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A WEAK APPROXIMATION WITH ASYMPTOTIC EXPANSION AND MULTIDIMENSIONAL MALLIAVIN WEIGHTS[1]

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This paper develops a new efficient scheme for approximations of expectations of the solutions to stochastic differential equations (SDEs). In particular, we present a method for connecting approximate operators based on an asymptotic expansion with multidimensional Malliavin weights to compute a target expectation value precisely. The mathematical validity is given based on Watanabe and Kusuoka theories in Malliavin calculus. Moreover, numerical experiments for option pricing under local and stochastic volatility models confirm the effectiveness of our scheme. Especially, our weak approximation substantially improves the accuracy at deep Out-of-The-Moneys (OTMs).

1. Introduction. Developing an approximation method for expectations of diffusion processes is an interesting topic in various research fields. In fact, it seems so useful that a precise approximation for the expectation would lead to substantial reduction of computational burden so that the subsequent analyses could be very easily implemented. Particularly, in finance it has drawn much attention for more than the past two decades since fast and precise computation is so important in terms of competition and risk management in practice such as in trading and investment.

An example among a large number of the related researches is an asymptotic expansion approach, which is mathematically justified by Watanabe theory [Watanabe (1987)] in Malliavin calculus [e.g., Malliavin (1997)]. Especially, the asymptotic expansion have been applied to a broad class of problems in finance; for instance, see Takahashi and Yamada (2012a, 2012b, 2013, 2015) and references therein.

Although the asymptotic expansion up to the fifth order is known to be sufficiently accurate for option pricing [e.g., Takahashi, Takehara and Toda (2012)], the main criticism against the method would be that the approximate density function deviates from the true density at its tails that is, some region of the very deep

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Out-of-The-Money (OTM). However, there exist similar problems, at least implicitly in other well-known approximation methods such as Hagan et al. (2002).

On the other hand, the Monte Carlo simulation method is quite popular mainly due to the ease of its implementation. Nevertheless, in order to achieve accuracy sufficient enough in practice, there exists an unavoidable drawback in computational cost under the standard weak approximation schemes of SDEs such as the Euler–Maruyama scheme.

To overcome this problem, Kusuoka (2001, 2003b, 2004) developed a high order weak approximation scheme for SDEs based on Malliavin calculus and Lie algebra, which opened the door for the possibility that the computational speed and the accuracy in the Monte Carlo simulation satisfies stringent requirements in financial business. Independently, Lyons and Victoir (2004) developed a cubature method on the Wiener space. Since then, there have been a large number of researches for weak approximations and its applications to the computational finance inspired by those pioneering works. For instance, see Crisan, Manolarakis and Nee (2013) for the Kusuoka's method and its related works [e.g., Bayer, Friz and Loeffen (2013)].

This paper develops a new weak approximation scheme for expectations of functions of the solutions to SDEs. In particular, the scheme connects approximate operators constructed based on the asymptotic expansion. More concretely, a diffusion semigroup is defined as the expectation of an appropriate function of the solution to a certain SDE, for example, $P_t^{\varepsilon} f(x) = E[f(X_t^{x,\varepsilon})]$ with the solution $X_t^{x,\varepsilon}$ of a SDE with perturbation parameter ε and a function f. Then we approximate P_t^{ε} by an operator $Q_t^{\varepsilon,m}$ which is constructed based on the asymptotic expansion up to a certain order m. Thus, given a partition of $[0, T]$, $\pi = \{(t_0, t_1, \dots, t_n) : 0 = t_0 < t_1 < \cdots < t_n = T\}$, we are able to approximate $P_T^{\varepsilon} f(x)$ by connecting the expansion-based approximations sequentially, that is, with $s_k = t_k - t_{k-1}$, $k = 1, \dots, n$,

$$P_T^{\varepsilon} f(x) \simeq Q_{s_n}^{\varepsilon,m} Q_{s_{n-1}}^{\varepsilon,m} \cdots Q_{s_1}^{\varepsilon,m} f(x).$$

This paper justifies this idea by applying Malliavin calculus, particularly, theories developed by Watanabe (1987) and Kusuoka (2001, 2003a, 2004).

