The Annals of Applied Probability
2016, Vol. 26, No. 2, 1082–1110
DOI: 10.1214/15-AAP1112
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STOCHASTIC PERRON FOR STOCHASTIC TARGET GAMES[1]

BY ERHAN BAYRAKTAR AND JIAQI LI

University of Michigan

We extend the stochastic Perron method to analyze the framework of stochastic target games, in which one player tries to find a strategy such that the state process almost surely reaches a given target no matter which action is chosen by the other player. Within this framework, our method produces a viscosity sub-solution (super-solution) of a Hamilton–Jacobi–Bellman (HJB) equation. We then characterize the value function as a viscosity solution to the HJB equation using a comparison result and a byproduct to obtain the dynamic programming principle.

1. Introduction. We will extend the stochastic Perron method to analyze a stochastic (semi) game where a controller tries to find a strategy such that the controlled state process almost surely *reaches a given target* at a given finite time, no matter which control is chosen by an adverse player (nature). More precisely, the controller has access to a filtration generated by a Brownian motion and can observe and react to nature, who may choose a parametrization of the model to be totally adverse to the controller, in a nonanticipative way. This stochastic target game was introduced and analyzed in [8].

In this paper, we will have a fresh look at the problem of Bouchard and Nutz [8] with a different methodology, namely the stochastic Perron method. Using this method we will be able to drop the assumption on the concavity of the Hamiltonian assumed in [8]. The stochastic Perron method was introduced in [3] for analyzing linear problems, in [5] for Dynkin games involving free-boundary games and in [4] for stochastic control problems. This method is a type of verification theorem, which identifies the value function as the unique solution to a corresponding HJB equation without going through the dynamic programming principle, but does not require the smoothness of the value function. It is a stochastic version of the Perron method [9] in that it creates classes of sub- and super-solutions that envelope the value function and are closed under maximization and minimization, respectively. More recently, the stochastic Perron method was adjusted to solve exit time problems in [12], state constraint problems in [11], singular control problems in [6], stochastic games in [14] and control problems with model uncertainty in

Received September 2014; revised January 2015.

[1]Supported in part by NSF Grant DMS-09-55463.

MSC2010 subject classifications. Primary 93E20, 49L20, 49L25, 60G46; secondary 60H30, 91B28, 35D05.

Key words and phrases. The stochastic target problem, stochastic Perron method, viscosity solutions, geometric dynamic programming principle.

[13] and [1]. In this paper, we show how the main ideas of this method can be modified to analyze the stochastic target games of Bouchard and Nutz [8].

The main difficulty of this analysis is identifying the correct collections of stochastic sub- and super-solutions. Once this is established, the technical contribution is in showing that in fact the supremum and the infimum of the respective families are viscosity super- and sub-solutions, respectively. Then a comparison result establishes the claim since the value function is already enveloped by these two families. The identification of these classes and the technical proofs turn out to be quite different from the works cited above because of the difference between nature of the stochastic target problems and the nature of the stochastic control problems. Unlike the usual stochastic control problems, the goal of the target problems is to beat a stochastic target almost surely by applying the admissible controls. These problems, which are generalizations of the super-hedging problems that appear in mathematical finance, were introduced in the seminal papers [16] and [15]; see [17] for a more recent exposition. Stochastic target games, on the other hand, were considered recently by Bouchard, Moreau and Nutz [7] when the target is of controlled loss type. The more difficult case of an almost sure target was then analyzed in [8].

In this paper we achieve the following:

- We give a proof of the result that the value function of the stochastic target game is the unique viscosity solution of the associated HJB equation without first going through the geometric dynamic programming principle. What we have is a new method for analyzing stochastic target problems.
- We give a more elementary proof of the result in [8]. This way we are able to avoid using Krylov's method of shaken coefficients, which requires the concavity of the Hamiltonian.

The rest of the paper is organized as follows: In Section 2, we present the setup of the stochastic target game, introduce the related HJB equation and the definitions of the sets of stochastic super- and sub-solutions (our conceptual contribution). The technical contribution of the paper is given in Section 3, where we characterize the infimum (supremum) of the stochastic super-solutions (sub-solutions) as the viscosity sub-solution (super-solution) of the HJB equation. A viscosity comparison argument concludes that the value function is the unique bounded continuous viscosity solution of the HJB equation. Finally, we obtain the dynamic programming principle as a byproduct. Some technical results are deferred to the Appendix.

2. Statement of the problem.

2.1. *The value function.* Let us denote

$$\mathcal{D} := [0, T] \times \mathbb{R}^d, \qquad \mathcal{D}_{<T} := [0, T) \times \mathbb{R}^d, \qquad \mathcal{D}_T := \{T\} \times \mathbb{R}^d.$$

Let Ω be the space of continuous functions $\omega : [0, T] \to \mathbb{R}^d$, and let $\mathbb{P}$ be the Wiener measure on Ω. We will denote by W the canonical process on Ω, that is, $W_t(\omega) = \omega_t$, and by $\mathbb{F} = (\mathcal{F}_s)_{0\le s\le T}$ the augmented filtration generated by W. For $0 \le t \le T$ let $\mathbb{F}^t = (\mathcal{F}^t_s)_{0\le s\le T}$ be the augmented filtration generated by $(W_s - W_t)_{s\ge t}$. By convention, $\mathcal{F}^t_s$ is trivial for $s \le t$.

We denote by $\mathcal{U}^t$ (resp., $\mathcal{A}^t$) the collection of all $\mathbb{F}^t$-predictable processes in $L^p(\mathbb{P} \otimes dt)$ with values in a given Borel subset U (resp., bounded set A) of $\mathbb{R}^d$, where $p \ge 2$ is fixed.

Given $(t, x, y) \in \mathcal{D} \times \mathbb{R}$ and $(u, \alpha) \in \mathcal{U}^t \times \mathcal{A}^t$, consider the stochastic differential equations (SDEs)

$$\begin{cases} dX(s) = \mu_X(s, X(s), \alpha_s)\, ds + \sigma_X(s, X(s), \alpha_s)\, dW_s, \\ dY(s) = \mu_Y(s, X(s), Y(s), u_s, \alpha_s)\, ds + \sigma_Y(s, X(s), Y(s), u_s, \alpha_s)\, dW_s, \end{cases} \tag{2.1}$$

with initial data $(X(t), Y(t)) = (x, y)$.

ASSUMPTION 2.1. The coefficients μ_X, μ_Y, σ_X and σ_Y are continuous in all variables and take values in $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{M}^d := \mathbb{R}^{d\times d}$, respectively. There exists $K > 0$ such that

$$\begin{aligned} |\mu_X(t, x, \cdot) - \mu_X(t', x', \cdot)| + |\sigma_X(t, x, \cdot) - \sigma_X(t', x', \cdot)| &\le K(|t - t'| + |x - x'|), \\ |\mu_X(\cdot, x, \cdot)| + |\sigma_X(\cdot, x, \cdot)| &\le K, \\ |\mu_Y(\cdot, y, \cdot) - \mu_Y(\cdot, y', \cdot)| + |\sigma_Y(\cdot, y, \cdot) - \sigma_Y(\cdot, y', \cdot)| &\le K|y - y'|, \\ |\mu_Y(\cdot, y, u, \cdot)| + |\sigma_Y(\cdot, y, u, \cdot)| &\le K(1 + |u| + |y|), \end{aligned}$$

for all $(x, y), (x', y') \in \mathbb{R}^d \times \mathbb{R}$ and $u \in U$.

This assumption ensures that the stochastic differential equations given in (2.1) are well posed. Denote the solutions to (2.1) by $(X^\alpha_{t,x}, Y^{u,\alpha}_{t,x,y})$. Let $t \le T$. We say that a map $\mathfrak{u} : \mathcal{A}^t \to \mathcal{U}^t$, $\alpha \mapsto \mathfrak{u}[\alpha]$ is a t-admissible strategy if it is nonanticipating in the sense that

$$\{\omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]}\} \subset \{\omega \in \Omega : \mathfrak{u}[\alpha](\omega)|_{[t,s]} = \mathfrak{u}[\alpha'](\omega)|_{[t,s]}\}\text{-a.s.}$$

for all $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$, where $|_{[t,s]}$ indicates the restriction to the interval $[t, s]$. We denote by $\mathfrak{U}(t)$ the collection of all t-admissible strategies; moreover, we write $Y^{\mathfrak{u},\alpha}_{t,x,y}$ for $Y^{\mathfrak{u}[\alpha],\alpha}_{t,x,y}$. Then we can introduce the value function of the stochastic target game,

$$v(t, x) := \inf\{y \in \mathbb{R} : \exists \mathfrak{u} \in \mathfrak{U}(t) \text{ s.t. } Y^{\mathfrak{u},\alpha}_{t,x,y}(T) \ge g(X^\alpha_{t,x}(T))\text{-a.s. } \forall \alpha \in \mathcal{A}^t\}, \tag{2.2}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a bounded and measurable function. We also need to define strategies starting at a family of stopping times. Let $\mathbb{S}^t$ be the set of $\mathcal{F}^t$-stopping times valued in $[t, T]$.

DEFINITION 2.1 (Nonanticipating family of stopping times). Let $\{\tau^\alpha\}_{\alpha\in\mathcal{A}^t} \subset \mathbb{S}^t$ be a family of stopping times. This family is t-nonanticipating if

$$\begin{aligned}&\{\omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]}\}\\&\quad\subset \{\omega \in \Omega : t \le \tau^\alpha(\omega) = \tau^{\alpha'}(\omega) \le s\} \cup \{\omega \in \Omega : s < \tau^\alpha(\omega), s < \tau^{\alpha'}(\omega)\}\text{-a.s.}\end{aligned}$$

Denote the set of t-nonanticipating families of stopping times by $\mathfrak{S}^t$.

We will use $\{\tau^\alpha\}$ for short to represent $\{\tau^\alpha\}_{\alpha\in\mathcal{A}^t}$, which will always denote a t-nonanticipating family of stopping times.

DEFINITION 2.2 (Strategies starting at a nonanticipating family of stopping times). Fix t, and let $\{\tau^\alpha\} \in \mathfrak{S}^t$. We say that a map $\mathfrak{u} : \mathcal{A}^t \to \mathcal{U}^t$, $\alpha \mapsto \mathfrak{u}[\alpha]$ is a $(t, \{\tau^\alpha\})$-admissible strategy if it is nonanticipating in the sense that

$$\begin{aligned}&\{\omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]}\}\\&\quad\subset \{\omega \in \Omega : s < \tau^\alpha(\omega), s < \tau^{\alpha'}(\omega)\}\\&\quad\cup \{\omega \in \Omega : t \le \tau^\alpha(\omega) = \tau^{\alpha'}(\omega) \le s,\\&\qquad \mathfrak{u}[\alpha](\omega)|_{[\tau^\alpha(\omega),s]} = \mathfrak{u}[\alpha'](\omega)|_{[\tau^{\alpha'}(\omega),s]}\}\text{-a.s.}\end{aligned}$$

for all $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$, denoted by $\mathfrak{u} \in \mathfrak{U}(t, \{\tau^\alpha\})$.

It is clear that from Definition 2.2 that if we set $\tau^\alpha = t$ for all α, then $\mathfrak{U}(t, \{\tau^\alpha\})$ is then the same as $\mathfrak{U}(t)$. Hence the above definitions are consistent.

DEFINITION 2.3 (Concatenation). Let $\alpha_1, \alpha_2 \in \mathcal{A}^t$, $\tau \in \mathbb{S}^t$ be a stopping time. The concatenation of α_1, α_2 is defined as follows:

$$\alpha_1 \otimes_\tau \alpha_2 := \alpha_1 \mathbb{1}_{[t,\tau)} + \alpha_2 \mathbb{1}_{[\tau,T]}.$$

The concatenation of elements in $\mathcal{U}^t$ is defined in a similar fashion.

