependence properties in the directions corresponding to $I_>$ and $I_<$, if not empty, are given below. Following ideas from [39], Definition 2.2, we state the following proposition. Recall that $|I_>| \le 1$.

PROPOSITION 5. *The random field W satisfies the following properties*:

(i) *When $I_> = \{j\}$, the random field W has invariant increments in the direction $\boldsymbol{e}_j$: for all $h,\delta\in\mathbb{R}$, $\boldsymbol{t}\in\mathbb{R}^d$, we have $\Delta_\delta^{(j)} W(\boldsymbol{t}+h\boldsymbol{e}_j) = \Delta_\delta^{(j)} W(\boldsymbol{t})$.*

(ii) *When $I_< \ne \varnothing$, the random field W has independent increments in any direction $\boldsymbol{e}_j$ with $j\in I_<$: for all $\delta > 0$, $\boldsymbol{t}\in\mathbb{R}^d$, $\Delta_\delta^{(j)} W(\boldsymbol{t})$ is independent from $W(\boldsymbol{t})$.*

PROOF. Let $\langle\boldsymbol{e}_j\rangle^\perp$ denote the subspace of $\mathbb{R}^d$ orthogonal to $\boldsymbol{e}_j$. Let $\pi_{\langle\boldsymbol{e}_j\rangle^\perp}$ and $\lambda_{\langle\boldsymbol{e}_j\rangle^\perp}$ denote the corresponding projection and Lebesgue measure, respectively. First, let us simply remark that for $I_> = \{j\}$, $\delta\in\mathbb{R}$, and $\boldsymbol{t}\in\mathbb{R}^d$,

$$\Delta_\delta^{(j)} W(\boldsymbol{t}) = \delta W(\pi_{\langle\boldsymbol{e}_j\rangle^\perp}(\boldsymbol{t}) + \boldsymbol{e}_j),$$

which does not depend on t_j. The desired statement then follows. For the second statement, when $j \in I_<$,

$$\Delta_\delta^{(j)} W(\boldsymbol{t}) = \sigma_X \Big(\prod_{k \in I_>} t_k \Big) \times \int_{\mathbb{R}^d} \frac{e^{i t_j y_j}(e^{i\delta y_j} - 1)}{i y_j} \Big(\prod_{k \in I_\le; k \neq j} \frac{e^{i t_k y_k} - 1}{i y_k} \Big) \big| \log \psi(\pi_\ge \boldsymbol{y}) \big|^{-1} \widetilde{\mathcal{M}}(d\boldsymbol{y}).$$

Therefore,

$$\operatorname{Cov}\big(\Delta_\delta^{(j)} W(\boldsymbol{t}), W(\boldsymbol{t})\big) = C_{\boldsymbol{e}_j}(\boldsymbol{t}) \int_{\mathbb{R}} \frac{(e^{i\delta y_j} - 1)(1 - e^{i t_j y_j})}{|y_j|^2} \, dy_j,$$

with

$$C_{\boldsymbol{e}_j}(\boldsymbol{t}) := \sigma_X^2 \Big(\prod_{k \in I_>} t_k \Big)^2 \int_{\langle \boldsymbol{e}_j \rangle^\perp} \prod_{k \in I_\le; k \neq j} \Big| \frac{e^{i t_k y_k} - 1}{i y_k} \Big|^2 \big| \log \psi(\pi_\ge \boldsymbol{y}) \big|^{-2} \, d\lambda_{\langle \boldsymbol{e}_j \rangle^\perp}(\boldsymbol{y}).$$

Hence,

$$\operatorname{Cov}\big(\Delta_\delta^{(j)} W(\boldsymbol{t}), W(\boldsymbol{t})\big) = 2\pi C_{\boldsymbol{e}_j}(\boldsymbol{t}) \operatorname{Cov}\big(B_{1/2}(t_j + \delta) - B_{1/2}(t_j), B_{1/2}(t_j)\big),$$

with $B_{1/2}$ a standard Brownian motion on $\mathbb{R}$. By independent increments of $B_{1/2}$, we obtain that $\operatorname{Cov}(\Delta_\delta^{(j)} W(\boldsymbol{t}), W(\boldsymbol{t})) = 0$ for $\delta \ge 0$. Since W is a Gaussian field, we conclude that $\Delta_\delta^{(j)} W(\boldsymbol{t})$ is independent from $W(\boldsymbol{t})$. □

