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RANDOM WALKS ON TORUS AND RANDOM INTERLACEMENTS: MACROSCOPIC COUPLING AND PHASE TRANSITION

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For $d \geq 3$, we construct a new coupling of the trace left by a random walk on a large d-dimensional discrete torus with the random interlacements on $\mathbb{Z}^d$. This coupling has the advantage of working up to *macroscopic* subsets of the torus. As an application, we show a sharp phase transition for the diameter of the component of the vacant set on the torus containing a given point. The threshold where this phase transition takes place coincides with the critical value $u_\star(d)$ of random interlacements on $\mathbb{Z}^d$. Our main tool is a variant of the *soft-local time* coupling technique of Popov and Teixeira [*J. Eur. Math. Soc. (JEMS)* **17** (2015) 2545–2593].

1. Introduction. In this paper, we study the trace of a simple random walk X_n on a large d-dimensional discrete torus $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$ for $d \geq 3$. In particular, we investigate the percolative properties of its vacant set

$$\mathcal{V}_N^u = \mathbb{T}_N^d \setminus \{X_0, \dots, X_{\lfloor uN^d \rfloor}\}, \tag{1.1}$$

for a fixed $u \in [0, \infty)$ as N tends to infinity.

Intuitively speaking, the parameter u plays the role of a density of the random walk trace. More precisely, for small values of u and as N grows, the vacant set occupies a large proportion of the torus. Therefore, $\mathcal{V}_N^u$ should consists of a single large cluster together with small finite components. In contrast, for large values of u, the asymptotic density of $\mathcal{V}_N^u$ should be small and it should have been fragmented into small pieces.

In analogy with the Bernoulli percolation behavior, it is actually expected that there is a phase transition. Namely, there is a critical value $u_c(d)$ such that the first behavior holds true for all $u < u_c(d)$ and the second for all $u > u_c(d)$, with high probability as N tends to infinity.

The percolative properties of $\mathcal{V}_N^u$ have been studied in several recent works. In [2], the authors showed that, for large dimensions d and small enough $u > 0$, the vacant set has a (unique, to some extent) connected component with a nonnegligible density. In order to understand the vacant set $\mathcal{V}_N^u$ more in detail, Sznitman introduced in [11] a model of random interlacements, which can be viewed as an

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analogue of the random walk trace in the torus, but constructed on the infinite lattice $\mathbb{Z}^d$. In [10, 11], it was then shown that the vacant set of random interlacements exhibit a percolation phase transition at some level $u_\star(d) \in (0, \infty)$. It is believed that the critical threshold of the torus, $u_c(d)$ coincides with $u_\star(d)$.

Later, in [15], it was established that as N grows, the set $\mathcal{V}_N^u$ converges locally in law to the vacant set of random interlacements $\mathcal{V}^u$, but this did not have immediate consequences on the percolative behavior of the $\mathcal{V}_N^u$. In [14], a more quantified control of $\mathcal{V}_N^u$ in terms of $\mathcal{V}^u$ improved our understanding of the behavior of the largest connected component $\mathcal{C}_{u,N}^{\max}$ of $\mathcal{V}_N^u$. In particular, it was shown that, for any dimension $d \geq 3$, with high probability as N goes to infinity:

- for u small enough, there is $\varepsilon > 0$ such that
$$|\mathcal{C}_{u,N}^{\max}| \geq \varepsilon N^d,$$
- for $u > u_\star(d)$,
$$|\mathcal{C}_{u,N}^{\max}| = o(N^d),$$
- for u large enough, for some $\lambda(u) > 0$
$$|\mathcal{C}_{u,N}^{\max}| = O(\log^\lambda N).$$

Note that this implies the existence of a certain transition in the asymptotic behavior of $\mathcal{V}_N^u$ as u varies. However, it was not known until now where this transition occurs, whether it is sharp, or whether it is related to the model of random interlacements. The results of this paper shed more light on this question.

Unfortunately, we are not able to control directly the volume of the largest connected component $\mathcal{C}_{u,N}^{\max}$. We thus define another observable that is better suited to our analysis. To this end, we let P to stand for the law of the simple random walk $(X_n)_{n\geq 0}$ on $\mathbb{T}_N^d$ started from its invariant distribution (which is uniform on $\mathbb{T}_N^d$), and write $\mathcal{C}_N(u)$ for the connected component of $\mathcal{V}_N^u$ containing some given point, say $0 \in \mathbb{T}_N^d$. We define the observable

$$\eta_N(u) = P\big[\operatorname{diam} \mathcal{C}_N(u) \geq N/4\big], \tag{1.2}$$

where the diameter is understood in the Euclidean sense, not in the one induced by the graph $\mathcal{C}_N(u)$.

Let us point out that the observable $\eta_N(u)$ is *macroscopic*, that is it depends on the properties of the vacant set $\mathcal{V}_N^u$ in the box of size comparable with N.

The next theorem establishes a phase transition for this observable and gives its asymptotic behavior in terms of related quantities for random interlacements.

THEOREM 1.1. *The observable $\eta_N(u)$ exhibits a phase transition at $u_\star(d)$. More precisely, for $u > u_\star(d)$,*

$$\lim_{N\to\infty} \eta_N(u) = 0, \tag{1.3}$$

and for $u < u_\star(d)$,

$$\lim_{N\to\infty} \eta_N(u) = \eta(u) > 0, \tag{1.4}$$

where $\eta(u)$ is the probability that $0 \in \mathbb{Z}^d$ is contained in the infinite component of the vacant set $\mathcal{V}^u$ of random interlacements at level u.

The main ingredient of the proof of Theorem 1.1 is a new coupling between $\mathcal{V}_N^u$ and $\mathcal{V}_u$ in macroscopic boxes of the torus which is of independent interest. This is stated precisely in the following result.

THEOREM 1.2. *Let $\mathcal{B}_N = [0, (1-\delta)N]^d$ for some $\delta > 0$. Then for every $u \geq 0$ and $\varepsilon > 0$ there exist couplings $\mathbb{Q}_N$ of the random walk on $\mathbb{T}_N^d$ with the random interlacements such that*

$$\lim_{N\to\infty} \mathbb{Q}_N\big[\big(\mathcal{V}^{u(1+\varepsilon)} \cap \mathcal{B}_N\big) \subset \big(\mathcal{V}_N^u \cap \mathcal{B}_N\big) \subset \big(\mathcal{V}^{u(1-\varepsilon)} \cap \mathcal{B}_N\big)\big] = 1. \tag{1.5}$$

We give a more quantitative version of this theorem later (see Theorem 4.1). Observe again that the box $\mathcal{B}_N$ is macroscopic, and that $|\mathcal{B}_N|/N^d$ can be made arbitrarily close to one. Theorem 1.2 thus improves considerably the best previously known couplings of the same objects, in particular [14] which couples them in one box of size $N^{1-\varepsilon}$ for fixed $u > 0$, and [1] which considers several distant boxes of size $N^{1-\varepsilon}$ and u may depend on N.

The principal tool for the construction of the above coupling is a streamlined version of the technique of *soft local times*, which was recently developed in [8] in order to prove new decorrelation inequalities for random interlacements. This technique allows us to couple two Markov chains so that their ranges almost coincide. Our formulation, stated as Theorem 3.2 below, provides more explicit bounds on the probability that the coupling fails, and more importantly, it is well adapted to situations where one can estimate the mixing time of the chains in question. See the introduction to Section 3 for more details.

Let us now briefly describe the organization of this paper. In Section 2, we introduce some basic notation and recall several useful known results. In Section 3, we extend the soft local times method and prove our main technical result on the coupling of ranges of Markov chains. The precise version of Theorem 1.2 giving a coupling between the random walk on $\mathbb{T}_N^d$ and the vacant set of random interlacements is stated in Theorem 4.1 in Section 4. Sections 5–9 provide estimates on the simple random walk, equilibrium measures, mixing times and the number of excursions of the walker which are needed in order to apply the results of Section 3. Finally, Section 10 contains the proofs of our main results. In the Appendix, we include a suitable version of classic Chernov bounds on the concentration of additive functionals of Markov chains.

2. Notation and some results. Let us first introduce some basic notation to be used in the sequel. We consider torus $\mathbb{T}_N^d = (\mathbb{Z}^d/N\mathbb{Z}^d)$ which we identify, for sake of concreteness, with the set $\{0, \dots, N-1\}^d \subset \mathbb{Z}^d$. On $\mathbb{Z}^d$, we respectively denote by $|\cdot|$ and $|\cdot|_\infty$ the Euclidean and ℓ^∞-norms. For any $x \in \mathbb{Z}^d$ and $r \geq 0$, we let $B(x,r) = \{y \in \mathbb{Z}^d : |y-x| \leq r\}$ stand for the Euclidean ball centered at x with radius r. Given $K, U \subset \mathbb{Z}^d$, $K^c = \mathbb{Z}^d \setminus K$ stands for the complement of K in $\mathbb{Z}^d$ and $\operatorname{dist}(K,U) = \inf\{|x-y| : x \in K, y \in U\}$ for the Euclidean distance of K and U. Finally, we define the inner boundary of K to be the set $\partial K = \{x \in K : \exists y \in K^c, |y-x| = 1\}$, and the outer boundary of K as $\partial_e K = \partial(K^c)$. Analogous notation is used on $\mathbb{T}_N^d$.

We endow $\mathbb{Z}^d$ and $\mathbb{T}_N^d$ with the nearest-neighbor graph structure. We write P_x for the law on $(\mathbb{T}_N^d)^{\mathbb{N}}$ of the canonical simple random walk on $\mathbb{T}_N^d$ started $x \in \mathbb{T}_N^d$, and denote the canonical coordinate process by X_n, $n \geq 0$. We use P to denote the law of the random walk with a uniformly chosen starting point, that is, $P = \sum_{x \in \mathbb{T}_N^d} N^{-d} P_x$. We write $P_x^{\mathbb{Z}^d}$ for the canonical law of the simple random walk on $\mathbb{Z}^d$ started from x, and (with slight abuse of notation) X_n for the coordinate process as well. Finally, θ_k denotes the canonical shifts of the walk, defined on either $(\mathbb{T}_N^d)^{\mathbb{N}}$ or $(\mathbb{Z}^d)^{\mathbb{N}}$,

$$\theta_k(x_0, x_1, \dots) = (x_k, x_{k+1}, \dots). \tag{2.1}$$

Throughout the text, we denote by c positive finite constants whose value might change during the computations, and which may depend on the dimension d. Starting from Section 5, the constants may additionally depend on γ, α which we will introduce later (this will be mentioned again when appropriate). Given two sequences a_N, b_N, we write $a_N \asymp b_N$ to mean that $c^{-1} a_N \leq b_N \leq c a_N$, for some constant $c \geq 1$.

For $K \subset \mathbb{Z}^d$ finite, as well as for $K \subset \mathbb{T}_N^d$, we use H_K, $\tilde{H}_K$ to denote entrance and hitting times of K

$$H_K = \inf\{k \geq 0 : X_k \in K\}, \qquad \tilde{H}_K = \inf\{k \geq 1 : X_k \in K\}. \tag{2.2}$$

For $K \subset \mathbb{Z}^d$, we define the equilibrium measure of K by

$$e_K(x) = P_x^{\mathbb{Z}^d}[\tilde{H}_K = \infty] \mathbf{1}\{x \in K\}, \qquad x \in \mathbb{Z}^d, \tag{2.3}$$

and the capacity of K

$$\operatorname{cap}(K) = e_K(K) = \sum_{x \in K} e_K(x). \tag{2.4}$$

For every finite K, $\operatorname{cap}(K) < \infty$, which allows us to introduce the normalized equilibrium measure

$$\bar{e}_K(\cdot) = \big(\operatorname{cap}(K)\big)^{-1} e_K(\cdot). \tag{2.5}$$

Finally, we give an explicit construction of the vacant set of random interlacements intersected with a finite set $K \subset \mathbb{Z}^d$. We build on some auxiliary probability space an i.i.d. sequence $X^{(i)}$, $i \geq 1$, of simple random walks on $\mathbb{Z}^d$ with the initial distribution $\bar{e}_K$, and an independent Poisson process $(J_u)_{u\geq 0}$ with intensity $\mathrm{cap}(K)$. The vacant set of the random interlacements (viewed as a process in $u \geq 0$) when intersected with K has the law characterized by

$$(\mathcal{V}^u \cap K)_{u\geq 0} \stackrel{\text{law}}{=} \Big(K \setminus \bigcup_{1\leq i\leq J_u} \bigcup_{k\geq 0} \{X_k^{(i)}\}\Big)_{u\geq 0}, \tag{2.6}$$

see, for instance, Proposition 1.3 and below (1.42) in [11].

3. Coupling the ranges of Markov chains. In this section, we construct a coupling of two Markov chains so that their ranges almost coincide. A method to construct such couplings was recently introduced in [8], based on the so-called *soft local times*. We will use the same method to construct the coupling, but propose a new method to estimate the probability that the coupling fails.

This is necessary since the estimates in [8] use considerably the fact that the Markov chains in consideration have "very strong renewals." More precisely the trajectory of the chain can easily be decomposed into i.i.d. blocks (of possibly random length). This, together with bounds on the moment generating function corresponding to one block, allows them to obtain very good bounds on the error of the coupling, that is, on the probability that the ranges of the Markov chains are considerably different.

In the present paper, we have in mind an application where this "very strong renewal" structure is not present. We hence need to find new estimates on the error of the coupling. These techniques combine the method of soft local times with quantitative Chernov-type estimates on deviations of additive functionals of Markov chains. An estimate of this type suitable for our purposes is proved in the Appendix.