LEMMA 2.1. *Fix t, and let $\{\tau^\alpha\} \in \mathfrak{S}^t$. For $\mathfrak{u} \in \mathfrak{U}(t)$ and $\tilde{\mathfrak{u}} \in \mathfrak{U}(t, \{\tau^\alpha\})$, define $\mathfrak{u}_*[\alpha] := \mathfrak{u}[\alpha] \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha]$. Then $\mathfrak{u}_* \in \mathfrak{U}(t)$. For the rest of the paper, we will use $\mathfrak{u} \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha]$ to represent $\mathfrak{u}[\alpha] \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha]$.*

PROOF. It is obvious that u_* maps $\mathcal{A}^t$ to $\mathcal{U}^t$. Let us check the nonanticipativity of the map. For any fixed $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$, $\omega' \in \{\omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]}\}$, by Definition 2.1,

$$\omega' \in \{t \le \tau^\alpha = \tau^{\alpha'} \le s\} \cup \{s < \tau^\alpha, s < \tau^{\alpha'}\}\text{-a.s.} \tag{2.3}$$

(i) If $\omega' \in \{t \le \tau^\alpha = \tau^{\alpha'} \le s\}$, by the definition of $\mathfrak{u}_*$,

$$\mathfrak{u}_*[\alpha](\omega')|_{[t,s]} = \mathfrak{u}[\alpha](\omega')\mathbb{1}_{[t,\tau^\alpha(\omega'))}|_{[t,s]} + \tilde{\mathfrak{u}}[\alpha](\omega')\mathbb{1}_{[\tau^\alpha(\omega'),T]}|_{[t,s]},$$
$$\mathfrak{u}_*[\alpha'](\omega')|_{[t,s]} = \mathfrak{u}[\alpha'](\omega')\mathbb{1}_{[t,\tau^{\alpha'}(\omega'))}|_{[t,s]} + \tilde{\mathfrak{u}}[\alpha'](\omega')\mathbb{1}_{[\tau^{\alpha'}(\omega'),T]}|_{[t,s]}.$$

Since $\tau^\alpha(\omega') = \tau^{\alpha'}(\omega')$, $\mathfrak{u} \in \mathfrak{U}(t)$ and by Definition 2.2, we know

$$\omega' \in \{\omega \in \Omega : \mathfrak{u}[\alpha](\omega)|_{[t,s]} = \mathfrak{u}[\alpha'](\omega)|_{[t,s]}\}\text{-a.s.}$$

(ii) If $\omega' \in \{s < \tau^\alpha, s < \tau^{\alpha'}\}$, using the definition of $\mathfrak{u}_*$,

$$\mathfrak{u}_*[\alpha](\omega')|_{[t,s]} = \mathfrak{u}[\alpha](\omega')|_{[t,s]},$$
$$\mathfrak{u}_*[\alpha'](\omega')|_{[t,s]} = \mathfrak{u}[\alpha'](\omega')|_{[t,s]}.$$

Since $\omega' \in \{\omega \in \Omega : \alpha(\omega)|_{[t,s]} = \alpha'(\omega)|_{[t,s]}\}$ and $\mathfrak{u} \in \mathfrak{U}(t)$, then $\omega' \in \{\omega \in \Omega : \mathfrak{u}_*[\alpha](\omega)|_{[t,s]} = \mathfrak{u}_*[\alpha'](\omega)|_{[t,s]}\}$-a.s. □

2.2. *The HJB equation.* Before giving the HJB equation, we will introduce some notation and an assumption, which was also assumed in [8]. Given $(t, x, y, z, a) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times A$, define the set

$$N(t, x, y, z, a) := \{u \in U : \sigma_Y(t, x, y, u, a) = z\}.$$

ASSUMPTION 2.2. $u \mapsto \sigma_Y(t, x, y, u, a)$ is invertible. More precisely, there exists a measurable map $\hat{u} : \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times A \to U$ such that $N = \{\hat{u}\}$. Moreover, the map $\hat{u}(\cdot, a)$ is continuous for each $a \in A$.

Let us define for $(t, x, y, p, M) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d$,

$$H(t, x, y, p, M) := \sup_{a \in A} \Big\{ -\mu_Y^{\hat{u}}(t, x, y, \sigma_X(t, x, a)p, a) + \mu_X(t, x, a)^\top p + \frac{1}{2}\operatorname{Tr}[\sigma_X \sigma_X^\top(t, x, a) M] \Big\},$$

where

$$\mu_Y^{\hat{u}}(t, x, y, z, a) := \mu_Y(t, x, y, \hat{u}(t, x, y, z, a), a), \qquad z \in \mathbb{R}^d.$$

Consider the equation

$$\begin{aligned} &\phi_t + H(t, x, \phi, D\phi, D^2\phi) = 0 \qquad \text{on } \mathcal{D}_{<T}, \\ &\phi = g \qquad \text{on } \mathcal{D}_T. \end{aligned} \tag{2.4}$$

2.3. *Stochastic solutions*. We will introduce weak solution concepts to the HJB equation that are stable under minimization and maximization, respectively, and envelope the value function v of the stochastic target game.

DEFINITION 2.4 (Stochastic super-solutions). A function $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is called a stochastic super-solution of (2.4) if:

(1) it is bounded, continuous and $w(T, \cdot) \geq g(\cdot)$;
(2) for fixed $(t, x, y) \in \mathcal{D} \times \mathbb{R}$ and $\{\tau^\alpha\} \in \mathfrak{S}^t$, for any $\mathfrak{u} \in \mathfrak{U}(t)$, there exists a strategy $\tilde{\mathfrak{u}} \in \mathfrak{U}(t, \{\tau^\alpha\})$ such that for any $\alpha \in \mathcal{A}^t$ and each stopping time $\rho \in \mathbb{S}^t$, $\tau^\alpha \leq \rho \leq T$ with the simplifying notation $X := X^\alpha_{t,x}$, $Y := Y^{\mathfrak{u} \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha], \alpha}_{t,x,y}$, we have

$$Y(\rho) \geq w\big(\rho, X(\rho)\big) \qquad \mathbb{P}\text{-a.s. on } \big\{Y\big(\tau^\alpha\big) > w\big(\tau^\alpha, X\big(\tau^\alpha\big)\big)\big\}.$$

The set of stochastic super-solutions is denoted by $\mathcal{U}^+$. Assume it is nonempty and $v^+ := \inf_{w \in \mathcal{U}^+} w$. For any stochastic super-solution w, choose $\tau^\alpha = t$ for all α and $\rho = T$. Then there exists $\tilde{\mathfrak{u}} \in \mathfrak{U}(t)$ such that, for any $\alpha \in \mathcal{A}^t$,

$$Y^{\tilde{\mathfrak{u}}, \alpha}_{t,x,y}(T) \geq w\big(T, X^\alpha_{t,x}(T)\big) \geq g\big(X^\alpha_{t,x}(T)\big) \qquad \mathbb{P}\text{-a.s. on } \big\{y > w(t, x)\big\}.$$

Hence, $y > w(t, x)$ implies $y \geq v(t, x)$ from (2.2). This gives $w \geq v$ and $v^+ \geq v$. Similarly, we could define the stochastic sub-solutions.

DEFINITION 2.5 (Stochastic sub-solutions). A function $w : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is called a stochastic sub-solution of (2.4) if:

(1) it is bounded, continuous and $w(T, \cdot) \leq g(\cdot)$;
(2) for fixed $(t, x, y) \in \mathcal{D} \times \mathbb{R}$ and $\{\tau^\alpha\} \in \mathfrak{S}^t$, for any $\mathfrak{u} \in \mathfrak{U}(t)$, $\alpha \in \mathcal{A}^t$, there exists $\widetilde{\alpha} \in \mathcal{A}^t$ (may depend on $\mathfrak{u}$, α and τ^α) such that for each stopping time $\rho \in \mathbb{S}^t$, $\tau^\alpha \leq \rho \leq T$ with the simplifying notation $X := X^\alpha_{t,x}$, $Y := Y^{\mathfrak{u}, \alpha \otimes_{\tau^\alpha} \widetilde{\alpha}}_{t,x,y}$, we have

$$\mathbb{P}\big(Y(\rho) < w\big(\rho, X(\rho)\big) | B\big) > 0,$$

for any $B \subset \{Y(\tau^\alpha) < w(\tau^\alpha, X(\tau^\alpha))\}$, $B \in \mathcal{F}^t_{\tau^\alpha}$ and $\mathbb{P}(B) > 0$.

The set of stochastic sub-solutions is denoted by $\mathcal{U}^-$. Assume it is nonempty, and let $v^- := \sup_{w \in \mathcal{U}^-} w$. For any stochastic sub-solution w, choose $\tau^\alpha = t$ for all α and $\rho = T$. Hence for any $\mathfrak{u} \in \mathfrak{U}(t)$, there exists $\widetilde{\alpha} \in \mathcal{A}^t$, such that

$$\mathbb{P}\big(Y^{\mathfrak{u}, \widetilde{\alpha}}_{t,x,y}(T) < w\big(T, X^{\widetilde{\alpha}}_{t,x}(T)\big) \leq g\big(X^{\widetilde{\alpha}}_{t,x}(T)\big) | y < w(t, x)\big) > 0.$$

Hence, $y < w(t, x)$ implies $y \leq v(t, x)$ from (2.2). This gives $w \leq v$ and $v^- \leq v$. As a result we have

$$v^- \triangleq \sup_{w \in \mathcal{U}^-} w \leq v \leq \inf_{w \in \mathcal{U}^+} w \triangleq v^+. \tag{2.5}$$

We will show in Section 3 that under some suitable assumptions, v^+ and v^- are viscosity sub- and super-solutions of (2.4), respectively.

2.4. *Additional technical assumptions.* We will need to make some more technical assumptions as in [8].

ASSUMPTION 2.3. The map $(t,x,y,z) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \mapsto \mu_Y^{\hat{u}}(t,x,y,z,a)$ is Lipschitz continuous, uniformly in $a \in A$, and $(y,z) \in \mathbb{R} \times \mathbb{R}^d \mapsto \mu_Y^{\hat{u}}(t,x,y,z,a)$ has linear growth, uniformly in $(t,x,a) \in \mathcal{D} \times A$.

For the derivation of the super-solution property of v^-, we will impose a condition on the growth of μ_Y relative to σ_Y.

ASSUMPTION 2.4.

$$\sup_{u \in U} \frac{|\mu_Y(\cdot, u, \cdot)|}{1 + \|\sigma_Y(\cdot, u, \cdot)\|} \qquad \text{is locally bounded,}$$

where $\|\cdot\|$ is the Euclidean norm.

In (2.5) we implicitly assume that the sets $\mathcal{U}^+$ and $\mathcal{U}^-$ are nonempty. The assumptions we made already imply that $\mathcal{U}^+$ is not empty, but the same may not be true when $\mathcal{U}^-$ is not empty.

ASSUMPTION 2.5. The collection $\mathcal{U}^-$ is not empty.

2.5. *When $\mathcal{U}^+$ and $\mathcal{U}^-$ are not empty.* As the next result shows, the assumptions above already guarantee that $\mathcal{U}^+$ is not empty.

PROPOSITION 2.1. *Under Assumptions 2.1, 2.2 and 2.3 the collection $\mathcal{U}^+$ is not empty.*

PROOF. See the Appendix. $\square$

In the above proposition the assumptions made can be replaced by the following natural assumption (although this is not the route we will take):

ASSUMPTION 2.6. There exists $\mathbf{u} \in U$ such that $\mu_Y(t,x,y,\mathbf{u},a) = 0$, $\sigma_Y(t,x,y,\mathbf{u},a) = 0$ for all $(t,x,y,a) \in \mathcal{D}_{<T} \times \mathbb{R} \times A$. (In these equations the right-hand sides are denoted by just 0 for simplicity, but they in fact are collections of 0's matching the dimension on the left-hand side.)

In the context of super-hedging in mathematical finance, in which Y represents the wealth of an investor and X the stock price, and $g(X_T)$ a financial contract, the last assumption is equivalent to allowing the investor not to trade in the risky assets.

PROPOSITION 2.2. *Under Assumptions 2.1 and 2.6 the collection $\mathcal{U}^+$ is not empty.*

PROOF. Choose the strategy $\tilde{u}[\alpha] = \mathbf{u}$. For any given $\{\tau^\alpha\} \in \mathfrak{S}^t$, we have $\tilde{\mathfrak{u}} \in \mathcal{U}(t, \{\tau^\alpha\})$, and from Assumption 2.6, it holds for any $\mathfrak{u} \in \mathfrak{U}(t)$ that

$$Y_{t,x,y}^{\mathfrak{u} \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha],\alpha}(\rho) = Y_{t,x,y}^{\mathfrak{u} \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha],\alpha}(\tau^\alpha) \qquad \forall \alpha \in \mathcal{A}^t \text{ and } \rho \in \mathbb{S}^t \text{ such that } \tau^\alpha \le \rho \le T.$$

From the boundedness of g, there exists a C, such that $g(x) < C$. Now take $w(t, x) \equiv C$, which clearly satisfies the first condition in Definition 2.4. On the other hand, on the set $\{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\}$, we clearly have that $\{Y(\rho) > w(\rho, X(\rho))\}$ for any ρ such that $\tau^\alpha \le \rho \le T$, which gives the second condition in Definition 2.4. □

PROPOSITION 2.3. *If in addition to Assumptions 2.1 there exists $a \in A$ such that $\mu_Y(t, x, y, u, a) = 0$, $\sigma_Y(t, x, y, u, a) = 0$ for all $(t, x, y, u) \in \mathcal{D}_{<T} \times \mathbb{R} \times U$, then $\mathcal{U}^-$ is not empty.*

PROOF. The proof is similar to that of Proposition 2.2. □

The additional assumption in the latter proposition is not very reasonable. Below we introduce an alternative assumption.

ASSUMPTION 2.7. $\frac{|\mu_Y|}{\|\sigma_Y\|}$ is bounded on $N = \{(t, x, y, u, a) : \sigma_Y(t, x, y, u, a) \neq 0\}$.

PROPOSITION 2.4. *Under Assumptions 2.1, 2.2, 2.6 and 2.7, the collection $\mathcal{U}^-$ is not empty.*

PROOF. See the Appendix. □

3. The main result and its proof. To prove the main theorem, we need some preparatory lemmas.

LEMMA 3.1. *The set of stochastic super/sub solutions is upwards/downwards directed; that is:*

(1) *if $w_1, w_2 \in \mathcal{U}^+$, then $w_1 \wedge w_2 \in \mathcal{U}^+$;*
(2) *if $w_1, w_2 \in \mathcal{U}^-$, then $w_1 \vee w_2 \in \mathcal{U}^-$.*

PROOF. This lemma is in the spirit of Lemma 3.7 in [14]. Here we only sketch the proof for (1). For $w_1, w_2 \in \mathcal{U}^+$, let $w = w_1 \wedge w_2$. Clearly w is bounded, continuous and $w(T, x) \ge g(x)$. For fixed $(t, x, y) \in \mathcal{D}_{<T} \times \mathbb{R}$ and $\{\tau^\alpha\} \in \mathfrak{S}^t$, let $\mathfrak{u}_1$ and $\mathfrak{u}_2$ be the strategies starting at $\{\tau^\alpha\}$ for w_1 and w_2, respectively. Let

$$\mathfrak{u}[\alpha] = \mathfrak{u}_1[\alpha] \mathbb{1}_{\{w_1(\tau^\alpha, X(\tau^\alpha)) < w_2(\tau^\alpha, X(\tau^\alpha))\}} + \mathfrak{u}_2[\alpha] \mathbb{1}_{\{w_1(\tau^\alpha, X(\tau^\alpha)) \ge w_2(\tau^\alpha, X(\tau^\alpha))\}}.$$

It is easy to show that $\mathfrak{u}$ works for w in the definition of stochastic super-solutions. □

LEMMA 3.2. *There exists a nonincreasing sequence $\mathcal{U}^+ \ni w_n \searrow v^+$ and a nondecreasing sequence $\mathcal{U}^- \ni v_n \nearrow v^-$.*

PROOF. The proof of the lemma follows directly from Proposition 4.1 in [3]. □

Let us also state the following well-known result without proof.