Moreover, we show through numerical examples for option pricing that very few partitions such as $n = 2$ is mostly enough to substantially improve the errors at deep OTMs of expansions with order $m = 1, 2$. For a related but different approach with similar motivation, see Section 5 in Fujii (2014).

The organization of the paper is as follows. The next section introduces the setup and the basic results necessary for the subsequent analysis. Section 3 shows our main result for a new weak approximation of the expectation of diffusion processes. Section 4 briefly describes an example for the implementation method of our scheme, Section 5 provides numerical experiments for option pricing under local and stochastic volatility models. Section 6 makes concluding remarks. The Appendix gives the proofs of Theorems 1, 2 and 3 as well as Lemma 2 and its proof.

2. Preparation. Let $(\mathcal{W}, H, \mathbb{P})$ be the d-dimensional Wiener space, that is, $\mathcal{W} = \{w \in C([0,T] \to \mathbf{R}^d); w(0) = 0\}$ which is a real Banach space under the supremum norm, $H = \{h \in \mathcal{W}; t \mapsto h(t)$ is absolutely continuous and $\|h\|_H^2 = \int_0^T |\frac{d}{dt}h(t)|^2\,dt < \infty\}$ is a real Hilbert space under $\|\cdot\|_H$ called the Cameron–Martin subspace and $\mathbb{P}$ is the d-dimensional Wiener measure. Let $B_t = (B_t^1, \ldots, B_t^d)^\top$ be a d-dimensional Brownian motion. In this paper, we consider the following general perturbed N-dimensional stochastic differential equation with $\varepsilon \in (0,1]$:

$$X_t^{x,\varepsilon} = x + \int_0^t V_0(\varepsilon, X_s^{x,\varepsilon})\,ds + \varepsilon \sum_{j=1}^d \int_0^t V_j(X_s^{x,\varepsilon})\,dB_s^j, \tag{2.1}$$

where $V_0 \in C_b^\infty((0,1] \times \mathbf{R}^N; \mathbf{R}^N)$ and $V_j \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, $j = 1, \ldots, d$ are bounded. Hereafter, we will use the notation $Vf(x) = \sum_{i=1}^N V^i(x)(\partial f/\partial x_i)(x)$ for $V \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$ and f a differentiable function $\mathbf{R}^N$ into $\mathbf{R}$. $X_t^{x,\varepsilon}$ can be written in the Stratonovich form:

$$X_t^{x,\varepsilon} = x + \int_0^t \tilde{V}_0(\varepsilon, X_s^{x,\varepsilon})\,ds + \varepsilon \sum_{j=1}^d \int_0^t \tilde{V}_j(X_s^{x,\varepsilon}) \circ dB_s^j, \tag{2.2}$$

where

$$\tilde{V}_0^i(\varepsilon, x) = V_0^i(\varepsilon, x) - \frac{\varepsilon^2}{2} \sum_{j=1}^d V_j V_j^i(x), \tag{2.3}$$

$$\varepsilon \tilde{V}_j^i(x) = \varepsilon V_j^i(x), \qquad j = 1, \ldots, d. \tag{2.4}$$

Here, we consider the case $V_0^i(\varepsilon, x) = \varepsilon^k \hat{V}_0^i(x)$, $\hat{V}_0 \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, $k = 0, 1, 2$, for $i = 1, \ldots, N$, which is useful in applications [see Takahashi and Toda (2013) for the details]. Moreover, we assume the following condition [**H**] on the vector fields, which ensures both the integration by parts on the Wiener space and the asymptotic expansion in the next section.

[**H**] The matrix $A(x) = (A^{i,i'}(x))_{i,i'}$ defined by

$$A^{i,i'}(x) = \sum_{j=1}^d V_j^i(x) V_j^{i'}(x) \qquad \text{for all } x \in \mathbf{R}^N, 1 \le i, i' \le N \tag{2.5}$$

is nondegenerate, that is, $\det(A(x)) > 0$.

