SUPER-REPLICATION WITH NONLINEAR TRANSACTION COSTS AND VOLATILITY UNCERTAINTY

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We study super-replication of contingent claims in an illiquid market with model uncertainty. Illiquidity is captured by nonlinear transaction costs in discrete time and model uncertainty arises as our only assumption on stock price returns is that they are in a range specified by fixed volatility bounds. We provide a dual characterization of super-replication prices as a supremum of penalized expectations for the contingent claim's payoff. We also describe the scaling limit of this dual representation when the number of trading periods increases to infinity. Hence, this paper complements the results in [*Finance Stoch.* **17** (2013) 447–475] and [*Ann. Appl. Probab.* **5** (1995) 198–221] for the case of model uncertainty.

1. Introduction. We study an illiquid discrete-time market with model uncertainty. As in [11], we consider the case where the size of the trade has an immediate but temporary effect on the price of the asset. This model captures the classical case of proportional transaction costs as well as other illiquidity models such as the discrete-time version of the model introduced by Çetin, Jarrow and Protter in [5] for continuous time. By contrast to [11], our sole assumption on the price dynamics of the traded security is that the absolute value of the log-returns is bounded from below and above. This is a natural discrete-time version of the widely studied uncertain volatility models; see, for example, [8, 22] and [25]. The paper [10] studies super-replication of game options in such a discrete-time model, but does not allow for any market frictions.

The benchmark problem in models with uncertain volatilities is the description of super-replication prices. In our version of such a result in a model with transaction costs, we provide a general duality for European options with an upper-semicontinuous payoff. Specifically, Theorem 2.2 provides a combination of the dual characterization for super-replication prices in frictionless uncertain volatility models (see [8]) with analogous duality formulae in binomial markets with frictions (see [11]). Let us emphasize that, under volatility uncertainty, we do not

define super-replication prices merely in an almost sure-sense because we actually insist on super-replication for *any* possible evolution of stock prices which respects the specified volatility bounds. Since, in contrast to [11], in our setup the set of all possible stock price evolutions is uncountable; we cannot find a dominating probability measure which will give positive weight to every possible evolution of stock prices. Hence, the duality result of Theorem 2.2 goes beyond the now classical results for almost sure super-replication, and it complements the findings of [4], who consider a probabilistically more general setting albeit without frictions. For a discussion of the fundamental theorem of asset pricing with proportional transaction costs under model uncertainty, see [3]. The key observation for the proof of Theorem 2.2 is that for continuous payoffs one can find approximative discrete models where classical duality results give us super-replication strategies that can be lifted to the original setting with uncertain volatilities in a way that allows us to control the difference in profits and losses.

In Theorem 2.3, we consider the special case of a convex payoff profile. In frictionless models with volatility uncertainty, it is well known from, for example, [18, 21] or Remark 7.20 in [15] that the super-replication price coincides with the one computed in the classical model where the volatility always takes the maximal value. We show that this result also holds in our framework with nonlinear transaction costs if these are deterministic.

Finally, we study the scaling limit of our super-replication prices when the number of trading periods becomes large. Theorem 2.7 describes this scaling limit as the value of a stochastic volatility control problem on the Wiener space. We use the duality result Theorem 2.2 and study the limit of the dual terms by applying the theory of weak convergence of stochastic processes. This extends the approach taken in [11], since we need to extend Kusuoka's construction of suitable martingales (see [19], Section 5) to our setting with volatility uncertainty. As a result, we get an understanding of how Kusuoka's finding of volatility uncertainty in the description of the scaling limit is extended to our setting which already starts with volatility uncertainty. In the special case of a frictionless setup, we recover Peng's [23] result that the limit is equal to the payoff's G-expectation with upper and lower bounds as in the discrete setup. In setups with market frictions, continuous-time models are known to produce trivial super-replication prices when one considers proportional transaction costs (cf., e.g., [20] and [24]) or no liquidity effect at all when one has nonlinear, differentiable transaction costs or market impact; see [5] or [2]. Our scaling limit, by contrast, gives a value in between these two extremes and can be viewed as a convex measure of risk for the payoff as in [14] or [16].

Our approach to the proof of the main results is purely probabilistic and based on the theory of weak convergence of stochastic processes. This approach allows us to study a quite general class of path dependent European options, and a general class of nonlinear transaction costs.

2. Preliminaries and main results.

2.1. *The discrete-time model.* Let us start by introducing a discrete time financial model with volatility uncertainty. We fix a time horizon $N \in \mathbb{N}$ and consider a financial market with a riskless savings account and a risky stock. The savings account will be used as a numeraire, and thus we normalize its value at time $n = 0, \ldots, N$ to $B_n = 1$. The stock price evolution starting from $s > 0$ will be denoted by $S_n > 0$, $n = 0, 1, \ldots, N$. Hence, by introducing the log-return $X_n \triangleq \log(S_n/S_{n-1})$ for period $n = 1, \ldots, N$ we can write

$$(2.1) \qquad S_n = s_0 \exp\left(\sum_{m=1}^{n} X_m \right), \qquad n = 0, \ldots, N.$$

Our sole assumption on these dynamics will be that there are volatility bounds on the stock's price fluctuations in the sense that the absolute values of these log-returns are bounded from below and above:

$$(2.2) \qquad \underline{\sigma} \le |X_n| \le \overline{\sigma}, \qquad n = 1, \ldots, N,$$

for some constants $0 \le \underline{\sigma} \le \overline{\sigma} < \infty$. In other words the log-returns will take values in the path-space

$$\Omega \triangleq \Omega_{\underline{\sigma},\overline{\sigma}} \triangleq \left\{ \omega = (x_1, \ldots, x_N) \in \mathbb{R}^N : \underline{\sigma} \le |x_n| \le \overline{\sigma}, n = 1, \ldots, N \right\}$$

and identifying these returns with the canonical process

$$X_k(\omega) \triangleq x_k \qquad \text{for } \omega = (x_1, \ldots, x_N) \in \Omega$$

we find that (2.1) allows us to view the stock's price evolution as a process $S = (S_n)_{n=0,\ldots,N}$ defined on Ω. Clearly, the canonical filtration

$$\mathcal{F}_n \triangleq \sigma(X_1, \ldots, X_n), \qquad n = 0, \ldots, N,$$

coincides with the one generated by $S = (S_n)_{n=0,\ldots,N}$. Similar models with volatility uncertainty have been considered in [10].

The aim of the present paper is to study the combined effects of volatility uncertainty and nonlinear transaction costs. Following [5, 11, 17], we assume these costs to be given by a penalty function

$$g : \{0, 1, \ldots, N\} \times \Omega \times \mathbb{R} \to \mathbb{R}_+,$$

where $g(n, \omega, \beta)$ denotes the costs (in terms of our numeraire B) of trading $\beta \in \mathbb{R}$ worth of stock at time n when the evolution of the stock price is determined by the returns from $\omega \in \Omega$.

ASSUMPTION 2.1. The cost function

$$g : \{0, 1, \ldots, N\} \times \Omega \times \mathbb{R} \to \mathbb{R}_+$$

is $(\mathcal{F}_n)_{n=0,\ldots,N}$-adapted. Moreover, for any $n = 0, \ldots, N$, the costs $g(n, \omega, \beta)$ are a nonnegative convex function in $\beta \in \mathbb{R}$ with $g(n, \omega, 0) = 0$ for any fixed $\omega \in \Omega$ and a continuous function in $\omega \in \Omega$ for any fixed $\beta \in \mathbb{R}$.

For simplicity of notation, we will often suppress the dependence of costs on ω and simply write $g_n(\beta)$ for $g(n, \omega, \beta)$. We will proceed similarly with other functions depending on $\omega \in \Omega$.

In our setup, a trading strategy is a pair $\pi = (y, \gamma)$ where y denotes the initial wealth and $\gamma : \{0, 1, \dots, N-1\} \times \Omega \to \mathbb{R}$ is an $(\mathcal{F}_n)$-adapted process specifying the number $\gamma_n = \gamma(n, \omega)$ of shares held at the beginning of any period $n = 0, \dots, N-1$ with the stock price evolution given by $\omega \in \Omega$. The set of all portfolios starting with initial capital y will be denoted by $\mathcal{A}(y)$.

The evolution of the mark-to-market value $Y^\pi = (Y_n^\pi(\omega))_{n=0,\dots,N}$ resulting from a trading strategy $\pi = (y, \gamma) \in \mathcal{A}(y)$ is given by $Y_0^\pi = y$ and the difference equation

$$Y_{n+1}^\pi - Y_n^\pi = \gamma_n(S_{n+1} - S_n) - g_n\big((\gamma_n - \gamma_{n-1})S_n\big), \qquad n = 0, \dots, N-1,$$

where we let $\gamma_{-1} \triangleq 0$. Hence, we start with zero stocks in our portfolio and trading to the new position γ_n to be held after time n incurs the transaction costs $g_n((\gamma_n - \gamma_{n-1})S_n)$, which is the only friction in our model. Hence, the value Y_{n+1}^π represents the portfolio's mark-to-market value *before* the transaction at time $n+1$ is made. Note that, focussing on the mark-to-market value rather than the liquidation value, we disregard in particular the costs of unwinding any nonzero position for simplicity.

2.2. *Robust super-replication with frictions.* The benchmark problem for models with uncertain volatility is the super-replication of a contingent claim. We investigate this problem in the presence of market frictions as specified by a function g satisfying Assumption 2.1. So consider a European option $\mathbb{F} : \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ which pays off $\mathbb{F}(S)$ when the stock price evolution is $S = (S_n)_{n=0,\dots,N}$. The super-replication price $V(\mathbb{F}) = V_g^{\underline{\sigma}, \overline{\sigma}}(\mathbb{F})$ is then defined as

$$V_g^{\underline{\sigma}, \overline{\sigma}}(\mathbb{F}) \triangleq \inf\big\{y \in \mathbb{R} : \exists \pi \in \mathcal{A}(y) \text{ with } Y_N^\pi(\omega) \geq \mathbb{F}\big(S(\omega)\big) \ \forall \omega \in \Omega_{\underline{\sigma}, \overline{\sigma}}\big\}.$$

We emphasize that we require the construction of a *robust* super-replication strategy π which leads to a terminal value Y_N^π that dominates the payoff X in *any* conceivable scenario $\omega \in \Omega$.

Our first result provides a dual description of super-replication prices.

THEOREM 2.2.   *Let $G_n : \Omega \times \mathbb{R} \to \mathbb{R}_+, n = 0, 1, \dots, N$, denote the Legendre–Fenchel transform (or convex conjugate) of g_n, that is,*

$$G_n(\alpha) \triangleq \sup_{\beta \in \mathbb{R}} \{\alpha\beta - g_n(\beta)\}, \qquad \alpha \in \mathbb{R}.$$

Then, under Assumption 2.1, the super-replication price of any contingent claim $\mathbb{F}$ with upper-semicontinuous payoff function $\mathbb{F} : \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ is given by

$$V(\mathbb{F}) = \sup_{\mathbb{P} \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}}} \mathbb{E}_{\mathbb{P}}\left[\mathbb{F}(S) - \sum_{n=0}^{N-1} G_n\left(\frac{\mathbb{E}_{\mathbb{P}}[S_N | \mathcal{F}_n] - S_n}{S_n}\right)\right],$$

where $\mathcal{P}_{\underline{\sigma},\overline{\sigma}}$ denotes the set of all Borel probability measures on $\Omega = \Omega_{\underline{\sigma},\overline{\sigma}}$ and where $\mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to such a probability measure $\mathbb{P}$.

The proof of this theorem will be carried out in Section 3.1 below. Observe that this result is a hybrid of the dual characterization for super-replication prices in frictionless uncertain volatility models and of analogous duality formulae in binomial markets with frictions; see [9] and [11], respectively.