Similarly as in [8], we will use the regularity of the transition probabilities of the Markov chain to improve the bounds on the error of the coupling. In contrast to [8], this regularity will be not expressed via comparing the transition probability with indicator functions of large balls (see Theorem 4.9 of [8]), but by controlling the variance of the transition probability.

Note also that the estimates on the error of the coupling provided by Theorems 3.1, 3.2 are weaker than the ones obtained by techniques of [8], when both techniques apply. This is due to the fact that the Chernov-type estimates mentioned above give the worst case asymptotic and are not optimal in many situations.

Let us now make precise the setting of this section. Let Σ be a finite state space, $P = (p(x, y))_{x,y\in\Sigma}$ a Markov transition matrix, and ν a distribution on Σ. We assume that P is irreducible, so there exists a unique P-invariant distribution π on Σ. The mixing time T corresponding to P is defined by

$$T = \min\Big\{n \geq 0 : \max_{x\in\Sigma} \| P^n(x, \cdot) - \pi(\cdot)\|_{\mathrm{TV}}\Big\} \leq \frac{1}{4}, \tag{3.1}$$

where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation distance $\|\nu-\nu'\|_{\mathrm{TV}} := (1/2)\sum_x |\nu(x)-\nu'(x)|$. We set

$$\pi_\star = \min_{z\in\Sigma} \pi(z). \tag{3.2}$$

Let μ be an a priori measure on Σ with full support. (This measure is introduced for convenience only; it will simplify some formulas later. The estimates that we obtain do not depend on the choice of μ.) Let $g:\Sigma\to[0,\infty)$ be the density of π with respect of μ,

$$g(x) = \frac{\pi(x)}{\mu(x)}, \qquad x\in\Sigma, \tag{3.3}$$

and let further $\rho:\Sigma^2\to[0,\infty)$ be the "transition density" with respect to μ,

$$\rho(x,y) = \frac{p(x,y)}{\mu(y)}, \qquad x,y\in\Sigma. \tag{3.4}$$

We use ρ_y to denote the function $x\mapsto\rho(x,y)$ giving the arrival probability density at y as we vary the starting point. For any function $f:\Sigma\to\mathbb{R}$, let $\pi(f)=\sum_{x\in\Sigma}\pi(x)f(x)$, and $\operatorname{Var}_\pi f = \pi((f-\pi(f))^2)$.

The following theorem provides a coupling of a Markov chain with transition matrix P with an i.i.d. sequence so that their ranges almost coincide.

THEOREM 3.1. *Let P, ν and π be as above. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{Q})$ where one can construct a Markov chain $(Z_i)_{i\ge 0}$ with transition matrix P and initial distribution ν and an i.i.d. sequence $(U_i)_{i\ge0}$ with marginal π satisfying the following property*: *Let ε be such that*

$$0<\varepsilon\le\frac{1}{2}\wedge\min_{z\in\Sigma}\frac{\operatorname{Var}_\pi \rho_z}{2\|\rho_z\|_\infty g(z)}. \tag{3.5}$$

Set

$$k(\varepsilon) = \left\lceil -\min_{z\in\Sigma}\log_2\frac{\pi_\star\varepsilon^2 g(z)^2}{6\operatorname{Var}_\pi(\rho_z)}\right\rceil, \tag{3.6}$$

and for $n\ge0$ define the "good" event

$$\mathcal{G}(n,\varepsilon) = \bigl\{\{U_i\}_{i=0}^{\lfloor n(1-\varepsilon)\rfloor}\subset\{Z_i\}_{i=0}^{n}\subset\{U_i\}_{i=0}^{\lceil n(1+\varepsilon)\rceil}\bigr\}. \tag{3.7}$$

Then for every $n\ge 2k(\varepsilon)T$

$$\begin{aligned}&\mathbb{Q}[\mathcal{G}(n,\varepsilon)^c]\\ &\quad\le C\sum_{z\in\Sigma}\left(e^{-cn\varepsilon^2}+e^{-cn\varepsilon(\pi(z)/\nu(z))}+\exp\left\{-\frac{c\varepsilon^2g(z)^2}{\operatorname{Var}_\pi\rho_z}\frac{n}{k(\varepsilon)T}\right\}\right),\end{aligned} \tag{3.8}$$

where $C,c\in(0,\infty)$ are universal constants (i.e., not depending on Σ, P, ν, π and ε).

The above coupling is constructed using a procedure that first appeared in [8], where the authors use a Poisson point process to "simulate" random elements of a probability space. For a more detailed explanation of this construction, see [8].

PROOF OF THEOREM 3.1. To construct the coupling, we use the same procedure as in [8]. Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which we are given a Poisson point process $\eta = (z_i, v_i)_{i\geq 1}$ on $\Sigma \times [0, \infty)$ with intensity measure $\mu \otimes dx$. On this probability space, we now construct a Markov chain $(Z_i)_{i\geq 0}$ and an i.i.d. sequence $(U_i)_{i\geq 0}$ with the required properties.

Let $G_{-1}(z) = 0$, $z \in \Sigma$, and define inductively random variables $\xi_k \geq 0$, $Z_k \in \Sigma$, $V_k \geq 0$, and random functions $G_k : \Sigma \to [0, \infty)$, $k \geq 0$,

$$\xi_k = \inf\bigl\{t \geq 0 : \exists (z, v) \in \eta \setminus \{(Z_i, V_i)\}_{i=1}^{k-1} \text{ s.t. } G_{k-1}(z) + t\rho(Z_{k-1}, z) \geq v\bigr\}, \tag{3.9}$$

$$G_k(z) = G_{k-1}(z) + \xi_k \rho(Z_{k-1}, z), \tag{3.10}$$

$$(Z_k, V_k) = \text{the unique point } (z, v) \in \eta \text{ such that } G_k(z) = v, \tag{3.11}$$

where we use the convention $\rho(Z_{-1}, z) = \nu(z)/\mu(z)$. If the point satisfying $G_k(z) = v$ in (3.11) is not unique, we pick one arbitrarily. The details of the choice are unimportant, as this occurs with zero probability.

Using a similar construction, on the same probability space, we further define random variables $U_k \in \Sigma$, $\tilde{\xi}_k \geq 0$, $W_k \geq 0$ and random functions $\tilde{G}_k : \Sigma \to [0, \infty)$, $k \geq 0$,

$$\tilde{\xi}_k = \inf\bigl\{t \geq 0 : \exists (z, v) \in \eta \setminus \{(U_i, W_i)\}_{i=1}^{k-1} \text{ s.t. } \tilde{G}_{k-1}(z) + t g(z) \geq v\bigr\}, \tag{3.12}$$

$$\tilde{G}_k(z) = \tilde{G}_{k-1}(z) + \tilde{\xi}_k g(z), \tag{3.13}$$

$$(U_k, W_k) = \text{the unique point } (z, v) \in \eta \text{ such that } \tilde{G}_k(z) = v, \tag{3.14}$$

where again $\tilde{G}_{-1} \equiv 0$.

It follows from [8], Section 4, that $Z = (Z_k)_{k\geq 0}$ is a Markov chain with the required distribution, and $U = (U_k)_{k\geq 0}$ an i.i.d. sequence with marginal π. Moreover, the sequences (ξ_k) and $(\tilde{\xi}_k)$ are i.i.d. with exponential mean-one marginal. The sequence (ξ_k) is independent of (Z_k), and similarly $(\tilde{\xi}_k)$ is independent of (U_k).

We now estimate the probability of $\mathcal{G}(n, \varepsilon)^c$. From the above construction, it follows that $\mathbb{Q}$-a.s.

$$\begin{aligned} \{Z_i\}_{i=0}^k &= \bigl\{z \in \Sigma : \text{ there exists } (z, v) \in \eta \text{ with } G_k(z) \geq v\bigr\}, \\ \{U_i\}_{i=0}^k &= \bigl\{z \in \Sigma : \text{ there exists } (z, v) \in \eta \text{ with } \tilde{G}_k(z) \geq v\bigr\}. \end{aligned} \tag{3.15}$$

Consider the following events:

$$A^- = \left\{\tilde{G}_{n(1-\varepsilon)} < \left(1 - \frac{\varepsilon}{2}\right)ng\right\},$$

$$A^+ = \left\{\tilde{G}_{n(1+\varepsilon)} > \left(1 + \frac{\varepsilon}{2}\right)ng\right\}, \tag{3.16}$$

$$B = \left\{n\left(1 - \frac{\varepsilon}{2}\right)g \le G_n \le \left(1 + \frac{\varepsilon}{2}\right)ng\right\}.$$

Using (3.15), it follows that $\mathcal{G}(n, \varepsilon)^c \subset (A^+)^c \cup (A^-)^c \cup B^c$.

To bound the probability of the events $(A^\pm)^c$ and B^c, observe first that, by construction, $\tilde{G}_n = g \sum_{i=1}^n \tilde{\xi}_i$. As $\tilde{\xi}_i$'s are i.i.d., the standard application of the exponential Chebyshev inequality yields the estimate

$$\mathbb{Q}\left[\left(A^\pm\right)^c\right] \le e^{-cn\varepsilon^2}. \tag{3.17}$$

To estimate $\mathbb{Q}[B^c]$, we write $G_n(z)$ as

$$G_n(z) = \xi_0 \frac{\nu(z)}{\mu(z)} + \sum_{i=1}^n \xi_i \rho_z(Z_{i-1}) = \xi_0 \frac{\nu(z)}{\mu(z)} + \int_0^{\tau_n} \rho_z(\bar{Z}_t)\,dt, \tag{3.18}$$

where $(\bar{Z}_t)_{t\ge 0}$ is a continuous-time Markov chain following the same trajectory as Z with mean-one exponential waiting times, and τ_n is the time of the nth jump of $\bar{Z}$. It follows that $\mathbb{Q}[B^c]$ can be estimated with help of quantitative estimates on the deviations of additive functionals of Markov chains. An estimate suitable for our purposes is proved in the Appendix.

To apply this estimate, we write

$$\begin{aligned}\mathbb{Q}[B^c] \le \sum_{z\in\Sigma} \Bigg\{ &\mathbb{Q}\left[\frac{\xi_0 \nu(z)}{\mu(z)} \ge \frac{1}{4}\varepsilon n g(z)\right] + \mathbb{Q}\left[|\tau_n - n| \ge \frac{1}{4} n\varepsilon\right] \\ &+ \mathbb{Q}\left[\int_0^{n(1+\varepsilon/4)} \rho_z(\bar{Z}_t)\,dt - n\left(1 + \frac{\varepsilon}{4}\right)g(z) \ge \frac{1}{4} n\varepsilon g(z)\right] \\ &+ \mathbb{Q}\left[\int_0^{n(1-\varepsilon/4)} \rho_z(\bar{Z}_t)\,dt - n\left(1 - \frac{\varepsilon}{4}\right)g(z) \le -\frac{1}{4} n\varepsilon g(z)\right]\Bigg\}.\end{aligned} \tag{3.19}$$

The first term satisfies

$$\mathbb{Q}\left[\xi_0 \nu(z)/\mu(z) \ge \varepsilon n g(z)/4\right] = e^{-cn\varepsilon(\pi(z)/\nu(z))}. \tag{3.20}$$

The second term can be bounded using a large deviation argument as in (3.17). The last two terms can be bounded using Corollary A.3 with $\delta = \varepsilon/(4 \pm \varepsilon)$, $t = n(1 \pm \varepsilon/4)$ and $f = \pm\rho_z$, using also the obvious identity $\pi(\rho_z) = g(z)$. The theorem then directly follows, the condition (3.5) is a direct consequence of the assumption (A.12) of Corollary A.3. $\square$

The same technique can trivially be adapted to couple the ranges of two Markov chains: Let P^1, P^2 be transition matrices of two Markov chains on a common finite state space Σ with respective mixing times T^1, T^2, but with the same invariant distribution π. Let further ν^1, ν^2 be two initial probability distributions on Σ. Similarly as above, we fix an a priori measure μ, and define $g(x) = \pi(x)/\mu(x)$, $\rho^i(x, y) = \mu(y)^{-1} p^i(x, y)$, $i = 1, 2$.

THEOREM 3.2. *Let P^1, P^2, ν^1, ν^2 and π be as above. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where one can define Markov chains Z^1, Z^2 with respective transition matrices P^1, P^2 and starting distributions ν^1, ν^2 satisfying the following property*: *Let ε be such that*

$$0 < \varepsilon \le \frac{1}{2} \wedge \min_{i=1,2} \min_{z\in\Sigma} \frac{\operatorname{Var}_\pi \rho_z^i}{2\|\rho_z^i\|_\infty g(z)}. \tag{3.21}$$

Set

$$k(\varepsilon) = \left\lceil -\min_{i=1,2} \min_{z\in\Sigma} \log_2 \frac{\pi_\star \varepsilon^2 g(z)^2}{6 \operatorname{Var}_\pi(\rho_z^i)} \right\rceil, \tag{3.22}$$

and for $n \ge 0$ define the "good" event

$$\tilde{\mathcal{G}}(n, \varepsilon) = \big\{ \{Z_i^1\}_{i=1}^{\lfloor n(1-\varepsilon)\rfloor} \subset \{Z_i^2\}_{i=1}^{n} \subset \{Z_i^1\}_{i=1}^{\lceil n(1+\varepsilon)\rceil} \big\}. \tag{3.23}$$

Then for every $n \ge 2k(\varepsilon)(T^1 \vee T^2)$

$$\begin{aligned} &\mathbb{Q}[\tilde{\mathcal{G}}(n, \varepsilon)^c] \\ &\quad \le C \sum_{i=1,2} \sum_{z\in\Sigma} \left(e^{-cn\varepsilon^2} + e^{-cn\varepsilon(\pi(z)/\nu^i(z))} + \exp\left\{ -\frac{c\varepsilon^2 g(z)^2}{\operatorname{Var}_\pi \rho_z^i} \frac{n}{k(\varepsilon) T^i} \right\} \right), \end{aligned} \tag{3.24}$$

where $C, c \in (0, \infty)$ are universal constants.