2.1. *The space $\mathcal{K}_r$.* Let $\mathbf{D}^{k,p}(E)$, $k \ge 1$, $p \in [1,\infty)$ be the space of k-times Malliavin differentiable Wiener functionals $F \in L^p(\mathcal{W}, E)$, where E is a separable Hilbert space. See Watanabe (1987), Ikeda and Watanabe (1989), Malliavin (1997), Malliavin and Thalmaier (2006) and Nualart (2006) for more details of the notation. This subsection introduces the space of Wiener functionals $\mathcal{K}_r$ developed by Kusuoka (2003a) and its properties. The element of $\mathcal{K}_r$ is called the *Kusuoka–Stroock function*. See Nee (2010, 2011) and Crisan, Manolarakis and Nee (2013) for more details of the notation and the proofs.

DEFINITION 1. Given $r \in \mathbf{R}$ and $n \in \mathbf{N}$, we denote by $\mathcal{K}_r(E,n)$ the set of functions $G:(0,1] \times \mathbf{R}^N \to \mathbf{D}^{n,\infty}(E)$ satisfying the following:

1. $G(t,\cdot)$ is n-times continuously differentiable and $[\partial^\alpha G/\partial x^\alpha]$ is continuous in $(t,x) \in (0,1] \times \mathbf{R}^N$ a.s. for any multiindex $\alpha = \alpha^{(l)} \in \{1,\dots,d\}^l$ with length $|\alpha| = l \le n$. Here, $[\partial^\alpha G/\partial x^\alpha]$ is the partial derivative of $G(t,x)$ given by $\frac{\partial^l}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}} G(t,x)$.
2. For all $k \le n - |\alpha|$, $p \in [1,\infty)$,

$$\sup_{t\in(0,1],x\in\mathbf{R}^N} t^{-r/2}\left\| \frac{\partial^\alpha G}{\partial x^\alpha}(t,x)\right\|_{\mathbf{D}^{k,p}} < \infty. \tag{2.6}$$

We write $\mathcal{K}_r$ for $\mathcal{K}_r(\mathbf{R},\infty)$.

Next, we show the basic properties of the Kusuoka–Stroock functions.

LEMMA 1 (Properties of Kusuoka–Stroock functions). 1. *The function* $(t,x) \in (0,1] \times \mathbf{R}^N \mapsto X_t^{x,\varepsilon}$ *belongs to* $\mathcal{K}_0$.

2. *Suppose* $G \in \mathcal{K}_r(n)$ *where* $r \ge 0$. *Then, for* $i = 1,\dots,d$,

$$\begin{aligned} &\text{(a)} \quad \int_0^\cdot G(s,x)\,dB_s^i \in \mathcal{K}_{r+1}(n) \quad \textit{and} \\ &\text{(b)} \quad \int_0^\cdot G(s,x)\,ds \in \mathcal{K}_{r+2}(n). \end{aligned} \tag{2.7}$$

3. *If* $G_i \in \mathcal{K}_{r_i}(n_i)$, $i = 1,\dots,l$, *then*

$$\begin{aligned} &\text{(a)} \quad \prod_i^l G_i \in \mathcal{K}_{r_1+\cdots+r_l}\Bigl(\min_i n_i\Bigr) \quad \textit{and} \\ &\text{(b)} \quad \sum_{i=1}^l G_i \in \mathcal{K}_{\min_i r_i}\Bigl(\min_i n_i\Bigr). \end{aligned} \tag{2.8}$$

Then we summarize the Malliavin's integration by parts formula using Kusuoka–Stroock functions. Hereafter, for any multiindex $\alpha = \alpha^{(k)} := (\alpha_1,\dots,\alpha_k) \in \{1,\dots,N\}^k$, $k \ge 1$ with the length $|\alpha^{(k)}| = k$, we denote by $\partial_{\alpha^{(k)}}$ the partial derivative $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^k}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_k}}$.