2.3. *Convex payoff functions.* Our next result deals with the special case where the payoff $\mathbb{F}$ is a nonnegative *convex* function of the stock price evolution $S = (S_n)_{n=0,\ldots,N}$. It is well known that in a frictionless binomial model, the price of a European option with a convex payoff is an increasing function of the volatility. This implies that super-replication prices in uncertain volatility models coincide with the replication costs in the model with maximal compatible volatility; see [21]. The next theorem gives a generalization of this claim for the setup of volatility uncertainty under friction.

THEOREM 2.3. *Suppose that the cost function $g = g(n, \omega, \beta)$ is deterministic in the sense that it does not depend on $\omega \in \Omega$.*

Then the super-replication price of any convex payoff $\mathbb{F} : \mathbb{R}_+^{N+1} \to \mathbb{R}_+$ is given by

$$V_g^{\underline{\sigma},\overline{\sigma}}(\mathbb{F}) = \overline{V}_g(\mathbb{F}),$$

where $\overline{V}_g(\mathbb{F}) \triangleq V_g^{\overline{\sigma},\overline{\sigma}}(\mathbb{F})$ denotes the super-replication price of $\mathbb{F} = \mathbb{F}(S)$ in the Binomial model with frictions for S with volatility $\overline{\sigma}$ and cost function g.

PROOF. The relation "$\geq$" holding true trivially, it suffices to construct, for any $\varepsilon > 0$, a strategy γ which super-replicates $\mathbb{F}(S)$ in every scenario from Ω starting with initial capital $y = \varepsilon + \overline{V}_g(\mathbb{F})$.

The binomial model with volatility $\overline{\sigma}$ can be formalized on $\overline{\Omega} \triangleq \{-1, 1\}^N$ with canonical process $\overline{X}_k(\overline{\omega}) \triangleq \overline{x}_k$ for $\overline{\omega} = (\overline{x}_1, \ldots, \overline{x}_N) \in \overline{\Omega}$ by letting the stock price evolution be given inductively by $\overline{S}_0 \triangleq s_0$ and $\overline{S}_n \triangleq \overline{S}_{n-1} \exp(\overline{\sigma}\overline{X}_n)$, $n = 1, \ldots, N$. With $(\overline{\mathcal{F}}_n)_{n=0,\ldots,N}$ denoting the corresponding canonical filtration, we get from the definition of $\overline{V}_g(\mathbb{F})$ that there is an $(\overline{\mathcal{F}}_n)_{n=0,\ldots,N}$-adapted process $\overline{\gamma}$ such that with $\overline{\gamma}_{-1} \triangleq 0$ we have

$$(2.3) \quad \varepsilon + \overline{V}_g(\mathbb{F}) + \sum_{n=0}^{N-1} \overline{\gamma}_n(\overline{S}_{n+1} - \overline{S}_n) - \sum_{n=0}^{N-1} g_n((\overline{\gamma}_n - \overline{\gamma}_{n-1})\overline{S}_n) \geq \mathbb{F}(\overline{S})$$

everywhere on $\overline{\Omega}$.

In view of (2.2), for any $\omega \in \Omega$ and $n = 1, \ldots, N$ there are unique weights $\lambda_n^{(+1)}(\omega), \lambda_n^{(-1)}(\omega) \geq 0$ with $\lambda_n^{(+1)}(\omega) + \lambda_n^{(-1)}(\omega) = 1$ such that

$$(2.4) \qquad\qquad e^{X_n(\omega)} = \lambda_n^{(+1)}(\omega)e^{\overline{\sigma}} + \lambda_n^{(-1)}(\omega)e^{-\overline{\sigma}}.$$

It is readily checked that also the weights

$$\lambda_n^{\overline{\omega}^n}(\omega) \triangleq \prod_{m=1}^{n} \lambda_m^{(\overline{\omega}_m^n)}(\omega), \qquad \overline{\omega}^n = (\overline{\omega}_1^n, \ldots, \overline{\omega}_n^n) \in \{-1, +1\}^n,$$

sum up to 1. Indeed,

$$(2.5) \qquad \sum_{\overline{\omega}^n \in \{-1,+1\}^n} \lambda_n^{\overline{\omega}^n} = \prod_{m=1}^{n} (\lambda_m^{(+1)}(\omega) + \lambda_m^{(-1)}(\omega)) = 1, \qquad n = 1, \ldots, N.$$

Moreover, for $n = 1, \ldots, N-1$ we have

$$(2.6) \quad \lambda_n^{\overline{\omega}^n} = \lambda_n^{\overline{\omega}^n} \prod_{m=n+1}^{N} (\lambda_m^{(+1)}(\omega) + \lambda_m^{(-1)}(\omega)) = \sum_{\overline{\omega}^{N-n} \in \{-1+,1\}^{N-n}} \lambda_N^{(\overline{\omega}^n, \overline{\omega}^{N-n})},$$

which in conjunction with (2.4) and the adaptedness of $\overline{S}$ entails the representation

$$(2.7) \quad \begin{aligned} S_n(\omega) &= s_0 \prod_{m=1}^{n} (\lambda_m^{(+1)}(\omega) e^{\overline{\sigma}} + \lambda_n^{(-1)}(\omega) e^{-\overline{\sigma}}) \\ &= \sum_{\overline{\omega}^n \in \{-1,+1\}^n} \overline{S}_n(\overline{\omega}^n, 1, \ldots, 1) \lambda_n^{\overline{\omega}^n}(\omega) = \sum_{\overline{\omega} \in \overline{\Omega}} \overline{S}_n(\overline{\omega}) \lambda_N^{\overline{\omega}}(\omega) \end{aligned}$$

for any $n = 1, \ldots, N, \omega \in \Omega$.

Now evaluate (2.3) at $\overline{\omega} \in \overline{\Omega}$, multiply by $\lambda_N^{\overline{\omega}}$ and then take the sum over all $\overline{\omega} \in \overline{\Omega}$.

The right-hand side of (2.3) then aggregates to

$$(2.8) \qquad R \triangleq \sum_{\overline{\omega} \in \overline{\Omega}} \mathbb{F}(\overline{S}(\overline{\omega})) \lambda_N^{\overline{\omega}} \geq \mathbb{F}\left(\sum_{\overline{\omega} \in \overline{\Omega}} \overline{S}(\overline{\omega}) \lambda_N^{\overline{\omega}} \right) = \mathbb{F}(S),$$

where the estimate follows from (2.5) in conjunction with the convexity of $\mathbb{F} : \mathbb{R}_+^{N+1} \to \mathbb{R}$ and where for the last identity we exploited (2.7).

On the left-hand side of (2.3) the contributions from the constant $\varepsilon + \overline{V}_g(\mathbb{F})$ just reproduce this very constant because of (2.5). Since $\overline{\gamma}$ and $\overline{S}$ are $(\overline{\mathcal{F}}_n)_{n=0,\ldots,N}$-adapted, the nth summand in the first sum of (2.3) contributes

$$\begin{aligned} I_n &\triangleq \sum_{\overline{\omega} \in \overline{\Omega}} \overline{\gamma}_n(\overline{\omega})(\overline{S}_{n+1}(\overline{\omega}) - \overline{S}_n(\overline{\omega})) \lambda_N^{\overline{\omega}} \\ &= \sum_{\overline{\omega}^n \in \{-1,1\}^n} \sum_{\overline{x} \in \{-1,1\}} (\overline{\gamma}_n \overline{S}_n)(\overline{\omega}^n, 1, \ldots, 1)(e^{\overline{\sigma x}} - 1) \lambda_n^{\overline{\omega}^n} \lambda_{n+1}^{(\overline{x})}. \end{aligned}$$

By definition of $\lambda_{n+1}^{(\pm 1)}$ we have

$$\sum_{\overline{x} \in \{-1,1\}} (e^{\overline{\sigma x}} - 1) \lambda_{n+1}^{(\overline{x})} = e^{X_{n+1}} - 1 = (S_{n+1} - S_n)/S_n$$

which entails

$$I_n = \sum_{\overline{\omega}^n \in \{-1,1\}^n} (\overline{\gamma}_n \overline{S}_n)(\overline{\omega}^n, 1, \ldots, 1) \lambda_n^{\overline{\omega}^n} (S_{n+1} - S_n)/S_n$$
$$= \gamma_n (S_{n+1} - S_n),$$

where the $(\mathcal{F}_n)_{n=0,\ldots,N}$-adapted process γ is given by $\gamma_0 \triangleq \overline{\gamma}_0$ and

$$\gamma_n(\omega) = \sum_{\overline{\omega}^n \in \{-1,1\}^n} (\overline{\gamma}_n \overline{S}_n)(\overline{\omega}^n, 1, \ldots, 1) \lambda_n^{\overline{\omega}^n}(\omega)/S_n(\omega), \qquad \omega \in \Omega,$$

for $n = 1, \ldots, N - 1$. In a similar fashion, the nth summand from the second sum in (2.3) gives

$$II_n \triangleq \sum_{\overline{\omega} \in \overline{\Omega}} g_n\big((\overline{\gamma}_n(\overline{\omega}) - \overline{\gamma}_{n-1}(\overline{\omega}))\overline{S}_n(\overline{\omega})\big)\lambda_N^{\overline{\omega}}$$

$$\geq g_n\left(\sum_{\overline{\omega} \in \overline{\Omega}} (\overline{\gamma}_n(\overline{\omega}) - \overline{\gamma}_{n-1}(\overline{\omega}))\overline{S}_n(\overline{\omega})\lambda_N^{\overline{\omega}}\right)$$

$$= g_n\big((\gamma_n - \gamma_{n-1})S_n\big),$$

where the estimate is due to (2.5) and the convexity of $g_n : \mathbb{R} \to \mathbb{R}$ and where the last identity is due to the adaptedness of $\overline{\gamma}$, $\overline{S}$ and to (2.6). As a consequence, the left-hand side of (2.3) aggregates in the above manner to

$$L \triangleq \varepsilon + \overline{V}_g(\mathbb{F}) + \sum_{n=0}^{N-1} I_n - \sum_{n=0}^{N-1} II_n$$

$$\leq \varepsilon + \overline{V}_g(\mathbb{F}) + \sum_{n=0}^{N-1} \gamma_n(S_{n+1} - S_n) - \sum_{n=0}^{N-1} g_n\big((\gamma_n - \gamma_{n-1})S_n\big).$$

In light of our estimate (2.8) for the analogously aggregated right-hand side of (2.3) this shows that γ super-replicates $\mathbb{F}(S)$ with initial capital $\varepsilon + \overline{V}_g(\mathbb{F})$. This completes our proof. □

2.4. *Scaling limit.* Our last result gives a dual description for the scaling limit of our super-replication prices when the number of periods N becomes large, stock returns are scaled by $1/\sqrt{N}$ and earned over periods of length $1/N$. The limiting trading costs will be specified in terms of a function

$$h : [0, 1] \times C[0, 1] \times \mathbb{R} \to \mathbb{R}_+,$$

$$(t, w, \beta) \mapsto h_t(w, \beta)$$

such that:

- for any $t \in [0, 1]$, $w \in C[0, 1]$, $\beta \mapsto h_t(w, \beta)$ is nonnegative and convex with $h_t(w, 0) = 0$;

- for any $\beta \in \mathbb{R}$ the process $h(\beta) = (h_t(w, \beta))_{t \in [0,1]}$ is progressively measurable in the sense that $h_t(w, \beta) = h_t(\tilde{w}, \beta)$ if $w_{[0,t]} = \tilde{w}_{[0,t]}$.

For technical reasons, we were able to establish our scaling limit only for a suitably truncated penalty function. For a given truncation level $c > 0$, we consider the linearly extrapolated costs h^c given by

$$h_t^c(w, \beta) \triangleq \begin{cases} h_t(w, \beta), & \text{for } \beta \in I_c(h), \\ \text{linear extrapolation with slope } c \text{ beyond } I_c(h), \end{cases}$$

where $I_c(h) \triangleq \{\beta \in \mathbb{R} : |\frac{\partial h}{\partial \beta}| \le c\}$ denotes the interval around zero where the slope of h has not yet exceeded c in absolute value. For a truncated penalty h^c, the dual formula of super-replication prices will only involve measures which satisfy a certain tightness condition; this allows us to establish the upper bound of our scaling limit in Section 3.2.2. For the original h (by taking c to infinity), we can only prove the lower bound of the super-replication prices; the proof for a tight upper bound remaining an open problem in this case.