Let us mention that our definitions of invariant and independent increments are not the ones used in [39], Definition 2.2. However, we remark that invariant increments in the direction $\boldsymbol{e}_j$ lead to invariant rectangular increments in the sense that, for all $\delta \in \mathbb{R}$, and $\boldsymbol{s} < \boldsymbol{t}$

$$W\big([\boldsymbol{s} + \delta \boldsymbol{e}_j, \boldsymbol{t} + \delta \boldsymbol{e}_j]\big) = W\big([\boldsymbol{s}, \boldsymbol{t}]\big).$$

This follows from the fact that

$$W\big([\boldsymbol{s} + \delta \boldsymbol{e}_j, \boldsymbol{t} + \delta \boldsymbol{e}_j]\big) = \Delta_{t_1 - s_1}^{(1)} \Delta_{t_2 - s_2}^{(2)} \cdots \Delta_{t_d - s_d}^{(d)} W(\boldsymbol{s} + \delta \boldsymbol{e}_j).$$

Indeed, computing first $\Delta_{t_j - s_j}^{(j)} W(\boldsymbol{s} + \delta e_j) = \Delta_{t_j - s_j}^{(j)} W(\boldsymbol{s})$, we obtain the desired result.

When the increments are either invariant or independent in at least one direction, we say that W has degenerate increments. Otherwise, we say that W has nondegenerate increments.

EXAMPLE 4. When $d = 2$, choosing $E' = \mathrm{diag}(1, \beta)$ for $\beta > 0$ as in [39] we obtain that $|I_=| = 2$ if and only if $\rho_1 = \rho_2$, that is $\beta = \frac{\alpha_2}{\alpha_1}$. It follows that for $\beta \neq \frac{\alpha_2}{\alpha_1}$, one has $|I_=| = 1$ and W has either independent or invariant increments in the orthogonal direction. However, when $\beta = \frac{\alpha_2}{\alpha_1}$ we get

$$W(\boldsymbol{t}) = \sigma_X \int_{\mathbb{R}^2} \left(\prod_{k=1}^{2} \frac{e^{i t_k y_k} - 1}{i y_k} \right) \left| \log \psi(\boldsymbol{y}) \right|^{-1} \widetilde{\mathcal{M}}(d\boldsymbol{y}) \qquad \text{for all } \boldsymbol{t} \in \mathbb{R}^2.$$

In this case, W has nondegenerate increments. Recall that all possible noncritical cases in $d = 2$ have been provided in Theorem 2 in the Introduction.

More generally, for $d \geq 2$ we can state the following scaling-transition property.

COROLLARY 1. *The random field $(X_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathbb{Z}^d}$, defined in Section 2.3, exhibits a scaling-transition in the sense that*:

(i) *If there exists $c > 0$ such that $E' = cE$, then W has nondegenerate increments*.

(ii) *Otherwise*, *W has degenerate increments*. *That is*, *there exists at least one direction in which the increments of the limit random field are either invariant or independent*.

In the sequel, we need to control the variance of the directional increments. By Proposition 5, for all $\boldsymbol{u} \in \mathbb{R}^d$, $\delta \in \mathbb{R}$,

$$\mathrm{Var}\big(\Delta_\delta^{(j)} W(\boldsymbol{u})\big) = \delta^2 \, \mathrm{Var}\big(W\big(\pi_{\langle \boldsymbol{e}_j \rangle^\perp}(\boldsymbol{u}) + \boldsymbol{e}_j\big)\big), \qquad j \in I_>$$

and

$$\mathrm{Var}\big(\Delta_\delta^{(j)} W(\boldsymbol{u})\big) = |\delta| \, \mathrm{Var}\big(W\big(\pi_{\langle \boldsymbol{e}_j \rangle^\perp}(\boldsymbol{u}) + \boldsymbol{e}_j\big)\big), \qquad j \in I_<. \tag{41}$$

The control for $j \in I_=$ is a little more involved, as summarized in the following lemma.