PROPOSITION 1. *Suppose that condition* [**H**] *holds. Let* $G:(0,1]\times\mathbf{R}^N\to\mathbf{D}^\infty=\mathbf{D}^{\infty,\infty}(\mathbf{R})$ *be an element of* $\mathcal{K}_r$ *and let* f *be a function that belongs to the space* $C_b^\infty(\mathbf{R}^N;\mathbf{R})$. *Then for any multiindex* $\alpha^{(k)}\in\{1,\dots,N\}^k$, $k\geq 1$, *there exists* $H_{\alpha^{(k)}}(X_t^{x,\varepsilon},G(t,x))\in\mathcal{K}_{r-k}$ *such that*

$$E\big[\partial_{\alpha^{(k)}}f\big(X_t^{x,\varepsilon}\big)G(t,x)\big]=E\big[f\big(X_t^{x,\varepsilon}\big)H_{\alpha^{(k)}}\big(X_t^{x,\varepsilon},G(t,x)\big)\big],\qquad t\in(0,1],\tag{2.9}$$

with

$$\sup_{x\in\mathbf{R}^N}\big\|H_{\alpha^{(k)}}\big(X_t^{x,\varepsilon},G(t,x)\big)\big\|_{L^p}\leq t^{(r-k)/2}C,\tag{2.10}$$

where $H_{\alpha^{(k)}}(X_t^{x,\varepsilon},G(t,x))$ *is recursively given by*

$$H_{(i)}\big(X_t^{x,\varepsilon},G(t,x)\big)=\delta\left(\sum_{j=1}^N G(t,x)\gamma_{ij}^{X_t^{x,\varepsilon}}DX_t^{x,\varepsilon,j}\right),\tag{2.11}$$

$$H_{\alpha^{(k)}}\big(X_t^{x,\varepsilon},G(t,x)\big)=H_{(\alpha_k)}\big(X_t^{x,\varepsilon},H_{\alpha^{(k-1)}}\big(X_t^{x,\varepsilon},G(t,x)\big)\big),\tag{2.12}$$

and a positive constant C. *Here,* δ *is the Skorohod integral and* $DX_t^{x,\varepsilon}$ *is the Malliavin derivative of* $X_t^{x,\varepsilon}$,

$$\begin{aligned}\big\langle DX_t^{x,\varepsilon},h\big\rangle_H&=\sum_{k=1}^d\int_0^t D_{s,k}X_t^{x,\varepsilon}\frac{d}{ds}h_k(s)\,ds\\&=\lim_{\lambda\to 0}\frac{X_t^{x,\varepsilon}(w+\lambda h)-X_t^{x,\varepsilon}(w)}{\lambda},\qquad h\in H,\end{aligned}\tag{2.13}$$

and $\gamma^{X_t^{x,\varepsilon}}=(\gamma_{ij}^{X_t^{x,\varepsilon}})_{1\leq i,j\leq N}$ *is the inverse matrix of the Malliavin covariance of* $X_t^{x,\varepsilon}$.

PROOF. By 1, 2, 3 of Lemma 1, we can see that the Malliavin covariance of $X_t^{x,\varepsilon}$ is given by

$$\sigma_{i,j}^{X_t^{x,\varepsilon}}=\sum_{k=1}^d\int_0^t D_{s,k}X_t^{x,\varepsilon,i}D_{s,k}X_t^{x,\varepsilon,j}\,ds\in\mathcal{K}_2,\tag{2.14}$$

since $D_{s,k}X_t^{x,\varepsilon,i}\in\mathcal{K}_0$, $s\leq t$, $k=1,\dots,d$, $i=1,\dots,N$. Under [**H**], it can be shown that the nondegenerate condition of the Malliavin covariance matrix is satisfied when $\varepsilon>0$ (but not satisfied when $\varepsilon=0$, that is, the Malliavin covariance matrix $\sigma^{X_t^{x,\varepsilon}}$ is not uniformly nondegenerate in ε) and then (2.9) holds [see the proofs of Proposition 5.8, Theorems 5.9 and 6.7 of Shigekawa (2004)]. Also, we have $\gamma^{X_t^{x,\varepsilon}}\in\mathcal{K}_{-2}$ since $\gamma^{X_t^{x,\varepsilon}}=(\sigma^{X_t^{x,\varepsilon}})^{-1}=\frac{\operatorname{adj}\sigma^{X_t^{x,\varepsilon}}}{\det\sigma^{X_t^{x,\varepsilon}}}$. Here, adj A is the adjugate matrix of A. By the property of the Skorohod integral [Proposition 1.3.3 of Nualart (2006) and Lemma 5.2 of Malliavin (1997) or (4.15) of proof of Lemma 4.10 of Malliavin and Thalmaier (2006)], we have