PROOF. We follow very closely the proof of Theorem 3.1 but we indicate the changes that need to be made in the argument.

Start by defining the Poisson point process $\eta = (z_i, v_i)$ on $\Sigma \times [0, \infty)$ with intensity $\mu \otimes dx$ as before. We now let ξ_k, $G_k(z)$ and (Z_k, V_k) be as in (3.9)–(3.11) and analogously for $\tilde{\xi}_k$, $\tilde{G}_k(z)$ and $(\tilde{Z}_k, \tilde{V}_k)$, where the transition densities ρ^2 (of to the chain Z_k^2) are used instead.

Then, using the same definition of $A^\pm$ and B, we see that $\mathcal{G}(n, \varepsilon)^c \subset (A^+)^c \cup (A^-)^c \cup B^c$ and the bound on $\mathbb{Q}(B^c)$ follows from exactly the same calculation as before. All we need to observe is that the bound on $\mathbb{Q}((A^\pm)^c)$ is proved exactly as that of B^c, since it concerns the Markov chain Z_k^2, which satisfies the same conditions as Z_k^1. This completes the proof of the theorem. $\square$

4. Coupling the vacant sets. In this section, we state the quantitative version of Theorem 1.2 giving the coupling between the vacant sets of the random walk and the random interlacements in the macroscopic subsets of the torus. We then show the connection between Theorem 3.2 and our main result by defining the relevant finite state space Markov chains.

For technical reasons, we should work with "rounded boxes" instead of the usual ones. Their advantage is that the common potential-theoretic quantities, like equilibrium measure and hitting probabilities, are smoother on them; similar smoothing was used in [8], Section 7. Let

$$\gamma \in \left(\frac{1}{d-1}, 1\right) \quad \text{and} \quad \alpha \in \left(0, \frac{1}{4}\right) \tag{4.1}$$

be two constants that remain fixed through the paper. Set $L = 2N^\gamma + \alpha N$, and define the box B with rounded corners

$$B = B_N = \bigcup_{x \in [L, N-L]^d \cap \mathbb{Z}^d} B(x, \alpha N). \tag{4.2}$$

Let further Δ be the set of points at distance at least N^γ from B,

$$\Delta = \Delta_N = \left(\bigcup_{x \in B_N} B(x, N^\gamma) \right)^c, \tag{4.3}$$

see Figure 1 for illustration. We view B and Δ as subsets of $\mathbb{Z}^d$ as well as of $\mathbb{T}_N^d$ (identified with $\{0, \dots, N-1\}^d$).

We can state the quantitative version of Theorem 1.2 now.

THEOREM 4.1. *Let $u > 0$ and ε_N be a sequence satisfying $\varepsilon_N \in (0, c_0)$ with c_0 sufficiently small. Set $\kappa = \gamma(d-1) - 1 > 0$ and assume that $\varepsilon_N^2 \geq c N^{\delta - \kappa}$ for some $\delta > 0$. Then there exists coupling $\mathbb{Q}$ of $\mathcal{V}_N^u$ with $\mathcal{V}^{u(1 \pm \varepsilon_N)}$ such that for every N large enough*

$$\mathbb{Q}\big[\big(\mathcal{V}^{u(1-\varepsilon_N)} \cap B_N\big) \supset \big(\mathcal{V}_N^u \cap B_N\big) \supset \big(\mathcal{V}^{u(1+\varepsilon_N)} \cap B_N\big)\big] \geq 1 - C_1 e^{-C_2 N^{\delta'}} \tag{4.4}$$

for some constants $\delta' > 0$, and $C_1, C_2 \in (0, \infty)$ depending on u, δ, γ and α.

Theorem 4.1 will be proved with help of Theorem 3.2. To this end, we now introduce relevant Markov chains which will be coupled together later.

The first Markov chain encodes the excursions of the random walk on the torus into the rounded box B. More precisely, let R_i, D_i be the successive excursion starting and ending times between B and Δ of the random walk X_n on $\mathbb{T}_N^d$ defined by $D_0 = H_\Delta$ and for $i \geq 1$ inductively

$$\begin{aligned} R_i &= H_B \circ \theta_{D_{i-1}} + D_{i-1}, \\ D_i &= H_\Delta \circ \theta_{R_i} + R_i, \end{aligned} \tag{4.5}$$

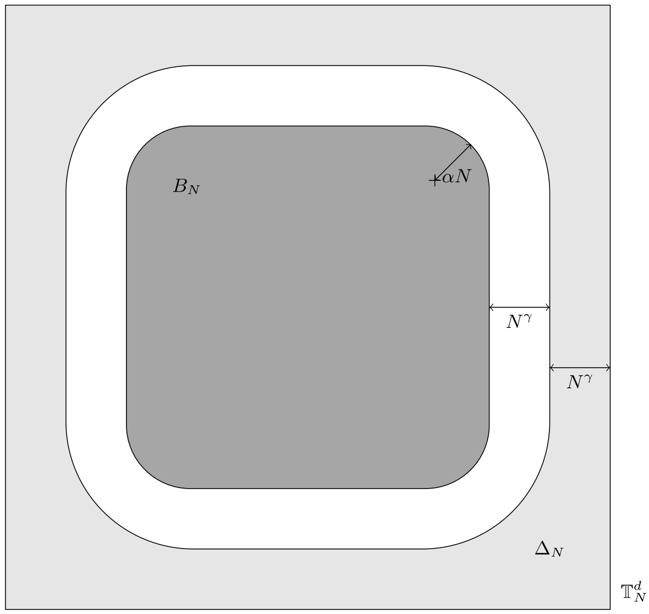

FIG. 1. *The rounded box* B_N (*dark gray*), *the 'security zone' of width* N^γ (*white*), *and the set* Δ_N (*light gray*) *in the torus* $\mathbb{T}_N^d$.

where θ_k is the shift operator introduced in (2.1). We define the process $Y_i = (X_{R_i}, X_{D_i}) \in \partial B \times \partial \Delta =: \Sigma$, $i \geq 1$. By the strong Markov property of X, $(Y_i)_{i\geq 1}$ is a Markov chain on Σ with transition probabilities

$$P[Y_{n+1} = \mathbf{y} | Y_n = \mathbf{x}] = P_{x_2}[X_{H_B} = y_1] P_{y_1}[X_{H_\Delta} = y_2], \tag{4.6}$$

for every $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2) \in \Sigma$, and with initial distribution

$$\nu_Y(\mathbf{x}) = P[X_{R_1} = x_1, X_{D_1} = x_2] = P[X_{R_1} = x_1] P_{x_1}[X_{H_\Delta} = x_2]. \tag{4.7}$$

The second Markov chain, encoding the behavior of the random interlacements in B, is defined similarly by considering separately the excursions of every random walk trajectory of random interlacements which enters B; cf. (2.6). Note that one random walk trajectory of random interlacements can, in general, induce more than one such excursions. Formally, let $(X^{(i)})_{i\geq 1}$ be a $P^{\mathbb{Z}^d}_{\bar{e}_B}$-distributed i.i.d. sequence, where $\bar{e}_B$ is the normalized equilibrium measure of B introduced in (2.5). For every $i \geq 1$, set $R_1^{(i)} = 0$ and define $D_j^{(i)}, R_j^{(i)}$, $j \geq 1$ analogously to (4.5) to be the successive departure and return times between B and Δ of the random walk $X^{(i)}$. Set

$$T^{(i)} = \sup\{j : R_j^{(i)} < \infty\} \tag{4.8}$$

to be the number of excursions of $X^{(i)}$ between B and Δ which is a.s. finite. Finally, let $(Z_k)_{k\geq 1}$ be the sequence of the starting and ending points of these excursions,

$$Z_k = \big(X^{(i)}_{R^{(i)}_j}, X^{(i)}_{D^{(i)}_j}\big) \tag{4.9}$$
$$\text{for } i \geq 1 \text{ and } 1 \leq j \leq T^{(i)} \text{ given by } k = \sum_{n=1}^{i-1} T^{(n)} + j.$$

The strong Markov property for $X^{(i)}$'s and their independence imply that Z_k is a Markov chain on Σ with transition probabilities

$$\begin{aligned} &P[Z_{n+1} = \mathbf{y} | Z_n = \mathbf{x}] \\ &\quad = \big(P^{\mathbb{Z}^d}_{x_2}[H_B < \infty, X_{H_B} = y_1] + P^{\mathbb{Z}^d}_{x_2}[H_B = \infty]\bar{e}_B(y_1)\big) P^{\mathbb{Z}^d}_{y_1}[X_{H_\Delta} = y_2] \end{aligned} \tag{4.10}$$

for every $\mathbf{x}, \mathbf{y} \in \Sigma$, and with initial distribution

$$\nu_Z(\mathbf{x}) = \bar{e}_B(x_1) P^{\mathbb{Z}^d}_{z_1}[X_{H_\Delta} = x_2]. \tag{4.11}$$

To apply Theorem 3.2, we need to estimate all relevant quantities for the Markov chains Y and Z. This is the content of the following four sections.

From now on, all constants c appearing in the text will possibly depend on the dimension d, and the constants α and γ defined in (4.1).

5. Technical estimates. In this section, we show several estimates on potential-theoretic quantities related to rounded boxes. Let $\bar{e}^\Delta_B$ be the normalized equilibrium measure on B for the walk killed on Δ,

$$\bar{e}^\Delta_B(x) = \frac{\mathbf{1}_{x \in \partial B}}{\mathrm{cap}_\Delta(B)} P_x[\tilde{H}_B > H_\Delta], \tag{5.1}$$

where

$$\underset{\Delta}{\mathrm{cap}}(B) = \sum_{x \in \partial B} P_x[\tilde{H}_B > H_\Delta] \tag{5.2}$$

is the associated capacity. We first show that $\bar{e}^\Delta_B$ is comparable with the uniform distribution on ∂B and give the order of $\mathrm{cap}_\Delta(B)$.

LEMMA 5.1. *The is $c \in (0, 1)$ such that*

$$c N^{d-1-\gamma} \leq \underset{\Delta}{\mathrm{cap}}(B) \leq c^{-1} N^{d-1-\gamma}, \tag{5.3}$$

and for every $x \in \partial B$

$$c N^{1-d} \leq \bar{e}^\Delta_B(x) \leq c^{-1} N^{1-d}. \tag{5.4}$$

PROOF. In view of (5.1), (5.2), to prove the lemma it is sufficient to show that uniformly in $x \in \partial B$,

$$c N^{-\gamma} \leq P_x[\tilde{H}_B > H_\Delta] \leq c^{-1} N^{-\gamma}. \tag{5.5}$$

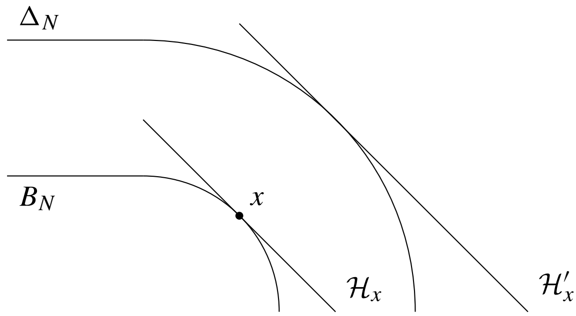

FIG. 2. *The planes $\mathcal{H}_x$ and $\mathcal{H}'_x$ from the proof of Lemma 5.1.*

For the lower bound, let $\mathcal{H}_x$ be the $(d-1)$-dimensional hyperplane "tangent" to ∂B containing x, and let $\mathcal{H}'_x$ be the hyperplane parallel to $\mathcal{H}_x$ tangent to $\partial\Delta$ (see Figure 2). Then

$$P_x[\tilde{H}_B > H_\Delta] \geq P_x[\tilde{H}_{\mathcal{H}_x} > H_{\mathcal{H}'_x}] \geq cN^{-\gamma}, \tag{5.6}$$

where the last inequality follows from observing the projection of X on the direction perpendicular to $\mathcal{H}_x$ and the usual martingale argument.

The upper bound in (5.5) is proved similarly. We consider a ball $\mathcal{G}_x$ contained in B with radius αN tangent to ∂B at x, and another ball $\mathcal{G}'_x$ with radius $\alpha N + N^\gamma$ concentric with $\mathcal{G}_x$. Then

$$P_x[\tilde{H}_B > H_\Delta] \leq P_x[\tilde{H}_{\mathcal{G}_x} > H_{\mathcal{G}'_x}] \leq cN^{-\gamma}, \tag{5.7}$$

using [5], Proposition 1.5.10, and the fact that $\alpha N \gg N^\gamma$. This completes the proof. □

For the usual equilibrium measure, we have similar estimates.

