

GAUSSIAN APPROXIMATION OF NONLINEAR HAWKES PROCESSES

BY GIOVANNI LUCA TORRISI

CNR—Istituto per le Applicazioni del Calcolo “Mauro Picone”

We give a general Gaussian bound for the first chaos (or innovation) of point processes with stochastic intensity constructed by embedding in a bivariate Poisson process. We apply the general result to nonlinear Hawkes processes, providing quantitative central limit theorems.

1. Introduction. In the seminal papers [27] and [29], Stein’s method and Malliavin’s calculus have been combined to derive explicit bounds in the Gaussian approximation of random variables on the Wiener and Poisson spaces. Further developments on the Poisson space include, for example, [19, 30, 32, 34]. In particular, in [32] the authors derive new Gaussian bounds for functionals of the one-dimensional homogeneous Poisson process by using the Clark–Ocone representation formula; see, for example, [31]. In contrast with covariance identities based on the inverse of the Ornstein–Uhlenbeck operator the Clark–Ocone representation formula only requires the computation of a gradient and a conditional expectation. For this reason, the Clark–Ocone representation formula is a valuable tool even for the probability approximation of random variables on spaces different from the Wiener and Poisson. We refer the reader to [33] for the use of the Clark–Ocone representation formula for the Gaussian and Poisson approximation of random variables on the Bernoulli space and to [26] for the use of the Clark–Ocone representation formula for the Gaussian approximation of solutions of some stochastic equations.

The contributions of this paper are the following. We provide a Gaussian bound for the first chaos of a large class of point processes with stochastic intensity; see Theorem 3.1. Particularly, we consider point processes on the line constructed by embedding in a bivariate Poisson process and provide a Gaussian approximation for the first chaos (or innovation) combining Stein’s method and Malliavin’s calculus via a Clark–Ocone type representation formula.

To the best of our knowledge, this is the first paper which provides Gaussian bounds for the innovation of a point process with stochastic intensity by the Malliavin–Stein method.

Received October 2014; revised June 2015.

MSC2010 subject classifications. 60F05, 60G55.

Key words and phrases. Clark–Ocone formula, Gaussian approximation, Hawkes process, Malliavin’s calculus, Poisson process, Stein’s method, stochastic intensity.

We apply our general result to nonlinear Hawkes processes, deriving an explicit Gaussian bound for the innovation; see Theorem 4.1. In the special case of self-exciting processes (or linear Hawkes processes), relying on the knowledge of the intensity of the process and the spectral theory of point processes, we are able to provide alternative Gaussian bounds for the innovation which, in some cases, improve those one obtained by directly applying Theorem 4.1; see Theorems 5.1, 5.2 and Proposition 5.3. We exploit such Gaussian bounds to provide new quantitative central limit theorems in the Wasserstein distance for the first chaos of Hawkes processes; see Corollaries 4.5 and 5.4. The quantitative nature of these Gaussian approximations allows, for example, to construct in a standard way confidence intervals for the corresponding innovations, we outlined this simple application in Example 4.6.

From the point of view of applications, the extension of our results to multivariate point processes with stochastic intensity and random marks is certainly of interest; see, for example, [1, 22] and [39]. This topic is presently under investigation by the author, as well as the topic concerning the Poisson approximation, via the Malliavin–Stein method, of first-order stochastic integrals with respect to point processes with stochastic intensity (note that for this latter argument some results are already known, see [3] and [4]).

In the last years, there has been a renewed interest on Hawkes processes, mainly due to their mathematical tractability and versatility in modeling contexts. Self-exciting processes were introduced in [15] and [16], while the wider class of nonlinear Hawkes processes was introduced in [7]. Various mathematical aspects of these processes (and their generalizations), such as stability, rate of convergence to equilibrium, perfect and approximate simulation, large deviations and limit theorems, are studied in [5, 7–9, 15–17, 22–25, 39, 42, 43]. Linear Hawkes processes are Poisson cluster processes with a simple self-exciting structure which makes them very appealing to account for situations where the occurrence of future events directly depends on the past history. Nonlinear Hawkes processes allow to account for inhibitory effects. For these reasons, Hawkes processes naturally and simply capture a causal structure of discrete events dynamics associated with endogenous triggering, contagion and self-activation phenomena. Typical fields where this kind of dynamics arise are seismology (occurrence of earthquakes), neuroscience (occurrence of neuron's spikes), genome analysis (occurrence of events along a DNA sequence), insurance (occurrence of claims) and finance (occurrence of market order arrivals); see, for example, [1, 2, 18, 28, 35–37, 41] for applications of Hawkes processes in these contexts.

The paper is organized as follows. In Section 2, we give some preliminaries on point processes including the notion of stochastic intensity, the Poisson embedding construction and a Clark–Ocone type representation formula. In Section 3, we prove a general upper bound for the Wasserstein distance between the first chaos of a point process with stochastic intensity (constructed by embedding on a bivariate Poisson process) and a standard normal random variable. In Section 4, we apply

the result in Section 3 to nonlinear Hawkes processes. Particularly, in Section 4.1 we provide an explicit Gaussian bound for the first chaos (and a suitable approximated version of it) of a stationary nonlinear Hawkes process. The corresponding quantitative central limit theorem is derived in Section 4.2. The special case of self-exciting processes is treated in Section 5.

2. Preliminaries on point processes. In this section, we give some preliminaries on point processes, and refer the reader to the books [6, 12, 13] for more insight into this subject.

Let $\{T_n\}_{n \in \mathbb{Z}}$ be a sequence of random times defined on a probability space (Ω, \mathcal{A}, P) . Given a Borel set $A \in \mathcal{B}(\mathbb{R})$, we define

$$N(A) := \sum_{n \in \mathbb{Z}} \mathbb{1}_A(T_n)$$

and we call $N := \{N(A)\}_{A \in \mathcal{B}(\mathbb{R})}$ the point process with times $\{T_n\}_{n \in \mathbb{Z}}$. We suppose that N has the following properties:

$$T_n \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}; \quad |T_n| < \infty \implies T_n < T_{n+1}; \quad T_0 \leq 0 < T_1;$$

$$N(A) < \infty, \quad \text{for all bounded } A.$$

These conditions guarantee that N is simple, that is, $N(\{a\}) \leq 1$ for any $a \in \mathbb{R}$, and locally finite.

Given a sequence $\{Z_n\}_{n \in \mathbb{Z}}$ of random variables on Ω with values in some measurable space (E, \mathcal{E}) , we define

$$\overline{N}(A) := \sum_{n \in \mathbb{Z}} \mathbb{1}_A(T_n, Z_n), \quad A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{E}$$

and

$$\int_A \psi(t, z) \overline{N}(dt \times dz) := \sum_{n \in \mathbb{Z}} \psi(T_n, Z_n) \mathbb{1}_A(T_n, Z_n)$$

for a measurable function $\psi : \mathbb{R} \times E \rightarrow \mathbb{R}$ for which the infinite sum is well defined.

2.1. Point processes with stochastic intensity. Let $\mathcal{F} := \{\mathcal{F}_t\}_{t \in \mathbb{R}} \subset \mathcal{A}$ be a filtration such that $\mathcal{F}_t \supseteq \mathcal{F}_t^N$ for any $t \in \mathbb{R}$, where $\mathcal{F}^N := \{\mathcal{F}_t^N\}_{t \in \mathbb{R}}$ is the natural filtration of the point process N , that is,

$$\mathcal{F}_t^N := \sigma\{N(A) : A \in \mathcal{B}(\mathbb{R}), A \subseteq (-\infty, t]\}.$$

Let $\{\lambda(t)\}_{t \in \mathbb{R}}$ be a nonnegative stochastic process defined on (Ω, \mathcal{A}, P) which is \mathcal{F} -adapted, that is, $\lambda(t)$ is \mathcal{F}_t -measurable for any $t \in \mathbb{R}$, and such that

$$\int_a^b \lambda(t) dt < \infty, \quad \text{a.s., for all } a, b \in \mathbb{R}.$$

We call $\{\lambda(t)\}_{t \in \mathbb{R}}$ \mathcal{F} -stochastic intensity of N if, for any $a, b \in \mathbb{R}$,

$$\mathbb{E}[N((a, b]) | \mathcal{F}_a] = \mathbb{E}\left[\int_a^b \lambda(t) dt | \mathcal{F}_a\right], \quad \text{a.s.}$$

Since one usually considers predictable stochastic intensities, we define the predictable σ -field. Given a filtration $\mathcal{G} := \{\mathcal{G}_t\}_{t \in \mathbb{R}} \subset \mathcal{A}$, we define the σ -field $\mathcal{P}(\mathcal{G})$ on $\mathbb{R} \times \Omega$ by

$$\mathcal{P}(\mathcal{G}) := \sigma\{(a, b] \times A : a, b \in \mathbb{R}, A \in \mathcal{G}_a\}.$$

We call $\mathcal{P}(\mathcal{G})$ predictable σ -field and say that a real-valued stochastic process $\{X(t)\}_{t \in \mathbb{R}}$ is \mathcal{G} -predictable if the mapping $X : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is $\mathcal{P}(\mathcal{G})$ -measurable. A typical \mathcal{G} -predictable process is a \mathcal{G} -adapted process with left-continuous trajectories.

2.2. *Point processes constructed by embedding in a bivariate Poisson process.* Hereafter, \bar{N} denotes a Poisson process on $\mathbb{R} \times \mathbb{R}_+$, defined on a probability space (Ω, \mathcal{A}, P) , with mean measure $dt dz$. Let $\mathcal{F}^{\bar{N}} := \{\mathcal{F}_t^{\bar{N}}\}_{t \in \mathbb{R}}$ be the natural filtration of \bar{N} , that is,

$$\mathcal{F}_t^{\bar{N}} := \sigma\{\bar{N}(A \times B) : A \in \mathcal{B}(\mathbb{R}), B \in \mathcal{B}(\mathbb{R}_+), A \subseteq (-\infty, t]\}.$$

Point processes with stochastic intensity may be constructed by embedding in a bivariate Poisson process as follows.

LEMMA 2.1. *Let $f, g : \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ be two nonnegative, $\mathcal{P}(\mathcal{F}^{\bar{N}})$ -measurable mappings such that*

$$\int_a^b |f(t) - g(t)| dt < \infty, \quad \text{a.s., for all } a, b \in \mathbb{R},$$

set $I_t := (\min\{f(t), g(t)\}, \max\{f(t), g(t)\}]$, $t \in \mathbb{R}$, and define the point process on \mathbb{R}

$$N(dt) := \bar{N}(dt \times I_t), \quad t \in \mathbb{R}.$$

Then N has $\mathcal{F}^{\bar{N}}$ -stochastic intensity $\{|f(t) - g(t)|\}_{t \in \mathbb{R}}$.

This result is an extension of the method proposed in [21] for the simulation of nonhomogeneous Poisson processes and was used, for example, in [7] and [22] to study the stability of various classes of point processes, including Hawkes processes.

Throughout this paper, we consider point processes N on \mathbb{R} defined by

$$(2.1) \quad N(dt) := \bar{N}(dt \times (0, \lambda(t)]),$$

where $\{\lambda(t)\}_{t \in \mathbb{R}}$ is a nonnegative process of the form

$$(2.2) \quad \lambda(t) := \varphi(t, \bar{N}|_{(-\infty, t)})$$

such that

$$(2.3) \quad \int_a^b \lambda(s) \, ds < \infty, \quad \text{a.s., for all } a, b \in \mathbb{R}.$$

Here, $\varphi : \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}_+$ is a measurable functional, \mathcal{N} denotes the space of simple and locally finite counting measures on $\mathbb{R} \times \mathbb{R}_+$ endowed with the vague topology (see, e.g., [13]) and, for simplicity, with a little abuse of notation, we denote by $\overline{N}|_{(-\infty, t)}$ the restriction of \overline{N} to $(-\infty, t) \times \mathbb{R}_+$, that is,

$$\overline{N}|_{(-\infty, t)}(A) := \overline{N}(A \cap ((-\infty, t) \times \mathbb{R}_+)), \quad A \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+).$$

Since the process $\{\overline{N}|_{(-\infty, t)}(A)\}_{t \in \mathbb{R}}$ is $\mathcal{F}^{\overline{N}}$ -adapted and left-continuous the mapping

$$(t, \omega) \rightarrow \overline{N}(\omega)|_{(-\infty, t)}(A)$$

is $\mathcal{P}(\mathcal{F}^{\overline{N}})$ -measurable for any fixed $A \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+)$. Therefore, $\{\lambda(t)\}_{t \in \mathbb{R}}$ is $\mathcal{F}^{\overline{N}}$ -predictable (see, e.g., Remark 1 in [22]). Consequently, by Lemma 2.1 we deduce that N defined by (2.1), (2.2) and (2.3) has $\mathcal{F}^{\overline{N}}$ -stochastic intensity $\{\lambda(t)\}_{t \in \mathbb{R}}$.

As we shall see more in detail later on, Hawkes processes may be constructed by embedding in a bivariate Poisson process; see [7].

2.3. *The finite difference operator on the Poisson space and a Clark–Ocone type representation formula.* Given a measurable functional $\psi : \mathcal{N} \rightarrow \mathbb{R}$, we define the finite difference operator D by

$$D_{(t,z)}\psi(\overline{N}) := \psi(\overline{N} + \varepsilon_{(t,z)}) - \psi(\overline{N}),$$

where $\varepsilon_{(t,z)}$ denotes the Dirac measure at $(t, z) \in \mathbb{R} \times \mathbb{R}_+$. We also define the σ -field

$$\mathcal{F}_t^{\overline{N}} := \sigma\{\overline{N}(A \times B) : A \in \mathcal{B}(\mathbb{R}), B \in \mathcal{B}(\mathbb{R}_+), A \subseteq (-\infty, t)\}, \quad t \in \mathbb{R}.$$

The following Clark–Ocone type representation formula holds; see Theorem 1.1 in [20] (see also Lemma 1.3 in [40]).

LEMMA 2.2. *For any measurable functional $\psi : \mathcal{N} \rightarrow \mathbb{R}$ such that $\psi(\overline{N}) \in L^2(\Omega, dP)$, we have*

$$\psi(\overline{N}) - \mathbb{E}[\psi(\overline{N})] = \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{E}[D_{(t,z)}\psi(\overline{N}) | \mathcal{F}_t^{\overline{N}}] (\overline{N}(dt \times dz) - dt \, dz).$$

As pointed out in [20] and [40], we can (and we will) work with a $\mathcal{P}(\mathcal{F}^{\overline{N}}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable version of the conditional expectation $\mathbb{E}[D_{(t,z)}\psi(\overline{N}) | \mathcal{F}_t^{\overline{N}}]$.