$$\begin{aligned} & H_{(i)}\big(X_t^{x,\varepsilon}, G(t,x)\big) \\ & \quad = \delta\left(\sum_{j=1}^{N} G(t,x)\gamma_{ij}^{X_t^\varepsilon} DX_t^{x,\varepsilon,j}\right) \\ & \quad = \Bigg[G(t,x)\sum_{j=1}^{N}\sum_{k=1}^{d}\int_0^t \gamma_{ij}^{X_t^\varepsilon}\big(J_t^{x,\varepsilon}(J_s^{x,\varepsilon})^{-1}\varepsilon V_k(X_s^{x,\varepsilon})\big)^j\,dB_s^k \\ & \qquad - \sum_{j=1}^{N}\sum_{k=1}^{d}\int_0^t [D_{s,k}G(t,x)]\gamma_{ij}^{X_t^\varepsilon}\big(J_t^{x,\varepsilon}(J_s^{x,\varepsilon})^{-1}\varepsilon V_k(X_s^{x,\varepsilon})\big)^j\,ds\Bigg]. \end{aligned} \tag{2.15}$$

Again, by Lemma 1, the first and the second terms in the second equality is characterized by

$$G(t,x)\sum_{j=1}^{N}\sum_{k=1}^{d}\int_0^t \gamma_{ij}^{X_t^\varepsilon}\big(J_t^{x,\varepsilon}(J_s^{x,\varepsilon})^{-1}\varepsilon V_k(X_s^{x,\varepsilon})\big)^j\,dB_s^k \in \mathcal{K}_{r-1}, \tag{2.16}$$

$$\int_0^t [D_{s,k}G(t,x)]\gamma_{ij}^{X_t^\varepsilon}\big(J_t^{x,\varepsilon}(J_s^{x,\varepsilon})^{-1}\varepsilon V_k(X_s^{x,\varepsilon})\big)^j\,ds \in \mathcal{K}_r, \tag{2.17}$$

since $J_t^{x,\varepsilon}, (J_t^{x,\varepsilon})^{-1} \in \mathcal{K}_0$, $\gamma_{ij}^{X_t^\varepsilon} \in \mathcal{K}_{-2}$ and

$$\int_0^t \gamma_{ij}^{X_t^\varepsilon}\big(J_t^{x,\varepsilon}(J_s^{x,\varepsilon})^{-1}\varepsilon V_k(X_s^{x,\varepsilon})\big)^j\,dB_s^k \in \mathcal{K}_{-2+1} = \mathcal{K}_{-1}. \tag{2.18}$$

Then $H_{(i)}(X_t^{x,\varepsilon}, G(t,x)) \in \mathcal{K}_{r-1}$ and $H_{\alpha^{(k)}}(X_t^{x,\varepsilon}, G(t,x)) \in \mathcal{K}_{r-k}$. Therefore, we have the assertion. □

3. Weak approximation with asymptotic expansion method. In the remainder of the paper, we use the following norms and seminorms:

$$\|f\|_\infty = \sup_{x\in\mathbf{R}^N}|f(x)|, \qquad \|\nabla f\|_\infty = \max_{i\in\{1,\dots,N\}}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty, \tag{3.1}$$

$$\|\nabla^i f\|_\infty = \max_{j_1,\dots,j_i\in\{1,\dots,N\}}\left\|\frac{\partial^i f}{\partial x_{j_1}\cdots\partial x_{j_i}}\right\|_\infty, \qquad f\in C_b^\infty(\mathbf{R}^N;\mathbf{R}). \tag{3.2}$$

In the first step, we give approximation results of an asymptotic expansion with Malliavin weights for $E[f(X_t^{x,\varepsilon})]$ where

$$X_t^{x,\varepsilon} = x + \int_0^t V_0(\varepsilon, X_s^{x,\varepsilon})\,ds + \varepsilon\sum_{j=1}^{d}\int_0^t V_j(X_s^{x,\varepsilon})\,dB_s^j. \tag{3.3}$$