LEMMA 11. *There exist some constants C such that for all $\boldsymbol{u} \in [-1, 1]^d$, $\delta \in \mathbb{R}$, $j \in I_=$, the following inequalities hold*:

(a) *If $|I_>| = 1$ or $I_> = \varnothing$ and $\alpha_j < 1/2$,*

$$\mathrm{Var}\big(\Delta_\delta^{(j)} W(\boldsymbol{u})\big) \leq C |\delta|^{2\beta_j} \qquad \text{with } \beta_j = \alpha_j \left(1 - \frac{q(\pi_> E)}{2}\right) + \frac{1}{2}. \tag{42}$$

(b) *If $I_> = \varnothing$, $\alpha_j = 1/2$, then*

$$\mathrm{Var}\big(\Delta_\delta^{(j)} W(\boldsymbol{u})\big) \leq C \max\big(\delta^2, |\delta|^{2H_j}\big) \qquad \text{for all } H_j \in (0, 1). \tag{43}$$

(c) *If* $I_> = \varnothing, \alpha_j > 1/2$, *then*

$$\operatorname{Var}\big(\Delta_\delta^{(j)} W(\boldsymbol{u})\big) \le C\delta^2. \tag{44}$$

PROOF. Recall (31). For $j \in I_=$, for all $\boldsymbol{u} \in [-1,1]^d$ and $\delta \in \mathbb{R}$,

$$\operatorname{Var}\big(\Delta_\delta^{(j)} W(\boldsymbol{u})\big) = \Big(\sigma_X \prod_{k \in I_>} u_k\Big)^2 \int_{\mathbb{R}} \left|\frac{e^{i\delta y_j} - 1}{i y_j}\right|^2 f_j(y_j)\, dy_j,$$

with

$$f_j(y_j) = \int_{\langle \boldsymbol{e}_j\rangle^\perp} \prod_{k \in I_\le; k \ne j} \left|\frac{e^{iu_k y_k} - 1}{i y_k}\right|^2 \big|\log \psi\big(\pi_\ge(\boldsymbol{y} + y_j \boldsymbol{e}_j)\big)\big|^{-2}\, d\lambda_{\langle \boldsymbol{e}_j\rangle^\perp}(\boldsymbol{y}).$$

This is a locally integrable function over $\mathbb{R}$ for all values of $\alpha_j \in (0,1)$ due to the fact that $|\log\psi(\pi_\ge \boldsymbol{y})|$ is a $\pi_\ge E$-homogeneous function, $q(\pi_\ge E) > 2$, and Lemma 3. Furthermore, by E-homogeneity and polar coordinate $\boldsymbol{x} = \tau(\boldsymbol{x})^E \boldsymbol{\theta}(\boldsymbol{x})$,

$$\begin{aligned}
&\big|\log\psi(\boldsymbol{x})\big|^{-1} \\
&\quad= \frac{|\log\psi(\pi_> \boldsymbol{x})| + |x_j|^{\alpha_j}}{|\log\psi(\boldsymbol{x})|}\big(\big|\log\psi(\pi_>\boldsymbol{x})\big| + |x_j|^{\alpha_j}\big)^{-1} \\
&\quad= \frac{\tau(\boldsymbol{x})|\log\psi(\pi_>\boldsymbol{\theta}(\boldsymbol{x}))| + \tau(\boldsymbol{x})|\theta_j(\boldsymbol{x})|^{\alpha_j}}{\tau(\boldsymbol{x})|\log\psi(\boldsymbol{\theta}(\boldsymbol{x}))|}\big(\big|\log\psi(\pi_>\boldsymbol{x})\big| + |x_j|^{\alpha_j}\big)^{-1} \\
&\quad\le c_1\big(\big|\log\psi(\pi_>\boldsymbol{x})\big| + |x_j|^{\alpha_j}\big)^{-1}
\end{aligned}$$

with $c_1 = \max_{\boldsymbol{\theta} \in S_E}(|\log\psi(\pi_>\boldsymbol{\theta})| + |\theta_j|^{\alpha_j})/|\log\psi(\boldsymbol{\theta})|$. Thus,

