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# A NOTE ON THE EXPANSION OF THE SMALLEST EIGENVALUE DISTRIBUTION OF THE LUE AT THE HARD EDGE[1]

BY FOLKMAR BORNEMANN

*Technische Universität München*

In a recent paper, Edelman, Guionnet and Péché conjectured a particular $n^{-1}$ correction term of the smallest eigenvalue distribution of the Laguerre unitary ensemble (LUE) of order $n$ in the hard-edge scaling limit: specifically, the derivative of the limit distribution, that is, the density, shows up in that correction term. We give a short proof by modifying the hard-edge scaling to achieve an optimal $O(n^{-2})$ rate of convergence of the smallest eigenvalue distribution. The appearance of the derivative follows then by a Taylor expansion of the less optimal, standard hard-edge scaling. We relate the $n^{-1}$ correction term further to the logarithmic derivative of the Bessel kernel Fredholm determinant in the work of Tracy and Widom.

**1. Introduction.** We recall from random matrix theory that the smallest eigenvalue distribution of the LUE of order $n$ with parameter $a > -1$ is given by the Fredholm determinant [6], Chapter 9,

$$\mathbb{P}(\lambda_{\min} \geq s) = \det\big(I - K_n^a|_{L^2(0,s)}\big),$$

induced by the Laguerre projection kernel

$$K_n^a(x,y) = \sum_{k=0}^{n-1} \phi_k^a(x)\phi_k^a(y), \qquad \phi_k^a(x) = \sqrt{\frac{k!}{\Gamma(k+a+1)}}\, e^{-x/2} x^{a/2} L_k^a(x).$$

Here, $L_k^a$ denotes the generalized Laguerre polynomial of degree $k$. By Christoffel–Darboux and the relation $L_n^{a-1}(x) + L_{n-1}^a(x) = L_n^a(x)$ (cf. [1], equation (22.7.30)), one gets the closed form

$$K_n^a(x,y) = \frac{n!\, e^{-(x+y)/2}(xy)^{a/2}}{\Gamma(n+a)} \cdot \frac{L_n^a(x)L_n^{a-1}(y) - L_n^{a-1}(x)L_n^a(y)}{x-y}.$$

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Using the Mehler–Heine type asymptotics ([10], Theorem 8.1.3) of the Laguerre polynomials, which holds uniformly for bounded $z$ in the complex plane,[2]

$$n^{-a} L_n^a(z/n) = z^{-a/2} J_a(2\sqrt{z}) + o(1) \qquad (n \to \infty)$$

one immediately obtains, as first done by Forrester ([5], equation (2.6); see also [6], Section 7.2.1), that in the hard-edge scaling

$$X = \frac{x}{4n},$$

likewise for $Y$ and $y$, there is the limit, locally uniform for positive $x$ and $y$,

$$K_n^a(X,Y)\,dX = \bigl(K_\infty^a(x,y) + o(1)\bigr)\,dx \qquad (n \to \infty)$$

with the Bessel kernel

$$\begin{aligned} K_\infty^a(x,y) &= \frac{\sqrt{y} J_a(\sqrt{x}) J_{a-1}(\sqrt{y}) - \sqrt{x} J_{a-1}(\sqrt{x}) J_a(\sqrt{y})}{2(x-y)} \\ &= \frac{\sqrt{y} J_a(\sqrt{x}) J_a'(\sqrt{y}) - \sqrt{x} J_a'(\sqrt{x}) J_a(\sqrt{y})}{2(x-y)}. \end{aligned}$$

Lifted to the convergence of the induced Fredholm determinants (see the next section for details), one thus gets the hard-edge scaling limit of the LUE as a limit of distributions, namely as $n \to \infty$

$$F_n^a(s) = \mathbb{P}\left(\lambda_{\min} \geq \frac{s}{4n}\right) \to F_\infty^a(s) = \det\bigl(I - K_\infty^a|_{L^2(0,s)}\bigr). \tag{1}$$

Based on an identity of finite-dimensional Bessel function determinants obtained from symbolical and numerical computer experiments, Edelman, Guionnet and Péché [4], page 14, *conjectured* the following refinement of (1):

$$F_n^a(s) = F_\infty^a(s) + \frac{a}{2n} s f_\infty^a(s) + O(n^{-2}), \qquad f_\infty^a(s) = \frac{d}{ds} F_\infty^a(s). \tag{2}$$

In this note, we will give a short proof that this expansion, in fact, holds true.

REMARK. At FoCM'14, Grégory Schehr [9] presented yet another proof (joint work with Anthony Perret) of this expansion which he had obtained as a byproduct of a new approach to the Painlevé III representation [11] of the Bessel kernel determinant.

[2]Hence, for all $a \in \mathbb{R}$,

$$z^{-a/2} J_a(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(a+k+1)} z^k$$

constitutes a uniquely defined *entire* function of the complex variable $z$.