LEMMA 5.2. *There is a constant c such that for every $x \in \partial B$*

$$cN^{1-d} \leq \bar{e}_B(x) \leq c^{-1}N^{1-d} \tag{5.8}$$

and

$$\inf_{y\in\partial\Delta} P_y^{\mathbb{Z}^d}[H_B = \infty] \geq c_0 N^{\gamma-1}. \tag{5.9}$$

PROOF. Since $\operatorname{cap}(B_N) \asymp N^{d-2}$ (see [5], (2.16) page 53), in order to prove the lower bound in (5.8), we need to show that $P_x[\tilde{H}_B = \infty] \geq cN^{-1}$. This can be proved by similar arguments as above. We fix the hyperplane $\mathcal{H}_x$ as previously, and let $\mathcal{H}'_x$ be the hyperplane parallel to $\mathcal{H}_x$ at distance N. Then

$$P_x[\tilde{H}_B = \infty] \geq P_x[\tilde{H}_{\mathcal{H}_x} > H_{\mathcal{H}'_x}] \cdot \inf_{y\in\mathcal{H}'_x} P_y[H_B = \infty]. \tag{5.10}$$

By the same reasoning as above, the first factor is bounded from below by cN^{-1} and the second factor is of order constant, as follows easily from [5], Proposition 1.5.10, again.

To prove the upper bound of (5.8), we need to show that $P_x[\tilde{H}_B = \infty] \le N^{-1}$. To this end, fix $\mathcal{G}_x$ as in the previous proof. Then

$$P_x[\tilde{H}_B = \infty] \le P_x[\tilde{H}_{\mathcal{G}_x} = \infty] \le cN^{-1} \tag{5.11}$$

by, for example, [8], Lemma 7.5.

Finally, using the same notation as in (5.10), for $y \in \partial\Delta$,

$$P_y[H_B = \infty] \ge P_y[H_{\mathcal{H}_x} > H_{\mathcal{H}'_x}] \inf_{y \in \mathcal{H}'_x} P_y[H_B = \infty]. \tag{5.12}$$

The first factor is larger than $cN^{\gamma-1}$ by a martingale argument and the second is of order constant which proves (5.9) and completes the proof. $\square$

Finally, we control hitting probabilities of boundary points of B.

LEMMA 5.3. *There is a $c < \infty$ such that for every $x \in \partial\Delta$ and $y \in \partial B$*

$$P_x[X_{H_B} = y] \le cN^{-\gamma(d-1)}, \tag{5.13}$$

$$P_x^{\mathbb{Z}^d}[X_{H_B} = y] \le cN^{-\gamma(d-1)}. \tag{5.14}$$

In addition, for every $y \in \partial B$, there are at least $c^{-1}N^{\gamma(d-1)}$ points $x \in \partial\Delta$ such that

$$P_x[X_{H_B} = y] \ge c^{-1}N^{-\gamma(d-1)}, \tag{5.15}$$

$$P_x^{\mathbb{Z}^d}[X_{H_B} = y] \ge c^{-1}N^{-\gamma(d-1)}. \tag{5.16}$$

For the proof of this lemma, we will need to use some facts about the entrance distribution of a random walk to a set. These estimates can be derived for a large collection of sets, assuming that the boundary is sufficiently regular. Let us state some results of this type that have been derived in [8] and are needed in this paper.

DEFINITION 5.4 (Definition 7.1 of [8]). Let $\mathsf{D} \subset \mathbb{R}^d$ be an open set (not necessarily connected) with smooth boundary $\partial\mathsf{D}$. We say that D is *s-regular* if for any $\mathsf{x} \in \partial\mathsf{D}$ there exist two balls $\mathsf{B}^{\mathsf{x}}_{\mathrm{in}} \subset \bar{\mathsf{D}}$ and $\mathsf{B}^{\mathsf{x}}_{\mathrm{out}} \subset \mathbb{R}^d \setminus \mathsf{D}$ of radius s, such that $\partial\mathsf{D} \cap \mathsf{B}^{\mathsf{x}}_{\mathrm{out}} = \partial\mathsf{D} \cap \mathsf{B}^{\mathsf{x}}_{\mathrm{in}} = \{\mathsf{x}\}$. Informally speaking, the above definition means that one can touch the boundary of D by spheres of radius s from inside and outside.

For $\mathsf{B}^{\mathsf{x}}_{\mathrm{in}}$ and $\mathsf{B}^{\mathsf{x}}_{\mathrm{out}}$ as above, let us denote by x^{in} and $\mathsf{x}^{\mathrm{out}}$ their respective centers. We also define x^{in} and x^{out} to be the closest points to x^{in} and $\mathsf{x}^{\mathrm{out}}$ in $\mathbb{Z}^d$ (chosen arbitrarily in case of ties). For $\mathsf{A} \subset \mathbb{R}^d$, we write A for $\mathsf{A} \cap \mathbb{Z}^d$.

LEMMA 5.5 (Lemma 7.6 of [8]). *There are constants $s_1 < \infty$ and $c < \infty$ such that the following hold*:

(i) *Suppose that* A *is an s-regular set for some $s \geq s_1$ and $y \in \partial A$, $x \in \mathbb{Z}^d \setminus A$ are such that $\|x - y\| \geq 2s$. Then $P_x[X_{H_A} = y] \leq cs^{-(d-1)}$.*

(ii) *Assume that* A *is s-regular with $s \geq s_1$ and $x \in \partial A$; then for every $y \in B(x^{\text{out}}, s/2)$, we have $P_y[X_{H_A} = x, H_A < H_{\mathbb{Z}^d \setminus B(x^{\text{out}}, s+\sqrt{d})}] \geq c^{-1} s^{-(d-1)}$.*

PROPOSITION 5.6 (Proposition 7.7 of [8]). *There exist constants $s_2, c_1 > c_2, c_3 \in (0, \infty)$ such that if $s \geq s_2$, $\mathsf{A} \subset \mathbb{R}^d$ is $c_1 s$-regular and if $y_1, y_2 \in \partial A$ are such that $\|y_1 - y_2\| \leq c_2 s$, then there exists a set $\hat{D}$ (depending on y_1, y_2) that separates $\{y_1, y_2\}$ from $\partial B(y_1, 2c_1 s)$ [i.e., any nearest-neighbor path starting at $\partial B(y_1, 2c_1 s)$ that enters A at $\{y_1, y_2\}$, must pass through $\hat{D}$] such that*

$$\sup_{\substack{x \in \hat{D}; \\ P_x[X_{H_A} = y_1] > 0}} \frac{P_x[X_{H_A} = y_2]}{P_x[X_{H_A} = y_1, H_A < H_{\mathbb{Z}^d \setminus B(y_1, (5/2)c_1 s)}]} \leq c_3. \tag{5.17}$$

PROOF OF LEMMA 5.3. The lower bounds (5.15), (5.16) follow directly from Lemma 5.5(ii) by taking $s = N^\gamma$. The upper bound (5.14) is a consequence of Lemma 5.5(i).

To show (5.13), let $y_1, y_2 \in \partial B$ be two points at distance smaller than δN^γ for some sufficiently small $\delta > 0$. By Proposition 5.6, there is a "surface" $\hat{D} = \hat{D}(y_1, y_2)$ in $\mathbb{Z}^d$ separating $\{y_1, y_2\}$ from x so that for every $z \in \hat{D} \setminus B$

$$c P_z[X_{H_B} = y_1] \leq P_z[X_{H_B} = y_2] \leq c^{-1} P_z[X_{H_B} = y_1] \tag{5.18}$$

for some sufficiently small c independent of y_1, y_2. Since every path in $\mathbb{T}_N^d \setminus B$ from x to $\{y_1, y_2\}$ must pass through $\hat{D} \setminus B$, using the strong Markov property on $H_{\hat{D}}$, it follows that z can be replaced by x in (5.18). As consequence, for every $y \in \partial B$ there are at least $c(\delta N^\gamma)^{(d-1)}$ points y' on ∂B with

$$P_x[X_{H_B} = y'] \geq c P_x[X_{H_B} = y], \tag{5.19}$$

from which (5.13) easily follows. □

6. Equilibrium measure. In this section, we show that the equilibrium measures of the Markov chains Y and Z that we defined in Section 4 coincide as required by Theorem 3.2. This may sound surprising at first, since the periodic boundary conditions in the torus are felt in the exit probabilities of macroscopic boxes.

LEMMA 6.1. *Let π be the probability measure on Σ given by*

$$\pi(\mathbf{x}) = \bar{e}_B^\Delta(x_1) P_{x_1}[X_{H_\Delta} = x_2], \qquad \mathbf{x} = (x_1, x_2) \in \Sigma. \tag{6.1}$$

Then π is the invariant measure for both Y and Z.

PROOF. To see that π is invariant for Y, consider the stationary random walk $(X_i)_{i\in\mathbb{Z}}$ (note the doubly infinite time indices) on $\mathbb{T}_N^d$. Let $\mathcal{R}$ be the set of "returns to B" for this walk,

$$\mathcal{R} = \{n \in \mathbb{Z} : X_n \in B, \exists m < n, X_m \in \Delta, \{X_{m+1}, \dots, X_{n-1}\} \subset (B \cup \Delta)^c\}, \tag{6.2}$$

$\mathcal{D}$ the set of "departures"

$$\mathcal{D} = \{n \in \mathbb{Z} : X_n \in \Delta, \exists m \in \mathcal{R}, m < n, \{X_m, \dots, X_{n-1}\} \in \Delta^c\}, \tag{6.3}$$

and write $\mathcal{R} = \{\bar{R}_i\}_{i\in\mathbb{Z}}$, $\mathcal{D} = \{\bar{D}_i\}_{i\in\mathbb{Z}}$ so that $\bar{R}_i < \bar{D}_i < \bar{R}_{i+1}$, $i \in \mathbb{Z}$, and

$$\bar{R}_0 < \inf\{i \geq 0 : X_i \in \Delta\} < \bar{R}_1. \tag{6.4}$$

Observe that by this convention the sequence $(\bar{R}_i, \bar{D}_i)_{i\geq 1}$ agrees with $(R_i, D_i)_{i\geq 1}$ defined in (4.5). Remark also that $\bar{R}_0$ might be nonnegative in general, but $\bar{R}_{-1} < 0$.

Due to the stationarity and the reversibility of X, for every $\mathbf{x} = (x_1, x_2)$,

$$\begin{aligned} &P[n \in \mathcal{R}, X_n = x_1] \\ &\quad = P[X_n = x_1, \exists m < n, X_m \in \Delta, \{X_{m+1}, \dots, X_{n-1}\} \subset (B \cup \Delta)^c] \\ &\quad = N^{-d} P_{x_1}[\tilde{H}_B > H_\Delta]. \end{aligned} \tag{6.5}$$

By the ergodic theorem, the stationary measure π_Y of Y satisfies

$$\begin{aligned} \pi_Y(\{x_1\} \times \partial\Delta) &= \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mathbf{1}\{X_{R_i} = x_1\} \\ &= \lim_{m\to\infty} \frac{m^{-1}\sum_{n=1}^{m} \mathbf{1}\{n \in \mathcal{R}, X_n = x_1\}}{m^{-1}\sum_{n=1}^{m} \mathbf{1}\{n \in \mathcal{R}\}}, \end{aligned} \tag{6.6}$$

where we used the observation below (6.4) for the last equality. Applying the ergodic theorem for the numerator and denominator separately, and then using (6.5) yields

$$\begin{aligned} \pi_Y(\{x_1\} \times \partial\Delta) &= \frac{P[n \in \mathcal{R}, X_n = x_1]}{P[n \in \mathcal{R}]} = \frac{P_{x_1}[\tilde{H}_B > H_\Delta]}{\sum_{y\in\partial B} P_y[\tilde{H}_B > H_\Delta]} \\ &= \bar{e}_B^\Delta(x_1). \end{aligned} \tag{6.7}$$

By the strong Markov property, $\pi_Y(\mathbf{x}) = \pi_Y(\{x_1\} \times \partial\Delta) P_{x_1}[H_\Delta = x_2]$, and thus $\pi_Y = \pi$ as claimed.

We now consider the Markov chain Z. This chain is defined from the i.i.d. sequence of random walks $X^{(i)}$. Each of these random walks give rise to a random-length block of excursions distributed as $\{(X^{(1)}_{R_i^{(1)}}, X^1_{D_i^{(1)}}) : i = 1, \dots, T^{(1)}\}$. The invariant measure π_Z of Z can thus be written as

$$\pi_Z(\mathbf{x}) = \frac{1}{E_{\bar{e}_B}^{\mathbb{Z}^d} T^{(1)}} E_{\bar{e}_B}^{\mathbb{Z}^d}\left[\sum_{i=1}^{T^{(1)}} \mathbf{1}_{X^{(1)}_{R_i^{(1)}} = x_1}\right] P_{x_1}[X_{H_\Delta} = x_2], \qquad \mathbf{x} = (x_1, x_2). \tag{6.8}$$

To show that $\pi_Z = \pi$, it is thus sufficient to show that the middle factor is proportional to $P_{x_1}[\tilde{H}_B > H_\Delta]$, since the first factor will then be the correct normalization.