The cost for the N period model with returns in

$$\Omega^N \triangleq \{\omega^N = (x_1, \ldots, x_N) : \underline{\sigma}/\sqrt{N} \le |x_n| \le \overline{\sigma}/\sqrt{N}, n = 1, \ldots, N\}$$

are given by

$$(2.9) \qquad g_n^{N,c}(\omega^N, \beta) \triangleq h_{n/N}^{c/\sqrt{N}}(\overline{S}^N(\omega^N), \beta),$$

where $\overline{S}^N(\omega^N) \in C[0,1]$ denotes the continuous linear interpolation in $C[0,1]$ of the points

$$\overline{S}_{n/N}^N(\omega^N) \triangleq s_0 \exp\left(\sum_{m=1}^{n} x_m\right), \qquad n = 0, \ldots, N,$$

for $\omega^N = (x_1, \ldots, x_N) \in \Omega^N$.

The technical assumption for our asymptotics to work out is the following.

ASSUMPTION 2.4.   The Legendre–Fenchel transform $H : [0,1] \times C[0,1] \times \mathbb{R} \to \mathbb{R}$ with

$$H_t(w, \alpha) = \sup_{\beta \in \mathbb{R}} \{\alpha\beta - h_t(w, \beta)\}, \qquad t \in [0,1], w \in C[0,1], \alpha \in \mathbb{R}$$

has polynomial growth in (w, α) uniformly in t in the sense that there are constants $C, p_1, p_2 \ge 0$ such that

$$H_t(w, \alpha) \le C(1 + \|w\|_\infty^{p_1})(1 + |\alpha|^{p_2}), \qquad (t, w, \alpha) \in [0,1] \times C[0,1] \times \mathbb{R}.$$

In addition, H is continuous in (t, w) and essentially quadratic in α asymptotically, that is, there is a function $\hat{H} : [0,1] \times \mathcal{C}[0,1] \to \mathbb{R}_+$ such that for any sequence $\{(t_N, w_N, \alpha_N)\}_{N=1,2,\ldots}$ converging to (t, w, α) in $[0,1] \times C[0,1] \times \mathbb{R}$ we have

$$\lim_{N \to \infty} |N H_{t_N}(w_N, \alpha_N/\sqrt{N}) - \hat{H}_t(w)\alpha^2| = 0.$$

Let us give two examples.

EXAMPLE 2.5.   (i) Proportional transaction costs: Fix $c > 0$ and consider the cost functions given by

$$g_n^{N,c}(\omega^N, \beta) \triangleq \frac{c}{\sqrt{N}} |\beta|$$

for our binomial market model with N trading periods. This example was studied in [19] for binomial models and corresponds in our setup to the case $\hat{H} = H \equiv 0$.

(ii) Quadratic costs. For a given constant $\Lambda > 0$, let

$$h_t(w, \beta) = \Lambda \beta^2.$$

Fix $c > 0$ and observe that our truncation h^c of h is then

$$h^c(w, \beta) = \begin{cases} \Lambda \beta^2, & \text{if } |\beta| \leq \dfrac{c}{2\Lambda}, \\ c\beta - \dfrac{c^2}{4\Lambda}, & \text{else.} \end{cases}$$

Thus, the penalty in the N-step binomial model is given by

$$g_n^{N,c}(\omega^N, \beta) = \begin{cases} \Lambda \beta^2, & \text{if } |\beta| \leq \dfrac{c}{2\Lambda\sqrt{N}}, \\ \dfrac{c}{\sqrt{N}} |\beta| - \dfrac{c^2}{4\Lambda N}, & \text{else.} \end{cases}$$

Hence, the marginal costs from trading slightly higher volumes are linear for a small total trading volume and constant for a large one. This example corresponds to the case where $H_t(w, \beta) = \frac{\beta^2}{4\Lambda}$ and $\hat{H}_t(w) \equiv \frac{1}{4\Lambda}$.

REMARK 2.6.   A sufficient condition for a limiting cost process h to satisfy our Assumption 2.4 is the joint validity of:

(i) there exists $\varepsilon > 0$ such that for any $(t, w) \in [0, T] \times C^+[0, T]$, the second derivative $\frac{\partial^2 h}{\partial \beta^2}(t, w, \beta)$ exists for any $-\varepsilon w(t) < \beta < \varepsilon w(t)$, and is continuous at $(t, w, 0)$, and
(ii) we have

$$\frac{\partial h}{\partial \beta}(t, w, 0) \equiv 0 \quad \text{and} \quad \inf_{(t,w) \in [0,T] \times C^+[0,T]} \inf_{|\beta| < \varepsilon w(t)} \frac{\partial^2 h}{\partial \beta^2}(t, w, \beta) > 0.$$

Indeed, sufficiency of these conditions can be verified by use of a Taylor expansion.

Under Assumption 2.4, the scaling limit for the discrete-time super-replication prices can be described as follows.

THEOREM 2.7. *Suppose that Assumption* 2.4 *holds and that* $\underline{\sigma} > 0$. *Furthermore assume that* $\mathbb{F} : C[0, 1] \to \mathbb{R}_+$ *is continuous with polynomial growth*: $0 \leq \mathbb{F}(S) \leq C(1 + \|S\|_\infty)^p$, $S \in C[0, 1]$, *for some constants* $C, p \geq 0$.
Then we have

$$\lim_{N \to \infty} V_{g^{N,c}}^{\underline{\sigma}/\sqrt{N}, \overline{\sigma}/\sqrt{N}}(\mathbb{F}) = \sup_{\sigma \in \Sigma(c)} \mathbb{E}^W \left[\mathbb{F}(S^\sigma) - \int_0^1 \hat{H}_t(S^\sigma) a^2(\sigma_t) \, dt \right],$$

where:

- $\mathbb{E}^W$ *denotes the expectation with respect to* $\mathbb{P}^W$, *the Wiener measure on* $(C[0, 1], \mathcal{B}(C[0, 1]))$, *for which the canonical process W is a Brownian motion*,
- $\Sigma(c)$ *is the class of processes $\sigma \geq 0$ on Wiener space which are progressively measurable with respect to the filtration generated by W and such that*

$$a(\sigma_t) \leq c$$

for

$$a(\sigma) \triangleq \begin{cases} \dfrac{1}{2} \dfrac{\underline{\sigma}^2 - \sigma^2}{\underline{\sigma}}, & 0 \leq \sigma \leq \underline{\sigma}, \\ 0, & \underline{\sigma} \leq \sigma \leq \overline{\sigma}, \\ \dfrac{1}{2} \dfrac{\sigma^2 - \overline{\sigma}^2}{\overline{\sigma}}, & \overline{\sigma} \leq \sigma, \end{cases}$$

- S^σ *denotes the stock price evolution with volatility σ*:

$$S_t^\sigma \triangleq s_0 \exp\left(\int_0^t \sigma_s \, dW_s - \frac{1}{2} \int_0^t \sigma_s^2 \, ds \right), \qquad 0 \leq t \leq 1.$$

The proof of this result is deferred to Section 3.2. One way to interpret Theorem 2.7 is that it describes the scaling limit of super-replication prices as a convex risk measure for the payoff; see [14] or [16]. The class $\Sigma(c)$ parametrizes the volatility models which one chooses to take into account and the integral term measures the relevance one associates with the payoff's mean under any such model. From this perspective, the most relevant models are those where the volatility stays within the prescribed uncertainty region $[\underline{\sigma}, \overline{\sigma}]$ (so that the integral term vanishes). One also considers models with higher or lower volatilities σ_t, though, [for as long as $a(\sigma_t) \leq c$], but diminishes their relevance according to the average difference of their local variances from $\underline{\sigma}^2$ on the low and $\overline{\sigma}^2$ on the high side as measured by $a(\sigma_t)$. In particular, the scaling limit of our super-replication prices is bounded from below by the G-expectation $\sup_{\sigma \in [\underline{\sigma}, \overline{\sigma}]} \mathbb{E}^W[\mathbb{F}(S^\sigma)]$. In the frictionless case (where $h \equiv 0$, $\hat{H} \equiv \infty$), the penalty for choosing a volatility model with values outside the interval $[\underline{\sigma}, \overline{\sigma}]$ is infinite and we recover the well-known frictionless characterization of super-replication prices under uncertainty as the payoff's G-expectation.

3. Proofs. In this section, we carry out the proofs of Theorems 2.2 and 2.7.

3.1. *Proof of Theorem* 2.2. Theorem 2.2 is concerned with the identity $V(\mathbb{F}) = U(\mathbb{F})$ where

$$U(\mathbb{F}) \triangleq \sup_{\mathbb{P} \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}}} \mathbb{E}_{\mathbb{P}}\left[\mathbb{F}(S) - \sum_{n=0}^{N-1} G_n\left(\frac{\mathbb{E}_{\mathbb{P}}[S_N | \mathcal{F}_n] - S_n}{S_n} \right) \right].$$

As a first step, we note the following.

LEMMA 3.1. *For any measurable* $\mathbb{F} : \mathbb{R}_+^{N+1} \to \mathbb{R}_+$, *we have that* $V(\mathbb{F}) \geq U(\mathbb{F})$.

PROOF. Let $\pi = (y, \gamma)$ super-replicate $\mathbb{F}(S)$. Then we have

$$\mathbb{F}(S) \leq Y_N^\pi = y + \sum_{n=0}^{N-1} \left(\gamma_n (S_{n+1} - S_n) - g_k((\gamma_n - \gamma_{n-1})S_n) \right)$$

$$= y + \sum_{n=0}^{N-1} \left((\gamma_n - \gamma_{n-1})(S_N - S_n) - g_n((\gamma_n - \gamma_{n-1})S_n) \right).$$

Taking (conditional) expectations with respect to any $\mathbb{P} \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}}$, this shows that

$$\mathbb{E}_{\mathbb{P}}[\mathbb{F}(S)] \leq y + \mathbb{E}_{\mathbb{P}}\left[\sum_{n=0}^{N-1} (\gamma_n - \gamma_{n-1})(\mathbb{E}_{\mathbb{P}}[S_N | \mathcal{F}_n] - S_n) - g_n((\gamma_n - \gamma_{n-1})S_n) \right]$$

$$= y + \mathbb{E}_{\mathbb{P}}\left[\sum_{n=0}^{N-1} (\gamma_n - \gamma_{n-1})S_n \frac{\mathbb{E}_{\mathbb{P}}[S_N | \mathcal{F}_n] - S_n}{S_n} - g_n((\gamma_n - \gamma_{n-1})S_n) \right]$$

$$\leq y + \mathbb{E}_{\mathbb{P}}\left[\sum_{n=0}^{N-1} G_n\left(\frac{\mathbb{E}_{\mathbb{P}}[S_N | \mathcal{F}_n] - S_n}{S_n} \right) \right],$$

where the final estimate follows from the definition of the dual functions G_n, $n = 0, \dots, N$.

Since this holds for arbitrary $\mathbb{P} \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}}$ and any initial wealth y for which we can find a super-replicating strategy, the preceding estimate yields $V(\mathbb{F}) \geq U(\mathbb{F})$. $\square$

We next observe that an identity analogous to $U(\mathbb{F}) = V(\mathbb{F})$ holds for multinomial models.

LEMMA 3.2. *For* $k \in \{1, 2, \dots\}$ *consider the finite set*

$$\Omega^k \triangleq \left\{ x \in \Omega : |x_i| = \frac{j}{k}\underline{\sigma} + \left(1 - \frac{j}{k}\right)\overline{\sigma} \text{ for some } j \in \{0, \dots, k\} \right\}$$

and let $\mathcal{P}^k_{\underline{\sigma},\overline{\sigma}}$ be the subset of $\mathcal{P}_{\underline{\sigma},\overline{\sigma}}$ which contains those discrete probability measures that are supported by Ω^k.