LEMMA 3.3. *Given $f: X \times Y \subset \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$, define $F(x) := \sup_{y\in Y} f(x, y)$. If $x \to f(x, y)$ is continuous, uniformly in y and $F(x) < \infty$ for all $x \in X$, then $x \to F(x)$ is continuous.*

THEOREM 3.1 (Stochastic Perron for stochastic target games). *Let Assumptions 2.1 and 2.2 hold.*

(1) *If in addition g is upper semi-continuous* (*USC*) *and Assumption 2.3 holds, the function v^+ is a bounded USC viscosity sub-solution of* (2.4).

(2) *On the other hand if g is lower semi-continuous* (*LSC*) *and Assumptions 2.4 and 2.5 hold, the function v^- is a bounded LSC viscosity super-solution of* (2.4).

PROOF. *Step* 1 (*v^+ is the viscosity sub-solution*). First due to Proposition 2.1 v^+ is well defined. We will first show the interior viscosity sub-solution property and then demonstrate the boundary condition.

Step 1.1. *The interior sub-solution property*: Let (t_0, x_0) be in the parabolic interior $[0, T) \times \mathbb{R}^d$ such that a smooth function φ strictly touches v^+ from above at (t_0, x_0). Assume, by contradiction, that

$$\varphi_t + H\big(t, x, \varphi, D\varphi, D^2\varphi\big) < 0 \qquad \text{at } (t_0, x_0).$$

From the uniform continuity of μ_X and σ_X in Assumption 2.1, the uniform continuity of $\mu_Y^{\hat{u}}$ in Assumption 2.3 and the smoothness of φ, the map $(t, x, y, a) \to -\mu_Y^{\hat{u}}(t, x, y, \sigma_X(t, x, a)D\varphi(t, x), a) + \mu_X(t, x, a)^\top D\varphi + \frac{1}{2}\mathrm{Tr}[\sigma_X\sigma_X^\top(t, x, a) \times D^2\varphi(t, x)]$ is uniformly continuous in (t, x, y). Hence the map $(t, x, y) \to H(t, x, y, D\varphi(t, x), D^2\varphi(t, x))$ is continuous due to Lemma 3.3. This implies that there exists a $\varepsilon > 0$ and $\delta > 0$ such that

$$\begin{aligned} &\varphi_t + H\big(t, x, y, D\varphi, D^2\varphi\big) < 0 \\ &\qquad \forall (t, x) \in \overline{B(t_0, x_0, \varepsilon)} \text{ and } \big|y - \varphi(t, x)\big| \le \delta, \end{aligned} \tag{3.1}$$

where $B(t_0, x_0, \varepsilon) = \{(t, x) \in \mathcal{D} : \max\{|t - t_0|, |x - x_0|\} < \varepsilon\}$. Now, on the compact torus $\mathbb{T} = \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2)$, we have that $\varphi > v^+$, and the min of $\varphi - v^+$

is attained since v^+ is USC. Therefore, $\varphi > v^+ + \eta$ on $\mathbb{T}$ for some $\eta > 0$. Since $w_n \searrow v^+$, a Dini-type argument shows that for large enough n, we have $\varphi > w_n + \eta/2$ on $\mathbb{T}$ and $\varphi > w_n - \delta$ on $\overline{B(t_0, x_0, \varepsilon/2)}$. For simplicity, fix such an n, and denote $w = w_n$. Now define, for small $\kappa < \frac{\eta}{2} \wedge \delta$,

$$w^\kappa \triangleq \begin{cases} (\varphi - \kappa) \wedge w, & \text{on } \overline{B(t_0, x_0, \varepsilon)}, \\ w, & \text{outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Since $\varphi > w + \eta/2 > w + \kappa$ on $\mathbb{T}$, then $w = w^\kappa$ on $\partial B(t_0, x_0; \varepsilon/2)$, which implies w^κ is continuous. Since $w^\kappa(t_0, x_0) < v^+(t_0, x_0)$, we would obtain a contradiction if we can show $w^\kappa \in \mathcal{U}^+$.

Fix t, $\{\tau^\alpha\} \in \mathfrak{S}^t$ and $\mathfrak{u} \in \mathfrak{U}(t)$. We need to construct a strategy $\tilde{\mathfrak{u}} \in \mathfrak{U}(t, \{\tau^\alpha\})$ in the definition of stochastic super-solutions for w^κ. This can be done as follows: since w is a stochastic super-solution, there exists an "optimal" strategy $\tilde{\mathfrak{u}}_1$ in Definition 2.4 for w starting at $\{\tau^\alpha\}$. We will construct $\tilde{\mathfrak{u}}$ in two steps:

(i) $w^\kappa(\tau^\alpha, X^\alpha_{t,x}(\tau^\alpha)) = w(\tau^\alpha, X^\alpha_{t,x}(\tau^\alpha))$: set $\tilde{\mathfrak{u}} = \tilde{\mathfrak{u}}_1$;

(ii) $w^\kappa(\tau^\alpha, X^\alpha_{t,x}(\tau^\alpha)) < w(\tau^\alpha, X^\alpha_{t,x}(\tau^\alpha))$: In this case we necessarily start inside the ball. Let $\overline{Y}$ be the unique strong solution (which is thanks in particular to Assumption 2.3) of the equation

$$\begin{aligned} \overline{Y}(l) = {} & Y^{\mathfrak{u},\alpha}_{t,x,y}(\tau^\alpha) \\ & + \int_{\tau^\alpha}^{\tau^\alpha \vee l} \mu_Y^{\hat{u}}\big(s, X^\alpha_{t,x}(s), \overline{Y}(s), \sigma_X\big(s, X^\alpha_{t,x}(s), \alpha_s\big) D\varphi\big(s, X^\alpha_{t,x}(s)\big), \alpha_s\big)\, ds \\ & + \int_{\tau^\alpha}^{\tau^\alpha \vee l} \sigma_X\big(s, X^\alpha_{t,x}(s), \alpha_s\big) D\varphi\big(s, X^\alpha_{t,x}(s)\big)\, dW_s, \qquad l \geq \tau^\alpha, \end{aligned}$$

for any $\mathfrak{u} \in \mathfrak{U}(t)$ and $\alpha \in \mathcal{A}^t$, and set $\overline{Y}(s) = Y^{\mathfrak{u},\alpha}_{t,x,y}(s)$ for $s < \tau^\alpha$. Define

$$\tilde{\mathfrak{u}}_0 := \tilde{\mathfrak{u}}_0[\alpha](s) = \hat{u}\big(s, X^\alpha_{t,x}(s), \overline{Y}(s), \sigma_X\big(s, X^\alpha_{t,x}(s), \alpha_s\big) D\varphi\big(s, X^\alpha_{t,x}(s)\big), \alpha_s\big).$$

Let θ^α_1 is the first exit time of $(s, X^\alpha_{t,x}(s))$ after τ^α from $B(t_0, x_0; \varepsilon/2)$ and θ^α_2 be the first time after τ^α when $|\overline{Y}(s) - \varphi(s, X^\alpha_{t,x}(s))| \geq \delta$. More precisely,

$$\theta^\alpha_1 := \inf\big\{s \in [\tau^\alpha, T] : \big(s, X^\alpha_{t,x}(s)\big) \notin B(t_0, x_0, \varepsilon/2)\big\}$$

and

$$\theta^\alpha_2 := \inf\big\{s \in [\tau^\alpha, T] : \big|\overline{Y}(s) - \varphi\big(s, X^\alpha_{t,x}(s)\big)\big| \geq \delta\big\}.$$

Let $\theta^\alpha = \theta^\alpha_1 \wedge \theta^\alpha_2$. We know that $\{\theta^\alpha\} \in \mathfrak{S}^t$ from Example 1 in [2]. We will set $\tilde{\mathfrak{u}}$ to be $\tilde{\mathfrak{u}}_0$ until θ^α. Starting at θ^α, we will then follow the strategy $\mathfrak{u}^\theta \in \mathfrak{U}(t, \{\theta^\alpha\})$ which is "optimal" for w.

In summary, (i) and (ii) together give us the following strategy:

$$\tilde{\mathfrak{u}}[\alpha] = \big(\mathbb{1}_A \tilde{\mathfrak{u}}_1[\alpha] + \mathbb{1}_{A^c}\big(\tilde{\mathfrak{u}}_0[\alpha]\mathbb{1}_{[t,\theta^\alpha)} + \mathfrak{u}^\theta[\alpha]\mathbb{1}_{[\theta^\alpha,T]}\big)\big)\mathbb{1}_{[\tau^\alpha,T]},$$

where

$$A = \{w^\kappa(\tau^\alpha, X^\alpha_{t,x}(\tau^\alpha)) = w(\tau^\alpha, X^\alpha_{t,x}(\tau^\alpha))\}.$$

We note that $\tilde{\mathfrak{u}}_0 \in \mathfrak{U}(t)$ by the pathwise uniqueness of X's, Y's and $\overline{Y}$'s equations. Then applying Lemma 2.1, $\tilde{\mathfrak{u}}_0[\alpha]\mathbb{1}_{[t,\theta^\alpha)} + \mathfrak{u}^\theta[\alpha]\mathbb{1}_{[\theta^\alpha,T]} \in \mathfrak{U}(t)$. Since $\tilde{\mathfrak{u}}_1 \in \mathfrak{U}(t, \{\tau^\alpha\})$, by Definition 2.2, it follows that $\tilde{\mathfrak{u}} \in \mathfrak{U}(t, \{\tau^\alpha\})$ by the pathwise uniqueness of X's equation. Now, let us show the above construction actually works. We need to show that for any $\rho \in \mathbb{S}^t$ such that $\tau^\alpha \le \rho \le T$,

$$Y(\rho) \ge w(\rho, X(\rho)) \qquad \mathbb{P}\text{-a.s. on } \{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\},$$

where

$$X := X^\alpha_{t,x} \quad \text{and} \quad Y := Y^{\mathfrak{u}\otimes_{\tau^\alpha}\tilde{\mathfrak{u}}[\alpha],\alpha}_{t,x,y}.$$

Note that $\overline{Y}(s) = Y^{\mathfrak{u}\otimes_{\tau^\alpha}\tilde{\mathfrak{u}}_0[\alpha],\alpha}_{t,x,y}(s)$ for $s \ge \tau^\alpha$ and

$$Y = \mathbb{1}_A Y^{\mathfrak{u}\otimes_{\tau^\alpha}\tilde{\mathfrak{u}}_1[\alpha],\alpha}_{t,x,y} + \mathbb{1}_{A^c} Y^{\mathfrak{u}\otimes_{\tau^\alpha}\tilde{\mathfrak{u}}_0[\alpha],\alpha}_{t,x,y} \qquad \text{for } \tau^\alpha \le s \le \theta^\alpha. \tag{3.2}$$

We will carry out the proof in two steps:

(i) *On the set* $A \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha))\}$, we have

$$Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha)).$$

From (3.2) and the "optimality" of $\tilde{\mathfrak{u}}_1$ (for w), we know

$$Y(\rho) = Y^{\mathfrak{u}\otimes_{\tau^\alpha}\tilde{\mathfrak{u}}_1[\alpha],\alpha}_{t,x,y}(\rho) \ge w(\rho, X(\rho)) \ge w^\kappa(\rho, X(\rho)) \qquad \mathbb{P}\text{-a.s on the above set.}$$

(ii) *On the set* $A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha))\}$, by the definition of $\tilde{\mathfrak{u}}_0$ and (3.2), using Itô's formula,

$$Y(\cdot \wedge \theta^\alpha) - \varphi(\cdot \wedge \theta^\alpha, X(\cdot \wedge \theta^\alpha)) = Y(\tau^\alpha) - \varphi(\tau^\alpha, X(\tau^\alpha)) + \int_{\tau^\alpha}^{\cdot\wedge\theta^\alpha} \gamma(s)\, ds,$$

where

$$\begin{aligned}\gamma := {}& \mu_Y^{\hat{u}}(\cdot, X, Y, \sigma_X(\cdot, X, \alpha) D\varphi(\cdot, X), \alpha) - \mu_X(\cdot, X, \alpha)^\top D\varphi(\cdot, X) \\ & - \tfrac{1}{2}\operatorname{Tr}[\sigma_X \sigma_X^\top(\cdot, X, \alpha) D^2\varphi(\cdot, X)] - \varphi_t(\cdot, X),\end{aligned}$$

since the definition of $\hat{u}$ allows us to cancel the Brownian motion terms on the right-hand side. On $[\tau^\alpha, \theta^\alpha]$, $(t, X) \in \overline{B(t_0, x_0, \varepsilon)}$ and $|Y(t) - \varphi(t, X(t))| \le \delta$, therefore from (3.1) we have that $\gamma > 0$. This implies that $Y(\cdot \wedge \theta^\alpha) - \varphi(\cdot \wedge \theta^\alpha, X(\cdot \wedge \theta^\alpha))$ is nondecreasing on $[\tau^\alpha, T]$ and

$$Y(\theta^\alpha) - \varphi(\theta^\alpha, X(\theta^\alpha)) + \kappa > Y(\tau^\alpha) - \varphi(\tau^\alpha, X(\tau^\alpha)) + \kappa > 0. \tag{3.3}$$