To simplify the notation, we write X, T, R_j for $X^{(1)}$, $T^{(1)}$, $R_j^{(1)}$, and extend X to a two-sided random walk on $\mathbb{Z}^d$ by requiring the law of $(X_{-i})_{i\geq 0}$ to be $P_{X_0}^{\mathbb{Z}^d}[\cdot|\tilde{H}_B = \infty]$, conditionally independent of $(X_i)_{i\geq 0}$. We denote by $L = \sup\{n : X_n \in B\}$ the time of the last visit of X to B. Then

$$\begin{aligned}
&E_{\bar{e}_B}^{\mathbb{Z}^d}\left[\sum_{j=1}^{T} \mathbf{1}_{X_{R_j}=x_1}\right] \\
&\quad = \sum_{y\in\partial B}\sum_{z\in\partial B} \bar{e}_B(y) E_y^{\mathbb{Z}^d}\left[\mathbf{1}_{X_L=z}\sum_{j=1}^{T}\mathbf{1}_{X_{R_j}=x_1}\right] \\
&\quad = \sum_{y\in\partial B}\sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(y) P_y^{\mathbb{Z}^d}\big[X_n = x_1, X_L = z, \exists m \in \mathbb{Z} : m < n, \\
&\quad\quad X_m \in \Delta, \{X_{m+1},\dots,X_{n-1}\} \subset (B\cup\Delta)^c\big].
\end{aligned} \tag{6.9}$$

According to [12], Proposition 1.8, under $P_{\bar{e}_B}^{\mathbb{Z}^d}$, X_L has also distribution $\bar{e}_B$. Hence, by reversibility, this equals

$$\begin{aligned}
&= \sum_{y\in\partial B}\sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(z) P_z^{\mathbb{Z}^d}\big[X_n = x_1, X_L = y, \exists m > n : X_m \in \Delta, \\
&\quad \{X_{n+1},\dots,X_{m-1}\} \subset (B\cup\Delta)^c\big] \\
&= \sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(z) P_z^{\mathbb{Z}^d}\big[X_n = x_1, \exists m > n : X_m \in \Delta, \\
&\quad \{X_{n+1},\dots,X_{m-1}\} \subset (B\cup\Delta)^c\big] \qquad (6.10)\\
&= \sum_{z\in\partial B}\sum_{n=0}^{\infty} \bar{e}_B(z) P_z^{\mathbb{Z}^d}[X_n = x_1] P_{x_1}[\tilde{H}_B > H_\Delta].
\end{aligned}$$

Introducing the Green function $g(x,y) = \sum_{n=0}^{\infty} P_x^{\mathbb{Z}^d}[X_n = y]$ and using the identity $\sum_z e_B(z) g(z,x) = 1$ (see [12], Proposition 1.8), this equals to

$$= \sum_{z\in\partial B} \bar{e}_B(z) g(z,x_1) P_{x_1}[\tilde{H}_B > H_\Delta] = P_{x_1}[\tilde{H}_B > H_\Delta]/\operatorname{cap}(B). \tag{6.11}$$

This shows the required proportionality and completes the proof of the lemma. □

We will need the following estimate on the measure π.

LEMMA 6.2. *For every $y \in \partial\Delta$,*

$$\pi\big(\partial B \times \{y\}\big) \le CN^{1-d}. \tag{6.12}$$

PROOF. By similar arguments as in the proof of Lemma 6.1, using the same notation,

$$\begin{aligned} &P[n \in \mathcal{D}, X_n = y] \\ &\quad = P\big[X_n = y, \exists m < n, X_m \in B, \{X_{m+1}, \dots, X_{n-1}\} \in (B \cup \Delta)^c\big] \\ &\quad = N^{-d} P_y[\tilde{H}_\Delta > H_B] \\ &\quad \le cN^{-d-\gamma}, \end{aligned} \tag{6.13}$$

since, by the same argument as in the proof of Lemma 5.1, $P_y[\tilde{H}_\Delta > H_B] \le cN^{-\gamma}$. Further,

$$P(n \in \mathcal{D}) = P(n \in \mathcal{R}) = \sum_{x \in \partial B} N^{-d} P_x[\tilde{H}_B > H_\Delta] \asymp cN^{-\gamma-1}, \tag{6.14}$$

by the estimates in the proof of Lemma 5.1 again. Therefore,

$$\pi\big(\partial B \times \{y\}\big) = P[X_n = y | n \in \mathcal{D}] \le cN^{1-d}, \tag{6.15}$$

and the proof is completed. □

7. Mixing times.

The next ingredient of Theorem 3.2 are the mixing times T_Y and T_Z of the Markov chains Y and Z. They are estimated in the following lemma.

LEMMA 7.1. *There is a constant c such that*

$$T_Z \le cN^{1-\gamma}, \tag{7.1}$$

$$T_Y \le cN^{1-\gamma}. \tag{7.2}$$

PROOF. To bound the mixing times we use repeatedly the following lemma which can be found, for example, in [6], Corollary 5.3.

LEMMA 7.2. *Let $(\mathcal{X}_i)_{i \ge 0}$ be an arbitrary Markov chain on a finite state space Σ. Assume that for every $x, y \in \Sigma$ there exist a coupling $Q_{x,y}$ of two copies $\mathcal{X}, \mathcal{X}'$ of $\mathcal{X}$ starting respectively from x and y, such that*

$$\max_{x,y \in \Sigma} Q_{x,y}\big[\mathcal{X}_n \neq \mathcal{X}'_n\big] \le 1/4. \tag{7.3}$$

Then $T_{\mathcal{X}} \le n$.

To show (7.1), we thus consider two copies Z_i, Z_i' of the chain Z starting respectively in $\mathbf{x}, \mathbf{x}' \in \Sigma$ and define the coupling $Q_{\mathbf{x},\mathbf{x}'}$ between them as follows. Let $(\xi_i)_{i\geq 0}$ be a sequence of i.i.d. Bernoulli random variables with $P[\xi_i = 1] = c_0 N^{\gamma-1} := p_N$ where the constant c_0 is as in (5.9). Given $Z_i = \mathbf{x}_i$, $Z_i' = \mathbf{x}_i'$, and given $\xi_i = 1$ we choose $Z_{i+1} = Z_{i+1}'$ distributed as $\nu(\mathbf{x}) = \bar{e}_B(x_1) P_{x_1}[X_{H_\Delta} = x_2]$. On the other hand, when $\xi_i = 0$, we choose Z_{i+1} and Z_{i+1}' independently with respective distributions $\mu_{\mathbf{x}_i}$ and $\mu_{\mathbf{x}_i'}$ where [cf. (4.10)]

$$\begin{aligned}\mu_{\mathbf{x}}(\mathbf{y}) = \{&P_{x_2}^{\mathbb{Z}^d}[H_B < \infty, X_{H_B} = y_1] \\ &+ (P_{x_2}^{\mathbb{Z}^d}[H_B = \infty] - p_N)\bar{e}_B(y_1)\} \frac{P_{y_1}^{\mathbb{Z}^d}[X_{H_\Delta} = y_2]}{1 - p_N}.\end{aligned} \tag{7.4}$$

The bound (5.9) ensures that all terms are nonnegative, and thus $\mu_{\mathbf{x}}$ is a well-defined probability distribution. If $Z_i = Z_i'$ for some i, then we let them move together, $Z_j = Z_j'$ for all $j \geq i$.

It follows that

$$\max_{\mathbf{x},\mathbf{x}'} Q_{\mathbf{x},\mathbf{x}'}[Z_i \neq Z_i'] \leq \mathbb{P}[\xi_j = 0\ \forall j < i] = (1 - p_N)^i. \tag{7.5}$$

Choosing now $i = cN^{1-\gamma}$ with c sufficiently large and using Lemma 7.2 yields (7.1).

To show (7.2), let $G = G_N = \{x \in B_N : \operatorname{dist}(x, \partial B_N) \geq \alpha N/2\}$. Intuitively, the excursions of the random walk into G will play the same role as the "excursions of the random interlacements to infinity" played in the proof of (7.1). We need two technical claims.

CLAIM 7.3. *For some constant $c_1 > 0$ and all N large,*

$$\inf_{x \in \partial B} P_x[H_G < H_\Delta] \geq c_1 N^{\gamma-1}. \tag{7.6}$$

PROOF. Similarly as in Section 5, let $\mathcal{G}_x$ be the ball with radius αN contained in B tangent to ∂B at x, and let $\mathcal{G}_x^1$, $\mathcal{G}_x^2$ be the balls concentric with $\mathcal{G}_x$ with radius $\alpha N/2$ and $\alpha N + N^\gamma$ respectively. Then $\mathcal{G}_x^2 \subset \mathbb{T}_N^d \setminus \Delta$, and $\mathcal{G}_x^1 \subset G$. Hence, using again [5], Proposition 1.5.10,

$$P_x[H_G < H_\Delta] \geq P_x[H_{\mathcal{G}_x^1} < H_{\mathcal{G}_x^2}] \geq cN^{1-\gamma} \tag{7.7}$$

which shows the claim. $\square$

CLAIM 7.4. *For some $c_2 < \infty$ and all N large,*

$$\sup_{x \in \partial G} P_x[X_{H_\Delta} = y] \leq c_2 \inf_{x \in \partial G} P_x[X_{H_\Delta} = y] \qquad \textit{for all } y \in \partial\Delta. \tag{7.8}$$

PROOF. For every $y \in \partial\Delta$, the function $x \mapsto P_x[X_{H_\Delta} = y]$ is harmonic on $\mathbb{T}_N^d \setminus \Delta$. The claim then follows by Harnack principle; see, for example, [5], Theorem 1.7.6. $\square$

We continue the proof of (7.2). For $x \in \partial B$, let $\nu_x(\cdot) = P_x[X_{H_{G\cup\Delta}} \in \cdot\,]$. By Claim 7.3, $\nu_x(\partial G) \geq c_1 N^{\gamma-1}$, so we can find a subprobability ν_x° on ∂G such that $\nu_x^\circ(\partial G) = c_1 N^{\gamma-1}$ and $\nu_x^\circ \leq \nu_x$. For any $x \in \mathbb{T}_N^d$, let $\mu_x(\cdot) = P_x[X_{H_\Delta} \in \cdot\,]$, and let μ be the subprobability on $\partial\Delta$ given by $\mu(y) = \inf_{x\in\partial G} \mu_x(y)$. It follows from Claim 7.4 that $\mu(\partial\Delta) \geq c_2^{-1}$. For any nontrivial subprobability measure κ, we denote by $\overline{\kappa}$ the probability measure obtained by normalizing κ.

We now construct the coupling required for the application of Lemma 7.2. Let $\mathbf{x}(0), \mathbf{x}'(0) \in \Sigma$ and define the coupling $Q_{\mathbf{x},\mathbf{x}'}$ of two copies Y, Y' of Y as follows. Let $Y_0 = \mathbf{x}$, $Y_0' = \mathbf{x}'$, and let $(\xi_i)_{i\geq 0}$, $(\tilde{\xi}_i)_{i\geq 0}$ be two independent sequences of i.i.d. Bernoulli random variables with $P[\xi_i = 1] = c_1 N^{\gamma-1}$ and $P[\tilde{\xi}_i = 1] = \mu(\partial\Delta)$. Now continue inductively through the following steps:

1. Given $Y_{k-1} = (Y_{k-1,1}, Y_{k-1,2})$ and $Y_{k-1}' = (Y_{k-1,1}', Y_{k-1,2}')$, $k \geq 1$, choose $Y_{k,1}$, resp. $Y_{k,1}'$, independently from $P_{Y_{k-1,2}}[X_{H_B} \in \cdot\,]$, resp. $P_{Y_{k-1,2}'}[X_{H_B} \in \cdot\,]$.
2. If $\xi_k = 0$, choose U_k according to $\overline{\nu_{Y_{k,1}} - \nu_{Y_{k,1}}^\circ}$, then $Y_{k,2}$ according to μ_{U_k}, and analogously U_k' according to $\overline{\nu_{Y_{k,1}'} - \nu_{Y_{k,1}'}^\circ}$ and then $Y_{k,2}'$ according to $\mu_{U_k'}$, independently.
3. Otherwise, if $\xi_k = 1$, choose U_k according to $\overline{\nu_{Y_{k,1}}^\circ}$, and U_k' according to $\overline{\nu_{Y_{k,1}'}^\circ}$, independently. If, in addition $\tilde{\xi}_k = 1$, choose $Y_{k,2} = Y_{k,2}'$ according to $\overline{\mu}$. Otherwise, if $\tilde{\xi}_k = 0$, choose $Y_{k,2}$ according to $\overline{\mu_{U_k} - \mu}$, and $Y_{k,2}'$ according to $\overline{\mu_{U_k'} - \mu}$, independently.
4. Finally, if for some k, $Y_{k,2} = Y_{k,2}'$, let Y and Y' follow the same trajectory after k.

It can be checked easily that these steps construct two copies of Y started from $\mathbf{x}$ and $\mathbf{x}'$ respectively. Moreover,

$$Q_{\mathbf{x},\mathbf{x}'}[Y_k \neq Y_k'] \leq \mathbb{P}[\xi_i\tilde{\xi}_i = 0\ \forall i < k] = \left(1 - c_1 N^{\gamma-1}\mu(\partial\Delta)\right)^{k-1}. \tag{7.9}$$

Observing that $\mu(\partial\Delta) \geq c_2^{-1}$, (7.2) follows by taking $k = cN^{1-\gamma}$ with c large enough and using Lemma 7.2. $\square$

8. Variance estimate. We continue to estimate the ingredients for the application of Theorem 3.2. Due to the form of the equilibrium measure π introduced in (6.1), it is suitable to fix the base measure μ on Σ as

$$\mu(\mathbf{x}) = P_{x_1}[X_{H_\Delta} = x_2], \qquad \mathbf{x} = (x_1, x_2) \in \Sigma. \tag{8.1}$$

Then [cf. (3.3), (3.4) for the notation]

$$g(\mathbf{x}) = \bar{e}_B^{\Delta}(x_1), \tag{8.2}$$

$$\rho^Y(\mathbf{x}, \mathbf{y}) = P_{x_2}[X_{H_B} = y_1] =: \tilde{\rho}^Y(x_2, y_1), \tag{8.3}$$

$$\rho^Z(\mathbf{x}, \mathbf{y}) = P_{x_2}^{\mathbb{Z}^d}[X_{H_B} = y_1] + P_{x_2}^{\mathbb{Z}^d}[H_B = \infty]\bar{e}_B(y_1) =: \tilde{\rho}^Z(x_2, y_1). \tag{8.4}$$

Recall that $\rho_{\mathbf{x}}^{\circ}$ denotes the function $\mathbf{y} \mapsto \rho^{\circ}(\mathbf{y}, \mathbf{x})$; we use $\circ$ to stand for either Y or Z.