Then we have

$$V^k(\mathbb{F}) = U^k(\mathbb{F}),$$

where

$$V^k(\mathbb{F}) = \inf\{y : Y^{(y,\gamma)}_N(\omega) \geq \mathbb{F}(S(\omega)) \ \forall \omega \in \Omega^k \text{ for some strategy } \gamma\}$$

and

$$U^k(\mathbb{F}) = \sup_{\mathbb{P} \in \mathcal{P}^k_{\underline{\sigma},\overline{\sigma}}} \mathbb{E}_{\mathbb{P}}\left[\mathbb{F}(S) - \sum_{n=0}^{N-1} G_n\left(\frac{\mathbb{E}_{\mathbb{P}}[S_N|\mathcal{F}_n] - S_n}{S_n}\right)\right].$$

PROOF. For $k = 1$, that is, in the binomial case, this is just Theorem 3.1 in [11]. This result is proved by observing that the identity can be cast as a finite dimensional convex duality claim. The same reasoning actually applies to the multinomial setup with $k > 1$ as well. This establishes our claim. □

In a third step, we argue how to pass to the limit $k \uparrow \infty$, first for continuous $\mathbb{F}$.

LEMMA 3.3. *With the notation of Lemma 3.2, we have*

$$(3.1) \qquad\qquad U^k(\mathbb{F}) \leq U(\mathbb{F}), \qquad k = 1, 2, \ldots.$$

If $\mathbb{F}$ is continuous, we have furthermore

$$(3.2) \qquad\qquad \liminf_{k \uparrow \infty} V^k(\mathbb{F}) \geq V(\mathbb{F}).$$

PROOF. Estimate (3.1) is immediate from the definitions of $U^k(\mathbb{F})$ and $U(\mathbb{F})$ as $\mathcal{P}^k_{\underline{\sigma},\overline{\sigma}} \subset \mathcal{P}_{\underline{\sigma},\overline{\sigma}}$.

To prove (3.2) take, for $k = 1, 2, \ldots$, a strategy $\tilde{\gamma}^k$ such that $Y^{V^k(\mathbb{F})+1/k, \tilde{\gamma}^k}_N \geq \mathbb{F}(S)$ on Ω^k. We will show below that without loss of generality we can assume that the sequence of $\tilde{\gamma}^k$s is uniformly bounded, that is,

$$(3.3) \qquad\qquad |\tilde{\gamma}^k| \leq \mathcal{C} \qquad \text{for some constant } \mathcal{C} > 0.$$

Now consider the strategies $\gamma^k \triangleq \tilde{\gamma}^k \circ p^k$ where $p^k : \Omega \to \Omega^k$ is the projection which maps $\omega = (x_1, \ldots, x_N)$ to $p^k(\omega) = \tilde{\omega} \triangleq (\tilde{x}_1, \ldots, \tilde{x}_N) \in \Omega^k$ with

$$\tilde{x}_i \triangleq \max\left\{x \leq x_i : |x| = \frac{j}{k}\underline{\sigma} + \left(1 - \frac{j}{k}\right)\overline{\sigma} \text{ for some } j \in \{0, \ldots, k\}\right\}.$$

For any initial capital y, we get

$$Y_N^{y,\gamma^k}(\omega) - Y_N^{y,\tilde{\gamma}^k}(\tilde{\omega})$$

$$= \sum_{n=0}^{N-1} \tilde{\gamma}_n^k(\tilde{\omega})\big((S_{n+1}(\omega) - S_n(\omega)) - (S_{n+1}(\tilde{\omega}) - S_n(\tilde{\omega}))\big)$$

$$- \sum_{n=0}^{N-1} \big(g_n(\omega, (\tilde{\gamma}_n^k(\tilde{\omega}) - \tilde{\gamma}_{n-1}^k(\tilde{\omega}))S_n(\omega))$$

$$- g_n(\tilde{\omega}, (\tilde{\gamma}_n^k(\tilde{\omega}) - \tilde{\gamma}_{n-1}^k(\tilde{\omega}))S_n(\tilde{\omega}))\big).$$

Because $|\tilde{\gamma}^k| \leq C$ uniformly in $k = 1, 2, \ldots$, the first of these two sums has absolute value less than

$$2C \sum_{n=0}^{N-1} w(S_n, |\omega - \tilde{\omega}|),$$

where for any function f we let $w(f, \delta)$, $\delta > 0$, denote the modulus of continuity over its domain. Similarly, we get for the second sum that its absolute value does not exceed

$$\sum_{n=0}^{N-1} w\big(g_n|_{\Omega \times [-2Cs_0 e^{\bar{\sigma}n}, 2Cs_0 e^{\bar{\sigma}n}]}, |\omega - \tilde{\omega}| + 2Cw(S_n, |\omega - \tilde{\omega}|)\big).$$

By continuity of S and g_n, $n = 0, \ldots, N - 1$, both of these bounds tend to 0 as $|\omega - \tilde{\omega}| \to 0$. It follows that there are $\varepsilon_k \downarrow 0$ as $k \uparrow \infty$ such that

$$Y_N^{y,\tilde{\gamma}^k}(\tilde{\omega}) \leq Y_N^{y,\gamma^k}(\omega) + \varepsilon_k \qquad \text{for all } |\omega - \tilde{\omega}| \leq 1/k, \ y \in \mathbb{R}.$$

By assumption, $\mathbb{F}$ is also continuous and so we get

$$\mathbb{F}(S)(\omega) \leq \mathbb{F}(S)(\tilde{\omega}) + w(\mathbb{F}(S), |\omega - \tilde{\omega}|)$$

$$\leq Y_N^{V^k(\mathbb{F})+1/k, \tilde{\gamma}^k}(\tilde{\omega}) + w(\mathbb{F}(S), |\omega - \tilde{\omega}|)$$

$$\leq Y_N^{V^k(\mathbb{F})+1/k, \gamma^k}(\omega) + \varepsilon_k + w(\mathbb{F}(S), 1/k).$$

It follows that $V(\mathbb{F}) \leq V^k(\mathbb{F}) + 1/k + \varepsilon_k + w(\mathbb{F}(S), 1/k)$ which implies our claim (3.2).

It remains to prove the uniform boundedness (3.3) of the sequence $(\tilde{\gamma}^k)_{k=1,2,\ldots}$. Clearly, $y^k \triangleq V^k(\mathbb{F}) + 1/k \leq A$, $k = 1, 2, \ldots$, for some $A > 0$. For any $\tilde{\pi}^k = (y^k, \tilde{\gamma}^k)$, $k = 1, 2, \ldots$, we will prove by induction over n that

$$(3.4) \qquad Y_n^{\tilde{\pi}^k}(\tilde{\omega}) \leq A(1 + e^{\bar{\sigma}})^n \quad \text{and} \quad |\tilde{\gamma}_n^k(\tilde{\omega})| \leq \frac{A(1 + e^{\bar{\sigma}})^n}{(1 - e^{-\bar{\sigma}})S_n(\tilde{\omega})}$$

for any $\tilde{\omega} = (\tilde{x}_1, \ldots, \tilde{x}_N) \in \Omega^k$ and $n = 0, 1, \ldots, N$. Since $S_n \geq s_0 e^{-\overline{\sigma} N}$, our claim (3.4) then holds for $\mathcal{C} \triangleq A(1 + e^{\overline{\sigma}})^N / ((1 - e^{-\overline{\sigma}}) s_0 e^{-\overline{\sigma} N})$.

Since each $\tilde{\pi}^k$ super-replicates a positive claim, we must have $Y_1^{\tilde{\pi}^k} \geq 0$ in any possible scenario. In particular, we have

$$y^k + \tilde{\gamma}_0^k s_0 (e^{\overline{\sigma}} - 1) \geq 0 \quad \text{and} \quad y^k + \tilde{\gamma}_0^k s_0 (e^{-\overline{\sigma}} - 1) \geq 0$$

which allows us to conclude that $|\tilde{\gamma}_0^k| \leq \frac{A}{s_0(1 - e^{-\overline{\sigma}})}$. Thus, (3.4) holds for $n = 0$. Next, assume that (3.4) holds for n and let us prove it for $n + 1$. From the induction assumption, we get

$$Y_{n+1}^{\tilde{\pi}^k}(\tilde{\omega}) \leq Y_n^{\tilde{\pi}^k}(\tilde{\omega}) + \tilde{\gamma}_n^k(\tilde{\omega}) (S_{n+1}(\tilde{\omega}) - S_n(\tilde{\omega}))$$

$$\leq A(1 + e^{\overline{\sigma}})^n + \frac{A(1 + e^{\overline{\sigma}})^n}{(1 - e^{-\overline{\sigma}}) S_n(\tilde{\omega})} S_n(\tilde{\omega})(e^{\overline{\sigma}} - 1) = A(1 + e^{\overline{\sigma}})^{n+1},$$

as required. Again, the portfolio valued at time $n + 2$ should be nonnegative, for any possible scenario. Thus,

$$Y_{n+1}^{\tilde{\pi}^k}(\tilde{\omega}) + \tilde{\gamma}_{n+1}^k(\tilde{\omega}) S_{n+1}(\tilde{\omega})(e^{\overline{\sigma}} - 1) \geq 0$$

and

$$Y_{n+1}^{\tilde{\pi}^k}(\tilde{\omega}) + \tilde{\gamma}_{n+1}^k(\tilde{\omega}) S_{n+1}(\tilde{\omega})(e^{-\overline{\sigma}} - 1) \geq 0$$

and so,

$$|\tilde{\gamma}_{n+1}^k(\tilde{\omega})| \leq \frac{Y_{n+1}^{\tilde{\pi}^k}(\tilde{\omega})}{(1 - e^{-\overline{\sigma}}) S_{n+1}(\tilde{\omega})} \leq \frac{A(1 + e^{\overline{\sigma}})^{n+1}}{(1 - e^{-\overline{\sigma}}) S_{n+1}(\tilde{\omega})}.$$

This completes the proof of (3.4). $\quad\square$

It is immediate from Lemmas 3.1–3.3 that $V(\mathbb{F}) = U(\mathbb{F})$ for continuous functions $\mathbb{F}$. For upper-semicontinuous $\mathbb{F}$, we can find continuous functions $\mathbb{F}^k$ with $\sup_\Omega \mathbb{F}(S) \geq \mathbb{F}^k(S) \geq \mathbb{F}(S)$ such that

$$\limsup_{k \uparrow \infty, \omega_k \to \omega} \mathbb{F}^k(S(\omega_k)) \leq \mathbb{F}(S(\omega))$$

for any $\omega \in \Omega$; see, for example, Lemma 5.3 in [12]. The proof of Theorem 2.2 will thus follow from the series of inequalities

$$V(\mathbb{F}) \geq U(\mathbb{F}) \geq \limsup_{k \uparrow \infty} U(\mathbb{F}^k) = \limsup_{k \uparrow \infty} V(\mathbb{F}^k) \geq V(\mathbb{F}),$$

where the first inequality is due to Lemma 3.1, the last holds because $\mathbb{F}^k \geq \mathbb{F}$ and the identity follows because our claim is already established for continuous $\mathbb{F}^k$. Hence, the only estimate still to be shown is the second one.