As a result, on the one hand, we have

$$\begin{aligned} 0 < (Y(\theta_1^\alpha) - \varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa) \le (Y(\theta_1^\alpha) - w(\theta_1^\alpha, X(\theta_1^\alpha))) \\ \text{on } \{\theta_1^\alpha < \theta_2^\alpha\}. \end{aligned} \tag{3.4}$$

On the other hand,

$$Y(\theta_2^\alpha) - \varphi(\theta_2^\alpha, X(\theta_2^\alpha)) = \delta \qquad \text{on } \{\theta_1^\alpha \geq \theta_2^\alpha\}.$$

Observe that the right-hand side of the above expression cannot be $-\delta$ due to (3.3). Therefore,

$$(3.5)\qquad \begin{aligned} &(Y(\theta_2^\alpha) - w(\theta_2^\alpha, X(\theta_2^\alpha))) = (\delta + \varphi(\theta_2^\alpha, X(\theta_2^\alpha)) - w(\theta_2^\alpha, X(\theta_2^\alpha))) > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \{\theta_1^\alpha \geq \theta_2^\alpha\}, \end{aligned}$$

since $\varphi > w - \delta$ on $\overline{B(t_0, x_0, \varepsilon/2)}$. Combining (3.4) and (3.5) we obtain

$$(3.6)\qquad Y(\theta^\alpha) - w(\theta^\alpha, X(\theta^\alpha)) > 0 \qquad \text{on } A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}.$$

It follows from this conclusion and the "optimality" of $\mathfrak{u}^\theta$ starting at $\{\theta^\alpha\}$ that

$$(Y(\rho \vee \theta^\alpha) - w^\kappa(\rho \vee \theta^\alpha, X(\rho \vee \theta^\alpha))) \geq (Y(\rho \vee \theta^\alpha) - w(\rho \vee \theta^\alpha, X(\rho \vee \theta^\alpha))) \geq 0,$$

on $A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}$.

Also, since $Y(\cdot \wedge \theta^\alpha) - \varphi(\cdot \wedge \theta^\alpha, X(\cdot \wedge \theta^\alpha))$ is nondecreasing on $[\tau^\alpha, T]$ it follows that $(Y(\rho \wedge \theta^\alpha) - \varphi(\rho \wedge \theta^\alpha, X(\rho \wedge \theta^\alpha)) + \kappa) > 0$, which further gives

$$(3.7)\qquad \begin{aligned} &(Y(\rho \wedge \theta^\alpha) - w^\kappa(\rho \wedge \theta^\alpha, X(\rho \wedge \theta^\alpha))) > 0 \\ &\qquad\qquad\qquad\qquad \text{on } A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}. \end{aligned}$$

From (3.6) and (3.7) we have

$$Y(\rho) - w^\kappa(\rho, X(\rho)) \geq 0 \qquad \text{on } A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}.$$

Step 1.2. *The boundary condition*:

Step A: In this step we will assume that $\mu_Y^{\hat{u}}$ is nondecreasing in its y-variable. Assume on the contrary that for some $x_0 \in \mathbb{R}^d$, we have

$$(3.8)\qquad v^+(T, x_0) > g(x_0).$$

Since g is USC, then from (3.8) there exists $\varepsilon > 0$ such that

$$(3.9)\qquad v^+(T, x_0) > g(x) + \varepsilon \qquad \text{for } |x - x_0| \leq \varepsilon.$$

Choose ε such that $\varepsilon < 1$. Since v^+ is USC, then v^+ is bounded above on the compact (rectangular) torus $\mathbb{T} = \overline{B(T, x_0; \varepsilon)} - B(T, x_0; \varepsilon/2)$, where $B(T, x_0; \varepsilon) = \{(t, x) \in \mathcal{D} : \max\{|T - t|, |x - x_0|\} < \varepsilon\}$. Choose $\beta > 0$ small enough, such that

$$v^+(T, x_0) + \frac{\varepsilon^2}{4\beta} > \varepsilon + \sup_{\mathbb{T}} v^+(t, x).$$

By a Dini-type argument there exists a $w \in \mathcal{U}^+$ such that

$$(3.10)\qquad v^+(T, x_0) + \frac{\varepsilon^2}{4\beta} > \varepsilon + \sup_{\mathbb{T}} w(t, x).$$

For $C > 0$ let us denote

$$\varphi^{\beta,C}(t,x) = v^+(T,x_0) + \frac{|x-x_0|^2}{\beta} + C(T-t).$$

Hence, $D\varphi^{\beta,C}(t,x) = \frac{2(x-x_0)}{\beta}$ and $D^2\varphi^{\beta,C}(t,x) = \frac{2}{\beta} I_{d\times d}$. From Assumption 2.2,

$$\big|\mu_X(t,x,a)^\top D\varphi^{\beta,C}(t,x)\big| \le 2K \frac{|x-x_0|}{\beta} \le \frac{2K}{\beta} \quad \text{for } (t,x) \in \overline{B(T,x_0;\varepsilon)} \text{ and } a \in A, \tag{3.11}$$

where we use $\varepsilon < 1$. Similarly,

$$\left|\frac{1}{2}\operatorname{Tr}\big[\sigma_X\sigma_X^\top(t,x,a)D^2\varphi^{\beta,C}(t,x)\big]\right| \le \frac{1}{2}K^2\frac{2d}{\beta} = \frac{K^2 d}{\beta} \quad \text{for } (t,x) \in \overline{B(T,x_0;\varepsilon)} \text{ and } a \in A, \tag{3.12}$$

where d is the dimension of the space where the variable x lives. From the linear growth condition of $\mu_Y^{\hat{u}}$ in Assumption 2.3, there exists a $L > 0$, such that

$$\begin{aligned} &-\mu_Y^{\hat{u}}\big(t,x,\varphi^{\beta,0}-\varepsilon, \sigma_X(t,x,a)D\varphi^{\beta,0}, a\big) \\ &\quad \le L\big(1 + \big|\varphi^{\beta,0}(t,x)-\varepsilon\big| + \big|\sigma_X(t,x,a)D\varphi^{\beta,0}(t,x)\big|\big) \\ &\quad \le L\big(1 + v^+(T,x_0) + 1/\beta + 1 + 2K/\beta\big) \\ &\qquad \text{for } (t,x) \in \overline{B(T,x_0;\varepsilon)} \text{ and } a \in A. \end{aligned} \tag{3.13}$$

Noting that $D\varphi^{\beta,C}(t,x) = D\varphi^{\beta,0}(t,x)$, from the monotonicity assumption of $\mu_Y^{\hat{u}}$, we have

$$\begin{aligned} &-\mu_Y^{\hat{u}}\big(t,x,\varphi^{\beta,C}-\varepsilon, \sigma_X(t,x,a)D\varphi^{\beta,C}, a\big) \\ &\quad \le -\mu_Y^{\hat{u}}\big(t,x,\varphi^{\beta,0}-\varepsilon, \sigma_X(t,x,a)D\varphi^{\beta,0}, a\big). \end{aligned}$$

The above equation, together with (3.11), (3.12) and (3.13), implies that $H(\cdot, \varphi^{\beta,C}-\varepsilon, D\varphi^{\beta,C}, D^2\varphi^{\beta,C})(t,x)$ is bounded from above on $\overline{B(T,x_0;\varepsilon)}$, and the bound is independent of C. As a result for a large enough C we have that

$$\varphi_t^{\beta,C} + H\big(\cdot, y, D\varphi^{\beta,C}, D^2\varphi^{\beta,C}\big)(t,x) < 0 \quad \forall (t,x) \in B(T,x_0;\varepsilon) \text{ and } y \ge \varphi^{\beta,C}(t,x)-\varepsilon, \tag{3.14}$$

where we used the monotonicity assumption of $\mu_Y^{\hat{u}}$. Making sure that $C \ge \varepsilon/2\beta$, we obtain from (3.10) that

$$\varphi^{\beta,C} \ge \varepsilon + w \qquad \text{on } \mathbb{T}.$$

Also,

$$\varphi^{\beta,C}(T,x) \ge v^+(T,x_0) > g(x) + \varepsilon \qquad \text{for } |x-x_0| \le \varepsilon. \tag{3.15}$$

Now we can choose $\kappa < \varepsilon$ and define

$$w^{\beta,C,\kappa} \triangleq \begin{cases} (\varphi^{\beta,C} - \kappa) \wedge w, & \text{on } \overline{B(T, x_0, \varepsilon)}, \\ w, & \text{outside } \overline{B(T, x_0, \varepsilon)}. \end{cases} \tag{3.16}$$

From (3.15) and (3.16) it is easy to see that $w^{\beta,C,\kappa}(T, x) \geq g(x)$. By applying similar arguments as in step 1.1, we can show that $w^{\beta,C,\kappa}$ is a stochastic super-solution with $w^{\beta,C,\kappa}(T, x_0) < v^+(T, x_0)$. This contradicts the definition of v^+.

Step B: We now turn to showing the same result for more general $\mu_Y^{\hat{u}}$ and follow a proof similar to that in [8]. Fix $c > 0$, and define $\widetilde{Y}_{t,x,y}^{\mathfrak{u},\alpha}$ as the strong solution of

$$d\widetilde{Y}(s) = \tilde{\mu}_Y\big(s, X_{t,x}^{\alpha}(s), \widetilde{Y}(s), \mathfrak{u}[\alpha]_s, \alpha_s\big)\,ds + \tilde{\sigma}_Y\big(s, X_{t,x}^{\alpha}(s), \widetilde{Y}(s), \mathfrak{u}[\alpha]_s, \alpha_s\big)\,dW_s$$

with initial data $\widetilde{Y}(t) = y$, where

$$\begin{aligned} \tilde{\mu}_Y(t, x, y, u, a) &:= cy + e^{ct}\mu_Y\big(t, x, e^{-ct}y, u, a\big), \\ \tilde{\sigma}_Y(t, x, y, u, a) &:= e^{ct}\sigma_Y\big(t, x, e^{-ct}y, u, a\big). \end{aligned}$$

Hence, $\widetilde{Y}_{t,x,y}^{\mathfrak{u},\alpha}(s)e^{-cs} = Y_{t,x,ye^{-ct}}^{\mathfrak{u},\alpha}(s)$ for any $s \in [t, T]$ by the strong uniqueness. Set $\widetilde{g}(x) := e^{cT}g(x)$, and define

$$\tilde{v}(t, x) := \inf\big\{y \in \mathbb{R} : \exists \mathfrak{u} \in \mathfrak{U}^t \text{ s.t. } \widetilde{Y}_{t,x,y}^{\mathfrak{u},\alpha}(T) \geq \widetilde{g}\big(X_{t,x}^{\alpha}(T)\big)\text{-a.s. } \forall \alpha \in \mathcal{A}^t\big\}.$$

Therefore, $\tilde{v}(t, x) = e^{ct}v(t, x)$. Since $\mu_Y^{\hat{u}}$ has linear growth in its second argument y, one can choose large enough $c > 0$ so that

$$\tilde{\mu}_Y^{\hat{u}} : (t, x, y, z, a) \mapsto cy + e^{ct}\mu_Y^{\hat{u}}\big(t, x, e^{-ct}y, e^{-ct}z, a\big) \tag{3.17}$$

is nondecreasing in its y-variable. This means that these dynamics satisfy the monotonicity assumption used in step A above. Moreover, all the assumptions needed to apply step A to this new problem are also satisfied. Let

$$\begin{aligned} &\widetilde{H}(t, x, y, p, M) \\ &\quad := \sup_{a \in A}\Big\{-cy - e^{ct}\mu_Y^{\tilde{u}}\big(t, x, e^{-ct}y, e^{-ct}\sigma_X(t, x, a)p, a\big) \\ &\qquad\qquad + \mu_X(t, x, a)^{\top}p + \frac{1}{2}\operatorname{Tr}\big[\sigma_X\sigma_X^{\top}(t, x, a)M\big]\Big\}, \end{aligned} \tag{3.18}$$

where $\tilde{u}$ is defined like $\hat{u}$ but now in terms of $\tilde{\sigma}_Y$. We will denote by $\widetilde{\mathcal{U}}^+$ be the set of stochastic super-solutions of

$$\begin{aligned} \varphi_t + \widetilde{H}\big(\cdot, \varphi, D\varphi, D^2\varphi\big) = 0 \qquad &\text{on } \mathcal{D}_{<T}, \\ \varphi = \widetilde{g} \qquad &\text{on } \mathcal{D}_T \end{aligned} \tag{3.19}$$

and $\tilde{v}^+(t, x) := \inf_{w \in \widetilde{\mathcal{U}}^+} w(t, x)$.

From step A, we know that $\tilde{v}^+$ is a viscosity sub-solution of the above PDE. Since any function $w(t,x)$ is a stochastic super-solution of (2.4) if and only if $\tilde{w}(t,x) = e^{ct}w(t,x)$ is a stochastic super-solution of (3.19), it follows that $\tilde{v}^+(t,x) = e^{ct}v^+(t,x)$. Now it is easy to conclude that v^+ is a viscosity sub-solution of (2.4).

Step 2 (v^- *is the viscosity super-solution*). Due to Assumption 2.5, v^- is well defined. Next we will show that it satisfies the interior viscosity super-solution property followed by the boundary condition.