LEMMA 8.1. *There exist constants $c, C \in (0, \infty)$ such that for every $\mathbf{x} \in \Sigma$*

$$cN^{1-d}N^{-\gamma(d-1)} \le \operatorname*{Var}_{\pi} \rho_{\mathbf{x}}^{\circ} \le CN^{1-d}N^{-\gamma(d-1)}. \tag{8.5}$$

PROOF. An easy computation yields, using Lemma 6.2 for the last inequality,

$$\begin{aligned} \operatorname*{Var}_{\pi} \rho_{\mathbf{x}}^{\circ} &\le \sum_{\mathbf{x}' \in \Sigma} \pi(\mathbf{x}')\rho^{\circ}(\mathbf{x}', \mathbf{x})^2 \\ &= \sum_{x_2' \in \partial\Delta} \pi(\partial B \times \{x_2'\})\tilde{\rho}^{\circ}(x_2', x_1)^2 \\ &\le CN^{1-d} \sum_{x_2' \in \partial\Delta} \tilde{\rho}^{\circ}(x_2', x_1)^2. \end{aligned} \tag{8.6}$$

Using Lemmas 5.2, 5.3 in (8.3) and (8.4), we obtain that

$$\max_{x \in \partial B, y \in \partial\Delta} \tilde{\rho}^{\circ}(y, x) \le cN^{-\gamma(d-1)} \tag{8.7}$$

for both chains $\circ \in \{Y, Z\}$. Therefore,

$$\begin{aligned} \operatorname*{Var}_{\pi} \rho_{\mathbf{x}}^{\circ} \le CN^{1-d} \sup\Bigg\{ &\sum_{z \in \partial B} h^2(z) : h : \partial B \to [0, cN^{-\gamma(d-1)}], \\ &\sum_{z \in \partial B} h(z) = 1 \Bigg\}. \end{aligned} \tag{8.8}$$

The supremum is achieved by a function h that takes the maximal value $cN^{-\gamma(d-1)}$ for as many points as it can, by a convexity argument. Hence,

$$\operatorname*{Var}_{\pi} \rho_{\mathbf{x}}^{\circ} \le CN^{1-d}N^{\gamma(d-1)}(N^{-\gamma(d-1)})^2, \tag{8.9}$$

and the upper bound follows.

Finally, by Lemma 5.3 and (8.3), (8.4), for every $x \in \partial B$ there are at least $cN^{\gamma(d-1)}$ points $y \in \partial\Delta$ such that $\tilde{\rho}^{\circ}(y, x) \ge c'N^{-\gamma(d-1)}$. Hence, $\pi((\rho_{\mathbf{x}}^{\circ})^2)$ is larger than the left-hand side of (8.5). Moreover, since π is invariant for both Markov chains, it follows that $\pi(\rho_{\mathbf{x}}^{\circ})^2 = g(x)^2 \asymp N^{2(1-d)}$, by Lemma 5.1. Combining the last two claims, the lower bound follows. $\square$

9. Number of excursions. The final ingredient needed for Theorem 3.2 is an estimate on the number of excursion that the random walk typically makes before the time uN^d, as well as on the corresponding quantity for the random interlacements at level u.

Consider first the random walk on the torus. Define

$$\mathcal{N}(t) = \sup\{i : R_i < t\} \tag{9.1}$$

to be the number of excursions starting before t. We show that $\mathcal{N}(t)$ concentrates around its expectation.

PROPOSITION 9.1. *Let $u > 0$ be fixed. There exist constants c, C depending only on γ and α such that for every $N \geq 1$*

$$P\Big[\Big|\mathcal{N}(uN^d) - u \operatorname*{cap}_{\Delta}(B)\Big| > \eta \operatorname*{cap}_{\Delta}(B)\Big] \leq C \exp\{-c\eta^2 N^c\}. \tag{9.2}$$

PROOF. To prove the proposition, we first compute the expectation of $\mathcal{N}(t)$.

LEMMA 9.2. *For every $t \in \mathbb{N}$,*

$$\Big|E\mathcal{N}(t) - tN^{-d} \operatorname*{cap}_{\Delta}(B)\Big| \leq 1. \tag{9.3}$$

Moreover, when starting from $\bar{e}_B^{\Delta}$, the stationary measure for R_i's, we have

$$E_{\bar{e}_B^{\Delta}}(R_1) = \frac{N^d}{\operatorname{cap}_{\Delta}(B)}. \tag{9.4}$$

PROOF. Recall from the proof of Lemma 6.1 that $(\bar{R}_i, \bar{D}_i)$ denote the returns and departures of the stationary random walk $(X_n)_{n\in\mathbb{Z}}$. Let $\bar{\mathcal{N}}(t) = \sup\{i : \bar{R}_i < t\}$. By the observation below (6.4), $|\bar{\mathcal{N}}(t) - \mathcal{N}(t)| \leq 1$. It is thus sufficient to show that $E\bar{\mathcal{N}}(t) = tN^{-d} \operatorname{cap}_{\Delta}(B)$. To this end, recall equality (6.5). Summing it over $x_1 \in \partial B$, we obtain

$$P[k = \bar{R}_j \text{ for some } j] = N^{-d} \operatorname*{cap}_{\Delta}(B), \qquad k \geq 0. \tag{9.5}$$

The required claim follows by summation over $0 \leq k < t$.

The second claim of the lemma is a consequence of the first claim, the fact that every X_{R_k} is $\bar{e}_B^{\Delta}$-distributed at stationarity, and the ergodic theorem. □

We proceed with proving Propositions 9.1. It is more convenient to show a concentration result for the return times R_i instead of $\mathcal{N}(t)$. Observing that for any $t > 0$ and $b > 0$,

$$\{|\mathcal{N}(t) - E(\mathcal{N}(t))| > b\} \subseteq \{R_{\lceil E(\mathcal{N}(t))-b\rceil} > t\} \cup \{R_{\lfloor E(\mathcal{N}(t))+b\rfloor} > t\} \tag{9.6}$$

we obtain easily that

$$P\Big[\Big|\mathcal{N}(uN^d) - u \operatorname*{cap}_{\Delta}(B)\Big| > \eta \operatorname*{cap}_{\Delta}(B)\Big] \le P\big[R_{k_-} > uN^d\big] + P\big[R_{k_+} < uN^d\big], \tag{9.7}$$

where $k_- = \lceil (u-\eta)\operatorname{cap}_\Delta(B)\rceil$ and $k_+ = \lfloor (u+\eta)\operatorname{cap}_\Delta(B)\rfloor$.

Let $\varepsilon > 0$ be a small constant that will be fixed later, and set $\ell = \lfloor N^\varepsilon T_Y \rfloor$, where T_Y stands for the mixing time of the chain Y estimated in (7.2). In order to estimate the right-hand side of (9.7), we study the typical size of $R_{m_\pm \ell}$ where

$$m_- = \Big\lceil \ell^{-1}(u-\eta)\operatorname*{cap}_{\Delta}(B)\Big\rceil \quad \text{and} \quad m_+ = \Big\lfloor \ell^{-1}(u+\eta)\operatorname*{cap}_{\Delta}(B)\Big\rfloor. \tag{9.8}$$

From Lemma 5.1 and (7.2), it follows that

$$m_\pm \ge cN^{d-2-\varepsilon}. \tag{9.9}$$

Let $\mathcal{G}_i = \sigma(X_i : i \le R_{i\ell})$. Using the standard properties of the mixing time (see, e.g., [6], Section 4.5) and the strong Markov property, it is easy to see that

$$\big\| P\big[(X_{R_{i\ell}}, X_{D_{i\ell}}) \in \cdot | \mathcal{G}_{i-1}\big] - \pi(\cdot)\big\|_{\mathrm{TV}} \le 2^{-N^\varepsilon}. \tag{9.10}$$

By Lemma 5.1, $\pi(\{y\}\times\partial\Delta) = \bar{e}_B^\Delta(y) \asymp N^{1-d}$ uniformly in $y \in \partial B$, and thus

$$\left|\frac{P[X_{R_{i\ell}} = y | \mathcal{G}_{i-1}]}{\bar{e}_B^\Delta(y)} - 1\right| \le c2^{-N^{\varepsilon/2}}, \qquad i \ge 1. \tag{9.11}$$

For m standing for m_+ or m_-, we write

$$R_{m\ell} = \sum_{j=1}^m Z_j \qquad \text{where } Z_j = R_{j\ell} - R_{(j-1)\ell} \text{ and } R_0 := 0. \tag{9.12}$$

For every $j \ge 2$, by (9.11),

$$P[Z_j > t | \mathcal{G}_{j-2}] \le \big(1 + c2^{-N^{\varepsilon/2}}\big) P_{\bar{e}_B^\Delta}[R_\ell > t] \le 2\ell P_{\bar{e}_B^\Delta}[R_1 > t/\ell]. \tag{9.13}$$

By the invariance principle $P[R_1 > N^2] \le c < 1$. Using this and Markov property iteratively yields $P[R_1 > N^{2+\delta}] \le e^{-cN^\delta}$ for any $\delta > 0$, and thus

$$P\big[Z_j > \ell N^{2+\delta} | \mathcal{G}_{j-2}\big] \le 2\ell P_{\bar{e}_B^\Delta}\big[R_1 > N^{2+\delta}\big] \le c\exp\{-N^{c'\delta}\}. \tag{9.14}$$

Analogous reasoning proves also that

$$P\big[Z_1 \ge \ell N^{2+\delta}\big] \le c\exp\{-N^{c'\delta}\}. \tag{9.15}$$

Observe also that for $j \ge 2$, by (9.11) again,

$$\big|E[Z_j] - E[Z_j | \mathcal{G}_{j-1}]\big| \le c2^{-N^{\varepsilon/2}} E(Z_j). \tag{9.16}$$

Hence,

$$
\begin{aligned}
&P\bigl[\bigl|R_{m\ell}-E(R_{m\ell})\bigr|>\eta E(R_{m\ell})\bigr]\\
&\quad=P\Biggl[\Biggl|\sum_{j=1}^{m}\bigl(Z_j-E[Z_j]\bigr)\Biggr|>\eta E(R_{m\ell})\Biggr]\\
&\quad\le P\bigl[Z_1\ge \eta E(R_{m\ell})/4\bigr]\\
&\qquad+\sum_{n\in\{0,1\}}P\Biggl[\Biggl|\sum_{\substack{1\le j\le m\\ j \bmod 2=n}}\bigl(Z_j-E[Z_j|\mathcal{G}_{j-2}]\bigr)\Biggr|>\eta E(R_{m\ell})/4\Biggr].
\end{aligned}
\tag{9.17}
$$

Setting $\hat{Z}_j=Z_j\wedge \ell N^{2+\delta}$, which by (9.14) satisfies

$$
\begin{aligned}
&\bigl|E[\hat{Z}_j|\mathcal{G}_{j-2}]-E[Z_j|\mathcal{G}_{j-2}]\bigr|\\
&\quad=\int_{\ell N^{2+\delta}}^{\infty}P[Z_j>t|\mathcal{G}_{j-2}]\,dt\le c\exp\{-N^{c'\delta}\},
\end{aligned}
\tag{9.18}
$$

the right-hand side of (9.17) can be bounded by

$$
\begin{aligned}
&\le cm\exp\{-N^{c'\delta}\}\\
&\quad+\sum_{n\in 0,1}P\Biggl[\Biggl|\sum_{\substack{1\le j\le m\\ j=n \bmod 2}}\bigl(\hat{Z}_j-E[\hat{Z}_j|\mathcal{G}_{j-2}]\bigr)\Biggr|>\eta E(R_{m\ell})/4\Biggr].
\end{aligned}
\tag{9.19}
$$

Azuma's inequality together with $E[R_{m\ell}]\asymp N^d$, (9.8), (9.9), and Lemma 5.1 then yield

$$
\begin{aligned}
&\le cm\exp\{-N^{c'\delta}\}+4\exp\biggl\{-\frac{2c(\eta E(R_{m\ell}))^2}{m(\ell N^{2+\delta})^2}\biggr\}\\
&\le cm\exp\{-N^{c'\delta}\}+4\exp\biggl\{-c\eta^2\frac{mN^{2d-4-2\delta}}{\mathrm{cap}_\Delta(B)^2}\biggr\}\\
&\le cm\exp\{-N^{c'\delta}\}+4\exp\{-c\eta^2N^{d-4+2\gamma-\varepsilon-2\delta}\}.
\end{aligned}
\tag{9.20}
$$

For every $d\ge 3$ and γ as in (4.1), it is possible to fix δ and ε sufficiently small so that the exponent of N on the right-hand side of the last display is positive. Therefore, the above decays at least as $C\exp\{-c\eta^2N^c\}$ as N tends to infinity, completing the proof of the proposition. □