Under the smoothness of the vector fields V_j, $j = 0, 1, \dots, d$, $X_t^{x,\varepsilon}$ is expanded as

$$X_t^{x,\varepsilon} = X_t^{x,0} + \varepsilon \frac{\partial}{\partial \varepsilon} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0} + \varepsilon^2 \frac{1}{2!}\frac{\partial^2}{\partial \varepsilon^2} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0} + \cdots \qquad \text{in } \mathbf{D}^\infty. \tag{3.4}$$

Here, the above expansion in the space $\mathbf{D}^\infty$ is given in the sense that for all $m \in \mathbf{N}$,

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{m+1}} \left\| X_t^{x,\varepsilon} - \left\{ X_t^{x,0} + \sum_{i=1}^m \varepsilon^i \frac{1}{i!}\frac{\partial^i}{\partial \varepsilon^i} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0} \right\} \right\|_{\mathbf{D}^{k,p}} < \infty \qquad \forall k \in \mathbf{N}, \forall p \in [1, \infty). \tag{3.5}$$

For instance, see Watanabe (1987) and Kunitomo and Takahashi (2003) for the details.

Let us define $\bar{X}_t^{x,\varepsilon}$ as the sum of the first two terms in the expansion (3.4) as follows:

$$\bar{X}_t^{x,\varepsilon} = X_t^{x,0} + \varepsilon \frac{\partial}{\partial \varepsilon} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0}. \tag{3.6}$$

We remark that $X_t^{x,0}$ is the solution to the following ODE:

$$X_t^{x,0} = x + \int_0^t V_0(0, X_s^{x,0})\, ds, \tag{3.7}$$

and $\frac{\partial}{\partial \varepsilon} X_s^{x,\varepsilon}|_{\varepsilon=0}$ satisfies the following linear SDE:

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} X_s^{x,\varepsilon,l}\bigg|_{\varepsilon=0} &= \int_0^t \frac{\partial}{\partial \varepsilon} V_0^l(\varepsilon, X_s^{x,0})\bigg|_{\varepsilon=0} ds + \sum_{j=1}^d \int_0^t V_j^l(X_s^{x,0})\, dB_s^j \\ &\quad + \sum_{k=1}^N \int_0^t \partial_k V_0^l(0, X_s^{x,\varepsilon})\bigg|_{\varepsilon=0} \frac{\partial}{\partial \varepsilon} X_s^{x,\varepsilon,k}\bigg|_{\varepsilon=0} ds, \end{aligned} \tag{3.8}$$

$$\frac{\partial}{\partial \varepsilon} X_0^{x,\varepsilon,l}\bigg|_{\varepsilon=0} = 0, \qquad l = 1, \dots, N. \tag{3.9}$$

The solution of $\frac{\partial}{\partial \varepsilon} X_s^{x,\varepsilon}|_{\varepsilon=0}$ is given by

$$\begin{aligned} &\sum_{j=1}^d \int_0^t J_t^{x,0} (J_u^{x,0})^{-1} V_j(X_u^{x,0})\, dB_u^j \\ &\quad + \int_0^t J_t^{x,0} (J_u^{x,0})^{-1} \frac{\partial}{\partial \varepsilon} V_0(\varepsilon, X_u^{x,0})\bigg|_{\varepsilon=0} du, \end{aligned} \tag{3.10}$$

where $J_t^{x,0} = \nabla_x X_t^{x,0}$ [see (6.6) on page 354 of Karatzas and Shreve (1991), e.g.]. Note that $\frac{\partial}{\partial \varepsilon} X_t^{x,\varepsilon}|_{\varepsilon=0}$ is a Gaussian random variable with a mean $\mu(t)$ and a covariance matrix $\Sigma(t) = (\Sigma_{i,j}(t))_{1\le i,j\le N}$

$$\mu(t) = \int_0^t J_t^{x,0}(J_u^{x,0})^{-1} \frac{\partial}{\partial\varepsilon} V_0(\varepsilon, X_u^{x,0})\Big|_{\varepsilon=0} du, \tag{3.11}$$

$$\Sigma_{i,j}(t) = \sum_{k=1}^{d} \int_0^t (J_t^{x,0}(J_s^{x,0})^{-1} V_k(X_s^{x,0}))^i (J_t^{x,0}(J_s^{x,0})^{-1} V_k(X_s^{x,0}))^j \, ds. \tag{3.12}$$

Here, we note that $t \mapsto \mu(t)$ and $t \mapsto \Sigma_{i,j}(t)$, $1 \le i, j \le N$, are deterministic functions. Therefore, $\bar{X}_t^{x,\varepsilon} = X_t^{x,0} + \varepsilon \frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}|_{\varepsilon=0}$ is a Gaussian random variable with a mean $X_t^{x,0} + \varepsilon\mu(t)$ and a covariance matrix $\varepsilon^2\Sigma(t) = (\varepsilon^2\Sigma_{i,j}(t))_{1\le i,j\le N}$.