3. Gaussian approximation of the first chaos of point processes with stochastic intensity. In this section, we provide a bound for the Wasserstein distance between a standard normal random variable Z and the first chaos

$$(3.1) \quad \delta(u) := \int_{\mathbb{R}} u(t)(N(dt) - \lambda(t) dt),$$

being $u : \mathbb{R} \rightarrow \mathbb{R}$ a measurable function and N defined by (2.1), (2.2) and (2.3). We recall that, given two random variables X, Y defined on the same probability space, the Wasserstein distance between X and Y is

$$d_W(X, Y) := \sup_{h \in \text{Lip}(1)} |E[h(X)] - E[h(Y)]|,$$

where $\text{Lip}(1)$ denotes the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1. We also recall that the topology induced by d_W on the class of probability measures over \mathbb{R} is finer than the topology of weak convergence (see, e.g., [14]).

Following [29], we give a general bound for $d_W(X, Z)$, where X is an integrable random variable. Given $h \in \text{Lip}(1)$, it turns out that there exists a twice differentiable function $f_h : \mathbb{R} \rightarrow \mathbb{R}$ so that

$$(3.2) \quad h(x) - E[h(Z)] = f'_h(x) - x f_h(x), \quad x \in \mathbb{R}.$$

For a function $g : \mathbb{R} \rightarrow \mathbb{R}$, we define $\|g\|_\infty := \sup_{x \in \mathbb{R}} |g(x)|$. Equation (3.2) is called Stein’s equation [38] and the function f_h has the following properties:

$$\|f_h\|_\infty \leq 2\|h'\|_\infty, \quad \|f'_h\|_\infty \leq \sqrt{2/\pi}\|h'\|_\infty, \quad \|f_h\|_\infty \leq 2\|h'\|_\infty;$$

see [11], Lemma 2.4. Since $\|h'\|_\infty \leq 1$ (indeed h has Lipschitz constant less than or equal to 1), letting \mathcal{F}_W denote the class of twice differentiable functions f so that $\|f\|_\infty \leq 2, \|f'\|_\infty \leq \sqrt{2/\pi}$ and $\|f''\|_\infty \leq 2$, we have

$$(3.3) \quad d_W(X, Z) \leq \sup_{f \in \mathcal{F}_W} |E[f'(X) - Xf(X)]|.$$

Note that the right-hand side of (3.3) is finite since the functions f, f' are bounded and X is integrable.

The following upper bound extends Corollary 3.4 in [29] to a class of not necessarily Poisson processes.

THEOREM 3.1. *Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that*

$$(3.4) \quad E\left[\int_{\mathbb{R}} |u(t)|\lambda(t) dt\right] < \infty,$$

$$(3.5) \quad E\left[\int_{\mathbb{R}} |u(t)|^2\lambda(t) dt\right] < \infty,$$

$$(3.6) \quad \int_{\mathbb{R} \times \mathbb{R}_+} \left(\int_t^\infty |u(s)|E[|D_{(t,z)}\lambda(s)|] ds\right) dt dz < \infty,$$

$$(3.7) \quad \int_{\mathbb{R} \times \mathbb{R}_+} \left(\int_t^\infty |u(s)|^2 \mathbb{E}[|D_{(t,z)}\lambda(s)|] ds \right) dt dz < \infty$$

and

$$(3.8) \quad \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)|^2 \left(\int_t^\infty |u(s)| \mathbb{E}[\mathbb{1}_{(0,\lambda(t))}(z) |D_{(t,z)}\lambda(s)|] ds \right) dt dz < \infty.$$

In addition, assume that, for $dx dy$ -almost all $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, the random function $|D_{(t,z)}\lambda(\cdot)|$ is a.s. locally integrable on (t, ∞) with respect to the Lebesgue measure. Then

$$(3.9) \quad \begin{aligned} & d_W(\delta(u), Z) \\ & \leq \sqrt{2/\pi} \mathbb{E} \left[\left| 1 - \int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right| \right] + \mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^3 \lambda(t) dt \right] \\ & + 2\sqrt{2/\pi} \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \left(\int_t^{+\infty} |u(s)| \mathbb{E}[\mathbb{1}_{(0,\lambda(t))}(z) |D_{(t,z)}\lambda(s)|] ds \right) dt dz \\ & + \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \left(\int_t^{+\infty} |u(s)|^2 \mathbb{E}[\mathbb{1}_{(0,\lambda(t))}(z) |D_{(t,z)}\lambda(s)|] ds \right) dt dz, \end{aligned}$$

where $\delta(u)$ is defined by (3.1).

REMARK 3.2. Note that if the function u is bounded, then conditions (3.4) and (3.6) imply (3.5), (3.7) and (3.8).

PROOF OF THEOREM 3.1. We may assume

$$(3.10) \quad \mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^3 \lambda(t) dt \right] < \infty,$$

$$(3.11) \quad \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \left(\int_t^{+\infty} |u(s)| \mathbb{E}[\mathbb{1}_{(0,\lambda(t))}(z) |D_{(t,z)}\lambda(s)|] ds \right) dt dz < \infty$$

and

$$(3.12) \quad \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \left(\int_t^{+\infty} |u(s)|^2 \mathbb{E}[\mathbb{1}_{(0,\lambda(t))}(z) |D_{(t,z)}\lambda(s)|] ds \right) dt dz < \infty.$$

Indeed, if one of the above terms is equal to infinity, then the claim is trivially true. We have

$$\begin{aligned} \delta(u) &= \int_{\mathbb{R}} u(t)(N(dt) - \lambda(t) dt) = \int_{\mathbb{R}} u(t)(\overline{N}(dt \times (0, \lambda(t))) - \lambda(t) dt) \\ &= \int_{\mathbb{R} \times \mathbb{R}_+} u(t) \mathbb{1}_{(0,\lambda(t))}(z) (\overline{N}(dt \times dz) - dt dz). \end{aligned}$$

For any $f \in \mathcal{F}_W$, we have $f(\delta(u)) \in L^2(\Omega, dP)$ since f is bounded. So by Lemma 2.2 we deduce

$$f(\delta(u)) - \mathbb{E}[f(\delta(u))] = \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{E}[D_{(t,z)} f(\delta(u)) | \mathcal{F}_{t-}^{\overline{N}}] (\overline{N}(dt \times dz) - dt dz).$$

For ease of notation, we set

$$g_1(t, \omega, z) := u(t)\mathbb{1}_{(0, \lambda(t, \omega)]}(z) \quad \text{and} \quad g_2(t, \omega, z) := \mathbb{E}[D_{(t,z)}f(\delta(u)) | \mathcal{F}_t^{\bar{N}}](\omega).$$

By the arguments at the end of Section 2.2 and the comment after the statement of Lemma 2.2, we have that g_1 and g_2 are $\mathcal{P}(\mathcal{F}^{\bar{N}}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. Note that, due to assumptions (3.4) and (3.5), g_1 is integrable and square integrable with respect to $dt dz dP(\omega)$. We shall check later on that

$$(3.13) \quad g_2 \text{ is integrable and square integrable with respect to } dt dz dP(\omega).$$

So by Theorem 3 in [8] [formulas (19) and (20)], we have

$$\begin{aligned} \mathbb{E}[\delta(u)f(\delta(u))] &= \mathbb{E}[\delta(u)(f(\delta(u)) - \mathbb{E}[f(\delta(u))])] \\ &= \mathbb{E}\left[\left(\int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z)(\bar{N}(dt \times dz) - dt dz)\right) \right. \\ &\quad \left. \times \left(\int_{\mathbb{R} \times \mathbb{R}_+} g_2(t, z)(\bar{N}(dt \times dz) - dt dz)\right)\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z)g_2(t, z) dt dz\right]. \end{aligned}$$

By the Taylor formula, we deduce

$$(3.14) \quad D_{(t,z)}f(\delta(u)) = f(\delta(u) + D_{(t,z)}\delta(u)) - f(\delta(u))$$

$$(3.15) \quad = f'(\delta(u))D_{(t,z)}\delta(u) + R(D_{(t,z)}\delta(u)),$$

where the rest R satisfies $|R(y)| \leq y^2$ since $\|f''\|_\infty \leq 2$. Since f is bounded, by (3.14) we have that g_2 is a.s. bounded, and so by the standard properties of the conditional expectation [note that $\lambda(t)$ is $\mathcal{F}_t^{\bar{N}}$ -measurable] and Fubini's theorem we deduce

$$\mathbb{E}[\delta(u)f(\delta(u))] = \mathbb{E}\left[\int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z)D_{(t,z)}f(\delta(u)) dt dz\right].$$

Consequently, by (3.15),

$$\begin{aligned} &|\mathbb{E}[f'(\delta(u)) - \delta(u)f(\delta(u))]| \\ &= \left| \mathbb{E}\left[f'(\delta(u)) - \int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z)D_{(t,z)}f(\delta(u)) dt dz \right] \right| \\ &\leq \left| \mathbb{E}\left[f'(\delta(u)) \left(1 - \int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z)D_{(t,z)}\delta(u) dt dz \right) \right] \right| \\ &\quad + \left| \mathbb{E}\left[\int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z)R(D_{(t,z)}\delta(u)) dt dz \right] \right| \end{aligned}$$

$$\begin{aligned} &\leq \sqrt{2/\pi} \mathbb{E} \left[\left| 1 - \int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z) D_{(t,z)} \delta(u) \, dt \, dz \right| \right] \\ &+ \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |D_{(t,z)} \delta(u)|^2 \, dt \, dz \right]. \end{aligned}$$

Therefore, using the basic inequality (3.3), we have

$$\begin{aligned} d_W(\delta(u), Z) &\leq \sqrt{2/\pi} \mathbb{E} \left[\left| 1 - \int_{\mathbb{R} \times \mathbb{R}_+} g_1(t, z) D_{(t,z)} \delta(u) \, dt \, dz \right| \right] \\ &+ \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |D_{(t,z)} \delta(u)|^2 \, dt \, dz \right]. \end{aligned}$$

We shall check later on that

$$(3.16) \quad \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |D_{(t,z)} \delta(u)| \, dt \, dz \right] < \infty$$

and

$$(3.17) \quad \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |D_{(t,z)} \delta(u)|^2 \, dt \, dz \right] < \infty.$$

So the above upper bound on $d_W(\delta(u), Z)$ is nontrivial. For $dx \, dy$ -almost all $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, we have

$$D_{(t,z)} \delta(u) = D_{(t,z)} \left(\int_{\mathbb{R} \times \mathbb{R}_+} g_1(s, v) \bar{N}(ds \times dv) \right) - D_{(t,z)} \left(\int_{\mathbb{R} \times \mathbb{R}_+} g_1(s, v) \, ds \, dv \right).$$

Computing separately these two finite differences and writing $\varphi_t(\bar{N}|_{(-\infty, t)})$ in place of $\varphi(t, \bar{N}|_{(-\infty, t)})$ for ease of notation, we have

$$\begin{aligned} &D_{(t,z)} \left(\int_{\mathbb{R} \times \mathbb{R}_+} g_1(s, v) \bar{N}(ds \times dv) \right) \\ &= D_{(t,z)} \left(\int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s \leq t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) \bar{N}(ds \times dv) \right. \\ &\quad \left. + \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s > t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) \bar{N}(ds \times dv) \right) \\ &= \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s \leq t} u(s) \mathbb{1}_{(0, \varphi_s((\bar{N} + \varepsilon_{(t,z)})|_{(-\infty, s)})]}(v) (\bar{N} + \varepsilon_{(t,z)})(ds \times dv) \\ &\quad + \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s > t} u(s) \mathbb{1}_{(0, \varphi_s((\bar{N} + \varepsilon_{(t,z)})|_{(-\infty, s)})]}(v) (\bar{N} + \varepsilon_{(t,z)})(ds \times dv) \\ &\quad - \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s \leq t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) \bar{N}(ds \times dv) \\ &\quad - \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s > t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) \bar{N}(ds \times dv) \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s \leq t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) (\bar{N} + \varepsilon_{(t, z)})(ds \times dv) \\
 &\quad + \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s > t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)} + \varepsilon_{(t, z)})]}(v) \bar{N}(ds \times dv) \\
 &\quad - \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s \leq t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) \bar{N}(ds \times dv) \\
 &\quad - \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{1}_{s > t} u(s) \mathbb{1}_{(0, \varphi_s(\bar{N}|_{(-\infty, s)})]}(v) \bar{N}(ds \times dv) \\
 &= g_1(t, z) + \int_{(t, \infty)} u_{(t, z)}(s) N_{(t, z)}(ds),
 \end{aligned}$$

where for $s > t$

$$\begin{aligned}
 u_{(t, z)}(s) &:= \text{sign}(\varphi_s(\bar{N}|_{(-\infty, s)} + \varepsilon_{(t, z)}) - \varphi_s(\bar{N}|_{(-\infty, s)})) u(s) \\
 &= \text{sign}(D_{(t, z)} \lambda(s)) u(s),
 \end{aligned}$$

$$\begin{aligned}
 N_{(t, z)}(ds) &:= \bar{N}(ds \times (\varphi_s(\bar{N}|_{(-\infty, s)} + \varepsilon_{(t, z)}) \wedge \varphi_s(\bar{N}|_{(-\infty, s)}), \\
 &\quad \varphi_s(\bar{N}|_{(-\infty, s)} + \varepsilon_{(t, z)}) \vee \varphi_s(\bar{N}|_{(-\infty, s)})).
 \end{aligned}$$

Here, for ease of notation, we denoted by $a \wedge b$ and $a \vee b$ the minimum and the maximum between $a, b \in \mathbb{R}$, respectively. Moreover,

$$\begin{aligned}
 D_{(t, z)} \left(\int_{\mathbb{R} \times \mathbb{R}_+} g_1(s, v) ds dv \right) &= \int_t^{+\infty} u(s) D_{(t, z)} \lambda(s) ds \\
 &= \int_t^{+\infty} u_{(t, z)}(s) |D_{(t, z)} \lambda(s)| ds.
 \end{aligned}$$