We now count the number of excursions of random interlacements at level u into B. Let J_u^N be the Poisson process with intensity $\mathrm{cap}(B_N)$ driving the excursions of random interlacements to B_N; cf. (2.6). From Section 4, recall the definition (4.8) of random variables $T^{(i)}$ giving the number of excursions of ith random walk between B and Δ. Given those, denote by $\mathcal{N}'(u)$ the number of steps of Markov chain Z corresponding to the level u of random interlacements,

$$\mathcal{N}'(u) = \sum_{i=1}^{J_u^N} T^{(i)}. \tag{9.21}$$

PROPOSITION 9.3. *There exist constants c, C depending only on γ and u such that for every $u > 0$*

$$P\Big[\Big|\mathcal{N}'(u) - u \operatorname*{cap}_{\Delta}(B)\Big| \geq \eta u \operatorname*{cap}_{\Delta}(B)\Big] \leq C \exp\{-c\eta^2 N^c\}. \tag{9.22}$$

PROOF. By definition of random interlacements, J_u^N is a Poisson random variable with parameter $u \operatorname{cap}(B) \asymp u N^{d-2}$, and thus, by Chernov estimate,

$$P\Big[\Big|J_u^N - u \operatorname{cap}(B)\Big| \geq \eta u \operatorname{cap}(B)\Big] \leq C \exp\{-c\eta^2 N^{d-2}\}. \tag{9.23}$$

The random variables $T^{(i)}$ are i.i.d. and stochastically dominated by the geometric distribution with parameter $\inf_{y\in\partial\Delta} P_y^{\mathbb{Z}^d}[H_B = \infty] \asymp N^{\gamma-1}$, by Lemma 5.2. Moreover, by summing (6.9)–(6.11) over $x_1 \in \partial B$ we obtain

$$E_{\bar{e}_B}^{\mathbb{Z}^d} T^{(i)} = \frac{\operatorname{cap}_\Delta(B)}{\operatorname{cap}(B)}. \tag{9.24}$$

Applying Chernov bound again for $v = (1 \pm \frac{\eta}{2}) u \operatorname{cap}(B)$,

$$P\left[\left|\sum_{i=1}^{v} T^{(i)} - \frac{v \operatorname{cap}_\Delta(B)}{\operatorname{cap}(B)}\right| \geq \frac{\eta}{2} \frac{v \operatorname{cap}_\Delta(B)}{\operatorname{cap}(B)}\right] \leq C \exp\{-c\eta^2 N^c\} \tag{9.25}$$

for some constants C and c depending on γ and u. The proof is completed by combining (9.23) and (9.25). □

10. Proofs of the main results. We can now finally show our main results: Theorem 4.1 giving the coupling between the vacant sets of the random walk and the random interlacements in macroscopic subsets of the torus, and Theorem 1.1 implying the phase transition in the behavior of the radius of the connected cluster of the vacant set of the random walk containing the origin.

PROOF OF THEOREM 4.1. As already announced several times, Theorem 3.2 is the key ingredient of this proof.

Recall the definitions and transition probabilities of the Markov chains $Y = (Y_i)_{i\geq 1}$ and $Z = (Z_i)_{i\geq 1}$ from Section 3. The state space Σ of these Markov chains is finite, so we can apply Theorem 3.2 to construct a coupling of those two chains on some probability space $(\Omega_N, \mathcal{F}_N, \mathbb{Q}_N)$ carrying a Poisson point process with intensity $\mu \otimes dx$ on $\Sigma \times [0, \infty)$, so that their ranges coincide in sense of (3.23). We will apply this theorem with

$$
\begin{aligned}
n = u \operatorname*{cap}_{\Delta}(B) &\asymp N^{d-1-\gamma} && \text{(cf. Lemma 5.1, Propositions 9.1, 9.3)}\\
|\Sigma| &\asymp N^{2(d-1)},\\
T_Y, T_Z &\le cN^{1-\gamma} && \text{(Lemma 7.1)}\\
g(\mathbf{z}) = \bar{e}^{\Delta}_B(z_1) &\asymp N^{1-d} && \text{(Lemma 5.1)}\\
\operatorname{Var} \rho^Y_{\mathbf{z}}, \operatorname{Var} \rho^Z_{\mathbf{z}} &\asymp N^{1-d} N^{-\gamma(d-1)} && \text{(Lemma 8.1)}\\
\|\rho^Y_{\mathbf{z}}\|_\infty, \|\rho^Z_{\mathbf{z}}\|_\infty &\asymp N^{-\gamma(d-1)} && \text{(Lemma 5.3, cf. (8.7) and below).}
\end{aligned}
\tag{10.1}
$$

In addition, it follows from Claims 7.3, 7.4, that $\pi^\star$ decays polynomially with N, and thus

$$k(\varepsilon_N) \sim c \log N - c' \log \varepsilon_N. \tag{10.2}$$

Substituting those into condition (3.21) of Theorem 3.2 implies that $\varepsilon_N < c_0 = c_0(d, \gamma, \alpha)$ as assumed in Theorem 4.1. If, in addition, ε_N satisfies $\varepsilon_N^2 \ge cN^{\delta-\kappa}$ for $\kappa = \gamma(d-1) - 1 > 0$ and $\delta > 0$, the, after some algebra, we obtain

$$\mathbb{Q}_N\Bigg[\bigcup_{i \le (1-\varepsilon_N)n} Z_i \subset \bigcup_{i \le n} Y_i \subset \bigcup_{i \le (1+\varepsilon_N)n} Z_i\Bigg] \ge 1 - c_1 e^{-c_2 N^{\delta'}} \tag{10.3}$$

for some δ' as in Theorem 4.1.

We now re-decorate Y and Z to obtain a coupling of the vacant sets restricted to B. Let Γ be the space of all finite-length nearest-neighbor paths on $\mathbb{T}^d_N$. For $\gamma \in \Gamma$, we use $\ell(\gamma)$ to denote its length and write γ as $(\gamma_0, \dots, \gamma_{\ell(\gamma)})$.

To construct the vacant set of the random walk, we define on the same probability space $(\Omega_N, \mathcal{F}_N, \mathbb{Q}_N)$ (by possibly enlarging it) two sequences of "excursions" $(\mathcal{E}_i)_{i\ge 1}$ and $(\tilde{\mathcal{E}}_i)_{i \ge 0}$, whose distribution is uniquely determined by the following properties:

- Given realization of $Y = ((Y_{i,1}, Y_{i,2}))_{i\ge 1}$ and $Z = ((Z_{i,1}, Z_{i,2}))_{i \ge 1}$, $(\mathcal{E}_i)$ and $(\tilde{\mathcal{E}}_i)$ are conditionally independent sequences of conditionally independent random variables.
- For every $i \ge 1$, the random variable $\mathcal{E}_i$ is Γ-valued and for every $\gamma \in \Gamma$,

$$\mathbb{Q}_N[\mathcal{E}_i = \gamma | Y, Z] = P_{Y_{i,1}}\big[H_\Delta = \ell(\gamma), X_i = \gamma_i\ \forall i \le \ell(\gamma) | X_{H_\Delta} = Y_{i,2}\big]. \tag{10.4}$$

- For every $i \ge 1$, the random variable $\tilde{\mathcal{E}}_i$ is Γ-valued and for every $\gamma \in \Gamma$,

$$
\begin{aligned}
&\mathbb{Q}_N[\tilde{\mathcal{E}}_i = \gamma | Y, Z]\\
&\quad = P_{Y_{i,2}}\big[H_B = \ell(\gamma), X_i = \gamma_i\ \forall i \le \ell(\gamma) | X_{H_B} = Y_{i+1,1}\big].
\end{aligned}
\tag{10.5}
$$

- The random variable $\tilde{\mathcal{E}}_0$ is Γ-valued and

$$\mathbb{Q}_N[\tilde{\mathcal{E}}_0 = \gamma | Y, Z] = P\big[R_1 = \ell(\gamma), X_i = \gamma_i \ \forall i \le \ell(\gamma) | X_{R_1} = Y_{1,1}\big]. \tag{10.6}$$

With slight abuse of notation, we construct on $(\Omega_N, \mathcal{F}_N, \mathbb{Q}_N)$ a process $(X_n)_{n\ge 0}$ defined by concatenation of $\tilde{\mathcal{E}}_0, \mathcal{E}_1, \tilde{\mathcal{E}}_1, \mathcal{E}_2, \dots$. From the construction, it follows easily that X is a simple random walk on $\mathbb{T}_N^d$ started from the uniform distribution. Finally, we write $R_1 = \ell(\tilde{\mathcal{E}}_0)$, $D_1 = \ell(\tilde{\mathcal{E}}_0) + \ell(\mathcal{E}_1)$, $\dots$, which is consistent with the previous notation, and set, as before, $\mathcal{N}(uN^d) = \sup\{i : R_i < uN^d\}$. Finally, we fix an arbitrary constant $\beta > 0$ and define the vacant set of random walk on $(\Omega_N, \mathcal{F}_N, \mathbb{Q}_N)$ by

$$\mathcal{V}_N^u = \mathbb{T}_N^d \setminus \{X_{\beta N^d}, \dots, X_{(\beta+u)N^d}\}, \tag{10.7}$$

which has the same distribution as the vacant set introduced in (1.1), since (X_i) is stationary Markov chain.

To construct the vacant set of random interlacements intersected with B, let $\mathcal{I}_0 = \varnothing$ and for $i \ge 1$ inductively

$$\begin{aligned} \iota_i &= \inf\{j \ge 1 : j \notin \mathcal{I}_{i-1}, Y_j = Z_i\}, \\ \mathcal{E}_i^{\mathrm{RI}} &= \mathcal{E}_{\iota_i}, \\ \mathcal{I}_i &= \mathcal{I}_{i-1} \cup \{\iota_i\}. \end{aligned} \tag{10.8}$$

Let further $(U_i)_{i\ge 1}$ be a sequence of conditionally independent Bernoulli random variables with [cf. (4.10)]

$$\begin{aligned} &P[U_i = 1] \\ &\quad = \frac{P_{Z_{i,2}}^{\mathbb{Z}^d}[H_B = \infty]\bar{e}_B(Z_{i+1,1})}{P_{Z_{i,2}}^{\mathbb{Z}^d}[H_B < \infty, X_{H_B} = Z_{i+1,1}] + P_{Z_{i,2}}^{\mathbb{Z}^d}[H_B = \infty]\bar{e}_B(Z_{i+1,1})}. \end{aligned} \tag{10.9}$$

The event $\{U_i = 1\}$ heuristically correspond to the event "after the excursion Z_i the random walk leaves to infinity and the excursion of random interlacements corresponding to Z_{i+1} is a part of another random walk trajectory." We set $V_0 = 0$ and inductively for $i \ge 1$. $V_i = \inf\{i > V_{i-1} : U_i = 1\}$. Then, by construction, for every $i \ge 1$, $(\mathcal{E}_j^{\mathrm{RI}})_{V_{i-1} < j \le V_i}$ has the same distribution as the sequence of excursions of random walk $X^{(i)}$ into B; cf. (2.6), (4.9). Finally, as in (2.6), we let $(J_u^N)_{u\ge 0}$ to stand for a Poisson process with intensity $\mathrm{cap}(B)$, defined on $(\Omega_N, \mathcal{F}_N, \mathbb{Q}_N)$, independent of all previous randomness, and set

$$\mathcal{N}'(u) = V_{J_u^N}. \tag{10.10}$$

This is again consistent with previous notation. Finally, for β as above, we can construct the random variables having the law of the vacant set of random interlacements at levels $u + \varepsilon_N$ and $u - \varepsilon_N$ intersected with B,

$$\mathcal{V}^{u\pm\varepsilon_N} = B \Big\backslash \bigcup_{i=\mathcal{N}'(\beta\mp\varepsilon N/2)}^{\mathcal{N}'(\beta+u\pm\varepsilon_N/2)} \mathrm{Range}(\mathcal{E}_i^{\mathrm{RI}}). \tag{10.11}$$

Denoting $\mathcal{K}_N = \mathrm{cap}_\Delta(B)$, by Proposition 9.1 the set $\mathcal{V}_N^u$ of (10.7) satisfies

$$\begin{aligned}&\mathbb{Q}_N\Bigg[B_N \Big\backslash \bigcup_{i=(\beta-\varepsilon_N/4)\mathcal{K}_N}^{(\beta+u+\varepsilon_N/4)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i) \subset \mathcal{V}_N^u \subset B_N \Big\backslash \bigcup_{i=(\beta+\varepsilon_N/4)\mathcal{K}_N}^{(\beta+u-\varepsilon_N/4)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i)\Bigg] \\ &\qquad \geq 1 - Ce^{-c\varepsilon_N^2 N^c}.\end{aligned} \tag{10.12}$$

Combining (10.3) and (10.8) yields

$$\begin{aligned}&\mathbb{Q}_N\Bigg[B_N \Big\backslash \bigcup_{i=(\beta-\varepsilon_N/4)\mathcal{K}_N}^{(\beta+u+\varepsilon_N/4)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i) \supset B_N \Big\backslash \bigcup_{i=(\beta-\varepsilon_N/3)\mathcal{K}_N}^{(\beta+u+\varepsilon_N/3)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i^{\mathrm{RI}})\Bigg] \\ &\qquad \geq 1 - Ce^{-c_2 N^{\delta'}}, \\ &\mathbb{Q}_N\Bigg[B_N \Big\backslash \bigcup_{i=(\beta+\varepsilon_N/4)\mathcal{K}_N}^{(\beta+u-\varepsilon_N/4)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i) \subset B_N \Big\backslash \bigcup_{i=(\beta+\varepsilon_N/3)\mathcal{K}_N}^{(\beta+u.\varepsilon_N/3)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i^{\mathrm{RI}})\Bigg] \\ &\qquad \geq 1 - Ce^{-c_2 N^{\delta'}}.\end{aligned} \tag{10.13}$$