$$\begin{aligned}
f_j(y_j) &\le c_1^2 \int_{\langle \boldsymbol{e}_j\rangle^\perp} \prod_{k \in I_\le; k \ne j} \left|\frac{e^{iu_k y_k} - 1}{i y_k}\right|^2 \big(\big|\log\psi(\pi_>\boldsymbol{y})\big| + |y_j|^{\alpha_j}\big)^{-2}\, d\lambda_{\langle \boldsymbol{e}_j\rangle^\perp}(\boldsymbol{y}) \\
&= c_1^2 \Big(\prod_{k \in I_\le; k \ne j} \int_{\mathbb{R}} \left|\frac{e^{iu_k y_k} - 1}{i y_k}\right|^2 dy_k\Big) \\
&\qquad \times \int_{\mathcal{H}_>} \big(\big|\log\psi(\pi_>\boldsymbol{y})\big| + |y_j|^{\alpha_j}\big)^{-2}\, d\lambda_>(\boldsymbol{y}),
\end{aligned}$$

where the last integral in the right-hand side has to be reduced to $|y_j|^{-2\alpha_j}$ if $\mathcal{H}_> = \{\boldsymbol{0}\}$ and otherwise is equal to

$$\begin{aligned}
&\int_{\mathcal{H}_>} |y_j|^{-2\alpha_j}\big(\big|\log\psi\big((|y_j|^{\alpha_j})^{-E}\pi_>\boldsymbol{y}\big)\big| + 1\big)^{-2}\, d\lambda_>(\boldsymbol{y}) \\
&\qquad= |y_j|^{-2\alpha_j + \alpha_j q(\pi_> E)} \int_{\mathcal{H}_>} \big(\big|\log\psi(\pi_>\boldsymbol{y})\big| + 1\big)^{-2}\, d\lambda_>(\boldsymbol{y}) =: |y_j|^{-2\beta_j + 1} c_2
\end{aligned}$$

with $\beta_j = \alpha_j(1 - q(\pi_> E)/2) + 1/2$. We have thus obtained

$$f_j(y_j) \le c_3 |y_j|^{-2\beta_j+1} \qquad \text{with } c_3 = c_1^2 c_2 \prod_{k \in I_\le; k \ne j} (2\pi u_k).$$

Recall that $|I_>| \le 1$.

(a) In case that $|I_>| = 1$, $q(\pi_> E) > 1$, and thus $\beta_j < 1$. Therefore, by the above calculation and (31),

$$\operatorname{Var}(\Delta_\delta^{(j)} W(\boldsymbol{u}^{(j)})) \le \sigma_X^2 c_3 \int \left| \frac{e^{i\delta y_j} - 1}{i y_j} \right|^2 |y_j|^{-2\beta_j+1} \, dy_j = \sigma_X^2 c_3 C_{\beta_j} |\delta|^{2\beta_j}. \tag{45}$$

In case that $|I_>| = 0$, $\beta_j = \alpha_j + 1/2$. If $\alpha_j < 1/2$, then the same bound (45) holds.

(b) If $\alpha_j = 1/2$, then for any $H_j \in (0, 1)$,

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{e^{i\delta y_j} - 1}{i y_j} \right|^2 f_j(y_j) \, dy_j \le \delta^2 &\int_{|y_j| \le 1} f_j(y_j) \, dy_j \\ &+ c_3 \int_{|y_j| > 1} \left| \frac{e^{i\delta y_j} - 1}{i y_j} \right|^2 |y_j|^{-2H_j+1} \, dy_j \\ \le c_4 \max(\delta^2, |\delta|^{2H_j}),& \end{aligned}$$

with

$$c_4 = \max_{\boldsymbol{u} \in [-1,1]^d} \int_{\langle \boldsymbol{e}_j \rangle^\perp} \prod_{k \in I_\le; k \ne j} \left| \frac{e^{i u_k y_k} - 1}{i u_k} \right|^2 |\log \psi(\pi_\ge \boldsymbol{y})|^{-2} \, d\lambda_{\langle \boldsymbol{e}_j \rangle^\perp}(\boldsymbol{y}) + c_3 C_{H_j}.$$

Therefore,

$$\operatorname{Var}(\Delta_\delta^{(j)} W(\boldsymbol{u}^{(j)})) \le \sigma_X^2 c_4 \max(\delta^2, |\delta|^{2H_j}).$$