Therefore,

$$(3.18) \quad D_{(t, z)} \delta(u) = g_1(t, z) + \delta_{(t, z)}(u),$$

where

$$\delta_{(t, z)}(u) := \int_{(t, \infty)} u_{(t, z)}(s) (N_{(t, z)}(ds) - |D_{(t, z)} \lambda(s)| ds).$$

Combining (3.18) with the previous bound on $d_W(\delta(u), Z)$, we deduce

$$\begin{aligned}
 (3.19) \quad d_W(\delta(u), Z) &\leq \sqrt{2/\pi} \mathbb{E} \left[\left| 1 - \int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right| \right] \\
 &\quad + \sqrt{2/\pi} \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \mathbb{1}_{(0, \lambda(t)]}(z) |\delta_{(t, z)}(u)| dt dz \right] \\
 &\quad + \mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^3 \lambda(t) dt \right]
 \end{aligned}$$

$$\begin{aligned}
 &+ 2\mathbb{E}\left[\int_{\mathbb{R}\times\mathbb{R}_+} |u(t)|u(t)\mathbb{1}_{(0,\lambda(t)]}(z)\delta_{(t,z)}(u) dt dz\right] \\
 &+ \mathbb{E}\left[\int_{\mathbb{R}\times\mathbb{R}_+} |u(t)|\mathbb{1}_{(0,\lambda(t)]}(z)|\delta_{(t,z)}(u)|^2 dt dz\right].
 \end{aligned}$$

We shall check later on that

$$(3.20) \quad \mathbb{E}\left[\int_{\mathbb{R}\times\mathbb{R}_+} |g_1(t, z)|^2 |D_{(t,z)}\delta(u)| dt dz\right] < \infty,$$

and so by (3.5), (3.10), (3.16), (3.17), (3.18) and (3.20) the bound (3.19) is nontrivial. By Lemma 2.1, for $dx dy$ -almost all $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, the point process $N_{(t,z)}$ on (t, ∞) has $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -stochastic intensity $\{|D_{(t,z)}\lambda(s)|\}_{s>t}$. Indeed, the mapping

$$(t, \infty) \times \Omega \ni (s, \omega) \mapsto |D_{(t,z)}\lambda(s, \omega)| \in \mathbb{R}$$

is $\mathcal{P}(\{\mathcal{F}_s^{\bar{N}}\}_{s>t})$ -measurable and (by assumption), for P -almost all ω , it is locally integrable in s with respect to the Lebesgue measure. We note that

$$\mathbb{1}_{(0,\lambda(t)]}(z)\delta_{(t,z)}(u) = \int_{(t,\infty)} \mathbb{1}_{(0,\lambda(t)]}(z)u_{(t,z)}(s)(N_{(t,z)}(ds) - |D_{(t,z)}\lambda(s)| ds)$$

and the mapping

$$(t, \infty) \times \Omega \ni (s, \omega) \rightarrow \mathbb{1}_{(0,\lambda(t,\omega)]}(z)u_{(t,z)}(s, \omega)$$

is $\mathcal{P}(\{\mathcal{F}_s^{\bar{N}}\}_{s>t})$ -measurable. By (3.6) and (3.7), we have

$$\int_t^{+\infty} |u(s)|\mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)|D_{(t,z)}\lambda(s)|] ds < \infty,$$

for $dx dy$ -almost all $(t, z) \in \mathbb{R} \times \mathbb{R}_+$

and

$$\int_t^{+\infty} |u(s)|^2\mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)|D_{(t,z)}\lambda(s)|] ds < \infty,$$

for $dx dy$ -almost all $(t, z) \in \mathbb{R} \times \mathbb{R}_+$.

Therefore, by Theorem 3 in [8] [formulas (19) and (20)], for $dx dy$ -almost all $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, we have

$$(3.21) \quad \mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)\delta_{(t,z)}(u)] = 0$$

and

$$(3.22) \quad \mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)|\delta_{(t,z)}(u)|^2] = \int_t^\infty |u(s)|^2\mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)|D_{(t,z)}\lambda(s)|] ds.$$

By the triangular inequality and formula (19) in [8], we have

$$(3.23) \quad \mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)|\delta_{(t,z)}(u)|] \leq 2 \int_t^\infty |u(s)|\mathbb{E}[\mathbb{1}_{(0,\lambda(t)]}(z)|D_{(t,z)}\lambda(s)|] ds.$$

Inequality (3.9) follows combining (3.19) with (3.21), (3.22) and (3.23).

It remains to prove the integrability conditions (3.13), (3.16), (3.17) and (3.20). Since $f \in \mathcal{F}_W$, then it is Lipschitz continuous with Lipschitz constant less than or equal to 1. Therefore, by (3.14) we have $|D_{(t,z)} f(\delta(u))| \leq |D_{(t,z)} \delta(u)|$, and so to prove (3.13) it suffices to check

$$\int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{E}[|D_{(t,z)} \delta(u)|] dt dz < \infty \quad \text{and} \quad \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{E}[|D_{(t,z)} \delta(u)|^2] dt dz < \infty.$$

Using relation (3.18) and formula (19) in [8], we have

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{E}[|D_{(t,z)} \delta(u)|] dt dz \\ & \leq \mathbb{E} \left[\int_{\mathbb{R}} |u(t)| \lambda(t) dt \right] + 2\mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} \left(\int_t^\infty |u(s)| |D_{(t,z)} \lambda(s)| ds \right) dt dz \right] \end{aligned}$$

and this latter term is finite due to assumptions (3.4) and (3.6). Using again relation (3.18) and formula (20) in [8], we have

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}_+} \mathbb{E}[|D_{(t,z)} \delta(u)|^2] dt dz \\ & \leq 2\mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right] + 2\mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} \left(\int_t^\infty |u(s)|^2 |D_{(t,z)} \lambda(s)| ds \right) dt dz \right] \end{aligned}$$

and this latter term is finite due to assumptions (3.5) and (3.7). By (3.18) and (3.23), we have

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |D_{(t,z)} \delta(u)| dt dz \right] \\ & \leq \mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right] + \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |\delta_{(t,z)}(u)| dt dz \right] \\ & \leq \mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right] \\ & \quad + 2 \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \left(\int_t^\infty |u(s)| \mathbb{E}[\mathbb{1}_{(0, \lambda(t))}(z) |D_{(t,z)} \lambda(s)|] ds \right) dt dz, \end{aligned}$$

and (3.16) follows by (3.5) and (3.11). Similarly, by (3.18) and (3.22) we have

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |D_{(t,z)} \delta(u)|^2 dt dz \right] \\ & \leq 2\mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^3 \lambda(t) dt \right] + 2\mathbb{E} \left[\int_{\mathbb{R} \times \mathbb{R}_+} |g_1(t, z)| |\delta_{(t,z)}(u)|^2 dt dz \right] \\ & = 2\mathbb{E} \left[\int_{\mathbb{R}} |u(t)|^3 \lambda(t) dt \right] \\ & \quad + 2 \int_{\mathbb{R} \times \mathbb{R}_+} |u(t)| \left(\int_t^\infty |u(s)|^2 \mathbb{E}[\mathbb{1}_{(0, \lambda(t))}(z) |D_{(t,z)} \lambda(s)|] ds \right) dt dz, \end{aligned}$$

and (3.17) follows by (3.10) and (3.12). Finally, (3.20) may be checked similarly to (3.16), but using (3.10) and (3.8) in place of (3.5) and (3.11), respectively. The proof is completed. \square

4. Application to stationary nonlinear Hawkes processes. A nonlinear Hawkes process with parameters (ϕ, h) is a point process N on \mathbb{R} with \mathcal{F}^N -stochastic intensity of the form

$$(4.1) \quad t \mapsto \phi \left(\int_{(-\infty, t)} h(t-s)N(ds) \right), \quad t \in \mathbb{R},$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ are measurable functions. A particular case is the self-exciting process (or linear Hawkes process) with parameters (ν, h) , for which $\phi(x) := \nu + x$, for some constant $\nu > 0$, and h is nonnegative.

In the seminal paper [7], the authors proved that if ϕ is Lipschitz continuous with Lipschitz constant α such that $\alpha\mu < 1$, where $\mu := \|h\|_{L^1(\mathbb{R}_+, dx)}$, then there exists a unique stationary distribution of N with dynamics (4.1) and finite intensity $\lambda := E[N((0, 1])]$. The stationary solution is constructed by embedding in a bivariate Poisson process, as follows. Define recursively the processes $\lambda^{(0)} \equiv 0$,

$$N^{(n)}(dt) := \bar{N}(dt \times (0, \lambda^{(n)}(t)])$$

and

$$\lambda^{(n+1)}(t) := \phi \left(\int_{(-\infty, t)} h(t-s)N^{(n)}(ds) \right),$$

$n \geq 0, t \in \mathbb{R}$, where \bar{N} is a Poisson process on $\mathbb{R} \times \mathbb{R}_+$ with mean measure $dt dz$. It turns out that, for any fixed $n \geq 0$, the point process $N^{(n)}$ is stationary and $\{\lambda^{(n)}(t)\}_{t \in \mathbb{R}}$ is an $\mathcal{F}^{\bar{N}}$ -stochastic intensity of $N^{(n)}$. It is then proved that $N^{(n)}((a, b]) \rightarrow N((a, b])$ and $\lambda^{(n)}(t) \rightarrow \lambda(t)$ a.s., for any $a, b, t \in \mathbb{R}$, and the limiting process is stationary and satisfies

$$N(dt) = \bar{N}(dt \times (0, \lambda(t)]), \quad \lambda(t) = \phi \left(\int_{(-\infty, t)} h(t-s)N(ds) \right), \quad t \in \mathbb{R}$$

and $\lambda \in (0, \infty)$. Note that $\lambda(t) = \varphi(t, \bar{N}|_{(-\infty, t)})$, for some functional $\varphi : \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}_+$ satisfying

$$(4.2) \quad \begin{aligned} &\varphi(t, \bar{N}|_{(-\infty, t)}) \\ &:= \phi \left(\int_{(-\infty, t) \times \mathbb{R}_+} \mathbb{1}_{(0, \varphi(s, \bar{N}|_{(-\infty, s)})]}(z)h(t-s)\bar{N}(ds \times dz) \right). \end{aligned}$$

Then by Lemma 2.1 it follows that N is a point process on \mathbb{R} with \mathcal{F}^N -stochastic intensity $\{\lambda(t)\}_{t \in \mathbb{R}}$. In conclusion, N is a stationary nonlinear Hawkes process with parameters (ϕ, h) and finite intensity.

4.1. *Explicit Gaussian bound for the first chaos of nonlinear Hawkes processes.* The following explicit Gaussian bound holds.

THEOREM 4.1. *Assume that $h : \mathbb{R}_+ \rightarrow [0, \infty)$ is locally bounded and $\phi : [0, \infty) \rightarrow [0, \infty)$, $\phi(0) > 0$, is nondecreasing and Lipschitz continuous, with Lipschitz constant α such that $\alpha\mu < 1$. Let N be a stationary nonlinear Hawkes process with parameters (ϕ, h) and finite intensity $\lambda \in (0, \infty)$. If $u \in L^1(\mathbb{R}, dx)$, then*

$$(4.3) \quad d_W(\delta(u), Z) \leq \mathfrak{N},$$

where

$$(4.4) \quad \begin{aligned} \mathfrak{N} := & \sqrt{2/\pi} \max \left\{ \left| 1 - \phi(0) \|u\|_{L^2(\mathbb{R}, dx)}^2, \left| 1 - \frac{\phi(0)}{1 - \alpha\mu} \|u\|_{L^2(\mathbb{R}, dx)}^2 \right| \right\} \\ & + \frac{\phi(0)}{1 - \alpha\mu} \|u\|_{L^3(\mathbb{R}, dx)}^3 + \frac{2\sqrt{2/\pi}\phi(0)\alpha\mu(2 - \alpha\mu)}{(1 - \alpha\mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^2 \\ & + \frac{\phi(0)\alpha\mu}{(1 - \alpha\mu)^2} \|u\|_{L^2(\mathbb{R}, dx)} \|u^2\|_{L^2(\mathbb{R}, dx)}. \end{aligned}$$

REMARK 4.2. Suppose that ϕ and h satisfy the assumptions of Theorem 4.1. One says that N' is a (nonstationary) nonlinear Hawkes process on \mathbb{R}_+ with parameters (ϕ, h) and initial condition \mathcal{IC} (see [7]) if N' has stochastic intensity

$$\lambda'(t) := \phi \left(\int_{(-\infty, t)} h(t-s) N'(ds) \right), \quad t > 0$$

on \mathbb{R}_+ and N' satisfies the condition \mathcal{IC} on \mathbb{R}_- . If $u' \in L^1(\mathbb{R}_+, dx)$, following the lines of the proof of Theorem 4.1, one can show (without major difficulties) that the Gaussian bound (4.3) holds replacing $\delta(u)$ with

$$\delta'(u') := \int_{\mathbb{R}_+} u'(t) (N'(dt) - \lambda'(t) dt)$$

and replacing u with u' (and \mathbb{R} with \mathbb{R}_+) in the expression of \mathfrak{N} .

Let N be a stationary nonlinear Hawkes process with parameters (ϕ, h) which satisfy the assumptions of Theorem 4.1. Since $h \geq 0$ and ϕ is nondecreasing and Lipschitz continuous with Lipschitz constant α , we have

$$(4.5) \quad \phi(0) \leq \lambda(t) \leq \phi(0) + \alpha \int_{(-\infty, t)} h(t-s) N(ds), \quad t \in \mathbb{R}.$$

Taking the mean, we deduce $\phi(0) \leq \lambda \leq \phi(0) + \lambda\alpha\mu$, and so

$$(4.6) \quad \phi(0) \leq \lambda \leq \frac{\phi(0)}{1 - \alpha\mu}.$$

Given an integrable function u , one may think of approximating the quantity

$$\int_{\mathbb{R}} u(t)\lambda(t) dt$$

with its expectation $\lambda \int_{\mathbb{R}} u(t) dt$. Unfortunately, in general the intensity λ is not known explicitly (unless we consider the linear case which is treated in the next section). However, it may be estimated by, for example, Monte Carlo simulation (using the ergodic theorem). For a fixed positive constant $\widehat{\lambda} \in [\phi(0), \phi(0)(1 - \alpha\mu)^{-1}]$, to be interpreted as an estimate of the intensity λ , we define the “approximated” first chaos by

$$\delta_a(u) := \int_{\mathbb{R}} u(t)(N(dt) - \widehat{\lambda} dt).$$

The following explicit Gaussian bound holds.