LEMMA 3.4.   *If $\mathbb{F}$ is approximated by $\mathbb{F}^k$, $k = 1, 2\ldots$, as above we have*

$$U(\mathbb{F}) \geq \limsup_{k \uparrow \infty} U(\mathbb{F}^k).$$

PROOF.   Without loss of generality, we can assume that $(U(\mathbb{F}^k))_{k=1,2,\ldots}$ converges in $\mathbb{R}$. By definition of $U(\mathbb{F}^k)$, there is $\mathbb{P}^k \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}}$ such that, for $k = 1, 2, \ldots$,

$$(3.5) \qquad U(\mathbb{F}^k) - \frac{1}{k} \leq \mathbb{E}_{\mathbb{P}^k}\left[\mathbb{F}^k(S) - \sum_{n=0}^{N-1} G_n\left(\frac{\mathbb{E}_{\mathbb{P}^k}[S_N|\mathcal{F}_n] - S_n}{S_n}\right)\right].$$

We wish to show that the lim sup of the right-hand side of (3.5) as $k \uparrow \infty$ is not larger than $U(\mathbb{F})$. To this end, denote by Π the set of Borel probability measures on $\Omega \times [0, s_0 e^{\overline{\sigma} N}]^N$. Since $\Omega \times [0, s_0 e^{\overline{\sigma} N}]^N$ is compact, so is Π when endowed with the weak topology. Now consider the sequence of probabilities measures in Π obtained by considering the law of

$$Z^k \triangleq (X, \mathbb{E}_{\mathbb{P}^k}[S_N], \mathbb{E}_{\mathbb{P}^k}[S_N|\mathcal{F}_1], \ldots, \mathbb{E}_{\mathbb{P}^k}[S_N|\mathcal{F}_{N-1}])$$

under $\mathbb{P}^k$ for $k = 1, 2, \ldots$. Due to Prohorov's theorem, by possibly passing to a subsequence, we can assume without loss of generality that this sequence converges weakly. By Skorohod's representation theorem, there thus exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with a $\hat{\mathbb{P}}$-almost surely convergent sequence of random variables $\hat{Z}^k$, $k = 1, 2, \ldots$, taking values in $\Omega \times [0, s_0 e^{\overline{\sigma} N}]^N$, whose laws under $\hat{\mathbb{P}}$ coincide, respectively, with those of Z^k under $\mathbb{P}^k$, $k = 1, 2, \ldots$. Let $\hat{Z}^\infty$ denote the $\hat{\mathbb{P}}$-a.s. existing limit of $\hat{Z}^k$, $k = 1, 2, \ldots$, and write it as

$$\hat{Z}^\infty = (\hat{X}^\infty, Y_0, \ldots, Y_{N-1}).$$

We will show that

$$(3.6) \qquad \mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^\infty)|\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty] = \mathbb{E}_{\hat{\mathbb{P}}}[Y_n|\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty], \qquad n = 0, \ldots, N.$$

By construction of $\hat{Z}^k$, we have for the right-hand side of (3.5):

$$\mathbb{E}_{\mathbb{P}^k}\left[\mathbb{F}^k(S) - \sum_{n=0}^{N-1} G_n\left(X, \frac{\mathbb{E}_{\mathbb{P}^k}[S_N|\mathcal{F}_n] - S_n}{S_n}\right)\right]$$

$$= \mathbb{E}_{\hat{\mathbb{P}}}\left[\mathbb{F}^k(S(\hat{X}^k)) - \sum_{n=0}^{N-1} G_n\left(\hat{X}^k, \frac{\mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^k)|\hat{X}_1^k, \ldots, \hat{X}_n^k] - S_n(\hat{X}^k)}{S_n(\hat{X}^k)}\right)\right].$$

The $\hat{\mathbb{P}}$-a.s. convergence of $\hat{Z}^k$ and the construction of the sequence of $\mathbb{F}^k$ imply that the $\limsup_{k \uparrow \infty}$ of the term inside this last expectation is $\hat{\mathbb{P}}$-a.s. not larger than

$$\mathbb{F}(S(\hat{X}^\infty)) - \sum_{n=0}^{N-1} G_n\left(\hat{X}^\infty, \frac{Y_n - S_n(\hat{X}^\infty)}{S_n(\hat{X}^\infty)}\right),$$

where we used the lower semi-continuity of G_n. Because of the boundedness of $\mathbb{F}$ on compact sets and because $G \geq 0$, it then follows by Fatou's lemma that the lim sup of the right-hand side of (3.5) is not larger than

$$\mathbb{E}_{\hat{\mathbb{P}}}\left[\mathbb{F}(S(\hat{X}^\infty)) - \sum_{n=0}^{N-1} G_n\left(\hat{X}^\infty, \frac{Y_n - S_n(\hat{X}^\infty)}{S_n(\hat{X}^\infty)}\right)\right].$$

From the definitions it follows that $G_n(\omega, \alpha)$ is adapted and $G_n(\omega, \cdot)$ is convex. This together with the Jensen inequality and (3.6) yields that for any $n < N$

$$\mathbb{E}_{\hat{\mathbb{P}}}\left[G_n\left(\hat{X}^\infty, \frac{Y_n - S_n(\hat{X}^\infty)}{S_n(\hat{X}^\infty)}\right)\Big|\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty\right]$$

$$\geq G_n\left(\hat{X}^\infty, \frac{\mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^\infty)|\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty] - S_n(\hat{X}^\infty)}{S_n(\hat{X}^\infty)}\right).$$

We conclude that the lim sup of the right-hand side of (3.5) is not larger than

$$\mathbb{E}_{\hat{\mathbb{P}}}\left[\mathbb{F}(S(\hat{X}^\infty)) - \sum_{n=0}^{N-1} G_n\left(\hat{X}^\infty, \frac{\mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^\infty)|\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty] - S_n(\hat{X}^\infty)}{S_n(\hat{X}^\infty)}\right)\right].$$

Since the distribution of $\hat{X}^\infty$ is an element in $\mathcal{P}_{\underline{\sigma},\overline{\sigma}}$, this last expectation is not larger than $U(\mathbb{F})$ as we had to show.

It remains to establish (3.6). Let $n < N$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous bounded function. From the dominated convergence theorem, it follows that

$$\mathbb{E}_{\hat{\mathbb{P}}}[Y_n f(\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty)] = \lim_{k \to \infty} \mathbb{E}_{\hat{\mathbb{P}}}[\mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^k)|\hat{X}_1^k, \ldots, \hat{X}_n^k]f(\hat{X}_1^k, \ldots, \hat{X}_n^k)]$$

$$= \lim_{k \to \infty} \mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^k)f(\hat{X}_1^k, \ldots, \hat{X}_n^k)]$$

$$= \mathbb{E}_{\hat{\mathbb{P}}}[S_N(\hat{X}^\infty)f(\hat{X}_1^\infty, \ldots, \hat{X}_n^\infty)].$$

Thus, by applying standard density arguments we obtain (3.6). This completes our proof. □

3.2. *Proof of Theorem* 2.7.   For the proof of the asserted limit,

$$(3.7) \quad \lim_{N \to \infty} V_{g^{N,c}}^{\underline{\sigma}/\sqrt{N}, \overline{\sigma}/\sqrt{N}}(\mathbb{F}) = \sup_{\sigma \in \Sigma(c)} \mathbb{E}^W\left[\mathbb{F}(S^\sigma) - \int_0^1 \hat{H}_t(S^\sigma)a^2(\sigma_t)\,dt\right]$$

we first have to go through some technical preparations in Section 3.2.1 before we can establish "≤" and then "≥" in (3.7) in Sections 3.2.2 and 3.2.3, respectively.

3.2.1. *Technical preparations.* Let us start by recalling that, for $N = 1, 2, \ldots,$

$$\Omega^N = \{\omega^N = (x_1, \ldots, x_N) : \underline{\sigma}/\sqrt{N} \leq |x_n| \leq \overline{\sigma}/\sqrt{N}, n = 1, \ldots, N\}$$

allows for the definition of the canonical process

$$X_n^N(\omega^N) = x_n \qquad \text{for } n = 1, \ldots, N, \omega^N = (x_1, \ldots, x_N) \in \Omega^N.$$

We thus can consider the canonical filtration

$$\mathcal{F}_n^N \triangleq \sigma(X_m^N, m = 1, \ldots, n), \qquad n = 0, \ldots, N,$$

which clearly is the same as the one generated by

$$S_n^N = s_0 \exp\left(\sum_{m=1}^n X_m\right), \qquad n = 0, \ldots, N,$$

since

$$X_n^N = \ln S_n^N - \ln S_{n-1}^N, \qquad n = 1, \ldots, N.$$

It will be convenient to let, for a vector $y = (y_0, \ldots, y_N) \in \mathbb{R}^{N+1}$, the function $\overline{y} \in C[0, 1]$ denote the continuous linear interpolation on $[0, 1]$ determined by $\overline{y}_{n/N} = y_n, n = 0, \ldots, N$. This gives us, in particular, the continuous time analog $(\overline{S}_t^N)_{0 \leq t \leq 1}$ of $(S_n^N)_{n=0,\ldots,N}$.

Our first observation is that the continuity of $\mathbb{F}$ allows us to write the supremum in (3.7) in different ways.

LEMMA 3.5. *Let*

$$R \triangleq \sup_{\sigma \in \Sigma(c)} \mathbb{E}^W\left[\mathbb{F}(S^\sigma) - \int_0^1 \hat{H}_t(S^\sigma) a^2(\sigma_t) \, dt\right]$$

denote the right-hand side of (3.7).

(i) *We have*

$$(3.8) \qquad R = \sup_{\mathbb{P} \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}, c}} \mathbb{E}_{\mathbb{P}}\left[\mathbb{F}(S) - \int_0^1 \hat{H}_t(S) a^2\left(\sqrt{\frac{d\langle S\rangle_t}{dt}/S_t^2}\right) dt\right],$$

where $\mathcal{P}_{\underline{\sigma}, \overline{\sigma}, c}$ *denotes the class of probabilities* $\mathbb{P}$ *on* $(C[0, 1], \mathcal{B}(C[0, 1]))$ *under which the coordinate process* $S = (S_t)_{0 \leq t \leq 1}$ *is a strictly positive martingale starting at* $S_0 = s_0$ *whose quadratic variation is absolutely continuous with*

$$a\left(\sqrt{\frac{d\langle S\rangle_t}{dt}/S_t^2}\right) \leq c, \qquad 0 \leq t \leq 1, \mathbb{P}\text{-almost surely.}$$

(ii) *The supremum defining* R *does not change when we take it over* $\tilde{\Sigma}(c) \subset \Sigma(c)$, *the class of progressively measurable processes* $\tilde{\sigma} : [0, 1] \times C[0, 1] \to \mathbb{R}_+$ *on the Wiener space* $(C[0, 1], \mathcal{B}(C[0, 1]), \mathbb{P}^W)$ *such that:*

- *there is $\delta > 0$ such that*

(3.9) $$\underline{\sigma}(\underline{\sigma} - 2c)^+ + \delta \leq \tilde{\sigma}^2 \leq \overline{\sigma}(\overline{\sigma} + 2c) - \delta$$

  uniformly on $[0, 1] \times C[0, 1]$ and such that, in addition,

(3.10) $$\tilde{\sigma} \equiv \underline{\sigma} \qquad on \ [1 - \delta, 1] \times C[0, 1].$$

- *$\tilde{\sigma}$ is Lipschitz continuous on $[0, 1] \times C[0, 1]$.*

PROOF.   The proof is done similar to the proof of Lemmas 7.1–7.2 in [11]. □

The following technical key lemma can be viewed as an adaption of Kusuoka's results from [19] on super-replication with proportional transaction costs to our uncertain volatility setting with nonlinear costs.