(c) At last, if $\alpha_j > 1/2$, then $\beta_j > 1$, the function f_j is integrable on $\mathbb{R}$ and

$$\int_{\mathbb{R}} \left| \frac{e^{i\delta y_j} - 1}{i y_j} \right|^2 f_j(y_j) \, dy_j \le \delta^2 \int_{\mathbb{R}} f_j(y_j) \, dy_j.$$

It then follows that

$$\operatorname{Var}(\Delta_\delta^{(j)} W(\boldsymbol{u}^{(j)})) \le c_5 \delta^2,$$

with

$$c_5 = \sigma_X^2 \sup_{\boldsymbol{u} \in [-1,1]^d} \int_{\mathbb{R}^d} \prod_{k \in I_\le; k \ne j} \left| \frac{e^{i u_k y_k} - 1}{i y_k} \right|^2 |\log \psi(\pi_\ge \boldsymbol{y})|^{-2} \, d\boldsymbol{y}.$$

□

5.2. *Fractional Brownian sheets.* Here, we give a complete characterization of when W is a fractional Brownian sheet. Recall that a zero-mean Gaussian random field $(X(\boldsymbol{t}))_{\boldsymbol{t}\in\mathbb{R}^d}$ is a standard fractional Brownian sheet with Hurst index $(H_1,\dots,H_d)\in(0,1]^d$ if

$$\operatorname{Cov}(X(\boldsymbol{t}),X(\boldsymbol{s}))=\frac{1}{2^d}\prod_{i=1}^d\bigl(|t_i|^{2H_i}+|s_i|^{2H_i}-|t_i-s_i|^{2H_i}\bigr).$$

Remark that we include the degenerate case that Hurst index equals 1.

For the limit random field W, the covariance function can be factorized according to different directions as

$$\begin{aligned}&\operatorname{Cov}(W(\boldsymbol{t}),W(\boldsymbol{s}))\\&\quad=\frac{\sigma_X^2}{(2\pi)^{|I_>|}}\Bigl(\prod_{k\in I_<}\operatorname{Cov}\bigl(B_{1/2}(t_k),B_{1/2}(s_k)\bigr)\Bigr)\Bigl(\prod_{k\in I_>}t_ks_k\Bigr)\Psi(\boldsymbol{t},\boldsymbol{s}),\end{aligned}$$

with $\Psi(\boldsymbol{t},\boldsymbol{s})$ only depending on $\{t_k,s_k\}_{k\in I_=}$, given by

$$\Psi(\boldsymbol{t},\boldsymbol{s}):=\int_{\mathcal{H}_\geq}\bigl|\log\psi(\boldsymbol{y})\bigr|^{-2}\prod_{k\in I_=}\frac{(e^{it_ky_k}-1)\overline{(e^{is_ky_k}-1)}}{2\pi|y_k|^2}\,d\lambda_\geq(\boldsymbol{y}).$$

Recall C_H in (31).

PROPOSITION 6. *The random field W is a fractional Brownian sheet, if and only if $|I_=|=1$. In this case, $\Psi(\boldsymbol{t},\boldsymbol{s})$ has the following expressions*:

in case $I_==\{j\}$, $I_>=\varnothing$,

$$\Psi(\boldsymbol{t},\boldsymbol{s})=\bigl|\log\psi(\boldsymbol{e}_j)\bigr|^{-2}C_{\alpha_j+1/2}\operatorname{Cov}\bigl(B_{\alpha_j+1/2}(t_j),B_{\alpha_j+1/2}(s_j)\bigr);\tag{46}$$

in case $I_==\{j\}$, $I_>=\{k\}$,

$$\Psi(\boldsymbol{t},\boldsymbol{s})=\int_{\mathcal{H}_>}\bigl|\log\psi(\boldsymbol{y}+\boldsymbol{e}_j)\bigr|^{-2}\,d\lambda_>(\boldsymbol{y})C_{H_j}\operatorname{Cov}\bigl(B_{H_j}(t_j),B_{H_j}(s_j)\bigr),\tag{47}$$

with $H_j=\alpha_j(1-1/(2\alpha_k))+1/2$.