Finally, by Proposition 9.3, for vacant sets as in (10.11),

$$\begin{aligned}&\mathbb{Q}_N\Bigg[\mathcal{V}^{u+\varepsilon_N/2} \cap B \subset B_N \Big\backslash \bigcup_{i=(\beta-\varepsilon_N/3)\mathcal{K}_N}^{(\beta+u+\varepsilon_N/3)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i^{\mathrm{RI}})\Bigg] \geq 1 - Ce^{-c\varepsilon_N^2 N^c}, \\ &\mathbb{Q}_N\Bigg[\mathcal{V}^{u-\varepsilon_N/2} \cap B \supset B_N \Big\backslash \bigcup_{i=(\beta+\varepsilon_N/3)\mathcal{K}_N}^{(\beta+u-\varepsilon_N/3)\mathcal{K}_N} \mathrm{Range}(\mathcal{E}_i^{\mathrm{RI}})\Bigg] \geq 1 - Ce^{-c\varepsilon_N^2 N^c}.\end{aligned} \tag{10.14}$$

Theorem 4.1 then follows by combining (10.12)–(10.14). □

PROOF OF THEOREM 1.1. Let us first introduce a simple notation. If $\mathcal{C}$ is a random subset of either $\mathbb{T}^d$ or $\mathbb{Z}^d$, let $A_N(\mathcal{C})$ stand for the event $[\mathrm{diam}(\mathcal{C}) > N/4]$, which appears in the definition of $\eta_N(u)$. We also denote by $\mathcal{C}_0(u)$ the connected component containing the origin of $\mathbb{Z}^d$ for random interlacements at level u.

We now turn to the proof of (1.3). Fix $u > u_\star(d)$. Letting $u' \in (u_\star, u)$ and writing $u' = (1-\varepsilon)u$, we estimate

$$\begin{aligned}&P[A_N(\mathcal{C}_N(u))] \\ &\qquad \leq 1 - \mathbb{Q}_N[(\mathcal{V}_N^u \cap \mathcal{B}_N) \subset (\mathcal{V}^{u(1-\varepsilon)} \cap \mathcal{B}_N)] + P[A_N(\mathcal{C}_0(u'))],\end{aligned} \tag{10.15}$$

which clearly tends to zero using Theorem 1.2 and the fact that $u' > u_\star$.

Now let us treat the supercritical case in (1.4). Given $u < u_\star$ and $\varepsilon > 0$, we use the continuity of $\eta(u)$ in $[0, u_\star)$ (see Corollary 1.2 of [13]), to find u' and u'' such that

$$(1-\varepsilon)u \leq u' < u < u'' \leq (1+\varepsilon)u \quad \text{and} \quad \eta(u') - \eta(u'') < \varepsilon. \tag{10.16}$$

We now observe that for $N > c$ we have $|\eta(u') - P[A_N(\mathcal{C}_0(u'))]| < \varepsilon$. Therefore, since η is nonincreasing function,

$$\begin{aligned}
&\bigl|P\bigl[A_N\bigl(\mathcal{C}_N(u)\bigr)\bigr] - \eta(u)\bigr| \\
&\quad \le \varepsilon + \bigl(P\bigl[A_N\bigl(\mathcal{C}_N(u)\bigr)\bigr] - \eta(u'')\bigr)_- + \bigl(P\bigl[A_N\bigl(\mathcal{C}_N(u)\bigr)\bigr] - \eta(u')\bigr)_+ \\
&\quad \overset{N>c}{\le} 2\varepsilon + \bigl(\mathbb{Q}\bigl[A_N\bigl(\mathcal{C}_N(u)\bigr)\bigr] - \mathbb{Q}\bigl[A_N\bigl(\mathcal{C}_0(u'')\bigr)\bigr]\bigr)_- \\
&\qquad + \bigl(\mathbb{Q}\bigl[A_N\bigl(\mathcal{C}_N(u)\bigr)\bigr] - \mathbb{Q}\bigl[A_N\bigl(\mathcal{C}_0(u')\bigr)\bigr]\bigr)_+ \\
&\quad \le 2\varepsilon + 1 - \mathbb{Q}_N\bigl[\bigl(\mathcal{V}^{u(1+\varepsilon)} \cap \mathcal{B}_N\bigr) \subset \bigl(\mathcal{V}_N^u \cap \mathcal{B}_N\bigr) \subset \bigl(\mathcal{V}^{u(1-\varepsilon)} \cap \mathcal{B}_N\bigr)\bigr].
\end{aligned} \tag{10.17}$$

Since the limsup of the right-hand side of the above equation is at most 2ε by Theorem 1.2 and $\varepsilon > 0$ is arbitrary, we have proved (1.4) and consequently Theorem 1.1. □

APPENDIX: A CHERNOV-TYPE ESTIMATE FOR ADDITIVE FUNCTIONALS OF MARKOV CHAINS

We show here a simple variant of Chernov bound for additive functionals of Markov chains. Many such bounds were obtained previously, but they do not suit our purposes, for example, Lezaud [7] (see also Theorems 2.1.8, 2.1.9 in [9]) provides such bounds in terms of the spectral gap of the Markov chain. Since the spectral gap of nonreversible Markov chains is not easy to estimate, and more importantly, it does not always reflect the mixing properties of the chain, it seems preferable to use the mixing time of the chain as the input. This idea was applied, for example, in [3], whose bounds, in contrary to [7], do not use the information about the variance of the additive functional under the equilibrium measure, and thus give worse estimates in the case where this variance is known. The theorems below can be viewed as combination of those two results.

We consider discrete time Markov chains first.

THEOREM A.1. *Let $(X_n)_{n\ge0}$ be a discrete-time Markov chain on a finite state space Σ with transition matrix P, initial distribution ν, mixing time T, and invariant distribution π. Then, for every $n \ge 1$, every function $f : \Sigma \to [-1, 1]$ satisfying $\pi(f) = 0$ and $\pi(f^2) \le \sigma^2$, and every $0 < \gamma \le \sigma^2 \wedge \frac{1}{2}$*

$$\mathbb{P}\biggl[\sum_{i<n} f(X_n) \ge n\gamma\biggr] \le 4\exp\biggl\{-\biggl\lfloor \frac{n}{k(\gamma)T} - 1\biggr\rfloor \frac{\gamma^2}{6\sigma^2}\biggr\}, \tag{A.1}$$

with

$$k(\gamma) = \bigl\lceil -\log_2\bigl(\pi_\star\gamma^2/(6\sigma^2)\bigr)\bigr\rceil \tag{A.2}$$

and $\pi_\star = \min_{x\in\Sigma} \pi(x)$.

PROOF. Let $\tau = k(\gamma)T$. From [6], Section 4.5, it follows that, for any initial distribution ν,

$$(1-\varepsilon)\pi(x) \le \mathbb{P}[X_\tau = x] \le (1+\varepsilon)\pi(x), \tag{A.3}$$

with $\varepsilon = \gamma^2/(6\sigma^2)$. For $0 \le k < \tau$, define $X_j^{(k)} = X_{k+\tau j}$, $j \ge 0$. For every k, $(X_j^{(k)})_{j\ge 0}$ is a Markov chain with transition matrix P^τ and invariant distribution π. In view of (A.3), $(X_j^{(k)})_{j\ge 1}$ are close to being i.i.d. with marginal π; the distribution of $X_0^{(k)}$ cannot be controlled in general.

Writing $Y_n^{(k)} = \sum_{0\le i<(n-k)/\tau} f(X_i^{(k)})$, with help of Jensen's inequality and the exponential Chebyshev bound, we have for every $\lambda > 0$

$$\mathbb{P}\Big[\sum_{j<n} f(X_j) \ge \gamma n\Big] \le \exp\{-\lambda\gamma n\tau^{-1}\}\frac{1}{\tau}\sum_{k<\tau}\mathbb{E}\big[\exp\{\lambda Y_n^{(k)}\}\big]. \tag{A.4}$$

Using (A.3), the Markov property recursively, and the fact $f \le 1$ for the summand $f(X_0^{(k)})$,

$$\mathbb{E}\big[\exp\{\lambda Y_n^{(k)}\}\big] \le e^\lambda \exp\Big\{\Big\lfloor\frac{n-k}{\tau}\Big\rfloor\big(\log\big(\pi\big(e^{\lambda f}\big)\big) + \log(1+\varepsilon)\big)\Big\}, \tag{A.5}$$

for all $0 \le k < \tau$. By Bennett's lemma (see, e.g., [4], Lemma 2.4.1),

$$\pi\big(e^{\lambda f}\big) \le \frac{1}{1+\sigma^2}e^{-\lambda\sigma^2} + \frac{\sigma^2}{1+\sigma^2}e^\lambda. \tag{A.6}$$

Inserting this bound back into (A.4) and optimizing over λ as in [4], Corollary 2.4.7, which amounts to choosing

$$e^\lambda = \frac{1}{\sigma^2}\cdot\frac{\gamma+\sigma^2}{1-\gamma} \le 4, \tag{A.7}$$

we obtain

$$\begin{aligned}&\mathbb{P}\Big[\sum_{j<n} f(X_j) \ge \gamma n\Big]\\ &\qquad\le 4\exp\Big\{-\Big\lfloor\frac{n}{\tau}-1\Big\rfloor\Big(H\Big(\frac{\gamma+\sigma^2}{1+\sigma^2}\Big|\frac{\sigma^2}{1+\sigma^2}\Big) - \log(1+\varepsilon)\Big)\Big\},\end{aligned} \tag{A.8}$$

where $H(x|p) = x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}$. Observing finally that for every $\sigma^2 \in (0,1)$ and $\gamma \in (0,\sigma^2)$

$$H\Big(\frac{\gamma+\sigma^2}{1+\sigma^2}\Big|\frac{\sigma^2}{1+\sigma^2}\Big) \ge \frac{\gamma^2}{3\sigma^2} \tag{A.9}$$

and $\log(1+\varepsilon) \le \varepsilon = \gamma^2/(6\sigma^2)$, we obtain the claim of the theorem. □

For continuous-time Markov chains, we have an analogous statement.

COROLLARY A.2. *Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on a finite state space Σ with generator L, initial distribution ν, mixing time T, and invariant distribution π. Then for every $t>0$, every function $f:\Sigma\to[-1,1]$ with $\pi(f)=0$ and $\pi(f^2)\leq\sigma^2$, and for $\gamma\leq\sigma^2\wedge\frac{1}{2}$,*

$$\mathbb{P}\left[\int_0^t f(X_s)\,ds\geq\gamma t\right]\leq 4\exp\left\{-\left\lfloor\frac{t}{k(\gamma)T}-1\right\rfloor\frac{\gamma^2}{6\sigma^2}\right\},\tag{A.10}$$

with $k(\gamma)$ as in Theorem A.1.

PROOF. The proof is a discretization argument: Consider a discrete-time Markov chain $Y_n^\delta=X_{\delta n}$. The mixing time $T(\delta)$ of Y^δ satisfies $T(\delta)=T\delta^{-1}(1+o(1))$ as $\delta\to 0$. The previous theorem applied with $n=\delta^{-1}t$, then implies

$$\mathbb{P}\left[\delta\sum_{j<t\delta^{-1}} f(X_{j\delta})\geq\gamma t\right]\leq 4\exp\left\{-\left\lfloor\frac{t}{k(\gamma)T}-1\right\rfloor\frac{\gamma^2}{6\sigma^2}\right\}.\tag{A.11}$$

Taking $\delta\to 0$ and using the fact that Σ is finite (that is the transition rates are bounded from below) yields the claim. □

The final corollary gives concentration for arbitrary function on Σ.

COROLLARY A.3. *Let $(X_t)_{t\geq 0}$ be as in Corollary A.2 and $h:\Sigma\to\mathbb{R}$ an arbitrary function such that $\mathrm{Var}_\pi(h)\leq\sigma^2$. Let further*

$$\delta\leq\frac{\sigma^2}{2\pi(h)\|h\|_\infty}\wedge 1\tag{A.12}$$

and set

$$k'(\delta)=\lceil-\log_2\bigl(\delta^2\pi(h)^2\pi_\star/(6\sigma^2)\bigr)\rceil.\tag{A.13}$$

Then, for every $t\geq 0$,

$$\begin{aligned}&\mathbb{P}\left[\int_0^t h(X_s)\,ds-t\pi(h)\geq\delta t\pi(h)\right]\\&\quad\leq 4\exp\left\{-\left\lfloor\frac{t}{k'(\delta)T}-1\right\rfloor\frac{\delta^2\pi(h)^2}{6\sigma^2}\right\}.\end{aligned}\tag{A.14}$$

PROOF. Define

$$f=\bigl(h-\pi(h)\bigr)/2\|h\|_\infty.\tag{A.15}$$

The function f then satisfies $\|f\|_\infty\leq 1$, $\pi(f)=0$ and $\pi(f^2)\leq\sigma^2/(4\|h\|_\infty^2)$. Corollary A.1 applied with $\gamma=\delta\pi(h)/2\|h\|_\infty$ then directly implies the result. □

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