THEOREM 4.3. *Under assumptions and notation of Theorem 4.1, we have*

$$(4.7) \quad d_W(\delta_a(u), Z) \leq \mathfrak{N} + \frac{2\phi(0)\alpha\mu}{1 - \alpha\mu} \|u\|_{L^1(\mathbb{R}, dx)}.$$

REMARK 4.4. In the case of stationary linear Hawkes processes, the bounds (4.3) and (4.7) may be (slightly) improved due to the knowledge of the intensity λ , see Theorem 5.1. Moreover, alternative bounds may be obtained by using the spectral theory of self-exciting processes; see Theorem 5.2.

The proofs of Theorems 4.1 and 4.3 are given in Section 4.3.

4.2. *A quantitative central limit theorem for nonlinear Hawkes processes.* The following quantitative central limit theorem in the Wasserstein distance is an immediate consequence of Theorems 4.1 and 4.3.

COROLLARY 4.5. *For $\varepsilon > 0$, assume that $h_\varepsilon : \mathbb{R}_+ \rightarrow [0, \infty)$ is locally bounded and $\phi_\varepsilon : [0, \infty) \rightarrow [0, \infty)$, $\phi_\varepsilon(0) > 0$, is nondecreasing and Lipschitz continuous, with Lipschitz constant α_ε such that $\alpha_\varepsilon\mu_\varepsilon < 1$, where $\mu_\varepsilon := \int_0^\infty h_\varepsilon(t) dt$. Let N_ε be a stationary nonlinear Hawkes process with parameters $(\phi_\varepsilon, h_\varepsilon)$ and finite intensity $\lambda_\varepsilon \in \mathbb{R}_+$, and take $u_\varepsilon \in L^1(\mathbb{R}, dx)$. Then:*

(i)

$$(4.8) \quad d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \leq \mathfrak{N}_\varepsilon, \quad \varepsilon > 0$$

and

$$(4.9) \quad d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \leq \mathfrak{N}_\varepsilon + \frac{2\phi_\varepsilon(0)\alpha_\varepsilon\mu_\varepsilon}{1 - \alpha_\varepsilon\mu_\varepsilon} \|u_\varepsilon\|_{L^1(\mathbb{R}, dx)}, \quad \varepsilon > 0.$$

Here, \mathfrak{N}_ε is defined as \mathfrak{N} in (4.4), with $\phi_\varepsilon, u_\varepsilon, \alpha_\varepsilon$ and μ_ε in place of ϕ, u, α and μ , respectively,

$$\begin{aligned} \delta^{(\varepsilon)}(u_\varepsilon) &:= \int_{\mathbb{R}} u_\varepsilon(t)(N_\varepsilon(dt) - \lambda_\varepsilon(t) dt), \\ \lambda_\varepsilon(t) &:= \phi_\varepsilon\left(\int_{(-\infty,t)} h_\varepsilon(t-s)N_\varepsilon(ds)\right), \\ \delta_a^{(\varepsilon)}(u_\varepsilon) &:= \int_{\mathbb{R}} u_\varepsilon(t)(N_\varepsilon(dt) - \widehat{\lambda}_\varepsilon dt) \end{aligned}$$

and $\widehat{\lambda}_\varepsilon \in [\phi_\varepsilon(0), \phi_\varepsilon(0)(1 - \alpha_\varepsilon\mu_\varepsilon)^{-1}]$.

(ii) If, as $\varepsilon \rightarrow 0$,

$$(4.10) \quad \alpha_\varepsilon\mu_\varepsilon \rightarrow 0,$$

$$(4.11) \quad \phi_\varepsilon(0)\|u_\varepsilon\|_{L^2(\mathbb{R},dx)}^2 \rightarrow 1,$$

$$(4.12) \quad \phi_\varepsilon(0)\|u_\varepsilon\|_{L^3(\mathbb{R},dx)}^3 \rightarrow 0,$$

$$(4.13) \quad \sqrt{\phi_\varepsilon(0)\alpha_\varepsilon\mu_\varepsilon}\|(u_\varepsilon)^2\|_{L^2(\mathbb{R},dx)} \rightarrow 0,$$

then

$$d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

If moreover, as $\varepsilon \rightarrow 0$,

$$(4.14) \quad \phi_\varepsilon(0)\alpha_\varepsilon\mu_\varepsilon\|u_\varepsilon\|_{L^1(\mathbb{R},dx)} \rightarrow 0,$$

then

$$d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

We conclude this subsection with an example.

EXAMPLE 4.6. Let $I_\varepsilon, \varepsilon > 0$, be a given family of bounded Borel sets, I_ε with Lebesgue measure ℓ_ε , and $\phi_\varepsilon : [0, \infty) \rightarrow [0, \infty), \phi_\varepsilon(0) > 0, \varepsilon > 0$, a family of nondecreasing and Lipschitz continuous functions with Lipschitz constant α_ε . Let $\mu_\varepsilon, \varepsilon > 0$, be a collection of positive numbers such that $\alpha_\varepsilon\mu_\varepsilon \in (0, 1), \varepsilon > 0$, and define the functions $h_\varepsilon(t) := \mu_\varepsilon f_\varepsilon(t), \varepsilon > 0, t > 0$, where f_ε is a locally bounded probability density (with respect to the Lebesgue measure) on $(0, \infty)$. Hereafter, we consider the family $N_\varepsilon, \varepsilon > 0$, of stationary nonlinear Hawkes processes with parameters $(\phi_\varepsilon, h_\varepsilon), \varepsilon > 0$, and the functions

$$u_\varepsilon(t) := \frac{1}{\sqrt{\frac{\phi_\varepsilon(0)\ell_\varepsilon}{1-\alpha_\varepsilon\mu_\varepsilon}}}\mathbb{1}_{I_\varepsilon}(t), \quad \varepsilon > 0, t \in \mathbb{R}.$$

We have

$$\|u_\varepsilon\|_{L^2(\mathbb{R}, dx)}^2 = \frac{1 - \alpha_\varepsilon \mu_\varepsilon}{\phi_\varepsilon(0)}, \quad \|u_\varepsilon\|_{L^3(\mathbb{R}, dx)}^3 = \left(\frac{1 - \alpha_\varepsilon \mu_\varepsilon}{\phi_\varepsilon(0)\ell_\varepsilon}\right)^{3/2} \ell_\varepsilon,$$

$$\|(u_\varepsilon)^2\|_{L^2(\mathbb{R}, dx)} = \frac{1 - \alpha_\varepsilon \mu_\varepsilon}{\phi_\varepsilon(0)\sqrt{\ell_\varepsilon}} \quad \text{and} \quad \|u_\varepsilon\|_{L^1(\mathbb{R}, dx)} = \left(\frac{1 - \alpha_\varepsilon \mu_\varepsilon}{\phi_\varepsilon(0)\ell_\varepsilon}\right)^{1/2} \ell_\varepsilon.$$

So by Corollary 4.5(i) we deduce

$$(4.15) \quad d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \leq \mathfrak{N}_\varepsilon, \quad \varepsilon > 0$$

and

$$(4.16) \quad d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \leq \mathfrak{N}_\varepsilon + \frac{2\sqrt{\phi_\varepsilon(0)\ell_\varepsilon\alpha_\varepsilon\mu_\varepsilon}}{\sqrt{1 - \alpha_\varepsilon\mu_\varepsilon}}, \quad \varepsilon > 0,$$

where

$$\mathfrak{N}_\varepsilon := \sqrt{2/\pi\alpha_\varepsilon\mu_\varepsilon} + \frac{2\sqrt{2/\pi\alpha_\varepsilon\mu_\varepsilon}(2 - \alpha_\varepsilon\mu_\varepsilon)}{1 - \alpha_\varepsilon\mu_\varepsilon} + \sqrt{\frac{1 - \alpha_\varepsilon\mu_\varepsilon}{\phi_\varepsilon(0)\ell_\varepsilon}} + \frac{\alpha_\varepsilon\mu_\varepsilon}{\sqrt{\phi_\varepsilon(0)\ell_\varepsilon(1 - \alpha_\varepsilon\mu_\varepsilon)}}.$$

If

$$(4.17) \quad \lim_{\varepsilon \rightarrow 0} \phi_\varepsilon(0)\ell_\varepsilon = +\infty \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon\mu_\varepsilon = 0,$$

then one may easily check conditions (4.10), (4.11), (4.12) and (4.13) and so $d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0$, as $\varepsilon \rightarrow 0$. To guarantee condition (4.14) and, therefore, $d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0$, as $\varepsilon \rightarrow 0$, we need to suppose

$$(4.18) \quad \alpha_\varepsilon\mu_\varepsilon = o\left(\frac{1}{\sqrt{\phi_\varepsilon(0)\ell_\varepsilon}}\right), \quad \text{as } \varepsilon \rightarrow 0.$$

Clearly, for specific choices of the sets I_ε , $\varepsilon > 0$, and the quantities $\phi_\varepsilon(0)$, α_ε and μ_ε , we can provide the rate of convergence to zero of the Wasserstein distances. For instance if, for $\varepsilon \in (0, 1)$, we take $I_\varepsilon = (0, 1/\varepsilon)$, $\phi_\varepsilon(0) = \nu$, $\alpha_\varepsilon = \alpha$, being ν and α positive constants, and $\mu_\varepsilon = \varepsilon$, then a straightforward computation shows that, as $\varepsilon \rightarrow 0$, the right-hand sides of (4.15) and (4.16) converge to a positive constant when divided by $\varepsilon^{1/2}$, and so

$$d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) = d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) = O(\sqrt{\varepsilon}), \quad \text{as } \varepsilon \rightarrow 0.$$

The bounds (4.15) and (4.16) may be used to construct confidence intervals for $\delta^{(\varepsilon)}(u_\varepsilon)$ and $\delta_a^{(\varepsilon)}(u_\varepsilon)$, respectively. For instance, let F_X denote the distribution function of a random variable X , assume (4.17), choose $b_i^{(\beta)}$, $i = 1, 2$, $\beta \in (0, 1/2)$, so that $b_1^{(\beta)} < b_2^{(\beta)}$ and

$$(4.19) \quad F_Z(b_1^{(\beta)}) \leq \beta/2 \quad \text{and} \quad F_Z(b_2^{(\beta)}) \geq 1 - \frac{\beta}{2}$$

and choose $\varepsilon > 0$ so small that $2\sqrt{\mathfrak{N}_\varepsilon} \leq \beta/2$, then

$$(4.20) \quad P(b_1^{(\beta)} < \delta^{(\varepsilon)}(u_\varepsilon) \leq b_2^{(\beta)}) \geq 1 - 2\beta.$$

Indeed, by (4.15) and the inequality

$$\sup_{x \in \mathbb{R}} |F_X(x) - F_Z(x)| \leq 2\sqrt{d_W(X, Z)}$$

(see, e.g., [11]), we have

$$|F_{\delta^{(\varepsilon)}(u_\varepsilon)}(b_i^{(\beta)}) - F_Z(b_i^{(\beta)})| \leq 2\sqrt{\mathfrak{N}_\varepsilon} \leq \beta/2, \quad i = 1, 2.$$

Relation (4.20) easily follows by this latter inequality and (4.19).

4.3. Proofs of Theorems 4.1 and 4.3.

PROOF OF THEOREM 4.1. We divide the proof in two main steps. In the first step, we prove

$$(4.21) \quad \begin{aligned} d_W(\delta(u), Z) &\leq \sqrt{2/\pi} \mathbb{E} \left[\left| 1 - \int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right| \right] \\ &\quad + \lambda \|u\|_{L^3(\mathbb{R}, dx)}^3 \\ &\quad + \frac{2\lambda\alpha\mu}{1 - \alpha\mu} \sqrt{2/\pi} \|u\|_{L^2(\mathbb{R}, dx)}^2 \\ &\quad + \frac{\lambda\alpha\mu}{1 - \alpha\mu} \|u\|_{L^2(\mathbb{R}, dx)} \|u^2\|_{L^2(\mathbb{R}, dx)}. \end{aligned}$$

In the second step, we complete the proof. If $u \notin L^2(\mathbb{R}, dx) \cap L^3(\mathbb{R}, dx) \cap L^4(\mathbb{R}, dx)$, then the claim is clearly true. So we shall assume $u \in L^2(\mathbb{R}, dx) \cap L^3(\mathbb{R}, dx) \cap L^4(\mathbb{R}, dx)$.