PROOF. We first prove the "if part." Suppose $I_==\{j\}$. In the case $I_>=\varnothing$,

$$\begin{aligned}\Psi(\boldsymbol{t},\boldsymbol{s})&=\int_{\mathbb{R}}\bigl|\log\psi(y_j\boldsymbol{e}_j)\bigr|^{-2}\frac{(e^{it_jy_j}-1)\overline{(e^{is_jy_j}-1)}}{2\pi|y_j|^2}\,dy_j\\&=\int_{\mathbb{R}}\bigl|\log\psi\bigl((|y_j|^{\alpha_j})^E\boldsymbol{e}_j\bigr)\bigr|^{-2}\frac{(e^{it_jy_j}-1)\overline{(e^{is_jy_j}-1)}}{2\pi|y_j|^2}\,dy_j\\&=\int_{\mathbb{R}}\bigl|\log\psi(\boldsymbol{e}_j)\bigr|^{-2}\frac{(e^{it_jy_j}-1)\overline{(e^{is_jy_j}-1)}}{2\pi|y_j|^{2+2\alpha_j}}\,dy_j.\end{aligned}$$

Thus, by (31), in case $I_= = \{j\}$, $I_> = \varnothing$, (46) follows. In the case $I_> \neq \varnothing$,

$$\begin{aligned}\Psi(\boldsymbol{t},\boldsymbol{s}) &= \int_{\mathbb{R}}\int_{\mathcal{H}_>} \left|\log\psi(\boldsymbol{y}+y_j\boldsymbol{e}_j)\right|^{-2} \frac{(e^{it_jy_j}-1)\overline{(e^{is_jy_j}-1)}}{2\pi|y_j|^2}\,d\lambda_>(\boldsymbol{y})\,dy_j \\ &= \int_{\mathbb{R}}\int_{\mathcal{H}_>} |y_j|^{-2\alpha_j}\left|\log\psi\big((|y_j|^{-\alpha_j})^E(\boldsymbol{y}+y_j\boldsymbol{e}_j)\big)\right|^{-2} \\ &\quad\times \frac{(e^{it_jy_j}-1)\overline{(e^{is_jy_j}-1)}}{2\pi|y_j|^2}\,d\lambda_>(\boldsymbol{y})\,dy_j \\ &= \int_{\mathcal{H}_>}\left|\log\psi(\boldsymbol{y}+\boldsymbol{e}_j)\right|^{-2}d\lambda_>(\boldsymbol{y})\int_{\mathbb{R}}\frac{(e^{it_jy_j}-1)\overline{(e^{is_jy_j}-1)}}{2\pi|y_j|^{2+2\alpha_j-\alpha_jq(\pi_>E)}}\,dy_j.\end{aligned}$$

That is, in case $I_= = \{j\}$, $I_> \neq \varnothing$, for $H_j = \alpha_j(1-q(\pi_>E)/2)+1/2$, (47) follows.

Next, we prove the "only if part." Suppose W is a fractional Brownian sheet with Hurst indices $H_1,\dots,H_d$. From Proposition 4, W is also (E',H)-operator-scaling with $H = \gamma_0 + q(E') - q(E'')/2$. Then it follows that

$$\frac{H_1}{\alpha'_1}+\cdots+\frac{H_d}{\alpha'_d} = \gamma_0 + q(E') - q(E'')/2,$$

or equivalently

$$\sum_{k\in I_\le}\frac{1}{\alpha'_k}(H_k-1/2) + \sum_{k\in I_>}\frac{1}{\alpha'_k}(H_k-1) = \gamma_0\biggl(1-\frac{1}{2}\sum_{k\in I_>}\frac{1}{\alpha_k}\biggr). \tag{48}$$

We consider the variance. By the assumption that W is a fractional Brownian sheet, and the fact that W has stationary directional increments, for all $j\in\{1,\dots,d\}$, for all $\delta\in\mathbb{R}$,

$$\operatorname{Var}\big(\Delta_\delta^{(j)}W(\boldsymbol{u})\big) = |\delta|^{2H_j}\operatorname{Var}\big(W\big(\pi_{\langle\boldsymbol{e}_j\rangle^\perp}(\boldsymbol{u})+\boldsymbol{e}_j\big)\big). \tag{49}$$

Recall that $|I_>|\le 1$. We first consider the case $I_> = \varnothing$. In this case:

- for $k\in I_<$, comparing (49) and (41) yields $H_k = 1/2$,
- for $k\in I_=$, $\alpha_k<1/2$, comparing (49) and (42) yields $H_k = \alpha_k+1/2$,
- for $k\in I_=$, $\alpha_k = 1/2$, comparing (49) and (43) yields $H_k = 1$,
- for $k\in I_=$, $\alpha_k > 1/2$, comparing (49) and (44) yields $H_k = 1$.