Step 2.1. *The interior super-solution property*: Let (t_0, x_0) in the parabolic interior $[0,T) \times \mathbb{R}^d$ such that a smooth function φ strictly touches v^- from below at (t_0, x_0). Assume by contradiction that

$$\varphi_t + H(\cdot, \varphi, D\varphi, D^2\varphi) > 0 \qquad \text{at } (t_0, x_0).$$

Hence there exists $a_0 \in A$, such that

$$\varphi_t + H^{u_0,a_0}(\cdot, \varphi, D\varphi, D^2\varphi) > 0 \qquad \text{at } (t_0, x_0), \tag{3.20}$$

where $u_0 = \hat{u}(t_0, x_0, \varphi(t_0,x_0), \sigma_X(t_0,x_0,a_0)D\varphi(t_0,x_0), D^2\varphi(t_0,x_0))$ and

$$\begin{aligned} &H^{u,a}(t,x,y,p,M) \\ &\quad := -\mu_Y(t,x,y,u,a) + \mu_X(t,x,a)^\top p + \tfrac{1}{2}\operatorname{Tr}[\sigma_X\sigma_X^\top(t,x,a)M]. \end{aligned} \tag{3.21}$$

From the continuity assumption on the coefficients in Assumption 2.1 and the continuity of $\hat{u}$ in Assumption 2.2, there exists $\varepsilon, \delta > 0$ such that

$$\begin{aligned} &\varphi_t + H^{u,a_0}(\cdot, y, D\varphi, D^2\varphi) > 0 \qquad \forall (t,x) \in \overline{B(t_0,x_0,\varepsilon)} \\ &\qquad \text{and } (y,u) \in R \times U \text{ s.t. } |y - \varphi(t,x)| \le \delta \\ &\qquad \text{and } \|\sigma_Y(t,x,y,u,a_0) - \sigma_X(t,x,a_0)D\varphi(t,x)\| \le \delta. \end{aligned}$$

Now, on the compact torus $\mathbb{T} = \overline{B(t_0,x_0,\varepsilon)} - B(t_0,x_0,\varepsilon/2)$, we have that $\varphi < v^-$ and the max of $\varphi - v^-$ is attained since v^- is LSC. Therefore, $\varphi + \eta < v^-$ on $\mathbb{T}$ for some $\eta > 0$. Since $w_n \nearrow v^-$, a Dini-type argument shows that for large enough n, we have $\varphi + \eta/2 < w_n$ on $\mathbb{T}$ and $\varphi < w_n + \delta$ on $\overline{B(t_0,x_0,\varepsilon/2)}$. For simplicity, fix such an n and denote $w = w_n$. Now define for small $\kappa < \frac{\eta}{2} \wedge \delta$,

$$w^\kappa \triangleq \begin{cases} (\varphi + \kappa) \vee w, & \text{on } \overline{B(t_0,x_0,\varepsilon)}, \\ w, & \text{outside } \overline{B(t_0,x_0,\varepsilon)}. \end{cases}$$

Since $w^\kappa(t_0,x_0) > v^-(t_0,x_0)$, we obtain a contradiction if we can show that $w^\kappa \in \mathcal{U}^-$.

In order to do so, fix t and $\{\tau^\alpha\} \in \mathfrak{S}^t$. For a given $\mathfrak{u} \in \mathfrak{U}(t)$ and $\alpha \in \mathcal{A}^t$, we will construct an "optimal" $\widetilde{\alpha} \in \mathcal{A}^t$ in the definition of stochastic sub-solutions for w^κ. We will divide the construction into two cases:

(i) $w(\tau^\alpha, X(\tau^\alpha)) = w^\kappa(\tau^\alpha, X(\tau^\alpha))$: Since w is a stochastic sub-solution, there exists an $\widetilde{\alpha}_1$ for w in the definition which is "optimal" for the nature given $\mathfrak{u}$, α and τ^α. Let $\widetilde{\alpha} = \widetilde{\alpha}_1$.

(ii) $w(\tau^\alpha, X(\tau^\alpha)) < w^\kappa(\tau^\alpha, X(\tau^\alpha))$: Let

$$\theta_1^\alpha := \inf\{s \in [\tau^\alpha, T] : (s, X_{t,x}^{\alpha \otimes_{\tau^\alpha} a_0}(s)) \notin B(t_0, x_0, \varepsilon/2)\}$$

and

$$\theta_2^\alpha := \inf\{s \in [\tau^\alpha, T] : |Y_{t,x,y}^{\mathfrak{u},\alpha \otimes_{\tau^\alpha} a_0}(s) - \varphi(s, X_{t,x}^{\alpha \otimes_{\tau^\alpha} a_0}(s))| \ge \delta\},$$

with the convention that $\inf \varnothing = T$. Denote $\theta^\alpha = \theta_1^\alpha \wedge \theta_2^\alpha$. Then let $\widetilde{\alpha} = a_0$ until θ^α. Starting from θ^α, choose $\widetilde{\alpha} = \alpha^*$, where the latter is "optimal" for nature given α and $\mathfrak{u}$ this time onward.

In summary, the above construction yields a candidate "optimal" control for w^κ given by

$$\widetilde{\alpha} = (\mathbb{1}_A \widetilde{\alpha}_1 + \mathbb{1}_{A^c}(a_0 \mathbb{1}_{[t,\theta^\alpha)} + \alpha^* \mathbb{1}_{[\theta^\alpha, T]}))\mathbb{1}_{[\tau^\alpha, T]},$$

where

$$A = \{w(\tau^\alpha, X_{t,x}^\alpha(\tau^\alpha)) = w^\kappa(\tau^\alpha, X_{t,x}^\alpha(\tau^\alpha))\}.$$

Let us check that what we constructed actually works: Let us abbreviate

$$(X, Y) = (X_{t,x}^{\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}}, Y_{t,x,y}^{\mathfrak{u},\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}}).$$

Note that

$$\begin{aligned} X(s) &= \mathbb{1}_A X_{t,x}^{\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}_1}(s) + \mathbb{1}_{A^c} X_{t,x}^{\alpha \otimes_{\tau^\alpha} a_0}(s) \qquad \text{for } \tau^\alpha \le s \le \theta^\alpha, \\ Y(s) &= \mathbb{1}_A Y_{t,x,y}^{\mathfrak{u},\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}_1}(s) + \mathbb{1}_{A^c} Y_{t,x,y}^{\mathfrak{u},\alpha \otimes_{\tau^\alpha} a_0}(s) \qquad \text{for } \tau^\alpha \le s \le \theta^\alpha. \end{aligned} \tag{3.22}$$

Again for brevity, let us introduce the following sets:

$$\begin{aligned} &E = \{Y(\tau^\alpha) < w^\kappa(\tau^\alpha, X(\tau^\alpha))\}, \qquad E_0 = E \cap A, \qquad E_1 = E \cap A^c, \\ &G = \{Y(\rho) < w^\kappa(\rho, X(\rho))\}, \qquad G_0 = \{Y(\rho) < w(\rho, X(\rho))\}. \end{aligned}$$

Observe that

$$E = E_0 \cup E_1, \qquad E_0 \cap E_1 = \varnothing \quad \text{and} \quad G_0 \subset G.$$

The proof will be complete if we can show that $P(G|B) > 0$ for any nonnull set $B \subset E$. In fact, it suffices to show that $\mathbb{P}(G \cap B) > 0$. Relying on the decomposition $\mathbb{P}(G \cap B) = \mathbb{P}(G \cap B \cap E_0) + \mathbb{P}(G \cap B \cap E_1)$ (recall that $B \subset E$), we will divide the proof into two steps:

(i) $\mathbb{P}(B \cap E_0) > 0$: Directly from the way $\widetilde{\alpha}_1$ is defined, the definition of the stochastic sub-solutions and $B \cap E_0 \subset A$, we get

$$\mathbb{P}(G_0 | B \cap E_0) = \mathbb{P}(Y_{t,x,y}^{\mathfrak{u},\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}_1}(\rho) < w(\rho, X_{t,x}^{\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}_1}(\rho)) | B \cap E_0) > 0.$$

This further implies that $\mathbb{P}(G \cap B \cap E_0) \geq \mathbb{P}(G_0 \cap B \cap E_0) > 0$.

(ii) $\mathbb{P}(B \cap E_1) > 0$: From (3.22) and $B \cap E_1 \subset A^c$,

$$\begin{aligned}
&\mathbb{P}(Y(\theta^\alpha) < w^\kappa(\theta^\alpha, X(\theta^\alpha)) | B \cap E_1) \\
&\quad = \mathbb{P}(Y_{t,x,y}^{\mathfrak{u},\alpha \otimes_{\tau^\alpha} a_0}(\theta^\alpha) < w^\kappa(\theta^\alpha, X_{t,x}^{\alpha \otimes_{\tau^\alpha} a_0}(\theta^\alpha)) | B \cap E_1).
\end{aligned}$$

The analysis in [8] shows that

$$\Delta(s) = Y(s \wedge \theta^\alpha) - (\varphi(s \wedge \theta^\alpha, X(s \wedge \theta^\alpha)) + \kappa)$$

is a super-martingale up to a change of measure. We will summarize these arguments here: Let

$$\begin{aligned}
\lambda(s) &:= \sigma_Y(s, X(s), Y(s), \mathfrak{u}[a_0]_s, a_0) - \sigma_X(s, X(s), a_0) D\varphi(s, X(s)), \\
\beta(s) &:= (\varphi_t(s, X(s)) + H^{\mathfrak{u}[a_0]_s, a_0}(s, X(s), Y(s), D\varphi(s, X(s)), D^2\varphi(s, X(s)))) \\
&\qquad \times \|\lambda(s)\|^{-2} \lambda(s) \mathbb{1}_{\{\|\lambda(s)\| > \delta\}}.
\end{aligned}$$

From the definition of θ^α and the regularity and growth conditions in Assumptions 2.1 and 2.4, β is uniformly bounded on $[\tau^\alpha, \theta^\alpha]$. This ensures that the positive exponential local martingale M defined by the SDE

$$M(\cdot) = 1 + \int_{\tau^\alpha}^{\cdot \wedge \theta^\alpha} M(s) \beta_s^\top \, dW_s$$

is a true martingale. An application of Itô's formula immediately implies that $M\Delta$ is a local super-martingale. By the definition of θ^α, Δ is bounded by $-\delta - \kappa$ from below and by $\delta - \kappa$ from above on $[\tau^\alpha, \theta^\alpha]$. Therefore, $M\Delta$ is bounded above by a martingale $2M\delta$, and below by another martingale $-2M\delta$. An application of Fatou's lemma implies that $M\Delta$ is a super-martingale.

From the definition of E_1 and w^κ, $\Delta(\tau^\alpha) < 0$ on $B \cap E_1$. The super-martingale property of $M\Delta$ implies that there exists a nonnull $H \subset B \cap E_1$, $H \in \mathcal{F}_{\tau^\alpha}^t$ such that $\Delta(\theta^\alpha \wedge \rho) < 0$ on H. Therefore, from the decomposition

$$\begin{aligned}
\Delta(\theta^\alpha \wedge \rho) \mathbb{1}_H &= (Y(\theta_1^\alpha) - (\varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa)) \mathbb{1}_{H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}} \\
&\quad + (Y(\theta_2^\alpha) - (\varphi(\theta_2^\alpha, X(\theta_2^\alpha)) + \kappa)) \mathbb{1}_{H \cap \{\theta_2^\alpha \leq \theta_1^\alpha \wedge \rho\}} \\
&\quad + (Y(\rho) - (\varphi(\rho, X(\rho)) + \kappa)) \mathbb{1}_{H \cap \{\rho < \theta^\alpha\}},
\end{aligned}$$

we see that

$$Y(\theta_1^\alpha) - (\varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa) < 0 \qquad \text{on } H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}, \tag{3.23}$$

$$Y(\theta_2^\alpha) - (\varphi(\theta_2^\alpha, X(\theta_2^\alpha)) + \kappa) < 0 \qquad \text{on } H \cap \{\theta_2^\alpha \leq \theta_1^\alpha \wedge \rho\} \tag{3.24}$$

and that

$$Y(\rho) - (\varphi(\rho, X(\rho)) + \kappa) < 0 \qquad \text{on } H \cap \{\rho < \theta^\alpha\}. \tag{3.25}$$

On the one hand, on $H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}$, $\varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa < w(\theta_1^\alpha, X(\theta_1^\alpha))$. Then from (3.23), we will have

$$Y(\theta_1^\alpha) < w(\theta_1^\alpha, X(\theta_1^\alpha)) \qquad \text{on } H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}. \tag{3.26}$$

On the other hand, on $H \cap \{\theta_2^\alpha \le \theta_1^\alpha \wedge \rho\}$, we get $Y(\theta_2^\alpha) - \varphi(\theta_2^\alpha, X(\theta_2^\alpha)) = -\delta$. [The right-hand side cannot be equal to δ; otherwise (3.24) would be contradicted.] Recalling the fact that $\varphi < w + \delta$ on $\overline{B(t_0, x_0, \varepsilon/2)}$, this observation gives that

$$\begin{aligned} Y(\theta_2^\alpha) - w(\theta_2^\alpha, X(\theta_2^\alpha)) = (\varphi - w)(\theta_2^\alpha, X(\theta_2^\alpha)) - \delta < 0 \\ \text{on } H \cap \{\theta_2^\alpha \le \theta_1^\alpha \wedge \rho\}. \end{aligned} \tag{3.27}$$

We have obtained in (3.26) and (3.27) that

$$Y(\theta^\alpha) < w(\theta^\alpha, X(\theta^\alpha)) \qquad \text{on } H \cap \{\theta^\alpha \le \rho\}.$$

Now from the definition of stochastic sub-solutions and of α^*, we have that

$$\mathbb{P}(G_0 | H \cap \{\theta^\alpha \le \rho\}) > 0 \qquad \text{if } \mathbb{P}(H \cap \{\theta^\alpha \le \rho\}) > 0. \tag{3.28}$$

On the other hand, (3.25) implies that

$$\mathbb{P}(G | H \cap \{\theta^\alpha > \rho\}) > 0 \qquad \text{if } \mathbb{P}(H \cap \{\theta^\alpha > \rho\}) > 0. \tag{3.29}$$

Since $\mathbb{P}(H) > 0$, $G_0 \subset G$, and $H \subset E_1 \cap B$, (3.28) and (3.29) imply $\mathbb{P}(G \cap E_1 \cap B) > 0$.