Step 1: Proof of (4.21). Hereafter, for ease of notation, we write $\varphi_t(\bar{N}|_{(-\infty, t)})$ in place of $\varphi(t, \bar{N}|_{(-\infty, t)})$, $t \in \mathbb{R}$. By (4.2), for $s, t \in \mathbb{R}$, we have

$$\begin{aligned} \lambda(s) &= \varphi_s(\bar{N}|_{(-\infty, s)}) \\ &= \phi \left(\int_{(-\infty, s) \times \mathbb{R}_+} \mathbb{1}_{u \leq t} h(s - u) \mathbb{1}_{(0, \varphi_u(\bar{N}|_{(-\infty, u)})]}(v) \bar{N}(du \times dv) \right. \\ &\quad \left. + \int_{(-\infty, s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s - u) \mathbb{1}_{(0, \varphi_u(\bar{N}|_{(-\infty, u)})]}(v) \bar{N}(du \times dv) \right). \end{aligned}$$

We shall show later on that $h \geq 0$ and ϕ nondecreasing imply

$$(4.22) \quad D_{(t, z)}\lambda(s) \geq 0, \quad \text{for } s, t \in \mathbb{R}, s > t \text{ and } z \in \mathbb{R}_+.$$

So, by the Lipschitz continuity of ϕ , for $s, t \in \mathbb{R}$, $s > t$, and $z \in \mathbb{R}_+$, we have

$$\begin{aligned}
 & 0 \leq D_{(t,z)}\lambda(s) \\
 & \leq \alpha \left(\int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u \leq t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N} + \varepsilon_{(t,z)})|_{(-\infty,u)})}(v) (\overline{N} + \varepsilon_{(t,z)})(du \times dv) \right. \\
 & \quad + \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N} + \varepsilon_{(t,z)})|_{(-\infty,u)})}(v) (\overline{N} + \varepsilon_{(t,z)})(du \times dv) \\
 & \quad - \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u \leq t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v) \overline{N}(du \times dv) \\
 & \quad \left. - \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v) \overline{N}(du \times dv) \right) \\
 & = \alpha \left(\int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u \leq t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v) (\overline{N} + \varepsilon_{(t,z)})(du \times dv) \right. \\
 & \quad + \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N}|_{(-\infty,u)} + \varepsilon_{(t,z)})}(v) (\overline{N} + \varepsilon_{(t,z)})(du \times dv) \\
 & \quad - \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u \leq t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v) \overline{N}(du \times dv) \\
 (4.23) \quad & \quad \left. - \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v) \overline{N}(du \times dv) \right) \\
 & = \alpha \left(h(s-t) \mathbb{1}_{(0, \varphi_t(\overline{N}|_{(-\infty,t)})}(z) \right. \\
 & \quad + \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N}|_{(-\infty,u)} + \varepsilon_{(t,z)})}(v) \overline{N}(du \times dv) \\
 & \quad \left. - \int_{(-\infty,s) \times \mathbb{R}_+} \mathbb{1}_{u > t} h(s-u) \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v) \overline{N}(du \times dv) \right) \\
 & = \alpha \left(h(s-t) \mathbb{1}_{(0, \varphi_t(\overline{N}|_{(-\infty,t)})}(z) \right. \\
 & \quad + \int_{(t,s) \times \mathbb{R}_+} h(s-u) (\mathbb{1}_{(0, \varphi_u(\overline{N}|_{(-\infty,u)} + \varepsilon_{(t,z)})}(v) \\
 & \quad \left. - \mathbb{1}_{(0, \varphi_u(\overline{N})|_{(-\infty,u)})}(v)) \overline{N}(du \times dv) \right) \\
 & = \alpha \left(h(s-t) \mathbb{1}_{(0, \varphi_t(\overline{N}|_{(-\infty,t)})}(z) + \int_{(t,s)} h(s-u) N_{(t,z)}(du) \right),
 \end{aligned}$$

where $N_{(t,z)}$ is the point process on (t, ∞) defined by

$$N_{(t,z)}(du) := \overline{N}(du \times (\varphi_u(\overline{N}|_{(-\infty,u)}), \varphi_u(\overline{N}|_{(-\infty,u)} + \varepsilon_{(t,z)}))).$$

The processes $\{\varphi_u(\bar{N}|_{(-\infty,u)} + \varepsilon_{(t,z)})\}_{u>t}$ and $\{\varphi_u(\bar{N}|_{(-\infty,u)})\}_{u>t}$ are $\{\mathcal{F}_u^{\bar{N}}\}_{u>t}$ -predictable, and we shall check later on that the mapping

$$(4.24) \quad (t, \infty) \ni u \mapsto E[D_{(t,z)}\lambda(u)] \in \mathbb{R} \text{ is locally bounded.}$$

Therefore,

$$\int_a^b D_{(t,z)}\lambda(u) du < \infty \quad \text{a.s., for any } a, b > t.$$

Consequently, by Lemma 2.1 we have that $N_{(t,z)}$ has $\{\mathcal{F}_u^{\bar{N}}\}_{u>t}$ -stochastic intensity $\{D_{(t,z)}\lambda(u)\}_{u>t}$. Taking the mean in (4.23), we deduce

$$E[D_{(t,z)}\lambda(s)] \leq \alpha \left(h(s-t)P(\lambda(t) \geq z) + \int_t^s h(s-u)E[D_{(t,z)}\lambda(u)] du \right).$$

Extending the definition of h for nonpositive times as $h(t) = 0, t \leq 0$, we rewrite the above inequality as

$$q_{(t,z)}(s) \leq p_{(t,z)}(s) + r * q_{(t,z)}(s), \quad s, t \in \mathbb{R}, z \in \mathbb{R}_+,$$

where for ease of notation we set $q_{(t,z)}(s) := E[D_{(t,z)}\lambda(s)], p_{(t,z)}(s) := \alpha h(s-t)P(\lambda(t) \geq z), r(s) := \alpha h(s)$ and $*$ denotes the convolution product between functions. Iterating this inequality, we deduce, for $n \geq 1$,

$$q_{(t,z)}(s) \leq \sum_{i=0}^{n-1} p_{(t,z)} * r^{*i}(s) + q_{(t,z)} * r^{*n}(s), \quad s, t \in \mathbb{R}, z \in \mathbb{R}_+,$$

where r^{*0} is by definition the Dirac delta function. By (4.24) and the stability condition $\alpha\mu < 1$, we deduce $q_{(t,z)} * r^{*n}(s) \rightarrow 0$, as $n \rightarrow \infty$, for any $t, s \in \mathbb{R}, z \in \mathbb{R}_+$. Indeed, for some constant $C_{t,z,s} > 0$,

$$\begin{aligned} q_{(t,z)} * r^{*n}(s) &= \int_{\mathbb{R}} r^{*n}(s-u)q_{(t,z)}(u) du = \int_t^s r^{*n}(s-u)q_{(t,z)}(u) du \\ &\leq C_{t,z,s} \int_{\mathbb{R}} r^{*n}(s-u) du \\ &\leq C_{t,z,s}(\alpha\mu)^n, \end{aligned}$$

where the latter inequality follows by a standard property of convolutions; see, for example, Theorem IV.15 in [10]. Therefore,

$$\begin{aligned} (4.25) \quad q_{(t,z)}(s) &\leq \sum_{i \geq 0} p_{(t,z)} * r^{*i}(s) \\ &= P(\lambda(t) \geq z) \sum_{i \geq 0} \alpha^{i+1} \int_{\mathbb{R}} h(s-u-t)h^{*i}(u) du \\ &= P(\lambda(t) \geq z) \sum_{i \geq 1} \alpha^i h^{*i}(s-t), \quad s, t \in \mathbb{R}, z \in \mathbb{R}_+. \end{aligned}$$

Consequently, for any f, g integrable and square integrable, defining $\check{f}(x) := f(-x)$, by the Cauchy–Schwarz inequality and the properties of the convolution product (see again Theorem IV.15 in [10]), we have

$$\begin{aligned}
 & \int_{\mathbb{R} \times \mathbb{R}_+} |f(t)| \left(\int_t^{+\infty} |g(s)| \mathbb{E}[\mathbb{1}_{(0, \lambda(t)]}(z) D_{(t,z)} \lambda(s)] ds \right) dt dz \\
 (4.26) \quad & \leq \lambda \sum_{i \geq 1} \alpha^i \int_{\mathbb{R}} |f(t)| (h^{*i} * |g|)(t) dt \\
 & \leq \lambda \|f\|_{L^2(\mathbb{R}, dx)} \sum_{i \geq 1} \alpha^i \|h^{*i} * |g|\|_{L^2(\mathbb{R}, dx)} \\
 & \leq \lambda \|f\|_{L^2(\mathbb{R}, dx)} \|g\|_{L^2(\mathbb{R}, dx)} \sum_{i \geq 1} \alpha^i \|h^{*i}\|_{L^1(\mathbb{R}, dx)} \\
 (4.27) \quad & \leq \lambda \|f\|_{L^2(\mathbb{R}, dx)} \|g\|_{L^2(\mathbb{R}, dx)} \sum_{i \geq 1} \alpha^i \mu^i \\
 & = \|f\|_{L^2(\mathbb{R}, dx)} \|g\|_{L^2(\mathbb{R}, dx)} \frac{\lambda \alpha \mu}{1 - \alpha \mu}.
 \end{aligned}$$

Assume for the moment that we may apply Theorem 3.1, then by (3.9) and the above inequality (applied first with $f = g = u$ and then with $f = u$ and $g = u^2$) we easily deduce (4.21).

It remains to prove (4.22), (4.24) and to check the assumptions of Theorem 3.1.

We first prove (4.24). Let $(t, z) \in \mathbb{R} \times \mathbb{R}_+$ be fixed. Since

$$\mathbb{E}[\lambda(u)] = \mathbb{E}[\varphi_u(\bar{N}|_{(-\infty, u)})] = \lambda \in (0, \infty)$$

for any $u \in \mathbb{R}$, to show (4.24) it suffices to prove that the map

$$u \mapsto \mathbb{E}[\varphi_u(\bar{N}|_{(-\infty, u)} + \varepsilon_{(t,z)})]$$

is locally bounded on (t, ∞) . We define recursively the processes $\lambda'_{(t,z)} \equiv 0$,

$$\begin{aligned}
 N'_{(t,z)}(ds) &= \bar{N}(ds \times (0, \lambda'_{(t,z)}(s)]), \\
 (4.28) \quad \lambda'_{(t,z)}(s) &= \phi \left(\int_{(-\infty, s)} h(s-u) (N'_{(t,z)}(du) \right. \\
 & \quad \left. + \varepsilon_{(t,z)}(du \times (0, \lambda'_{(t,z)}(u)])) \right), \quad n \geq 0, s > t.
 \end{aligned}$$

We are going to check by induction that, for any $n \geq 0$,

$$(4.29) \quad \int_a^b \lambda'_{(t,z)}(s) ds < \infty, \quad \text{a.s., for any } a, b > t$$

and $\{\lambda'_{(t,z)}(s)\}_{s>t}$ is $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -predictable. The basis of the induction is clearly verified. So assume the claim for $\lambda'_{(t,z)}$ and let $\{T'_{(t,z),m}\}_{m \in \mathbb{Z}}$ be the points of $N'_{(t,z)}$

on (t, ∞) . By Lemma 2.1, we have that $N'_{(t,z)}^{(n)}$ has $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -stochastic intensity $\{\lambda'_{(t,z)}^{(n)}(s)\}_{s>t}$. By the Lipschitz property of ϕ and the nonnegativity of h , we deduce

$$\begin{aligned}
 \lambda'^{(n+1)}_{(t,z)}(s) &\leq \phi(0) + \alpha \int_{(-\infty,s)} h(s-u) N'_{(t,z)}^{(n)}(du) \\
 &\quad + \alpha \int_{(-\infty,s)} h(s-u) \varepsilon_{(t,z)}(du \times (0, \lambda'^{(n)}_{(t,z)}(u))) \\
 &= \phi(0) + \alpha \sum_{m \in \mathbb{Z}} h(s - T'_{(t,z),m}{}^{(n)}) \mathbb{1}_{(-\infty,s)}(T'_{(t,z),m}{}^{(n)}) \\
 (4.30) \quad &\quad + \alpha \int_{(-\infty,s) \times \mathbb{R}_+} h(s-u) \mathbb{1}_{(0, \lambda'^{(n)}_{(t,z)}(u))}(v) \varepsilon_{(t,z)}(du \times dv) \\
 &= \phi(0) + \alpha \sum_{m \in \mathbb{Z}} h(s - T'_{(t,z),m}{}^{(n)}) \mathbb{1}_{(-\infty,s)}(T'_{(t,z),m}{}^{(n)}) \\
 &\quad + \alpha h(s-t) \mathbb{1}_{(0, \lambda'^{(n)}_{(t,z)}(t))}(z).
 \end{aligned}$$

Integrating over the finite interval $(a, b) \subset (t, \infty)$, we have

$$\begin{aligned}
 \int_a^b \lambda'^{(n+1)}_{(t,z)}(s) ds &\leq \phi(0)(b-a) + \alpha \sum_{m \in \mathbb{Z}} \mathbb{1}\{t < T'_{(t,z),m}{}^{(n)} < b\} \int_{0 \vee (a - T'_{(t,z),m}{}^{(n)})}^{b - T'_{(t,z),m}{}^{(n)}} h(u) du \\
 &\quad + \alpha \int_a^b h(s-t) ds,
 \end{aligned}$$

and this latter quantity is finite since h is integrable and $N'_{(t,z)}^{(n)}$ has an a.s. finite number of points in any bounded interval of (t, ∞) [due to (4.29)]. Moreover, the process $\{\lambda'^{(n+1)}_{(t,z)}(s)\}_{s>t}$ is $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -predictable. Indeed,

$$\lambda'^{(n+1)}_{(t,z)}(s) = \phi\left(h(s-t) \mathbb{1}_{(0, \lambda'^{(n)}_{(t,z)}(t))}(z) + \int_{(-\infty,s)} h(s-u) N'_{(t,z)}^{(n)}(du)\right)$$

and the processes

$$(4.31) \quad \left\{h(s-t) \mathbb{1}_{(0, \lambda'^{(n)}_{(t,z)}(t))}(z)\right\}_{s>t}, \quad \left\{\int_{(-\infty,s)} h(s-u) N'_{(t,z)}^{(n)}(du)\right\}_{s>t}$$

are $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -predictable. To justify the predictability of the first process in (4.31), one may first note that it is $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -adapted [since $\lambda'_{(t,z)}^{(n)}(t)$ is $\mathcal{F}_t^{\bar{N}}$ -measurable and h is deterministic] and then conclude by applying, for example, Theorem T34 in [6]. To justify the predictability of the second process in (4.31), one notes that it is left-continuous and $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -adapted. The induction is therefore completed and by Lemma 2.1, for any $n \geq 0$ and $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, the point process $N'_{(t,z)}^{(n)}$

on (t, ∞) has $\{\mathcal{F}_s^{\overline{N}}\}_{s>t}$ -stochastic intensity $\{\lambda_{(t,z)}^{(n)}(s)\}_{s>t}$. For fixed $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, since h is nonnegative and ϕ is nondecreasing, we have that $\lambda_{(t,z)}^{(n)}(s, \omega)$ and $N_{(t,z)}^{(n)}(C)(\omega)$ increase with $n \geq 0$, for all $\omega, s > t$ and Borel sets $C \subseteq (t, \infty)$. So the limiting processes $\{\lambda_{(t,z)}^{(\infty)}(s)\}_{s>t}$ and $N_{(t,z)}^{(\infty)}$ are defined for all ω . Setting $h \equiv 0$ on $(-\infty, 0]$, by (4.30), for any $n \geq 0, s, t \in \mathbb{R}$ and $z \in \mathbb{R}_+$, we have

$$\lambda_{(t,z)}^{(n+1)}(s) \leq \phi(0) + \alpha h(s - t) + \alpha \int_{\mathbb{R}} h(s - u) N_{(t,z)}^{(n)}(du).$$