Then (48) becomes

$$\sum_{k\in I_=,\alpha_k>1/2}\frac{\gamma_0}{2\alpha_k} + \sum_{k\in I_=,\alpha_k\le 1/2}\gamma_0 = \gamma_0.$$

Since $\alpha_k < 1$, it then follows that $|I_=| = 1$. Similarly, in the case $I_> \neq \varnothing$, say $I_> = \{1\}$, it follows from comparing the corresponding inequalities that:

- $H_1 = 1$,
- for $k \in I_<$, $H_k = 1/2$,
- for $k \in I_=$, $H_k = \alpha_k(1 - 1/(2\alpha_1)) + 1/2$.

Then (48) becomes

$$\sum_{k \in I_=} \gamma_0 \left(1 - \frac{1}{2\alpha_1}\right) = \gamma_0 \left(1 - \frac{1}{2\alpha_1}\right),$$

which implies $|I_=| = 1$. □

REMARK 7. When the limit is a fractional Brownian sheet, in directions corresponding to $I_>$, $I_<$ (if not empty) and $I_=$, the Hurst indices equals 1, 1/2 and some value in (1/2, 1), respectively. Thus, W exhibits long-range dependence in the directions corresponding to $I_\geq$.

As a concrete example, we prove Theorem 2.

PROOF OF THEOREM 2. Case (i): when $\alpha_2' > \alpha_2, \alpha_2 \in (0, 1/2)$. In this case, $\gamma_0 = \rho_2 = \alpha_2/\alpha_2'$, $E'' = \mathrm{diag}(1/\alpha_1, 1/\alpha_2')$, $I_< = \{1\}, I_= = \{2\}$, $\beta = \alpha_2/\alpha_2' + \frac{1}{2}(\frac{1}{\alpha_1} + \frac{1}{\alpha_2'})$ and $H_1 = 1/2$ are straightforward. Then, by (46), $H_2 = \frac{1}{2} + \alpha_2$ and $\sigma^2 = C_{H_2} |\log \psi(0, 1)|^{-2}$.

Case (ii): when $\alpha_2' > \alpha_2, \alpha_2 \in (1/2, 1)$. In this case, $\gamma_0 = \rho_1 = 1$, $E'' = E$, $I_> = \{2\}$, $I_= = \{1\}$, $\beta = 1 + \frac{1}{2\alpha_1} + \frac{1}{\alpha_2'} - \frac{1}{2\alpha_2}$ and $H_2 = 1$ are straightforward. Then, by (47), $H_1 = \frac{1}{2} + \alpha_1(1 - \frac{1}{2\alpha_2})$ and $\sigma^2 = C_{H_1} \int_{\mathbb{R}} |\log \psi(1, y)|^{-2}\, dy$.

The calculation of cases (iii) and (iv) are similar, and thus omitted. One obtains that $\sigma^2 = C_{H_1} |\log \psi(1, 0)|^{-2}$ for case (iii) and $\sigma^2 = C_{H_2} \int_{\mathbb{R}} |\log \psi(y, 1)|^{-2}\, dy$ for case (iv). □

5.3. *Sample-path properties.* We conclude this section by the following general sample-path properties for the random field W that is a consequence of [7], Proposition 5.3.