Step 2.2. *The boundary condition*:

Assume that for some $x_0 \in \mathbb{R}^d$, we have

$$v^-(T, x_0) < g(x_0). \tag{3.30}$$

Since g is LSC, then from (3.30) there exists $\varepsilon > 0$ such that

$$v^-(T, x_0) < g(x) - \varepsilon \qquad \text{for } |x - x_0| \le \varepsilon. \tag{3.31}$$

Since v^- is LSC, then v^- is bounded below on the compact (rectangular) torus $\mathbb{T} = \overline{B(T, x_0; \varepsilon)} - B(T, x_0; \varepsilon/2)$. Choose $\beta > 0$ small enough, such that

$$v^-(T, x_0) - \frac{\varepsilon^2}{4\beta} < \inf_{\mathbb{T}} v^-(t, x) - \varepsilon.$$

By a Dini-type argument, there exists a $w \in \mathcal{U}^-$, such that

$$v^-(T, x_0) - \frac{\varepsilon^2}{4\beta} < \inf_{\mathbb{T}} w(t, x) - \varepsilon. \tag{3.32}$$

We now define for $C > 0$,

$$\varphi^{\beta,C} = v^-(T, x_0) - \frac{|x - x_0|^2}{\beta} - C(T - t).$$

For any a_0 we can choose large enough C,[2]

$$\varphi_t^{\beta,C} + H^{u_0,a_0}(\cdot, \varphi^{\beta,C}, D\varphi^{\beta,C}, D^2\varphi^{\beta,C}) > 0 \qquad \text{on } \overline{B(T, x_0; \varepsilon)},$$

where $H^{u,a}$ is the same as that in (3.21), $u_0 = \hat{u}(T, x_0, \varphi(T, x_0), \sigma_X(T, x_0, a_0) \times D\varphi(T, x_0), a_0)$. Then from the continuity of the coefficients in Assumption 2.1 and the continuity of $\hat{u}$ in Assumption 2.2, for any a_0, and there exists a small enough $\delta > 0$ such that

$$\begin{aligned} &\varphi_t^{\beta,C} + H^{u,a_0}(\cdot, y, D\varphi^{\beta,C}, D^2\varphi^{\beta,C}) > 0 \qquad \forall (t,x) \in \overline{B(T, x_0, \varepsilon)} \\ &\qquad \text{and } (y,u) \in R \times U \text{ s.t. } |y - \varphi^{\beta,C}(t,x)| \le \delta \\ &\qquad \text{and } \|\sigma_Y(t,x,y,u,a_0) - \sigma_X(t,x,a_0) D\varphi^{\beta,C}(t,x)\| \le \delta. \end{aligned}$$

Choosing C at least as large as $\varepsilon/2\beta$, we obtain from (3.32) that

$$\varphi^{\beta,C} \le w - \varepsilon \qquad \text{on } \mathbb{T}.$$

Also we have that

$$\varphi^{\beta,C}(T,x) \le v^-(T,x_0) < g(x) - \varepsilon \qquad \text{for } |x - x_0| \le \varepsilon. \tag{3.33}$$

Now for $\kappa < \varepsilon \wedge \delta$ define

$$w^{\beta,C,\kappa} \triangleq \begin{cases} (\varphi^{\beta,C} + \kappa) \vee w, & \text{on } \overline{B(T, x_0, \varepsilon)}, \\ w, & \text{outside } \overline{B(T, x_0, \varepsilon)}. \end{cases} \tag{3.34}$$

From (3.33) and (3.34) it is easy to see that $w^{\beta,C,\kappa}(T,x) \le g(x)$. By applying arguments similar to those in step 2.1, we can show that $w^{\beta,C,\kappa}$ is a stochastic sub-solution with $w^{\beta,C,\kappa}(T,x_0) > v^-(T,x_0)$. This contradicts the definition of v^-. □

To characterize v as the unique viscosity solution of (2.4), we need a comparison principle.

PROPOSITION 3.1 (Comparison principle). *Under Assumptions* 2.1, 2.2 *and* 2.3, *the comparison principle for* (2.4) *holds*. *More precisely*, *let* U (*resp*., V) *be a bounded USC viscosity sub-solution* (*resp*., *LSC viscosity super-solution*) *to* (2.4). *If* $U \le V$ *on* $\mathcal{D}_T$, *then* $U \le V$ *on* $\mathcal{D}$.

PROOF. *Step* 1: Without loss of generality, assume that

$$\begin{aligned} &\exists \gamma > 0, \text{ such that } H(t,x,y,p,M) - H(t,x,y',p,M) < -\gamma(y - y') \\ &\qquad \text{for all } y > y'. \end{aligned} \tag{3.35}$$

[2]Similar analysis for (3.14) will guarantee that choosing C is possible.

Otherwise, let $\widetilde{U}(t,x) = e^{ct}U(t,x)$ and $\widetilde{V}(t,x) = e^{ct}V(t,x)$. Then a straightforward calculation shows that $\widetilde{U}$ (resp., $\widetilde{V}$) is a sub-solution (resp., super-solution) to

$$\begin{aligned} -\varphi_t - \widetilde{H}(\cdot, \varphi, D\varphi, D^2\varphi) = 0 \qquad &\text{on } \mathcal{D}_{<T}, \\ \varphi = \widetilde{g} \qquad &\text{on } \mathcal{D}_T, \end{aligned} \tag{3.36}$$

where $\widetilde{g}(x) = e^{cT}g(x)$ and $\widetilde{H}$ is the same as that in (3.18). We can choose c large enough such that (3.35) holds for $\widetilde{H}$. In fact, from the Lipschitz continuity of $\mu_Y^{\hat{u}}$ in Assumption 2.3, for $y > y'$,

$$\begin{aligned} &\widetilde{H}^a(t,x,y,p,M) - \widetilde{H}^a(t,x,y',p,M) \\ &\quad = -c(y-y') + e^{ct}\big(\mu_Y^{\tilde{u}}(t,x,e^{-ct}y', e^{-ct}\sigma_X(t,x,a)p,a) \\ &\qquad\qquad - \mu_Y^{\tilde{u}}(t,x,e^{-ct}y, e^{-ct}\sigma_X(t,x,a)p,a)\big) \\ &\quad \le -c(y-y') + e^{ct}L \cdot e^{-ct}(y-y') \\ &\quad = -(c-L)(y-y'), \end{aligned}$$

where L is the Lipschitz constant and

$$\begin{aligned} \widetilde{H}^a(t,x,y,p,M) := {}& -cy - e^{ct}\mu_Y^{\tilde{u}}(t,x,e^{-ct}y, e^{-ct}\sigma_X(t,x,a)p,a) \\ & + \mu_X(t,x,a)^\top p + \tfrac{1}{2}\operatorname{Tr}[\sigma_X\sigma_X^\top(t,x,a)M]. \end{aligned}$$

Then $\gamma := c - L > 0$ for large enough c. Since $\widetilde{H}(\cdot) = \sup_{a\in A}\widetilde{H}^a(\cdot)$, equation (3.35) holds for $\widetilde{H}$.

Step 2: In this step, we claim that for large enough λ, $V_\delta := V + \delta e^{-\lambda t}(1+|x|^2)$ is a LSC viscosity super-solution to (2.4) for $\delta > 0$. Then, if we can show that $U - V_\delta \le 0$ on $\mathcal{D}$ for all $\delta > 0$, we will get the required result by sending δ to zero. Now we prove the above claim.

Obviously, the boundary condition is satisfied. Let φ be a smooth function which strictly touches V_δ from below at $(t_0,x_0) \in \mathcal{D}_{<T}$. Let $\varphi^\delta = \varphi - \delta e^{-\lambda t}(1+|x|^2)$. Then $V - \varphi^\delta$ has a strict minimum at (t_0,x_0). Since V is a viscosity super-solution, then it holds that

$$\varphi_t^\delta + H(t,x,\varphi^\delta, D\varphi^\delta, D^2\varphi^\delta) \le 0. \tag{3.37}$$

Note that

$$\begin{aligned} &\varphi_t^\delta = \varphi_t + \lambda\delta e^{-\lambda t}(1+|x|^2), \qquad D\varphi^\delta = D\varphi - 2\delta e^{-\lambda t}x, \\ &D^2\varphi^\delta = D^2\varphi - 2\delta e^{-\lambda t}I_{d\times d}. \end{aligned} \tag{3.38}$$

Consider the difference of $H(t,x,\varphi^\delta, D\varphi^\delta, D^2\varphi^\delta)$ and $H(t,x,\varphi, D\varphi, D^2\varphi)$. From (3.38) and Assumption 2.1, we get

$$\begin{aligned} \big|\mu_X^\top(t,x,a)D\varphi(t,x) - \mu_X^\top(t,x,a)D\varphi^\delta(t,x)\big| &\le K\big|D\varphi(t,x) - D\varphi^\delta(t,x)\big| \\ &= 2K\delta e^{-\lambda t}|x|. \end{aligned} \tag{3.39}$$

Similarly,

$$\begin{aligned}(3.40)\qquad &\left|\tfrac{1}{2}\operatorname{Tr}\bigl(\sigma_X\sigma_X^\top(t,x,a)\bigr)D^2\varphi(t,x)-\tfrac{1}{2}\operatorname{Tr}\bigl(\sigma_X\sigma_X^\top(t,x,a)\bigr)D^2\varphi^\delta(t,x)\right|\\ &\quad\le K^2 d\delta e^{-\lambda t}.\end{aligned}$$

From the Lipschitz continuity of $\mu_Y^{\hat{u}}$ in Assumption 2.3,

$$\begin{aligned}(3.41)\qquad &\left|\mu_Y^{\hat{u}}(t,x,\varphi,\sigma_X(t,x,a)D\varphi,a)-\mu_Y^{\hat{u}}(t,x,\varphi^\delta,\sigma_X(t,x,a)D\varphi^\delta,a)\right|\\ &\quad\le L\bigl(\delta e^{-\lambda t}(1+|x|^2)+2K\delta e^{-\lambda t}|x|\bigr).\end{aligned}$$

From (3.39), (3.40) and (3.41),

$$\begin{aligned}&\left|H\bigl(t,x,\varphi^\delta,D\varphi^\delta,D^2\varphi^\delta\bigr)-H\bigl(t,x,\varphi,D\varphi,D^2\varphi\bigr)\right|\\ &\quad\le\delta e^{-\lambda t}\bigl(1+|x|^2\bigr)\bigl(L+LK+K^2d+K\bigr).\end{aligned}$$

Taking $\lambda>\lambda^*:=L+LK+K^2d+K$, from the above inequality, we get

$$\begin{aligned}\varphi_t+H\bigl(t,x,\varphi,D\varphi,D^2\varphi\bigr)&\le\varphi_t^\delta+H\bigl(t,x,\varphi^\delta,D\varphi^\delta,D^2\varphi^\delta\bigr)-\lambda\delta e^{-\lambda t}\bigl(1+|x|^2\bigr)\\ &\quad+\left|H\bigl(t,x,\varphi^\delta,D\varphi^\delta,D^2\varphi^\delta\bigr)-H\bigl(t,x,\varphi,D\varphi,D^2\varphi\bigr)\right|\\ &\le\varphi_t^\delta+H\bigl(t,x,\varphi^\delta,D\varphi^\delta,D^2\varphi^\delta\bigr)\le 0.\end{aligned}$$

Step 3: In this step, we show that $U-V_\delta\le 0$ on $\mathcal{D}$ for all $\delta>0$. From boundedness of U and V, for all $\delta>0$,

$$(3.42)\qquad \lim_{|x|\to\infty}\sup_{[0,T]}(U-V_\delta)(t,x)=-\infty.$$

This implies the supremum of $U-V_\delta$ on $\mathcal{D}$ is attained on $[0,T]\times\mathcal{O}$ for some open bounded set $\mathcal{O}$ of $\mathbb{R}^d$. We assume

$$M^*:=\sup_{\mathcal{D}}(U-V_\delta)=\max_{[0,T)\times\mathcal{O}}(U-V_\delta)>0,$$

and we will obtain a contradiction to the above equation. We consider a bounded sequence $(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon)_\varepsilon$ that maximizes Φ_ε on $[0,T]^2\times\mathbb{R}^d\times\mathbb{R}^d$ with $\Phi_\varepsilon=U(t,x)-V_\delta(s,y)-\phi_\varepsilon(t,s,x,y)$ and $\phi_\varepsilon(t,s,x,y):=\frac{1}{2\varepsilon}(|t-s|^2+|x-y|^2)$. By arguments similar to those in Theorem 4.4.4 of [10], we know that $(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon)_\varepsilon$ converges to (t_0,t_0,x_0,x_0) for some $(t_0,x_0)\in[0,T]\times\mathcal{O}$ and

$$(3.43)\qquad M_\varepsilon=\Phi(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon)\to M^*\quad\text{and}\quad\phi_\varepsilon(t_\varepsilon,s_\varepsilon,x_\varepsilon,y_\varepsilon)\to 0.$$