Taking the mean over this inequality, we have

$$q_{(t,z)}^{(n+1)}(s) \leq p_t'(s) + r * q_{(t,z)}^{(n)}(s),$$

where for ease of notation we set $q_{(t,z)}^{(n)}(s) := E[\lambda_{(t,z)}^{(n)}(s)]$, $p_t'(s) := \phi(0) + r(s - t)$ and the function r is defined as above. Iterating this latter inequality and using that $q_{(t,z)}^{(0)} \equiv 0$, we deduce

$$q_{(t,z)}^{(n+1)}(s) \leq \sum_{i \geq 0} r^{*i} * p_t'(s) = \phi(0) \sum_{i \geq 0} \|r^{*i}\|_{L^1(\mathbb{R}, dx)} + \sum_{i \geq 1} r^{*i}(s - t).$$

Passing to the limit as $n \rightarrow \infty$, by the monotone convergence theorem, a standard property of the convolution and the stability condition $\alpha\mu < 1$, we have

$$\begin{aligned} E[\lambda_{(t,z)}^{(\infty)}(s)] &=: q_{(t,z)}^{(\infty)}(s) \leq \phi(0) \sum_{i \geq 0} \|r^{*i}\|_{L^1(\mathbb{R}, dx)} + \sum_{i \geq 1} r^{*i}(s - t) \\ &\leq \frac{\phi(0)}{1 - \alpha\mu} + \sum_{i \geq 1} r^{*i}(s - t) \\ &= \frac{\phi(0)}{1 - \alpha\mu} + \sum_{i \geq 1} \alpha^i \int_{\mathbb{R}} h^{*i-1}(s - t - u) h(u) du \\ &\leq \frac{\phi(0)}{1 - \alpha\mu} + \sum_{i \geq 1} \alpha^i \int_0^{s-t} h^{*i-1}(s - t - u) h(u) du \\ &\leq \frac{\phi(0)}{1 - \alpha\mu} + \alpha(1 - \alpha\mu)^{-1} \mathbb{1}_{s>t} \max_{u \in [0, s-t]} h(u), \end{aligned}$$

and so by the local boundedness of h we have

$$(4.32) \quad \max_{s \in [a, b]} q_{(t,z)}^{(\infty)}(s) < \infty, \quad \text{for any } a < b, a, b > t.$$

In particular,

$$\int_a^b \lambda_{(t,z)}^{(\infty)}(s) ds < \infty \quad \text{a.s., for any } a < b, a, b > t.$$

Moreover, $\{\lambda'_{(t,z)}(\infty)(s)\}_{s>t}$ is $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -predictable as limit of $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -predictable processes and

$$N'_{(t,z)}(\infty)(ds) = \bar{N}(ds \times (0, \lambda'_{(t,z)}(\infty)(s)]), \quad s > t.$$

So by Lemma 2.1 $N'_{(t,z)}(\infty)$ has $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -stochastic intensity $\{\lambda'_{(t,z)}(\infty)(s)\}_{s>t}$. Taking the limit as $n \rightarrow \infty$ in (4.28), we have

$$(4.33) \quad \lambda'_{(t,z)}(\infty)(s) = \phi \left(\int_{(-\infty,s)} h(s-u) (N'_{(t,z)}(\infty)(du) + \varepsilon_{(t,z)}(du \times (0, \lambda'_{(t,z)}(\infty)(u)])) \right), \quad s > t.$$

Therefore,

$$E[\varphi_s((\bar{N} + \varepsilon_{(t,z)})|_{(-\infty,s)})] = E[\varphi_s(\bar{N}|_{(-\infty,s)} + \varepsilon_{(t,z)})] = E[\lambda'_{(t,z)}(\infty)(s)], \quad s > t$$

and (4.24) follows by (4.32).

We now prove (4.22). Let $(t, z) \in \mathbb{R} \times \mathbb{R}_+$ be fixed. We define recursively the processes $\lambda''_{(t,z)}(0) \equiv 0$,

$$N''_{(t,z)}(n)(ds) = \bar{N}(ds \times (0, \lambda''_{(t,z)}(n)(s)]),$$

$$\lambda''_{(t,z)}(n+1)(s) = \phi \left(\int_{(-\infty,s)} h(s-u) N''_{(t,z)}(n)(du) \right), \quad n \geq 0, s > t$$

and note that since $h \geq 0$ and ϕ is nondecreasing we have

$$(4.34) \quad \lambda''_{(t,z)}(n)(s, \omega) \leq \lambda'_{(t,z)}(n)(s, \omega), \quad \text{for all } \omega, n \geq 0 \text{ and } s > t,$$

where $\lambda'_{(t,z)}(0) \equiv 0$ and $\lambda'_{(t,z)}(n+1), n \geq 0$, is defined by (4.28). Arguing as above, we have that, for fixed $(t, z) \in \mathbb{R} \times \mathbb{R}_+$, $\lambda''_{(t,z)}(n)(s, \omega)$ and $N''_{(t,z)}(n)(C)(\omega)$ increase with $n \geq 0$, for all $\omega, s > t$ and Borel sets $C \subseteq (t, \infty)$, and the limiting processes $\{\lambda''_{(t,z)}(\infty)(s)\}_{s>t}$ and $N''_{(t,z)}(\infty)$ are such that

$$\int_a^b \lambda''_{(t,z)}(\infty)(s) ds < \infty \quad \text{a.s., for any } a < b, a, b > t$$

$\{\lambda''_{(t,z)}(\infty)(s)\}_{s>t}$ is $\{\mathcal{F}_s^{\bar{N}}\}_{s>t}$ -predictable and

$$N''_{(t,z)}(\infty)(ds) = \bar{N}(ds \times (0, \lambda''_{(t,z)}(\infty)(s)]), \quad s > t,$$

$$\lambda''_{(t,z)}(\infty)(s) = \phi \left(\int_{(-\infty,s)} h(s-u) N''_{(t,z)}(\infty)(du) \right), \quad s > t.$$

Inequality (4.22) follows noticing that taking the limit as $n \rightarrow \infty$ in (4.34) we have

$$\lambda''_{(t,z)}(\infty)(s, \omega) \leq \lambda'_{(t,z)}(\infty)(s, \omega), \quad \text{for almost all } \omega \text{ and any } s > t.$$

We now check the assumptions of Theorem 3.1. Since N is stationary with a finite intensity and u is integrable and square integrable, conditions (3.4) and (3.5) hold. Arguing similarly to (4.26), for an integrable function g we have

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}_+} \left(\int_t^\infty |g(s)| \mathbb{E}[D_{(t,z)} \lambda(s)] ds \right) dt dz &\leq \lambda \sum_{i \geq 1} \alpha^i \int_{\mathbb{R}} \check{h}^{*i} * |g|(t) dt \\ &\leq \frac{\lambda \alpha \mu}{1 - \alpha \mu} \|g\|_{L^1(\mathbb{R}, dx)}. \end{aligned}$$

Taking first $g = u$ and then $g = u^2$, one then has that conditions (3.6) and (3.7) are satisfied. Condition (3.8) follows by (4.27) with $f = u^2$ and $g = u$. Finally, the local integrability on (t, ∞) of the random function $D_{(t,z)} \lambda(\cdot)$ is a consequence of (4.24). The proof is complete.

Step 2: Proof of (4.3). We have

$$\begin{aligned} (4.35) \quad &\mathbb{E} \left[\left| 1 - \int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right| \right] \\ &\leq |1 - \lambda \|u\|_{L^2(\mathbb{R}, dx)}^2| + \int_{\mathbb{R}} |u(t)|^2 \mathbb{E}[|\lambda(t) - \lambda|] dt. \end{aligned}$$

By (4.5) and (4.6), it follows

$$|\lambda(t) - \lambda| \leq \max \left\{ \alpha \int_{(-\infty, t)} h(t-s) N(ds), \frac{\phi(0)\alpha\mu}{1 - \alpha\mu} \right\} \quad \text{a.s., for all } t \in \mathbb{R}.$$

Taking the expectation on this relation and using the rightmost inequality in (4.6), we have

$$(4.36) \quad \mathbb{E}[|\lambda(t) - \lambda|] \leq \frac{2\phi(0)\alpha\mu}{1 - \alpha\mu} \quad \text{a.s., for all } t \in \mathbb{R}.$$

Combining this latter inequality with (4.35) and (4.6), we deduce

$$\begin{aligned} (4.37) \quad &\mathbb{E} \left[\left| 1 - \int_{\mathbb{R}} |u(t)|^2 \lambda(t) dt \right| \right] \\ &\leq |1 - \lambda \|u\|_{L^2(\mathbb{R}, dx)}^2| + \frac{2\phi(0)\alpha\mu}{1 - \alpha\mu} \|u\|_{L^2(\mathbb{R}, dx)}^2 \\ &\leq \max_{x \in \{\phi(0), \phi(0)(1-\alpha\mu)^{-1}\}} |1 - x \|u\|_{L^2(\mathbb{R}, dx)}^2| + \frac{2\phi(0)\alpha\mu}{1 - \alpha\mu} \|u\|_{L^2(\mathbb{R}, dx)}^2. \end{aligned}$$

The claim follows by (4.21), (4.37) and the rightmost inequality in (4.6). \square

PROOF OF THEOREM 4.3. Without loss of generality, we may assume $u \in L^2(\mathbb{R}, dx) \cap L^3(\mathbb{R}, dx) \cap L^4(\mathbb{R}, dx)$ (otherwise the claim trivially holds). By the triangular inequality, we have

$$d_W(\delta_a(u), Z) \leq d_W(\delta_a(u), \delta(u)) + d_W(\delta(u), Z).$$

So, due to Theorem 4.1, we only need to prove

$$(4.38) \quad d_W(\delta_a(u), \delta(u)) \leq \frac{2\phi(0)\alpha\mu}{1 - \alpha\mu} \|u\|_{L^1(\mathbb{R}, dx)}.$$

We have

$$\begin{aligned} d_W(\delta_a(u), \delta(u)) &= \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(\delta_a(u))] - \mathbb{E}[h(\delta(u))]| \\ &\leq \mathbb{E}[|\delta_a(u) - \delta(u)|] = \mathbb{E}\left[\left|\int_{\mathbb{R}} u(t)\lambda(t) dt - \widehat{\lambda} \int_{\mathbb{R}} u(t) dt\right|\right] \\ &\leq \int_{\mathbb{R}} |u(t)| \mathbb{E}[|\lambda(t) - \widehat{\lambda}|] dt. \end{aligned}$$

Inequality (4.38) then follows by bounding the term $\mathbb{E}[|\lambda(t) - \widehat{\lambda}|]$, $t \in \mathbb{R}$, with the quantity $2\phi(0)\alpha\mu/(1 - \alpha\mu)$ (since $\widehat{\lambda} \in [\phi(0), \phi(0)(1 - \alpha\mu)^{-1}]$ the same arguments for (4.36) work). \square

5. The case of stationary linear Hawkes processes. Let N be a stationary linear Hawkes process with parameters (ν, h) and $\mu := \int_0^\infty h(t) dt < 1$. Taking the mean of its stochastic intensity we easily see that the intensity of N is equal to

$$(5.1) \quad \lambda = \frac{\nu}{1 - \mu}$$

and so the ‘‘approximated’’ first chaos reads

$$\delta_a(u) := \int_{\mathbb{R}} u(t)(N(dt) - \nu(1 - \mu)^{-1} dt).$$

5.1. *Explicit Gaussian bounds for the first chaos of linear Hawkes processes.* The knowledge of the intensity allows to improve the bounds (4.3) and (4.7) specialized to the linear case. More precisely, the following theorem holds.

THEOREM 5.1. *Assume $h : \mathbb{R}_+ \rightarrow [0, \infty)$ locally bounded and $\mu < 1$. Let N be a stationary linear Hawkes process with parameters (ν, h) . If $u \in L^1(\mathbb{R}, dx)$, then*

$$(5.2) \quad d_W(\delta(u), Z) \leq \mathfrak{L}$$

and

$$(5.3) \quad d_W(\delta_a(u), Z) \leq \mathfrak{L} + \frac{2\nu\mu}{1 - \mu} \|u\|_{L^1(\mathbb{R}, dx)},$$

where

$$\begin{aligned} \mathfrak{L} &:= \sqrt{2/\pi} \left| 1 - \frac{\nu}{1 - \mu} \|u\|_{L^2(\mathbb{R}, dx)}^2 \right| + \frac{\nu}{1 - \mu} \|u\|_{L^3(\mathbb{R}, dx)}^3 \\ &\quad + \frac{2\sqrt{2/\pi} \nu \mu (2 - \mu)}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^2 + \frac{\nu\mu}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)} \|u^2\|_{L^2(\mathbb{R}, dx)}. \end{aligned}$$

In the linear case, alternative explicit Gaussian bounds may be obtained using the spectral theory of self-exciting processes; see [15]. See also [12], pages 303–309.

THEOREM 5.2. *Under assumptions and notation of Theorem 5.1, if moreover $h \in L^2(\mathbb{R}_+, dx)$, then*

$$(5.4) \quad d_W(\delta(u), Z) \leq \mathcal{L}'$$

and

$$(5.5) \quad \begin{aligned} & d_W(\delta_a(u), Z) \\ & \leq \mathcal{L}' + \frac{\sqrt{v}}{(1-\mu)^{3/2}} \min\{\mu\|u\|_{L^2(\mathbb{R}, dx)}, \|h\|_{L^2(\mathbb{R}_+, dx)}\|u\|_{L^1(\mathbb{R}, dx)}\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}' := & \sqrt{2/\pi} \left(\left(1 - \frac{v}{1-\mu} \|u\|_{L^2(\mathbb{R}, dx)}^2 \right)^2 \right. \\ & \left. + \frac{v}{(1-\mu)^3} \min\{\mu^2\|u^2\|_{L^2(\mathbb{R}, dx)}^2, \|h\|_{L^2(\mathbb{R}_+, dx)}^2\|u^2\|_{L^1(\mathbb{R}, dx)}^2\} \right)^{1/2} \\ & + \frac{v}{1-\mu} \|u\|_{L^3(\mathbb{R}, dx)}^3 + \frac{2\sqrt{2/\pi}v\mu}{(1-\mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^2 \\ & + \frac{v\mu}{(1-\mu)^2} \|u\|_{L^2(\mathbb{R}, dx)} \|u^2\|_{L^2(\mathbb{R}, dx)}. \end{aligned}$$

The proofs of Theorems 5.1 and 5.2 are given in Section 5.3.