PROPOSITION 7. *There exists a modification W^* of W on $[0, 1]^d$ such that for all $\varepsilon > 0$, almost surely there exists a finite random variable Z such that for all $\boldsymbol{s}, \boldsymbol{t} \in [0, 1]^d$,*

$$|W^*(\boldsymbol{t}) - W^*(\boldsymbol{s})| \leq Z\rho(\boldsymbol{t}, \boldsymbol{s}) \log\left(1 + \rho(\boldsymbol{s}, \boldsymbol{t})^{-1}\right)^{1/2+\varepsilon},$$

with

$$\rho(\boldsymbol{s}, \boldsymbol{t}) = \sum_{j \in I>} |t_j - s_j| + \sum_{j \in I_<} |t_j - s_j|^{1/2} + \sum_{j \in I_=} |t_j - s_j|^{H_j}$$

where, for $j \in I_=$:

(a) $H_j = \alpha_j(1 - q(\pi_> E)/2) + 1/2$ *if either* $|I_>| = 1$ *or* $I_> = \varnothing$ *and* $\alpha_j < 1/2$,
(b) H_j *can take any value in* $(0, 1)$ *if* $I_> = \varnothing$ *and* $\alpha_j = 1/2$, *and*
(c) $H_j = 1$ *if* $I_> = \varnothing$ *and* $\alpha_j > 1/2$.

PROOF. Let us consider E''' the diagonal matrix with entries corresponding to 1 for $j \in I_>$, 2 for $j \in I_<$ and $1/H_j$ for $j \in I_=$. Let $\tau_{E'''}$ be the radial part with respect to E''' according to [7], equation (9). Let us quote that since $\boldsymbol{t} \mapsto \rho(\mathbf{0}, \boldsymbol{t})$ is E''' homogeneous and strictly positive on $\mathbb{R}^d \setminus \{\mathbf{0}\}$, following ideas of Clausel and Vedel [13], Theorem 3.2, the function $\boldsymbol{t} \mapsto \rho(\mathbf{0}, \boldsymbol{t})/\tau_{E'''}(\boldsymbol{t})$ is continuous and strictly positive on the compact set $S_{E'''}$. It follows that we may find $C, C' > 0$ such that for all $\boldsymbol{t} \in \mathbb{R}^d$,

$$C\tau_{E'''}(\boldsymbol{t}) \le \rho(\mathbf{0}, \boldsymbol{t}) \le C'\tau_{E'''}(\boldsymbol{t}).$$

Therefore, by [7], Proposition 5.3 (with $\beta = 0$), to show Proposition 7 we prove for $\boldsymbol{t}, \boldsymbol{s} \in [0, 1]^d$ that

$$\sqrt{\mathbb{E}\big(\big(W(\boldsymbol{t}) - W(\boldsymbol{s})\big)^2\big)} = \sqrt{\operatorname{Var}\big(W(\boldsymbol{t}) - W(\boldsymbol{s})\big)} \le C\rho(\boldsymbol{s}, \boldsymbol{t}). \tag{50}$$

For $\boldsymbol{t}, \boldsymbol{s} \in [0, 1]^d$, considering as in [26], the sequence $(\boldsymbol{u}^{(j)})_{0 \le j \le d}$ defined by $\boldsymbol{u}^{(0)} = \boldsymbol{s}$ and $\boldsymbol{u}^{(j+1)} = \boldsymbol{u}^{(j)} + (t_j - s_j)\boldsymbol{e}_j$ for $0 \le j \le d - 1$, we get $W(\boldsymbol{t}) - W(\boldsymbol{s}) = \sum_{j=1}^d \Delta^{(j)}_{(t_j - s_j)} W(\boldsymbol{u}^{(j)})$. Hence,

$$\sqrt{\operatorname{Var}\big(W(\boldsymbol{t}) - W(\boldsymbol{s})\big)} \le \sum_{j=1}^d \sqrt{\operatorname{Var}\big(\Delta^{(j)}_{(t_j - s_j)} W(\boldsymbol{u}^{(j)})\big)}.$$

Now to obtain (50), it suffices to apply the bounds on the directional increments established in Lemma 11. Observe that in the case $j \in I_=, I_> = \varnothing$, since $\delta = t_j - s_j \in [-1, 1]$, the right-hand side of (43) becomes $C|\delta|^{2H_j}$. The details are omitted. The proof is thus complete. □

Let us mention that we probably could improve this result. Actually, following [52], it is sufficient to get a similar lower bound on the variance on $[\varepsilon, 1]^d$ to establish condition (C_1), from which Theorem 4.2 follows, saying that the inequality is true for $\varepsilon = 0$ and Z has finite moments of any order.

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