In view of Ishii's lemma (Lemma A.2), there exist $M,N\in\mathcal{S}^d$ such that

$$\left(\frac{1}{\varepsilon}(t_\varepsilon-s_\varepsilon),\frac{1}{\varepsilon}(x_\varepsilon-y_\varepsilon),M\right)\in\overline{P}^{2,+}U(t,x),$$

$$\left(\frac{1}{\varepsilon}(t_\varepsilon-s_\varepsilon),\frac{1}{\varepsilon}(x_\varepsilon-y_\varepsilon),N\right)\in\overline{P}^{2,-}V_\delta(t,x).$$

From the viscosity sub-solution and super-solution characterization of U and V_δ in terms of super-jets and sub-jets, we then have

$$-\frac{1}{\varepsilon}(t_\varepsilon - s_\varepsilon) - H\left(t_\varepsilon, x_\varepsilon, U(t_\varepsilon, x_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), M\right) \le 0,$$
$$-\frac{1}{\varepsilon}(t_\varepsilon - s_\varepsilon) - H\left(s_\varepsilon, y_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), N\right) \ge 0.$$

By subtracting the two inequalities above, we get

$$H\left(t_\varepsilon, x_\varepsilon, U(t_\varepsilon, x_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), M\right) \ge H\left(s_\varepsilon, y_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), N\right).$$

Subtracting $H(t_\varepsilon, x_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), M)$ from both sides of the equation above, we get

$$\begin{aligned}
\text{LHS} &:= H\left(t_\varepsilon, x_\varepsilon, U(t_\varepsilon, x_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), M\right) \\
&\quad - H\left(t_\varepsilon, x_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), M\right) \\
&\ge H\left(s_\varepsilon, y_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), N\right) \\
&\quad - H\left(t_\varepsilon, x_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), M\right) \\
&=: \text{RHS}.
\end{aligned} \tag{3.44}$$

On the one hand, since $U(t_\varepsilon, x_\varepsilon) - V_\delta(s_\varepsilon, y_\varepsilon) \ge M^*$,

$$\text{LHS} \le -\gamma\big(U(t_\varepsilon, x_\varepsilon) - V_\delta(s_\varepsilon, y_\varepsilon)\big) \le -\gamma M^*. \tag{3.45}$$

On the other hand, applying inequality (A.5) to $C = \sigma_X(t_\varepsilon, x_\varepsilon, a)$ and $D = \sigma_X(s_\varepsilon, y_\varepsilon, a)$, we get

$$\begin{aligned}
I_1 &:= \left|\frac{1}{2}\operatorname{Tr}\big[\sigma_X\sigma_X^\top(t_\varepsilon, x_\varepsilon, a)M\big] - \frac{1}{2}\operatorname{Tr}\big[\sigma_X\sigma_X^\top(s_\varepsilon, y_\varepsilon, a)N\big]\right| \\
&\le \frac{3}{2\varepsilon}\operatorname{Tr}\big[\big(\sigma_X(t_\varepsilon, x_\varepsilon) - \sigma_X(s_\varepsilon, y_\varepsilon)\big)\big(\sigma_X(t_\varepsilon, x_\varepsilon) - \sigma_X(s_\varepsilon, y_\varepsilon)\big)^\top\big] \\
&\le \frac{1}{2\varepsilon}O\big(|t_\varepsilon - s_\varepsilon|^2 + |x_\varepsilon - y_\varepsilon|^2\big) \to 0.
\end{aligned}$$

In the last inequality, we use (3.43) and Lipschitz continuity of σ_X (uniformly in a). Therefore,

$$I_1 \to 0 \qquad \text{as } \varepsilon \to 0, \text{ uniformly in } a \in A. \tag{3.46}$$

Similarly, from (3.43) and Lipschitz continuity of μ_X (uniformly in a)

$$I_2 := \left|\frac{1}{\varepsilon}\mu_X^\top(t_\varepsilon, x_\varepsilon, a)(x_\varepsilon - y_\varepsilon) - \frac{1}{\varepsilon}\mu_X^\top(s_\varepsilon, y_\varepsilon, a)(x_\varepsilon - y_\varepsilon)\right| \to 0 \tag{3.47}$$
$$\text{uniformly in } a \in A.$$

From (3.43) and Lipschitz continuity of σ_X (Assumption 2.1) and $\mu_Y^{\hat{u}}$ (Assumption 2.3), we get

$$\begin{aligned} I_3 := \bigg|\mu_Y^{\hat{u}}\bigg(& t_\varepsilon, x_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \sigma_X(t_\varepsilon, x_\varepsilon, a)\bigg(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\bigg), a\bigg) \\ &- \mu_Y^{\hat{u}}\bigg(s_\varepsilon, y_\varepsilon, V_\delta(s_\varepsilon, y_\varepsilon), \sigma_X(s_\varepsilon, y_\varepsilon, a)\bigg(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\bigg), a\bigg)\bigg| \\ &\le \nu\big(|t_\varepsilon - s_\varepsilon| + |x_\varepsilon - y_\varepsilon|\big) + \frac{1}{2\varepsilon}O\big(|t_\varepsilon - s_\varepsilon|^2 + |x_\varepsilon - y_\varepsilon|^2\big) \to 0 \qquad \text{as } \varepsilon \to 0, \end{aligned}$$

where $\nu(z) \to 0$ as $z \to 0$. The first term in the last inequality above is the modulus of continuity of $\mu_Y^{\hat{u}}$ in the variables (t, x) (uniformly in a) and the second term comes from similar arguments for I_1 and I_2. Therefore,

$$I_3 \to 0 \qquad \text{uniformly in } a \in A. \tag{3.48}$$

Then (3.46), (3.47) and (3.48) imply that

$$\text{RHS} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.49}$$

From (3.44), (3.45) and (3.49), we obtain a contradiction. □

COROLLARY 3.1. *If g is continuous and Assumptions 2.1–2.5 hold, then v is the unique bounded continuous viscosity solution of* (2.4).

PROOF. From Theorem 3.1, v^+ (resp., v^-) is a bounded USC viscosity subsolution (resp., LSC viscosity super-solution) to (2.4). Then $v^+(T, x) \le g(x) \le v^-(T, x)$. This implies $v^+ \le v^-$ on $\mathcal{D}$ from Proposition 3.1. Since $v^+ \ge v \ge v^-$ by definition, $v^+ = v = v^-$. We have shown that v is continuous and a bounded viscosity solution of (2.4).

To check the uniqueness, let w be a bounded continuous viscosity solution of (2.4). Note that w is a LSC viscosity super-solution and v is an USC viscosity sub-solution of (2.4). From Proposition 3.1, $v \le w$ on $\mathcal{D}$. Similarly, $w \le v$ on $\mathcal{D}$. This implies $w = v$ on $\mathcal{D}$. □

From Theorem 3.1 and Corollary 3.1, we obtain dynamic programming principle as a byproduct.

COROLLARY 3.2 (Dynamic programming principle). *Assume g is continuous and Assumptions 2.1–2.5 hold. For any $(t, x) \in \mathcal{D}$, the following two statements hold*:

DPP 1. *For any* $y > v(t,x)$, *there exists* $\mathfrak{u} \in \mathfrak{U}(t)$ *such that for all* $\alpha \in \mathcal{A}^t$ *and* $\theta \in \mathbb{S}^t$,

$$Y^{\mathfrak{u},\alpha}_{t,x,y}(\theta) \geq v\big(\theta, X^{\alpha}_{t,x}(\theta)\big).$$

DPP 2. *For any* $y < v(t,x)$ *and* $\mathfrak{u} \in \mathfrak{U}(t)$, *there exists* $\alpha \in \mathcal{A}^t$ *such that for all* $\theta \in \mathbb{S}^t$,

$$\mathbb{P}\big(Y^{\mathfrak{u},\alpha}_{t,x,y} \geq v\big(\theta, X^{\alpha}_{t,x}(\theta)\big)\big) < 1.$$

PROOF. DPP 1: If $y > v(t,x) = v^+(t,x)$ (due to Corollary 3.1), there exists a $w \in \mathcal{U}^+$ such that $y > w(t,x)$. From the definition of stochastic super-solution, there exists $\mathfrak{u} \in \mathfrak{U}(t)$ such that

$$Y^{\mathfrak{u},\alpha}_{t,x,y}(\theta) \geq w\big(\theta, X^{\alpha}_{t,x}(\theta)\big) \geq v\big(\theta, X^{\alpha}_{t,x}(\theta)\big)$$

for all $\theta \in \mathbb{S}^t$ and $\alpha \in \mathcal{A}^t$.

DPP 2: If $y < v(t,x) = v^-(t,x) = \sup_{w \in \mathcal{U}^-} w(t,x)$, there exists a $w \in \mathcal{U}^-$ such that $y < w(t,x)$. From the definition of stochastic sub-solution, for any $\mathfrak{u} \in \mathfrak{U}(t)$, there exits an $\alpha \in \mathcal{A}^t$ such that

$$\mathbb{P}\big(Y^{\mathfrak{u},\alpha}_{t,x,y}(\theta) < w\big(\theta, X^{\alpha}_{t,x}(\theta)\big)\big) > 0$$

for all $\theta \in \mathbb{S}^t$. Since $w(\theta, X^{\alpha}_{t,x}(\theta)) \leq v(\theta, X^{\alpha}_{t,x}(\theta))$, this gives us the desired result. □

APPENDIX

A.1. Proof of Proposition 2.1. We carry out the proof in two steps. First under Assumptions 2.2 and 2.3, we will show that there exists a classical solution to (2.4). Next, we will show that if we additionally have Assumption 2.1, then every classical super-solution is a stochastic super-solution, which implies in particular that $\mathcal{U}^+$ is not empty.

Step 1. *Existence of a classical super-solution to (2.4):*

Step 1A. In this step we will assume that $\mu^{\hat{u}}_Y$ is nondecreasing in its y-variable. Letting $\phi(t,x) = -e^{\lambda t}$ we have that

$$\phi_t + H\big(t,x,\phi,D\phi,D^2\phi\big) = -\lambda e^{\lambda t} + \sup_{a\in A}\big\{-\mu^{\hat{u}}_Y\big(t,x,\phi(t,x),0,a\big)\big\}. \tag{A.1}$$

From the linear growth condition of $\mu^{\hat{u}}_Y$ in Assumption 2.3, we know there exists an $L > 0$, such that $-\mu^{\hat{u}}_Y(t,x,\phi(t,x),0,a) \leq L(1 + |\phi(t,x)|) = L(1 + e^{\lambda t})$. Therefore, from (A.1),

$$\phi_t + H\big(t,x,\phi,D\phi,D^2\phi\big) \leq -\lambda e^{\lambda t} + L\big(1 + e^{\lambda t}\big) \leq 0 \qquad \text{in } \mathcal{D}, \text{ for } \lambda > 2L.$$

Fix $\lambda > 2L$, and choose N_2 such that $-e^{\lambda T} + N_2 \geq \|g\|_\infty$. Then $\phi'(T,x) = \phi(T,x) + N_2 \geq g(x)$. From the assumption that $\mu^{\hat{u}}_Y$ is nondecreasing in its y-variable, it holds that

$$\phi'_t + H\big(t,x,\phi',D\phi',D^2\phi'\big) \leq 0 \qquad \text{on } \mathcal{D}_{<T}.$$

Therefore, ϕ' is a classical super-solution.

Step 1B. We now show the same result for more general $\mu_Y^{\hat{u}}$. This follows the same reparameterization argument outlined in step 1.2B in the proof of the main theorem.

Step 2. *Classical super-solutions are stochastic super-solutions.* Let w be a classical super-solution. Fix $(t,x,y) \in \mathcal{D} \times \mathbb{R}$ and $\{\tau^\alpha\} \in \mathfrak{S}^t$. Let $\overline{Y}$ be the unique strong solution (which is thanks to Assumption 2.3) of the equation

$$\begin{aligned}\overline{Y}(l) = {}& Y_{t,x,y}^{\mathfrak{u},\alpha}(\tau^\alpha)\\ &+ \int_{\tau^\alpha}^{\tau^\alpha \vee l} \mu_Y^{\hat{u}}\big(s, X_{t,x}^\alpha(s), \overline{Y}(s), \sigma_X\big(s, X_{t,x}^\alpha(s), \alpha_s\big) Dw\big(s, X_{t,x}^\alpha(s)\big), \alpha_s\big)\, ds\\ &+ \int_{\tau^\alpha}^{\tau^\alpha \vee l} \sigma_X\big(s, X_{t,x}^\alpha(s), \alpha_s\big) Dw\big(s, X_{t,x}^\alpha(s)\big)\, dW_s, \qquad l \ge \tau^\alpha,\end{aligned}$$

for any $\mathfrak{u} \in \mathfrak{U}(t)$ and $\alpha \in \mathcal{A}^t$, and set $\overline{Y}(s) = Y_{t,x,y}^{\mathfrak{u},\alpha}(s)$ for $s < \tau^\alpha$. We will set $\tilde{\mathfrak{u}}$ to be

$$\tilde{\mathfrak{u}} := \tilde{\mathfrak{u}}[\alpha](s) = \hat{u}\big(s, X_{t,x}^\alpha(s), \overline{Y}(s), \sigma_X\big(s, X_{t,x}^\alpha(s), \alpha_s\big) Dw\big(s, X_{t,x}^\alpha(s)\big), \alpha_s\big).$$

It is not difficult to check that $\tilde{\mathfrak{u}} \in \mathfrak{U}(t, \{\tau^\alpha\})$. We will show that for any $\mathfrak{u} \in \mathfrak{U}(t)$, $\alpha \in \mathcal{A}^t$ and each stopping time $\rho \in \mathbb{S}^t$, $\tau^\alpha \le \rho \le T$ with the simplifying notation $X := X_{t,x}^\alpha$, $Y := Y_{t,x,y}^{\mathfrak{u} \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha], \alpha}$, we have

$$Y(\rho) \ge w\big(\rho, X(\rho)\big) \qquad \mathbb{P}\text{-a.s. on } \big\{Y\big(\tau^\alpha\big) > w\big(\tau^\alpha, X\big(\tau^\alpha\big)\big)\big\}.$$

Note that $\overline{Y} = Y_{t,x,y}^{\mathfrak{u} \otimes_{\tau^\alpha} \tilde{\mathfrak{u}}[\alpha], \alpha}$ for $s \ge \tau^\alpha$. We will carry out the rest of the proof in two steps.