LEMMA 3.6.   *Under the assumptions of Theorem 2.7 the following holds true*:

(i) *Let $c > 0$ and, for $N = 1, 2, \ldots$, let $\mathbb{Q}^N$ be a probability measure on $(\Omega^N, \mathcal{F}_N^N)$ for which*

(3.11) $$M_n^N \triangleq \mathbb{E}^{\mathbb{Q}^N}[S_N^N | \mathcal{F}_n^N], \qquad n = 0, \ldots, N,$$

is close to S^N in the sense that $\mathbb{Q}^N$-almost surely

(3.12) $$\left| \frac{M_n^N - S_n^N}{S_n^N} \right| \leq \frac{c}{\sqrt{N}}, \qquad n = 0, \ldots, N.$$

Then we have

(3.13) $$\sup_{N=1,2,\ldots} \mathbb{E}_{\mathbb{Q}^N} \left[\left(\max_{n=0,\ldots,N} S_n^N \right)^p \right] < \infty \qquad for \ any \ p > 0$$

and, with

(3.14) $$Q_n^N \triangleq \sum_{m=1}^n (X_m^N)^2 + 2 \sum_{m=1}^n \frac{M_m^N - S_m^N}{S_m^N} X_m^N, \qquad n = 0, \ldots, N,$$

there is a subsequence, again denoted by N, such that, for $N \uparrow \infty$,

(3.15) $$\mathrm{Law}(\overline{S}^N, \overline{M}^N, \overline{Q}^N | \mathbb{Q}^N) \quad \Rightarrow \quad \mathrm{Law}\left(S, S, \int_0^\cdot \frac{d\langle S \rangle_s}{S_s^2} \Big| \mathbb{P} \right),$$

where $\mathbb{P}$ is a probability measure in $\mathcal{P}_{\underline{\sigma}, \overline{\sigma}, c}$, S is as in Lemma 3.5(i), and where, as before, $\overline{S}^N$ etc. denote the continuous interpolations on $[0, 1]$ induced by the vector $S^N = (S_n^N)_{n=0,\ldots,N}$ etc.

(ii) *For any $c > 0$ and $\tilde{\sigma} \in \tilde{\Sigma}(c)$ as in Lemma* 3.5(ii), *there exists a sequence of probability measures* $\mathbb{Q}^N$, $N = 1, 2, \ldots$, *as in* (i) *such that the weak convergence in* (3.15) *holds with* $\mathbb{P} \triangleq \text{Law}(S^{\tilde{\sigma}}|\mathbb{P}^W)$. *In addition, we get the weak convergence* ($as\ N \uparrow \infty$)

$$(3.16) \quad \text{Law}\left(\overline{S}^N, \overline{M}^N, \overline{\sqrt{N}\frac{|M^N - S^N|}{S^N}}\Big|\mathbb{Q}^N\right) \quad \Rightarrow \quad \text{Law}\left(S^{\tilde{\sigma}}, S^{\tilde{\sigma}}, a(\tilde{\sigma})|\mathbb{P}^W\right).$$

PROOF. Let us first focus on claim (i). It obviously suffices to prove (3.13) only for $p \in \{1, 2, \ldots\}$. For this, we proceed similar to Kusuoka for his claim (4.23) in [19] and write

$$(M_{n+1}^N)^p = (M_n^N)^p \left(1 + \frac{M_{n+1}^N - M_n^N}{M_n^N}\right)^p$$
$$= (M_n^N)^p \left(1 + \sum_{j=1}^p \binom{p}{j}\left(\frac{M_{n+1}^N - M_n^N}{M_n^N}\right)^j\right).$$

Now observe that, when taking the $\mathbb{Q}^N$-expectation, the contribution from the summand for $j = 1$ can be dropped since it has vanishing $\mathcal{F}_n^N$-conditional expectation due to the martingale property of M^N under $\mathbb{Q}^N$. From (3.12) and $S_{n+1}^N/S_n^N = \exp(X_n^N) \in [e^{-\overline{\sigma}/\sqrt{N}}, e^{\overline{\sigma}/\sqrt{N}}]$, it follows that $M_{n+1}^N/M_n^N = 1 + O(1/\sqrt{N})$ where the random $O(1/\sqrt{N})$-term becomes small uniformly in n and ω^N. Therefore, the summands for $j = 2, \ldots, p$ are uniformly of the order $O(1/N)$. Thus, we obtain

$$\mathbb{E}_{\mathbb{Q}^N}[(M_{n+1}^N)^p] = (1 + O(1/N))\mathbb{E}_{\mathbb{Q}^N}[(M_n^N)^p]$$

and, so upon iteration,

$$\mathbb{E}_{\mathbb{Q}^N}[(M_N^N)^p] = (1 + O(1/N))^N (M_0^N)^p.$$

Clearly, $(1 + O(1/N))^N$ is bounded in N. The same holds for $M_0^N = s_0(1 + O(1/\sqrt{N}))$, where we used (3.12)) and $S_0^N = s_0$. Hence,

$$\sup_{N=1,2,\ldots} \mathbb{E}_{\mathbb{Q}^N}[(M_N^N)^p] < \infty$$

which by Doob's inequality entails that even

$$(3.17) \qquad\qquad \sup_{N=1,2,\ldots} \mathbb{E}_{\mathbb{Q}^N}\left[\left(\max_{n=0,\ldots,N} M_n^N\right)^p\right] < \infty.$$

Because of (3.12), this yields our claim (3.13).

Let us next focus on $\text{Law}(\overline{M}^N|\mathbb{Q}^N)_{N=1,2,\ldots}$ for which we will verify Kolmogorov's tightness criterion (see [6]) on $C[0, 1]$. To this end, recall that

$M_{n+j}^N / M_{n+j-1}^N - 1 = O(1/\sqrt{N})$ and so the quadratic variation of M^N satisfies

$$\langle M^N \rangle_{n+l} - \langle M^N \rangle_n$$
$$= \sum_{j=1}^{l} \mathbb{E}_{\mathbb{Q}^N} \big[(M_{n+j}^N - M_{n+j-1}^N)^2 | \mathcal{F}_{n+j-1}^N \big]$$
$$= \Big(\max_{0 \le n \le N} M_n^N \Big)^2 O(l/N).$$

From the Burkholder–Davis–Gundy inequality and the bound (3.17) we thus get

$$\mathbb{E}_{\mathbb{Q}^N} \big[(M_{n+l}^N - M_n^N)^4 \big] = O\big((l/N)^2 \big)$$

which readily gives Kolmogorov's criterion for our continuous interpolations $\overline{M}^N$, $N = 1, 2, \ldots$.

Having established tightness, we can find a subsequence, again denoted by N, such that $\text{Law}(\overline{M}^N | \mathbb{Q}^N)_{N=1,2,\ldots}$ converges to the law of a continuous process M on a suitable probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$. We will show next that this process M is a strictly positive martingale. In fact, by Skorohod's representation theorem, we can assume that there are processes $\hat{M}^N$, $N = 1, 2, \ldots$, on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with

$$\text{Law}(\overline{M}^N | \mathbb{Q}^N) = \text{Law}(\hat{M}^N | \hat{\mathbb{P}})$$

which converge $\hat{\mathbb{P}}$-almost surely to M as $N \uparrow \infty$. It is then immediate from (3.17) that the martingale property of M^N under $\mathbb{Q}^N$ gives the martingale property of $\hat{M}$ under $\hat{\mathbb{P}}$. To see that M is strictly positive, we follow Kusuoka's argument for (4.24) and (4.25) in his paper [19] and establish

$$(3.18) \qquad \sup_{N=1,2,\ldots} \mathbb{E}_{\mathbb{Q}^N} \Big[\max_{n=0,\ldots,N} |\ln M_n^N|^2 \Big] < \infty$$

since this entails the $\hat{\mathbb{P}}$-integrability of $\max_{0 \le t \le 1} |\ln M_t|$ by Fatou's lemma. For (3.18), recall that $M_m^N / M_{m-1}^N = 1 + O(1/\sqrt{N})$ uniformly in m and ω^N. This allows us to use Taylor's expansion to obtain

$$\ln M_m^N - \ln M_{m-1}^N = \frac{M_m^N - M_{m-1}^N}{M_{m-1}^N} + O(1/N).$$

Upon summation over $m = 1, \ldots, n$, this gives in conjunction with $M_0^N = s_0(1 + O(1/\sqrt{N}))$:

$$\max_{n=0,\ldots,N} |\ln M_n^N| \le \max_{n=0,\ldots,N} \left| \sum_{m=1}^{n} \frac{M_m^N - M_{m-1}^N}{M_{m-1}^N} \right| + O(1).$$

Hence,

$$
\mathbb{E}_{\mathbb{Q}^N}\Big[\max_{n=0,\dots,N}|\ln M_n^N|^2\Big] \le 2\mathbb{E}_{\mathbb{Q}^N}\Big[\max_{n=0,\dots,N}\Big|\sum_{m=1}^n \frac{M_m^N - M_{m-1}^N}{M_{m-1}^N}\Big|^2\Big] + O(1)
$$

$$
\le 8\mathbb{E}_{\mathbb{Q}^N}\Big[\sum_{m=1}^N \Big(\frac{M_m^N - M_{m-1}^N}{M_{m-1}^N}\Big)^2\Big] + O(1),
$$

where for the second estimate we used Doob's inequality for the martingale given by the sum for which we take the maximum in the above expression. Recalling again that $M_m^N/M_{m-1}^N - 1 = O(1/\sqrt{N})$ uniformly in m and ω^N the above expectation is of order $O(1)$ and we obtain (3.18).

Let us finally turn to the weak convergence (3.13) and introduce, for $N = 1, 2, \dots$, the auxiliary discrete stochastic integrals

$$
Y_n^N \triangleq \sum_{m=1}^n \frac{M_m^N - M_{m-1}^N}{S_{m-1}^N}, \qquad n = 0, \dots, N.
$$

By applying Theorem 4.3 in [13], (3.12)–(3.13) and the already established weak convergence of $\mathrm{Law}(\overline{M}^N|\mathbb{Q}^N)_{N=1,2,\dots}$ on $C[0,1]$, we deduce the weak convergence

(3.19)
$$
\mathrm{Law}\Big(\Big(\frac{1}{S_{[Nt]}^N}, M_{[Nt]}^N, Y_{[Nt]}^N\Big)_{0\le t\le 1}\Big|\mathbb{Q}^N\Big)
$$
$$
\Rightarrow \quad \mathrm{Law}\Big(\Big(\frac{1}{M_t}, M_t, Y_t\Big)_{0\le t\le 1}\Big|\hat{\mathbb{P}}\Big) \qquad \text{as } N \uparrow \infty
$$

on the Skorohod space $D[0,1] \times D[0,1] \times D[0,1]$ where $Y \triangleq \int_0^{\cdot} dM_s/M_s$. Hence, $M = M_0\exp(Y - \langle Y\rangle/2)$ and, in particular, $\langle \ln M\rangle = \langle Y\rangle$. Moreover, again by Skorohod's representation theorem, we can assume that there are processes $1/\hat{S}^N$, $\hat{M}^N$ and $\hat{Y}^N$ on $(\hat{\Omega}, \hat{F}, \hat{\mathbb{P}})$ whose joint law under $\hat{\mathbb{P}}$ coincides with that in (3.19) and which converge $\hat{\mathbb{P}}$-almost surely to $1/M$, M and Y, respectively. Now, recalling that $|X_m^N| \le \overline{\sigma}/\sqrt{N}$ we can Taylor expand $e^x = 1 + x + x^2/2 + O(x^3)$ so that with (3.12) we can write

$$
\frac{M_m^N - M_m^N}{S_{m-1}^N} = \Big(1 + \frac{M_m^N - S_m^N}{S_m^N}\Big)e^{X_m^N} - \frac{M_{m-1}^N}{S_{m-1}^N}
$$

$$
= \frac{M_m^N}{S_m^N} - \frac{M_{m-1}^N}{S_{m-1}^N} + X_m^N + \frac{1}{2}(X_m^N)^2 + \frac{M_m^N - S_m^N}{S_m^N}X_m^N
$$

$$
+ O\big(1/\sqrt{N}^3\big).
$$

Thus, with Q^N as defined in (3.14), we obtain upon summing over $m = 1, \ldots, n$:

$$Y_n^N = \frac{M_n^N}{S_n^N} - \frac{M_0^N}{S_0^N} + \ln S_n^N - \ln S_0^N + \frac{1}{2} Q_n^N + O(1/\sqrt{N}).$$