Next proposition (whose proof is a simple consequence of the elementary inequality $\sqrt{a^2 + b^2} \leq |a| + |b|$, $a, b \in \mathbb{R}$, and therefore omitted) provides sufficient conditions under which the bounds of Theorem 5.2 improve the bounds of Theorem 5.1. Hereafter, for ease of notation, we denote by $\tilde{\mathcal{L}}$ the right-hand side of (5.3) and by $\tilde{\mathcal{L}}'$ the right-hand side of (5.5).

PROPOSITION 5.3. *Under assumptions and notation of Theorem 5.2, we have:*

(i) *If*

$$(5.6) \quad v \geq \frac{1}{4(1-\mu)} \min\left\{ \frac{\|u^2\|_{L^2(\mathbb{R}, dx)}^2}{\|u\|_{L^2(\mathbb{R}, dx)}^4}, \frac{\|h\|_{L^2(\mathbb{R}_+, dx)}^2}{\mu^2} \right\}$$

then $\mathcal{L}' \leq \mathcal{L}$.

(ii) If

$$(5.7) \quad v \geq \frac{1}{4(1-\mu)} \max \left\{ \min \left\{ \frac{\|u^2\|_{L^2(\mathbb{R}, dx)}^2}{\|u\|_{L^2(\mathbb{R}, dx)}^4}, \frac{\|h\|_{L^2(\mathbb{R}_+, dx)}^2}{\mu^2} \right\}, \right. \\ \left. \min \left\{ \frac{\|u\|_{L^2(\mathbb{R}, dx)}^2}{\|u\|_{L^1(\mathbb{R}, dx)}^2}, \frac{\|h\|_{L^2(\mathbb{R}_+, dx)}^2}{\mu^2} \right\} \right\}$$

then $\tilde{\mathfrak{L}}' \leq \tilde{\mathfrak{L}}$.

5.2. A quantitative central limit theorem for linear Hawkes processes. The following quantitative central limit theorem in the Wasserstein distance is an immediate consequence of Theorems 5.1 and 5.2.

COROLLARY 5.4. For $\varepsilon > 0$, assume $h_\varepsilon : \mathbb{R}_+ \rightarrow [0, \infty)$ locally bounded functions and such that $\mu_\varepsilon := \int_0^\infty h_\varepsilon(x) dx < 1$. Let N_ε be a stationary linear Hawkes process with parameters $(v_\varepsilon, h_\varepsilon)$ and take $u_\varepsilon \in L^1(\mathbb{R}, dx)$. Then:

$$(i) \quad (5.8) \quad d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \leq \min\{\mathfrak{L}_\varepsilon, \mathfrak{L}'_\varepsilon \mathbb{1}_{\{h_\varepsilon \in L^2(\mathbb{R}_+, dx)\}}\} + \mathfrak{L}_\varepsilon \mathbb{1}_{\{h_\varepsilon \notin L^2(\mathbb{R}_+, dx)\}},$$

$\varepsilon > 0$

and

$$(5.9) \quad d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \leq \min\{\tilde{\mathfrak{L}}_\varepsilon, \tilde{\mathfrak{L}}'_\varepsilon \mathbb{1}_{\{h_\varepsilon \in L^2(\mathbb{R}_+, dx)\}}\} + \tilde{\mathfrak{L}}_\varepsilon \mathbb{1}_{\{h_\varepsilon \notin L^2(\mathbb{R}_+, dx)\}},$$

$\varepsilon > 0$.

Here, $\mathfrak{L}_\varepsilon, \mathfrak{L}'_\varepsilon, \tilde{\mathfrak{L}}_\varepsilon$ and $\tilde{\mathfrak{L}}'_\varepsilon$ are defined as $\mathfrak{L}, \mathfrak{L}', \tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{L}}'$, respectively, with $v_\varepsilon, \mu_\varepsilon, u_\varepsilon$ and h_ε in place of v, μ, u and h , respectively;

$$\delta^{(\varepsilon)}(u_\varepsilon) := \int_{\mathbb{R}} u_\varepsilon(t) (N_\varepsilon(dt) - \lambda_\varepsilon(t) dt),$$

$$\lambda_\varepsilon(t) := v_\varepsilon + \int_{(-\infty, t)} h_\varepsilon(t-s) N_\varepsilon(ds),$$

$$\delta_a^{(\varepsilon)}(u_\varepsilon) := \int_{\mathbb{R}} u_\varepsilon(t) (N_\varepsilon(dt) - v_\varepsilon(1 - \mu_\varepsilon)^{-1} dt).$$

(ii) If, as $\varepsilon \rightarrow 0$,

$$(5.10) \quad \mu_\varepsilon \rightarrow 0,$$

$$(5.11) \quad v_\varepsilon \|u_\varepsilon\|_{L^2(\mathbb{R}, dx)}^2 \rightarrow 1,$$

$$(5.12) \quad v_\varepsilon \|u_\varepsilon\|_{L^3(\mathbb{R}, dx)}^3 \rightarrow 0,$$

$$(5.13) \quad v_\varepsilon (\mu_\varepsilon)^2 \|(u_\varepsilon)^2\|_{L^2(\mathbb{R}, dx)}^2 \rightarrow 0,$$

then

$$d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

If moreover, as $\varepsilon \rightarrow 0$,

$$(5.14) \quad \nu_\varepsilon \mu_\varepsilon \|u_\varepsilon\|_{L^1(\mathbb{R}, dx)} \rightarrow 0,$$

then

$$d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

This latter limit holds even if we replace condition (5.14) with

$$(5.15) \quad h_\varepsilon \in L^2(\mathbb{R}_+, dx), \quad \varepsilon > 0.$$

REMARK 5.5. In this remark, we compare Corollary 4.5, specialized to the case of a self-exciting process N_ε with parameters $(\nu_\varepsilon, h_\varepsilon)$, with Corollary 5.4. First, we note that the upper bounds (5.8) and (5.9) improve the upper bounds (4.8) and (4.9), respectively. Second, we note that conditions (5.10)–(5.14) coincide with conditions (4.10)–(4.14). Finally, we note that in Corollary 5.4 we deduce the convergence to zero of the family $\{d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z)\}_{\varepsilon>0}$ even replacing condition (5.14) with the alternative condition (5.15).

We conclude this subsection with an example.

EXAMPLE 5.6. Let $I_\varepsilon, \varepsilon > 0$, be a given family of bounded Borel sets, I_ε with Lebesgue measure ℓ_ε , and $\nu_\varepsilon > 0, \varepsilon > 0$, be a family of positive constants. Let $\mu_\varepsilon, \varepsilon > 0$, be a collection of positive numbers such that $\mu_\varepsilon < 1, \varepsilon > 0$, and define the functions $h_\varepsilon(t) := \mu_\varepsilon f_\varepsilon(t), \varepsilon > 0, t > 0$, where f_ε is a locally bounded probability density (with respect to the Lebesgue measure) on $(0, \infty)$ such that $f_\varepsilon \in L^2(\mathbb{R}_+, dx), \varepsilon > 0$. Hereafter, we consider the family $N_\varepsilon, \varepsilon > 0$, of stationary linear Hawkes processes with parameters $(\nu_\varepsilon, h_\varepsilon), \varepsilon > 0$, and the functions

$$u_\varepsilon(t) := \frac{1}{\sqrt{(\nu_\varepsilon \ell_\varepsilon)/(1 - \mu_\varepsilon)}} \mathbb{1}_{I_\varepsilon}(t), \quad \varepsilon > 0, t \in \mathbb{R}.$$

Using the expressions of the L^p -norms of u_ε computed in the Example 4.6 [clearly setting $\alpha_\varepsilon = 1$ and $\phi_\varepsilon(0) = \nu_\varepsilon$ therein], one may easily see that conditions (5.6) and (5.7) are both equivalent to

$$(5.16) \quad \nu_\varepsilon \geq \frac{1}{4(1 - \mu_\varepsilon)} \min\{\ell_\varepsilon^{-1}, \|f_\varepsilon\|_{L^2(\mathbb{R}_+, dx)}^2\}, \quad \varepsilon > 0.$$

Note also that the square integrability of f_ε implies $h_\varepsilon \in L^2(\mathbb{R}_+, dx)$. Therefore, under (5.16), by Proposition 5.3 we deduce $\mathfrak{L}'_\varepsilon \leq \mathfrak{L}_\varepsilon$ and $\widetilde{\mathfrak{L}}'_\varepsilon \leq \widetilde{\mathfrak{L}}_\varepsilon, \varepsilon > 0$. So, under

(5.16), by Corollary 5.4 we have

$$\begin{aligned}
 & d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \\
 & \leq \mathfrak{L}'_\varepsilon = (\sqrt{2/\pi} \min\{(\ell_\varepsilon)^{-1/2}, \|f_\varepsilon\|_{L^2(\mathbb{R}_+, dx)}\} + (\ell_\varepsilon)^{-1/2}) \frac{\mu_\varepsilon}{\sqrt{v_\varepsilon(1 - \mu_\varepsilon)}} \\
 & \quad + \sqrt{\frac{1 - \mu_\varepsilon}{v_\varepsilon \ell_\varepsilon}} + 2\sqrt{2/\pi} \frac{\mu_\varepsilon}{1 - \mu_\varepsilon}, \quad \varepsilon > 0
 \end{aligned}$$

and

$$\begin{aligned}
 & d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \\
 & \leq \widetilde{\mathfrak{L}}'_\varepsilon = (\sqrt{2/\pi} \min\{(\ell_\varepsilon)^{-1/2}, \|f_\varepsilon\|_{L^2(\mathbb{R}_+, dx)}\} + (\ell_\varepsilon)^{-1/2}) \frac{\mu_\varepsilon}{\sqrt{v_\varepsilon(1 - \mu_\varepsilon)}} \\
 & \quad + \sqrt{\frac{1 - \mu_\varepsilon}{v_\varepsilon \ell_\varepsilon}} + (2\sqrt{2/\pi} + \min\{1, \sqrt{\ell_\varepsilon} \|f_\varepsilon\|_{L^2(\mathbb{R}_+, dx)}\}) \frac{\mu_\varepsilon}{1 - \mu_\varepsilon}, \quad \varepsilon > 0.
 \end{aligned}$$

Finally, one easily sees that if

$$(5.17) \quad \mu_\varepsilon \rightarrow 0 \quad \text{and} \quad v_\varepsilon \ell_\varepsilon \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0$$

then

$$d_W(\delta^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0 \quad \text{and} \quad d_W(\delta_a^{(\varepsilon)}(u_\varepsilon), Z) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

It has to be noticed that a straightforward computation shows that condition (5.17) implies conditions (5.10)–(5.13), but does not imply condition (5.14).

5.3. Proofs of Theorems 5.1 and 5.2.

PROOF OF THEOREM 5.1. The claim follows by an obvious modification of the proofs of Theorems 4.1 and 4.3. For instance, to get (5.2) it suffices to modify the proof of Theorem 4.1 as follows. We combine inequality (4.21) [taking therein $\lambda = v(1 - \mu)^{-1}$ and $\alpha = 1$] with inequalities (4.35) and (4.36) [taking therein $\lambda = v(1 - \mu)^{-1}$, $\alpha = 1$ and $\phi(0) = v$]. Note that, due to the knowledge of λ , we do not need anymore to further bound the quantity $|1 - \lambda \|u\|_{L^2(\mathbb{R}, dx)}^2|$ as in (4.37). □

PROOF OF THEOREM 5.2. The claim is clearly true if $u \notin L^2(\mathbb{R}, dx) \cap L^3(\mathbb{R}, dx) \cap L^4(\mathbb{R}, dx)$. So we shall assume these integrability conditions. We first prove the bound (5.4). By (4.21) [with $\lambda = v(1 - \mu)^{-1}$ and $\alpha = 1$], the Cauchy–

Schwarz inequality and the stationarity of N , we have

$$\begin{aligned}
 d_W(\delta(u), Z) &\leq \sqrt{2/\pi} \sqrt{1 - 2 \frac{\nu}{1 - \mu} \|u\|_{L^2(\mathbb{R}, dx)}^2} + \int_{\mathbb{R}^2} |u(t)u(s)|^2 \mathbb{E}[\lambda(t)\lambda(s)] dt ds \\
 (5.18) \quad &+ \frac{\nu}{1 - \mu} \|u\|_{L^3(\mathbb{R}, dx)}^3 + \frac{2\nu\mu}{(1 - \mu)^2} \sqrt{2/\pi} \|u\|_{L^2(\mathbb{R}, dx)}^2 \\
 &+ \frac{\nu\mu}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)} \|u^2\|_{L^2(\mathbb{R}, dx)}.
 \end{aligned}$$

By (5.1) and again the stationarity of N , we deduce

$$\begin{aligned}
 (5.19) \quad \mathbb{E}[\lambda(t)\lambda(s)] &= \lambda^2 + \text{Cov}(\lambda(t), \lambda(s)) \\
 &= \left(\frac{\nu}{1 - \mu}\right)^2 + \text{Cov}\left(\int_{\mathbb{R}} h_t(u)N(du), \int_{\mathbb{R}} h_s(u)N(du)\right),
 \end{aligned}$$

where we set $h_t(u) := \mathbb{1}_{(-\infty, t)}(u)h(t - u)$. In the following, for $f \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$, we denote by $\widehat{f}(\omega) := \int_{\mathbb{R}} e^{i\omega t} f(t) dt$ the Fourier transform of f , and we extend the definition of h on $(-\infty, 0]$ setting $h(t) := 0$ for $t \leq 0$. By the results in [15], we have (see also formulas (8) and (24) in [8])

$$\begin{aligned}
 (5.20) \quad &\text{Cov}\left(\int_{\mathbb{R}} h_t(u)N(du), \int_{\mathbb{R}} h_s(u)N(du)\right) \\
 &= \frac{\nu}{2\pi(1 - \mu)} \int_{\mathbb{R}} \widehat{h}_t(\omega)\widehat{h}_s(\omega) \frac{1}{|1 - \widehat{h}(\omega)|^2} d\omega.
 \end{aligned}$$