**2. A short proof of the expansion.** Lemma 4.1 of [4] easily implies a refinement of the Mehler–Heine type asymptotics in the form of an expansion (see also [8], page 29, for $a = 0$ and [12], page 156, for the general case), namely,

$$(3)\quad \begin{aligned} &(n+a)^{-a} L_n^a\big(z/(n+a)\big) \\ &\quad = z^{-a/2} J_a(2\sqrt{z}) - \frac{1}{2n} z^{-(a-2)/2} J_{a-2}(2\sqrt{z}) + O\big(n^{-2}\big), \end{aligned}$$

uniformly for bounded complex $z$. Subject to the following modified hard-edge scaling, likewise for $Y$ and $y$,

$$X = \frac{x}{4n}\left(1 - \frac{a+c}{2n}\right),$$

which transforms the Laguerre kernel according to

$$K_n^a(X, Y)\, dX = \tilde{K}_n^a(x, y)\, dx,$$

this gives after some routine calculation[3] the expansion

$$(4)\quad \begin{aligned} &(xy)^{-a/2} \tilde{K}_n^a(x, y) \\ &\quad = (xy)^{-a/2} K_\infty^a(x, y) - \frac{c}{8n} x^{-a/2} J_a(\sqrt{x}) y^{-a/2} J_a(\sqrt{y}) + O\big(n^{-2}\big), \end{aligned}$$

uniformly for bounded complex $x$ and $y$; in particular, uniformly for $x, y \in [0, s]$. Because of the pre-factor $(xy)^{-a/2}$, all the terms in (4) represent *entire* kernels; see the footnote in the last section. Now, such a pre-multiplication of a kernel $K$ by $(xy)^{-a/2}$ leaves its Fredholm determinant invariant if we transform the measure defining the underlying $L^2$-space accordingly:

$$\det(I - K|_{L^2(0,s)}) = \det\big(I - (xy)^{-a/2} K|_{L^2((0,s);\nu_a)}\big), \qquad \nu_a(dx) = x^a\, dx,$$

which follows simply from conjugating $K$ with the *unitary* transformation

$$U : L^2(0, s) \to L^2\big((0, s), \nu_a\big), \qquad f(x) \mapsto x^{-a/2} f(x).$$

Because of $a > -1$, the transformed measure $\nu_a$ on $[0, s]$ has finite mass. Since the Fredholm determinant, when defined with respect to a measure $\nu$ of finite mass, is locally Lipschitz continuous with respect to the *uniform* convergence of the kernels (see [2], Lemma 3.4.5), the *uniform* kernel expansion (4) on $(0, s)^2$ immediately lifts to an expansion of the induced Fredholm determinants. For the particular choice $c = 0$, which eliminates the $O(n^{-1})$ term, we thus get

$$(5)\quad F_n^a\left(\left(1 - \frac{a}{2n}\right) s\right) = F_\infty^a(s) + O\big(n^{-2}\big).$$

Now, a simple Taylor expansion readily establishes (2):

$$F_n^a(s) = F_\infty^a\left(\left(1 - \frac{a}{2n}\right)^{-1} s\right) + O\big(n^{-2}\big) = F_\infty^a(s) + \frac{a}{2n} s f_\infty^a(s) + O\big(n^{-2}\big).$$

[3] A *Mathematica* notebook checking this result can be found at arXiv:1504.00235.

REMARK. The choice $c = -a$ of the additional scaling parameter which is implicitly used in (2) is not the best possible one; the optimally modified hard-edge scaling is given by (5). Obtaining such a second-order convergence rate by appropriately modifying the scaling was stimulated by the corresponding work of Johnstone and Ma [7] for the *largest* eigenvalue distributions of the Gaussian unitary ensemble (GUE) at the soft edge; cf. also the work of Choup [3] on the largest eigenvalue distributions of GUE and LUE.

**3. Relation to the Tracy–Widom theory.** Writing $\phi_a(x) = J_a(\sqrt{x})$ for short and, as integral operators acting on $L^2(0, s)$,

$$\hat{K}_n^a = K_\infty^a + \frac{a}{8n}\phi_a \otimes \phi_a,$$

$$I - \hat{K}_n^a = (I - K_\infty^a)\left(I - \frac{a}{8n}(I - K_n^a)^{-1}\phi_a \otimes \phi_a\right),$$

we get from (4) with $c = -a$, by the same reasoning as in the last section,

$$\begin{aligned} F_n^a(s) &= \det(I - \hat{K}_n^a) + O(n^{-2}) \\ &= \det(I - K_\infty^a) \cdot \det\left(I - \frac{a}{8n}(I - K_n^a)^{-1}\phi_a \otimes \phi_a\right) + O(n^{-2}) \\ &= F_\infty^a(s) \cdot \left(1 - \frac{a}{8n}\langle (I - K_n^a)^{-1}\phi_a, \phi_a\rangle_{L^2(0,s)}\right) + O(n^{-2}). \end{aligned}$$

Comparing the $n^{-1}$ term of the expansion with (2) establishes the relation

$$-\frac{1}{4}\langle (I - K_n^a)^{-1}\phi_a, \phi_a\rangle_{L^2(0,s)} = s\frac{f_\infty^a(s)}{F_\infty^a(s)} = s\frac{d}{ds}\log F_\infty^a(s), \tag{6}$$

which can already be found in the work of Tracy and Widom. In fact, from the relation $\log\det = \operatorname{tr}\log$, one gets that the logarithmic derivative of a Fredholm determinant $F(s)$ of a trace class operator $K$ acting on $L^2(0, s)$ is generally given by [11], equation (1.5),

$$\frac{d}{ds}F(s) = \frac{d}{ds}\log\det(I - K|_{L^2(0,s)}) = -R(s, s),$$

where $R(x, y)$ is the kernel of the operator $K(I - K)^{-1}$. Specifically, in the case of the Bessel kernel, Tracy and Widom calculated [11], equations (2.5) and (2.21), that

$$sR(s, s) = \tfrac{1}{4}\langle (I - K_n^a)^{-1}\phi_a, \phi_a\rangle_{L^2(0,s)},$$

which finally reproves (6).

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ZENTRUM MATHEMATIK—M3
TECHNISCHE UNIVERSITÄT MÜNCHEN
80290 MÜNCHEN
GERMANY
E-MAIL: bornemann@tum.de