In terms of $\hat{Y}^N$, $\hat{M}^N$ and $\hat{Q}^N \triangleq (Q_{[Nt]}^N)_{0 \le t \le 1}$ this amounts to

$$\hat{Y}_t^N = \frac{\hat{M}_t^N}{\hat{S}_t^N} - \frac{\hat{M}_0^N}{\hat{S}_0^N} + \ln \hat{S}_t^N - \ln s_0 + \frac{1}{2} \hat{Q}_t^N + O(1/\sqrt{N})$$

for $t \in \{0, 1/N, \ldots, 1\}$. Since all other terms in this expression converge $\hat{\mathbb{P}}$-almost surely as $N \uparrow \infty$, so does $\hat{Q}^N$ and its limit is given by

$$\lim_N \hat{Q}_t^N = 2(Y_t - \ln M_t + \ln s_0) = \langle Y \rangle_t = \langle \ln M \rangle_t, \qquad 0 \le t \le 1.$$

Now, fix $0 \le s < t \le 1$ and observe that

$$\langle \ln M \rangle_t - \langle \ln M \rangle_s = \lim_N (\hat{Q}_t^N - \hat{Q}_s^N)$$

$$= \lim_N \sum_{[Ns] < n \le [Nt]} \hat{X}_{n/N}^N \left(\hat{X}_{n/N}^N + 2 \frac{\hat{M}_{n/N}^n - \hat{S}_{n/N}^N}{\hat{S}_{n/N}^N} \right)$$

$$\in \left[\underline{\sigma}(\underline{\sigma} - 2c)^+ (t - s), \overline{\sigma}(\overline{\sigma} + 2c)(t - s) \right].$$

Hence, $\langle \ln M \rangle$ is absolutely continuous with density

$$\frac{d \langle \ln M \rangle_t}{dt} \in \left[\underline{\sigma}(\underline{\sigma} - 2c)^+, \overline{\sigma}(\overline{\sigma} + 2c) \right], \qquad 0 \le t \le 1,$$

which readily implies

$$a \left(\sqrt{\frac{d \langle \ln M \rangle_t}{dt}} \right) \le c.$$

It thus follows that $\mathbb{P} \triangleq \mathrm{Law}(M | \hat{\mathbb{P}})$ lies in the class $\mathcal{P}_{\underline{\sigma}, \overline{\sigma}, c}$ as considered in Lemma 3.5(i). By the construction and $\hat{\mathbb{P}}$-almost sure convergence of $\hat{S}^M$, $\hat{M}^N$, $\hat{Q}^N$, this proves (3.15) for this $\mathbb{P}$.

Let us now turn to the proof of item (ii) of our lemma and take a $\overline{\sigma} : [0, 1] \times C[0, 1] \to \mathbb{R}$ from the class $\tilde{\Sigma}(c)$ introduced in Lemma 3.5. Fix $N \in \{1, 2, \ldots\}$ and define on Ω^N the processes σ^N, κ^N, B^N and ξ^N by the following recursion:

$$\sigma_0^N \triangleq \underline{\sigma} \vee \tilde{\sigma}_0(0) \wedge \overline{\sigma}, \qquad B_0^N \triangleq 0$$

and, for $n = 1, \ldots, N$,

$$\sigma_n^N \triangleq \underline{\sigma} \vee \tilde{\sigma}_{(n-1)/N}(\overline{B}^N) \wedge \overline{\sigma},$$

$$\kappa_n^N \triangleq \frac{1}{2}\left(\frac{\tilde{\sigma}_{(n-1)/N}^2(\overline{B}^N)}{(\sigma_n^N)^2} - 1\right),$$

$$\xi_n^N \triangleq \sqrt{N}\frac{\ln S_n^N - \ln S_{n-1}^N}{\sigma_{n-1}^N} = \sqrt{N}X_n^N/\sigma_n^N,$$

$$B_n^N \triangleq B_{n-1}^N + \frac{\exp((1+\kappa_n^N)X_n^N - \kappa_{n-1}^N X_{n-1}^N) - 1}{\sqrt{1+2\kappa_n^N}\sigma_n^N}.$$

Observe that the progressive measurability of $\tilde{\sigma}$ ensures that its evaluation in the definition of σ_n^N and κ_n^N depends on $B^N = (B_m^N)_{m=0,\dots,N}$ only via its already constructed values for $m = 0,\dots,n-1$.

Next, define the process q^N by

(3.20) $$q_n^N = \frac{\exp(\kappa_{n-1}^N X_{n-1}^N) - \exp(-(1+\kappa_n^N)\sigma_n^N/\sqrt{N})}{\exp((1+\kappa_n^N)\sigma_n^N/\sqrt{N}) - \exp(-(1+\kappa_n^N)\sigma_n^N/\sqrt{N})}.$$

Consider the probability measure $\mathbb{P}^N$ on $(\Omega^N, \mathcal{F}_N^N)$ for which the random variables $\xi_1^N,\dots,\xi_N^N$ are i.i.d. with $\mathbb{P}^N(\xi_1^N = 1) = \mathbb{P}^N(\xi_1^N = -1) = 1/2$. From (3.10), it follows that there exists $\varepsilon > 0$ for which $\kappa_n^N > \varepsilon - 1/2$. Thus $|B_n^N - B_{n-1}^N| = O(N^{-1/2})$ and also $|\kappa_n^N - \kappa_{n-1}^N| = O(N^{-1/2})$ because of the Lipschitz continuity of $\tilde{\sigma}$. We conclude that, for sufficiently large N, $q_n^N \in (0,1)$ $\mathbb{P}^N$-almost surely. For such N we consider $\mathbb{Q}^N$ given by the Radon–Nikodym derivatives

$$\frac{d\mathbb{Q}^N}{d\mathbb{P}^N}\Big|_{\mathcal{F}_n^N} = 2^N \prod_{m=1}^n \big(q_m^N \mathbb{I}_{\{\xi_m^N = 1\}} + (1 - q_m^N)\mathbb{I}_{\{\xi_m^N = -1\}}\big).$$

Since $\mathbb{P}^N[\xi_n^N = \pm 1] = 1/2$, our choice of q^N (3.20) ensures that B^N is a martingale under $\mathbb{Q}^N$. Now consider the stochastic process

$$M_n^N \triangleq S_n^N \exp(\kappa_n^N X_n^N), \qquad n = 0,\dots,N.$$

From (3.10), it follows that $\kappa_N^N = 0$ for sufficiently large N and so $M_N^N = S_N^N$. Furthermore, from (3.9) we have $\frac{|M_n^N - S_n^N|}{S_n^N} \leq \frac{c}{\sqrt{N}}$. Observe also that

(3.21) $$M_n^N = M_{n-1}^N\Big(1 + \sqrt{1+2\kappa_n^N}\sigma_n^N(B_n^N - B_{n-1}^N)\Big).$$

Hence, the predictability of σ^N, κ^N ensures that, along with B^N, also M^N is a martingale under $\mathbb{Q}^N$. Hence, M^N and $\mathbb{Q}^N$ are as requested in part (i) of our present lemma.

It thus remains to establish the weak convergence (3.16). By applying Taylor's expansion, we get

$$\left|q_n^N - \frac{\kappa_{n-1}^N \sigma_{n-1}^N \xi_{n-1}^N + (1+\kappa_n^N)\sigma_n^N}{2(1+\kappa_n^N)\sigma_n^N}\right| = O(N^{-1/2}) \qquad \mathbb{P}^N\text{-almost surely.}$$

Thus, $|(2q_n^N - 1)\xi_{n-1}^N - \frac{\kappa_{n-1}^N \sigma_{n-1}^N}{(1+\kappa_n^N)\sigma_n^N}| = O(N^{-1/2})$ $\mathbb{P}^N$-almost surely. From the last equality and the definition of the measure $\mathbb{Q}^N$, we get that

$$\mathbb{E}_{\mathbb{Q}^N}\big[((1 + \kappa_n^N)\sigma_n^N \xi_n^N - \kappa_{n-1}^N \sigma_{n-1}^N \xi_{n-1}^N)^2 | \mathcal{F}_{n-1}^N\big]$$
$$= (\sigma_n^N)^2(1 + \kappa_n^N)^2 + (\sigma_{n-1}^N)^2(\kappa_{n-1}^N)^2$$
$$\quad - 2(2q_n^N - 1)(1 + \kappa_n^N)\sigma_n^N \kappa_{n-1}^N \sigma_{n-1}^N \xi_{n-1}^N$$
$$= (\sigma_n^N)^2(1 + 2\kappa_n^N) + O(N^{-1/2}).$$

This together with applying the Taylor expansion yields

$$\mathbb{E}_{\mathbb{Q}^N}\big[(B_n^N - B_{n-1}^N)^2 | \mathcal{F}_{n-1}^N\big] = \frac{1}{N} + O(N^{-3/2}).$$

From Theorem 8.7 in [1], we get the convergence of $\mathrm{Law}(\overline{B}^N | \mathbb{Q}^N)$ to Wiener measure on $C[0, 1]$. From the continuity of $\tilde{\sigma}$, it follows that the continuous interpolation on $[0, 1]$ of $(\{\sqrt{1 + 2\kappa_n^N}\sigma_n^N\})_{n=0}^N$, that is, the process $\tilde{\sigma}_{[Nt]/N}(\overline{B}^N)$ under $\mathbb{Q}^N$ converges in law to $\tilde{\sigma}$ under $\mathbb{P}^W$ on $C[0, 1]$. Thus, Theorem 5.4 in [13] and (3.21) give the convergence

$$\mathrm{Law}(\overline{B}^N, \overline{M}^N | \mathbb{Q}^N) \quad \Rightarrow \quad \mathrm{Law}(W, S^{\tilde{\sigma}} | \mathbb{P}^W)$$

on the space $C[0, 1] \times C[0, 1]$. Finally, observe that we have the joint convergence

$$\mathrm{Law}(\overline{S}^N, \overline{M}^N, \sqrt{N}|\kappa^N \sigma^N| | \mathbb{Q}^N) \quad \Rightarrow \quad \mathrm{Law}(S^{\tilde{\sigma}}, S^{\tilde{\sigma}}, a(\tilde{\sigma}) | \mathbb{P}^W) \qquad \text{as } N \uparrow \infty$$

on $C[0, 1] \times C[0, 1] \times C[0, 1]$ and (3.16) follows as required. $\square$

3.2.2. *Proof of "$\leq$" in* (3.7).   Applying Theorem 2.3 with $g \triangleq g^{N,c}$ of (2.9) shows that for $N = 1, 2, \ldots$ there exists a measure $\mathbb{Q}^N$ on $(\Omega^N, \mathcal{F}_N^N)$ such that

$$(3.22) \quad V_{g^{N,c}}^{\underline{\sigma}/\sqrt{N}, \overline{\sigma}/\sqrt{N}}(\mathbb{F}) \leq \frac{1}{N} + \mathbb{E}_{\mathbb{Q}^N}\bigg[\mathbb{F}(\overline{S}^N) - \sum_{n=0}^{N-1} G_n^{N,c}\bigg(\frac{M_n^N - S_n^N}{S_n^N}\bigg)\bigg],$$

where $G^{N,c}$ is the Legendre–Fenchel transform of $g^{N,c}$ and where M^N is defined as in Lemma 3.6. Since by construction h^c, and thus also $g^{N,c}$ has maximum slope c, we have

$$G_n^{N,c}(\alpha) = \begin{cases} H_{n/N}(\overline{S}^N, \alpha), & \text{if } |\alpha| \leq c, \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, the above sequence of probabilities $(\mathbb{Q}^N)_{N=1,2,\ldots}$ is as required in the first part of Lemma 3.6. We thus obtain the weak convergence (3.15) with some probability $\mathbb{P}^* \in \mathcal{P}_{\underline{\sigma}, \overline{\sigma}, c}$.