Note that

$$(5.21) \quad |1 - \widehat{h}(\omega)| \geq |1 - |\widehat{h}(\omega)|| \geq 1 - |\widehat{h}(\omega)| \geq 1 - \mu > 0, \quad \omega \in \mathbb{R}$$

and that $\widehat{h}_t(\omega) = e^{i\omega t}\widehat{h}(-\omega)$ (since h has a positive support). Therefore,

$$\begin{aligned}
 &\int_{\mathbb{R}^2} |u(t)u(s)|^2 \mathbb{E}[\lambda(t)\lambda(s)] dt ds \\
 (5.22) \quad &= \left(\frac{\nu}{1 - \mu}\right)^2 \|u\|_{L^2(\mathbb{R}, dx)}^4 \\
 &+ \frac{\nu}{2\pi(1 - \mu)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(t)|^2 \widehat{h}_t(\omega) dt\right)^2 \frac{1}{|1 - \widehat{h}(\omega)|^2} d\omega \\
 (5.23) \quad &\leq \left(\frac{\nu}{1 - \mu}\right)^2 \|u\|_{L^2(\mathbb{R}, dx)}^4 + \frac{\nu}{2\pi(1 - \mu)^3} \int_{\mathbb{R}} \left|\int_{\mathbb{R}} |u(t)|^2 \widehat{h}_t(\omega) dt\right|^2 d\omega.
 \end{aligned}$$

In (5.22), we used Fubini’s theorem, which is applicable since

$$\int_{\mathbb{R}^3} |u(t)u(s)|^2 |\widehat{h}_t(\omega)\widehat{h}_s(\omega)| \frac{1}{|1 - \widehat{h}(\omega)|^2} ds dt d\omega$$

$$\begin{aligned} &\leq \frac{1}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^4 \int_{\mathbb{R}} |\widehat{h}(-\omega)|^2 d\omega \\ &= \frac{2\pi}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^4 \|h\|_{L^2(\mathbb{R}, dx)}^2 < \infty, \end{aligned}$$

where in the latter equality we used Parseval’s identity. Setting $\check{f}(x) := f(-x)$, $u^2(\cdot) := u(\cdot)^2$ and letting the symbol $*$ denote the convolution product, we have

$$\int_{\mathbb{R}} |u(t)|^2 \widehat{h}_t(\omega) dt = \widehat{h}(-\omega) \widehat{u^2}(\omega) = \widehat{\check{h}}(\omega) \widehat{u^2}(\omega) = \widehat{\check{h} * u^2}(\omega).$$

Consequently, using again the Parseval identity, we deduce

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} |u(t)|^2 \widehat{h}_t(\omega) dt \right|^2 d\omega = \int_{\mathbb{R}} |\widehat{\check{h} * u^2}(\omega)|^2 d\omega = 2\pi \|\check{h} * u^2\|_{L^2(\mathbb{R}, dx)}^2.$$

By this relation, (5.23) and standard properties of the convolution (see, e.g., Theorem IV.15 in [10]), we have

$$\begin{aligned} &\int_{\mathbb{R}^2} |u(t)u(s)|^2 \mathbb{E}[\lambda(t)\lambda(s)] dt ds \\ &\leq \frac{v^2}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^4 + \frac{v}{(1 - \mu)^3} \|\check{h} * u^2\|_{L^2(\mathbb{R}, dx)}^2 \\ &\leq \frac{v^2}{(1 - \mu)^2} \|u\|_{L^2(\mathbb{R}, dx)}^4 \\ &\quad + \frac{v}{(1 - \mu)^3} \min\{\mu^2 \|u^2\|_{L^2(\mathbb{R}, dx)}^2, \|h\|_{L^2(\mathbb{R}, dx)}^2 \|u^2\|_{L^1(\mathbb{R}, dx)}^2\}. \end{aligned}$$

The claim follows combining this inequality with (5.18). We now prove the bound (5.5). By the triangular inequality and (5.4), we only need to prove

$$d_W(\delta_a(u), \delta(u)) \leq \frac{\sqrt{v}}{(1 - \mu)^{3/2}} \min\{\mu \|u\|_{L^2(\mathbb{R}, dx)}, \|h\|_{L^2(\mathbb{R}, dx)} \|u\|_{L^1(\mathbb{R}, dx)}\}.$$

Note that

$$\begin{aligned} d_W(\delta_a(u), \delta(u)) &= \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(\delta_a(u))] - \mathbb{E}[h(\delta(u))]| \\ &\leq \mathbb{E}[|\delta_a(u) - \delta(u)|] = \mathbb{E}\left[\left| \int_{\mathbb{R}} u(t)\lambda(t) dt - \lambda \int_{\mathbb{R}} u(t) dt \right|\right] \end{aligned}$$

and so by the Cauchy–Schwarz inequality we have

$$\begin{aligned} &d_W(\delta_a(u), \delta(u)) \\ (5.24) \quad &\leq \left(\mathbb{E}\left[\left| \int_{\mathbb{R}} u(t)\lambda(t) dt - \frac{v}{1 - \mu} \int_{\mathbb{R}} u(t) dt \right|^2\right] \right)^{1/2} \end{aligned}$$

$$= \sqrt{\int_{\mathbb{R}^2} u(t)u(s)E[\lambda(t)\lambda(s)] dt ds - \left(\frac{\nu}{1-\mu}\right)^2 \left(\int_{\mathbb{R}} u(t) dt\right)^2}.$$

To upper bound the first addend inside the square root, we repeat the arguments above. So, by (5.20), (5.20), (5.21), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} u(s)u(t)E[\lambda(s)\lambda(t)] ds dt \\ & \leq \left(\frac{\nu}{1-\mu}\right)^2 \left(\int_{\mathbb{R}} u(t) dt\right)^2 + \frac{\nu}{2\pi(1-\mu)^3} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} u(t)\widehat{h}_t(\omega) dt \right|^2 d\omega \\ (5.25) \quad & = \left(\frac{\nu}{1-\mu}\right)^2 \left(\int_{\mathbb{R}} u(t) dt\right)^2 + \frac{\nu}{(1-\mu)^3} \|\check{h} * u\|_{L^2(\mathbb{R}, dx)}^2 \\ & \leq \left(\frac{\nu}{1-\mu}\right)^2 \left(\int_{\mathbb{R}} u(t) dt\right)^2 \\ & \quad + \frac{\nu}{(1-\mu)^3} \min\{\mu^2 \|u\|_{L^2(\mathbb{R}, dx)}^2, \|h\|_{L^2(\mathbb{R}, dx)} \|u\|_{L^1(\mathbb{R}, dx)}\}. \end{aligned}$$

Note that in (5.24) we used Fubini's theorem to exchange the double integral with the expectation. This is justified by the fact that inequality (5.25) holds replacing u with $|u|$ and the resulting right-hand side is finite. The proof is complete. \square

Acknowledgments. The author thanks the Editor and two anonymous reviewers for a careful reading of the paper.

REFERENCES

- [1] BACRY, E., DELATTRE, S., HOFFMANN, M. and MUZY, J. F. (2013). Modelling microstructure noise with mutually exciting point processes. *Quant. Finance* **13** 65–77. [MR3005350](#)
- [2] BACRY, E., DELATTRE, S., HOFFMANN, M. and MUZY, J. F. (2013). Some limit theorems for Hawkes processes and application to financial statistics. *Stochastic Process. Appl.* **123** 2475–2499. [MR3054533](#)
- [3] BARBOUR, A. D. and BROWN, T. C. (1992). The Stein–Chen method, point processes and compensators. *Ann. Probab.* **20** 1504–1527. [MR1175275](#)
- [4] BARBOUR, A. D. and BROWN, T. C. (1992). Stein's method and point process approximation. *Stochastic Process. Appl.* **43** 9–31. [MR1190904](#)
- [5] BORDENAVE, C. and TORRISI, G. L. (2007). Large deviations of Poisson cluster processes. *Stoch. Models* **23** 593–625. [MR2362700](#)
- [6] BRÉMAUD, P. (1981). *Point Processes and Queues*. Springer, New York. [MR0636252](#)
- [7] BRÉMAUD, P. and MASSOULIÉ, L. (1996). Stability of nonlinear Hawkes processes. *Ann. Probab.* **24** 1563–1588. [MR1411506](#)
- [8] BRÉMAUD, P. and MASSOULIÉ, L. (2002). Power spectra of general shot noises and Hawkes point processes with a random excitation. *Adv. in Appl. Probab.* **34** 205–222. [MR1895338](#)

- [9] BRÉMAUD, P., NAPPO, G. and TORRISI, G. L. (2002). Rate of convergence to equilibrium of marked Hawkes processes. *J. Appl. Probab.* **39** 123–136. [MR1895148](#)
- [10] BREZIS, H. (2005). *Analyse Fonctionnelle*. Dunod, Paris.
- [11] CHEN, L. H. Y., GOLDSTEIN, L. and SHAO, Q.-M. (2011). *Normal Approximation by Stein's Method*. Springer, Heidelberg. [MR2732624](#)
- [12] DALEY, D. J. and VERE-JONES, D. (2003). *An Introduction to the Theory of Point Processes. Vol. I*, 2nd ed. Springer, New York. [MR1950431](#)
- [13] DALEY, D. J. and VERE-JONES, D. (2008). *An Introduction to the Theory of Point Processes. Vol. II*, 2nd ed. Springer, New York. [MR2371524](#)
- [14] DUDLEY, R. M. (2002). *Real Analysis and Probability. Cambridge Studies in Advanced Mathematics 74*. Cambridge Univ. Press, Cambridge. [MR1932358](#)
- [15] HAWKES, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. *Biometrika* **58** 83–90. [MR0278410](#)
- [16] HAWKES, A. G. and OAKES, D. (1974). A cluster process representation of a self-exciting process. *J. Appl. Probab.* **11** 493–503. [MR0378093](#)
- [17] JAISSON, T. and ROSENBAUM, M. (2015). Limit theorems for nearly unstable Hawkes processes. *Ann. Appl. Probab.* **25** 600–631. [MR3313750](#)
- [18] KAGAN, Y. Y. (2010). Statistical distributions of earthquake numbers: Consequence of branching process. *Geophys. J. Int.* **180** 1313–1328.
- [19] LAST, G., PECCATI, G. and SCHULTE, M. (2016). Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. *Probab. Theory Related Fields* **165** 667–723. [MR3520016](#)
- [20] LAST, G. and PENROSE, M. D. (2011). Martingale representation for Poisson processes with applications to minimal variance hedging. *Stochastic Process. Appl.* **121** 1588–1606. [MR2802467](#)
- [21] LEWIS, P. A. W. and SHEDLER, G. S. (1976). Simulation of nonhomogeneous Poisson processes with log linear rate function. *Biometrika* **63** 501–505. [MR0464448](#)
- [22] MASSOULIÉ, L. (1998). Stability results for a general class of interacting point processes dynamics, and applications. *Stochastic Process. Appl.* **75** 1–30. [MR1629010](#)
- [23] MØLLER, J. and RASMUSSEN, J. G. (2005). Perfect simulation of Hawkes processes. *Adv. in Appl. Probab.* **37** 629–646. [MR2156552](#)
- [24] MØLLER, J. and RASMUSSEN, J. G. (2006). Approximate simulation of Hawkes processes. *Methodol. Comput. Appl. Probab.* **8** 53–64. [MR2253076](#)
- [25] MØLLER, J. and TORRISI, G. L. (2007). The pair correlation function of spatial Hawkes processes. *Statist. Probab. Lett.* **77** 995–1003. [MR2370426](#)
- [26] NGUYEN, T. D., PRIVAULT, N. and TORRISI, G. L. (2015). Gaussian estimates for the solutions of some one-dimensional stochastic equations. *Potential Anal.* **43** 289–311. [MR3374113](#)
- [27] NOURDIN, I. and PECCATI, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. [MR2520122](#)
- [28] OGATA, Y. (1999). Seismicity analysis through point process modeling: A review. *Pure Appl. Geophys.* **155** 471–507.
- [29] PECCATI, G., SOLÉ, J. L., TAQQU, M. S. and UTZET, F. (2010). Stein's method and normal approximation of Poisson functionals. *Ann. Probab.* **38** 443–478. [MR2642882](#)
- [30] PECCATI, G. and ZHENG, C. (2010). Multi-dimensional Gaussian fluctuations on the Poisson space. *Electron. J. Probab.* **15** 1487–1527. [MR2727319](#)
- [31] PRIVAULT, N. (2009). *Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales. Lecture Notes in Math. 1982*. Springer, Berlin. [MR2531026](#)
- [32] PRIVAULT, N. and TORRISI, G. L. (2013). Probability approximation by Clark–Ocone covariance representation. *Electron. J. Probab.* **18** 1–25. [MR3126574](#)

- [33] PRIVAULT, N. and TORRISI, G. L. (2015). The Stein and Chen–Stein methods for functionals of non-symmetric Bernoulli processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **12** 309–356. [MR3357130](#)
- [34] REITZNER, M. and SCHULTE, M. (2013). Central limit theorems for U -statistics of Poisson point processes. *Ann. Probab.* **41** 3879–3909. [MR3161465](#)
- [35] REYNAUD-BOURET, P., RIVOIRARD, V., GRAMMONT, F. and TULEAU-MALOT, C. (2014). Goodness-of-fit tests and nonparametric adaptive estimation for spike train analysis. *J. Math. Neurosci.* **4** Art. 3, 41. [MR3197017](#)
- [36] REYNAUD-BOURET, P. and SCHBATH, S. (2010). Adaptive estimation for Hawkes processes; application to genome analysis. *Ann. Statist.* **38** 2781–2822. [MR2722456](#)
- [37] STABILE, G. and TORRISI, G. L. (2010). Risk processes with non-stationary Hawkes claims arrivals. *Methodol. Comput. Appl. Probab.* **12** 415–429. [MR2665268](#)
- [38] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability Theory* 583–602. Univ. California Press, Berkeley, CA. [MR0402873](#)
- [39] TORRISI, G. L. (2002). A class of interacting marked point processes: Rate of convergence to equilibrium. *J. Appl. Probab.* **39** 137–160. [MR1895149](#)
- [40] WU, L. (2000). A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probab. Theory Related Fields* **118** 427–438. [MR1800540](#)
- [41] ZHU, L. (2013). Ruin probabilities for risk processes with non-stationary arrivals and subexponential claims. *Insurance Math. Econom.* **53** 544–550. [MR3130449](#)
- [42] ZHU, L. (2013). Central limit theorem for nonlinear Hawkes processes. *J. Appl. Probab.* **50** 760–771. [MR3102513](#)
- [43] ZHU, L. (2015). Large deviations for Markovian nonlinear Hawkes processes. *Ann. Appl. Probab.* **25** 548–581. [MR3313748](#)

CNR—ISTITUTO PER LE APPLICAZIONI
DEL CALCOLO “MAURO PICONE”
VIA DEI TAURINI 19
I-00185 ROME
ITALY
E-MAIL: torrisi@iac.rm.cnr.it