Step 2A. In this step we will assume that $\mu_Y^{\hat{u}}$ is nondecreasing in its y-variable. Let

$$\begin{aligned}&A = \big\{Y\big(\tau^\alpha\big) > w\big(\tau^\alpha, X\big(\tau^\alpha\big)\big)\big\}, \qquad Z(s) = w\big(s, X(s)\big),\\ &\Gamma(s) = \big(Z(s) - Y(s)\big)\mathbb{1}_A.\end{aligned}$$

Therefore, for $s \ge \tau^\alpha$,

$$\begin{aligned}dY &= \mu_Y^{\hat{u}}\big(s, X(s), Y(s), \sigma_X\big(s, X(s), \alpha_s\big) Dw\big(s, X(s)\big), \alpha_s\big)\, ds\\ &\qquad + \sigma_X\big(s, X(s), \alpha_s\big) Dw\big(s, X(s)\big)\, dW_s,\\ dZ &= \big\{w_t\big(s, X(s)\big) + \mu_X\big(s, X(s), \alpha_s\big)^\top Dw\big(s, X(s)\big)\\ &\qquad\quad + \tfrac{1}{2}\operatorname{Tr}\big[\sigma_X \sigma_X^\top\big(s, X(s), \alpha_s\big) D^2 w\big(s, X(s)\big)\big]\big\}\, ds\\ &\qquad + \sigma_X\big(s, X(s), \alpha_s\big) Dw\big(s, X(s)\big)\, dW_s.\end{aligned}$$

From above equations,

$$\Gamma(s) = \mathbb{1}_A \int_{\tau^\alpha}^{s} \big(\xi(u) - \gamma'(u)\big)\, du \qquad \text{for } s \ge \tau^\alpha, \tag{A.2}$$

where

$$\gamma' := \mu_Y^{\hat{u}}\big(\cdot, X, w(\cdot, X), \sigma_X(\cdot, X, \alpha) Dw(\cdot, X), \alpha\big) - \mu_X(\cdot, X, \alpha)^\top Dw(\cdot, X) \\ - \tfrac{1}{2}\operatorname{Tr}\big[\sigma_X \sigma_X^\top(\cdot, X, \alpha) D^2 w(\cdot, X)\big] - w_t(\cdot, X)$$

and

$$\xi := \mu_Y^{\hat{u}}\big(\cdot, X, Z, \sigma_X(\cdot, X, \alpha) Dw(\cdot, X), \alpha\big) - \mu_Y^{\hat{u}}\big(\cdot, X, Y, \sigma_X(\cdot, X, \alpha) Dw(\cdot, X), \alpha\big).$$

Since w is a classical super-solution $\gamma' \geq 0$. Then from (A.2) it follows that

$$\Gamma(s) \leq \mathbb{1}_A \int_{\tau^\alpha}^s \xi(u)\,du \quad \text{and} \quad \Gamma^+(s) \leq \mathbb{1}_A \int_{\tau^\alpha}^s \xi^+(u)\,du \qquad \text{for } s \geq \tau^\alpha.$$

From the Lipschitz continuity of $\mu_Y^{\hat{u}}$ in y-variable in Assumption 2.3,

$$\Gamma^+(s) \leq \mathbb{1}_A \int_{\tau^\alpha}^s \xi^+(u)\,du \leq \int_{\tau^\alpha}^s L\Gamma^+(u)\,du \qquad \text{for } s \geq \tau^\alpha,$$

where we also use the assumption that $\mu_Y^{\hat{u}}$ is nondecreasing in its y-variable to obtain the second inequality. Since $\mathbb{E}\Gamma^+(\tau^\alpha) = 0$, an application of Gronwall's inequality implies that $\mathbb{E}\Gamma^+(\rho) \leq 0$.

Step 2B: Now we will show the same result for more general $\mu_Y^{\hat{u}}$. However, this again follows the same reparameterization argument outlined in step 1.2B in the proof of the main theorem.

A.2. Proof of Proposition 2.4. Take $w(t,x) = m$ for any $(t,x) \in \mathcal{D}$, where the constant m is a lower bound of g. For any given $\mathfrak{u} \in \mathfrak{U}(t)$, $\alpha \in \mathcal{A}^t$, choose any $\widetilde{\alpha} \in \mathcal{A}^t$. Let $B \subset \{Y(\tau^\alpha) < w(\tau, X(\tau^\alpha))\}$ and $\mathbb{P}(B) > 0$. Set

$$\theta_s \triangleq \begin{cases} \dfrac{\mu_Y \sigma_Y}{\|\sigma_Y\|^2}\big(s, X(s), Y(s), \mathfrak{u}[\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}]_s, [\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}]_s\big), \\ \qquad\qquad \text{if } \sigma_Y\big(s, X(s), Y(s), \mathfrak{u}[\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}]_s, [\alpha \otimes_{\tau^\alpha} \widetilde{\alpha}]_s\big) \neq 0, \\ C, \qquad\quad \text{otherwise,} \end{cases}$$

for some constant vector C in $\mathbb{R}^d$. Therefore, θ_s satisfies Novikov's condition due to Assumption 2.7, and $\widetilde{W}(s) = W(s) - \int_0^s \theta_u\,du$ is a Brownian motion under the probability measure $\mathbb{Q}$, where

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(Z_T \mathbb{1}_A) \qquad \text{for all } A \in \mathcal{F} \quad \text{and}$$
$$Z_s := \exp\biggl(\int_0^s \theta_u\,dW_u - \frac{1}{2}\int_0^s \|\theta_u\|^2\,du\biggr).$$

$Z_T \in \mathbb{L}^q(\mathbb{P})$ for any $q \geq 1$ since θ is a bounded. From Assumption 2.6 and the assumption that σ_Y is invertible in its u-variable (Assumption 2.2), it follows that $\sigma_Y(t,x,y,u,a) = 0$ implies $\mu_Y(t,x,y,u,a) = 0$. Therefore under $\mathbb{Q}$

$$dY(s) = \sigma_Y\big(s, X(s), Y(s), \mathfrak{u}[\widetilde{\alpha}]_s, \widetilde{\alpha}_s\big)\,d\widetilde{W}_s \qquad \text{for } s \geq \tau^\alpha,$$

where $Y := Y_{t,x,y}^{u,\alpha\otimes_{\tau^\alpha}\widetilde{\alpha}}$. We will show that the $\mathbb{Q}$-local martingale Y is actually a $\mathbb{Q}$-martingale. Assumption 2.1 implies that

$$\mathbb{E}_{\mathbb{P}}\Big[\sup_{0\le s\le T}|Y(s)|^2\Big]<\infty; \tag{A.3}$$

see, for example, Theorem 1.3.5 in [10] or Theorem 2.2 in [17]. As a result an application of Hölder's inequality yields

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\le s\le T}|Y(s)|\Big]&=\mathbb{E}_{\mathbb{P}}\Big[\sup_{0\le s\le T}|Y(s)|\cdot Z_T\Big]\\&\le\mathbb{E}_{\mathbb{P}}\Big[\sup_{0\le s\le T}|Y(s)|^2\Big]\mathbb{E}_{\mathbb{P}}\big[Z_T^2\big]<\infty.\end{aligned} \tag{A.4}$$

From (A.4), Y is a martingale on $[\tau^\alpha, T]$ under $\mathbb{Q}$. Moreover, since $\mathbb{Q}$ is equivalent to $\mathbb{P}$ we have $\mathbb{Q}(B)>0$. As a result of the latter two statements, for any $\rho\ge\tau^\alpha$,

$$Y(\rho)\le Y(\tau^\alpha)\qquad\text{on some set } H\subset B \text{ with } \mathbb{Q}(H)>0.$$

Since $H\subset B$,

$$Y(\rho)\le Y(\tau^\alpha)<m=w(t,x)\qquad\text{on } H.$$

This implies $\mathbb{Q}(Y(\rho)<m|B)>0$ and by equivalence of the measures $\mathbb{P}(Y(\rho)<m|B)>0$. Therefore, $w(t,x)=m$ is a stochastic sub-solution. □

A.3. Some well-known results from the theory of viscosity solutions. In this subsection, we introduce an alternative definition of viscosity solutions and Ishii's lemma following [10]. First, we define the second-order super-jet of an USC function U at a point $(\bar{t},\bar{x})\in[0,T)\times\mathbb{R}^d$ as the set of elements $(\overline{q},\overline{p},\overline{M})\in\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$ satisfying

$$\begin{aligned}U(t,x)\le{}&U(\bar{t},\bar{x})+\overline{q}(t-\bar{t})+\overline{p}\cdot(x-\bar{x})+\tfrac{1}{2}M(x-\bar{x})\cdot(x-\bar{x})\\&+o\big(|t-\bar{t}|+|x-\bar{x}|^2\big).\end{aligned}$$

This set is denoted by $P^{2,+}U(\bar{t},\bar{x})$. Similarly, $P^{2,-}V(\bar{t},\bar{x})$, the second-order sub-jet of a LSC function V at the point $(\bar{t},\bar{x})\in[0,T)\times\mathbb{R}^d$ is defined as the set of elements $(\overline{q},\overline{p},\overline{M})\in\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$ satisfying

$$\begin{aligned}V(t,x)\ge{}&V(\bar{t},\bar{x})+\overline{q}(t-\bar{t})+\overline{p}\cdot(x-\bar{x})+\tfrac{1}{2}M(x-\bar{x})\cdot(x-\bar{x})\\&+o\big(|t-\bar{t}|+|x-\bar{x}|^2\big).\end{aligned}$$

For technical reasons related to Ishii's lemma, we also need to consider the limiting super-jets and sub-jets. More precisely, we define $\overline{P}^{2,+}U(t,x)$ as the set of elements $(q,p,M)\in\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}^d$ for which there exists a sequence $(t_\varepsilon,x_\varepsilon,q_\varepsilon,p_\varepsilon,M_\varepsilon)_\varepsilon$ satisfying $(q_\varepsilon,p_\varepsilon,M_\varepsilon)\in P^{2,+}U(t_\varepsilon,x_\varepsilon)$ and $(t_\varepsilon,x_\varepsilon,U(t_\varepsilon,x_\varepsilon),q_\varepsilon,p_\varepsilon,M_\varepsilon)\to(t,x,U(t,x),q,p,M)$. The set $\overline{P}^{2,-}V(t,x)$ is defined similarly. Now we state the alternative definition of viscosity solutions to (2.4).

LEMMA A.1. *A USC (resp., LSC) function w on $\mathcal{D}_{<T}$ is a viscosity sub-solution (resp., super-solution) to (2.4) if and only if for all $(t,x) \in \mathcal{D}_{<T}$, and all $(q,p,M) \in \overline{P}^{2,+}w(t,x)$ [resp., $\overline{P}^{2,-}w(t,x)$],*

$$-q - H\big(t,x,w(t,x),p,M\big) \leq (\text{resp.}, \geq)\ 0.$$

Finally, we state Ishii's lemma used in [10] without proof and refer the reader to Theorem 8.3 in [9].

LEMMA A.2 (Ishii's lemma). *Let U (resp., V) be an USC (resp., LSC) function on $\mathcal{D}_{<T}$, $\varphi \in C^{1,1,2,2}([0,T)^2 \times \mathbb{R}^d \times \mathbb{R}^d)$, and $(t_0,s_0,x_0,y_0) \in [0,T)^2 \times \mathbb{R}^d \times \mathbb{R}^d$ a local maximum of $U(t,x) - V(s,y) - \varphi(t,s,x,y)$. Then, for all $\eta > 0$, there exist $M, N \in \mathcal{S}^d$ satisfying*

$$\big(\varphi_t(t_0,s_0,x_0,y_0), D_x\varphi(t_0,s_0,x_0,y_0), M\big) \in \overline{P}^{2,+}U(t,x),$$

$$\big(-\varphi_s(t_0,s_0,x_0,y_0), -D_y\varphi(t_0,s_0,x_0,y_0), N\big) \in \overline{P}^{2,-}V(t,x)$$

and

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \leq D^2_{x,y}\varphi(t_0,s_0,x_0,y_0) + \eta\big(D^2_{x,y}\varphi(t_0,s_0,x_0,y_0)\big)^2.$$

REMARK A.1. From Remark 4.4.9 in [10], by choosing $\varphi_\varepsilon(t,s,x,y) := \frac{1}{2\varepsilon}(|t-s|^2 + |x-y|^2)$ and $\eta = \varepsilon$, for any $d \times n$ matrices C, D, we get

$$\operatorname{Tr}\big(CC^\top M - DD^\top N\big) \leq \frac{3}{\varepsilon}\operatorname{Tr}\big((C-D)(C-D)^\top\big). \tag{A.5}$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
530 CHURCH STREET
ANN ARBOR, MICHIGAN 48109
USA
E-MAIL: erhan@umich.edu
lijiaqi@umich.edu