Due to Skorohod's representation theorem, there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with processes $\hat{S}^N$, $\hat{M}^N$, and $\hat{Q}^N$, $N = 1, 2, \ldots$ and a continuous martingale $M > 0$ such that

$$(3.23) \quad \text{Law}(\overline{S}^N, \overline{M}^N, \overline{Q}^N | \mathbb{Q}^N) = \text{Law}(\hat{S}^N, \hat{M}^N, \hat{Q}^N | \hat{\mathbb{P}}), \qquad N = 1, 2, \ldots,$$

$$(3.24) \qquad\qquad \text{Law}(S | \mathbb{P}^*) = \text{Law}(M | \hat{\mathbb{P}}),$$

and such that

$$(3.25) \quad (\hat{S}^N, \hat{M}^N, \hat{Q}^N) \to (M, M, \langle \ln M \rangle) \qquad \hat{\mathbb{P}}\text{-almost surely as } N \uparrow \infty.$$

Due to (3.13) of Lemma 3.6, $\max_{0 \leq t \leq 1} \hat{S}_t^N$ is bounded in $L^p(\hat{\mathbb{P}})$ for any $p > 0$. By Lebesgue's theorem, the assumed continuity and polynomial growth of $\mathbb{F}$ in conjunction with (3.23) and (3.25) thus suffices to conclude that

$$(3.26) \qquad\qquad \mathbb{E}_{\mathbb{Q}^N}[\mathbb{F}(\overline{S}^N)] \to \mathbb{E}_{\hat{\mathbb{P}}}[\mathbb{F}(M)] \qquad \text{as } N \uparrow \infty.$$

Below we will argue that

$$(3.27) \quad \liminf_{N \uparrow \infty} \mathbb{E}_{\mathbb{Q}^N}\left[\sum_{n=0}^{N-1} G_n^{N,c}\left(\frac{M_n^N - S_n^N}{S_n^N} \right) \right]$$
$$\geq \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 \hat{H}_t(M) a^2\left(\sqrt{\frac{d\langle \ln M \rangle_t}{dt}} \right) dt \right].$$

Combining (3.22) with (3.26) and (3.27) then gives

$$(3.28) \quad \limsup_{N \uparrow \infty} V_{g^{N,c}}^{\underline{\sigma}/\sqrt{N}, \overline{\sigma}/\sqrt{N}}(\mathbb{F}) \leq \mathbb{E}_{\hat{\mathbb{P}}}\left[\mathbb{F}(M) - \int_0^1 \hat{H}_t(M) a^2\left(\sqrt{\frac{d\langle \ln M \rangle_t}{dt}} \right) dt \right].$$

Because of (3.24) the right-hand side of (3.28) can be viewed as one of the expectations considered in (3.8). We deduce from Lemma 3.5(i) that "$\leq$" holds in (3.7).

Let us conclude by proving (3.27) and write

$$(3.29) \quad \mathbb{E}_{\mathbb{Q}^N}\left[\sum_{n=0}^{N-1} G_n^{N,c}\left(\frac{M_n^N - S_n^N}{S_n^N} \right) \right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 N H_{[Nt]/N}(\hat{S}^N, \Delta_t^N / \sqrt{N}) \, dt \right],$$

where

$$-c \leq \Delta_t^N \triangleq \sqrt{N} \frac{\hat{M}_{[Nt]/N}^N - \hat{S}_{[Nt]/N}^N}{\hat{S}_{[Nt]/N}^N} \leq c.$$

Let $b : \mathbb{R} \to \mathbb{R}$ be a convex function such that $b(u) = a^2(\sqrt{u})$ for $u \geq 0$, for instance

$$(3.30) \qquad\qquad b(u) \triangleq \begin{cases} -u, & u \leq -\underline{\sigma}^2, \\ \dfrac{1}{4} \dfrac{(\underline{\sigma}^2 - u)^2}{\underline{\sigma}^2}, & -\underline{\sigma}^2 < u < 0, \\ a(\sqrt{u})^2, & u \geq 0. \end{cases}$$

By Assumption 2.4 and the $\hat{\mathbb{P}}$-almost sure convergence of $\hat{S}^N$ to M, the integrand $NH_{[Nt]/N}(\hat{S}^N, \Delta/\sqrt{N})$ converges uniformly in $t \in [0, 1]$ and $\Delta \in [-c, c]$ to $\hat{H}_t(M)\Delta^2$. Moreover, from Lemma 3.7 below we have $(\Delta_t^N)^2 \geq b(N\Delta\hat{Q}_{[Nt]/N}^N)$. Hence, the lim inf in (3.27) is bounded from below by

$$
\begin{aligned}
& \liminf_{N\uparrow\infty} \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 \hat{H}_t(M)(\Delta_t^N)^2 \, dt\right] \\
(3.31) \quad & \geq \liminf_{N\uparrow\infty} \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 \hat{H}_t(M)b(N\Delta\hat{Q}_{[Nt]/N}^N) \, dt\right].
\end{aligned}
$$

From their definition, it follows that the processes $(N\Delta\hat{Q}_{[Nt]/N}^N)_{0\leq t\leq 1}$, $N = 1, 2, \ldots$ take values in the interval $[\underline{\sigma}^2 - 2c\overline{\sigma}, \overline{\sigma}^2 + 2c\overline{\sigma}]$, and so they are bounded uniformly. Hence, we can use Lemma A1.1 in [7] to find, for $N = 1, 2, \ldots$, convex combinations δ^N of elements in this sequence of processes with index in $\{N, N+1, \ldots\}$ which converge $\hat{\mathbb{P}} \otimes dt$-almost every. Denote the limit process by $(\delta_t)_{0\leq t\leq 1}$. Observe that

$$
\int_0^t \delta_u \, du = \lim_{N\to\infty} \int_0^t N\Delta\hat{Q}_{[Nu]/N}^N \, du = \langle \ln M \rangle_t
$$

and so we conclude that $\delta_t = d\langle \ln M \rangle_t/dt$, $\hat{\mathbb{P}} \otimes dt$-almost every. Hence, by Fatou's lemma,

$$
\begin{aligned}
\mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 \hat{H}_t(M)a^2(\sqrt{\delta_t}) \, dt\right] &= \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 \hat{H}_t(M)b(\delta_t) \, dt\right] \\
&\leq \liminf_{N\uparrow\infty} \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 \hat{H}_t(M)b(\delta_t^N) \, dt\right].
\end{aligned}
$$

By convexity of b and by the construction of δ^N this last lim inf is not larger than the lim inf on the right-hand side of (3.31). Hence, we can combine these estimates to obtain our assertion (3.27).

3.2.3. *Proof of "$\geq$" in (3.7).*   Because of Lemma 3.5 it suffices to show that

$$
(3.32) \quad \liminf_{N\uparrow\infty} V_{g^{N,c}}^{\underline{\sigma}/\sqrt{N}, \overline{\sigma}/\sqrt{N}}(\mathbb{F}) \geq \mathbb{E}_{\mathbb{P}^W}\left[\mathbb{F}(S^{\tilde{\sigma}}) - \int_0^1 \hat{H}_t(S^{\tilde{\sigma}})a^2(\tilde{\sigma}_t) \, dt\right]
$$

for $\tilde{\sigma} \in \tilde{\Sigma}(c)$. For such $\tilde{\sigma}$, we can apply Lemma 3.6(ii) which provides us with probabilities $\mathbb{Q}^N$ and martingales M^N on $(\Omega^N, \mathcal{F}_N^N)$, $N = 1, 2, \ldots$, such that

$$
\text{Law}\left(\overline{S}^N, \overline{M}^N, \overline{\sqrt{N}\frac{|M^N - S^N|}{S^N}}\middle|\mathbb{Q}^N\right) \quad \Rightarrow \quad \text{Law}(S^{\tilde{\sigma}}, S^{\tilde{\sigma}}, a(\tilde{\sigma})|\mathbb{P}^W)
$$

(3.33)

$$
\text{as } N \uparrow \infty.
$$

We can now use Skorohod's representation theorem exactly as in Section 3.2.2 to obtain, in analogy with (3.26), that

$$\text{(3.34)} \qquad \mathbb{E}_{\mathbb{Q}^N}\big[\mathbb{F}(\overline{S}^N)\big] \to \mathbb{E}_{\mathbb{P}^W}\big[\mathbb{F}(S^{\tilde{\sigma}})\big] \qquad \text{as } N \uparrow \infty.$$

Set

$$-c \le \Delta_t^N \triangleq \sqrt{N} \frac{\hat{M}_{[Nt]/N}^N - \hat{S}_{[Nt]/N}^N}{\hat{S}_{[Nt]/N}^N} \le c.$$

In analogy with (3.29), we have that

$$\lim_{N \uparrow \infty} \mathbb{E}_{\mathbb{Q}^N}\left[\sum_{n=0}^{N-1} G_n^{N,c}\left(\frac{M_n^N - S_n^N}{S_n^N}\right)\right]$$

$$\text{(3.35)} \qquad = \lim_{N \uparrow \infty} \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^1 N H_{[Nt]/N}\big(\hat{S}^N, \Delta_t^N/\sqrt{N}\big)\,dt\right]$$

$$= \mathbb{E}_{\mathbb{P}^W}\left[\int_0^1 \hat{H}_t(S^{\tilde{\sigma}}) a^2(\tilde{\sigma}_t)\,dt\right],$$

where the last two equalities follow from our assumptions on H, the moment estimate in Lemma 3.6 (which gives uniform integrability) and (3.33) (together with the Skorohod representation theorem).

Combining (3.34) with (3.35) then allows us to write the right-hand side of (3.32) as

$$\mathbb{E}_{\mathbb{P}^W}\left[\mathbb{F}(S^{\tilde{\sigma}}) - \int_0^1 \hat{H}_t(S^{\tilde{\sigma}}) a^2(\tilde{\sigma}_t)\,dt\right]$$

$$= \lim_{N \uparrow \infty} \mathbb{E}_{\mathbb{Q}^N}\left[\mathbb{F}(\overline{S}^N) - \sum_{n=0}^{N-1} G_n^{N,c}\left(\frac{M_n^N - S_n^N}{S_n^N}\right)\right].$$

On the other hand, Theorem 2.2 reveals that such a limit is a lower bound for the left side of (3.32). This accomplishes our proof that "$\ge$" holds in (3.7).

We complete the section with the following elementary inequality which we used in the previous proof:

LEMMA 3.7. *For $x, y \in \mathbb{R}$, such that $|x| \in [\underline{\sigma}, \overline{\sigma}]$ we have*

$$b(x^2 + 2xy) \le y^2,$$

where b denotes the function of (3.30).

PROOF. We distinguish four cases:

(i) $x^2 + 2xy \ge \overline{\sigma}^2$. In this case $0 \le x^2 + 2xy - \overline{\sigma}^2 \le 2xy \le 2\overline{\sigma}|y|$ and so

$$b(x^2 + 2xy) = \frac{1}{4}\frac{(x^2 + 2xy - \overline{\sigma}^2)^2}{\overline{\sigma}^2} \le y^2.$$

(ii) $\underline{\sigma}^2 \leq x^2 + 2xy \leq \overline{\sigma}^2$. In this case $b(x^2 + 2xy) = 0$ and the statement is trivial.

(iii) $|x^2 + 2xy| \leq \underline{\sigma}^2$. Set $z = x^2 + 2xy$. Assume that z is fixed and introduce the function $f_z(u) = \frac{(z-u)^2}{u}$, $u \geq \underline{\sigma}^2$. Observe that for $z \in [-\underline{\sigma}^2, \underline{\sigma}^2]$ the derivative $f_z'(u) = 1 - z^2/u^2 \geq 0$ and so

$$b(x^2 + 2xy) = \tfrac{1}{4} f_z(\underline{\sigma}^2) \leq \tfrac{1}{4} f_z(x^2) = y^2.$$

(iv) Finally, assume that $x^2 + 2xy \leq -\underline{\sigma}^2$. Clearly $x^2 + 2xy + y^2 \geq 0$ and so

$$b(x^2 + 2xy) = -x^2 - 2xy \leq y^2. \qquad \square$$

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