REMARK 1. 1. When $V_0(\varepsilon, x) = \varepsilon V_0(x)$, $\bar{X}_t^{x,\varepsilon}$ is given by

$$\bar{X}_t^{x,\varepsilon} = x + \varepsilon \sum_{i=0}^{d} V_i(x) \int_0^t dB_s^i, \tag{3.13}$$

where $B_t^0 = t$.

2. When $V_0(\varepsilon, x) = V_0(x)$, $\bar{X}_t^{x,\varepsilon}$ is given by

$$\bar{X}_t^{x,\varepsilon} = X_t^{x,0} + \varepsilon \sum_{j=1}^{d} \int_0^t J_t^{x,0}(J_u^{x,0})^{-1} V_j(X_u^{x,0}) \, dB_u^j. \tag{3.14}$$

The next theorem shows the local approximation errors for $E[f(X_t^{x,\varepsilon})]$ using Malliavin weights.

THEOREM 1. *Under condition* [**H**], *we have the following*:

1. *For any* $t \in (0, 1]$ *and* $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, *there exists* $C > 0$ *such that*

$$\begin{aligned} &\sup_{x\in\mathbf{R}^N} \left| E[f(X_t^{x,\varepsilon})] - \left\{ E[f(\bar{X}_t^{x,\varepsilon})] + \sum_{j=1}^{m} \varepsilon^j E[f(\bar{X}_t^{x,\varepsilon})\Phi_t^j] \right\} \right| \\ &\quad \le \varepsilon^{m+1} C \left(\sum_{k=1}^{m+1} t^{(m+1+k)/2} \|\nabla^k f\|_\infty \right), \end{aligned} \tag{3.15}$$

where Φ_t^j, $j \ge 1$, *is the Malliavin weights defined by*

$$\begin{aligned} \Phi_t^j = {} & \sum_{k=1}^{j} \sum_{\beta_1+\cdots+\beta_k=j+k, \beta_l\ge 2} \sum_{\alpha^{(k)}\in\{1,\dots,N\}^k} \frac{1}{k!} \\ & \times H_{\alpha^{(k)}}\left(\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}\Big|_{\varepsilon=0}, \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}} X_t^{x,\varepsilon,\alpha_l}\Big|_{\varepsilon=0} \right). \end{aligned} \tag{3.16}$$

2. *For any $t \in (0,1]$ and Lipschitz continuous function $f : \mathbf{R}^N \to \mathbf{R}$, there exists $C > 0$ such that*

$$\sup_{x\in\mathbf{R}^N}\left|E[f(X_t^{x,\varepsilon})] - \left\{E[f(\bar{X}_t^{x,\varepsilon})] + \sum_{j=1}^m \varepsilon^j E[f(\bar{X}_t^{x,\varepsilon})\Phi_t^j]\right\}\right| \le \varepsilon^{m+1} C t^{(m+2)/2}, \tag{3.17}$$

with same weights in (3.16).

3. *For any $t \in (0,1]$ and bounded Borel function $f : \mathbf{R}^N \to \mathbf{R}$, there exists $C > 0$ such that*

$$\sup_{x\in\mathbf{R}^N}\left|E[f(X_t^{x,\varepsilon})] - \left\{E[f(\bar{X}_t^{x,\varepsilon})] + \sum_{j=1}^m \varepsilon^j E[f(\bar{X}_t^{x,\varepsilon})\Phi_t^j]\right\}\right| \le \varepsilon^{m+1} C t^{(m+1)/2}, \tag{3.18}$$

with same weights in (3.16).

PROOF. See Appendix A. □

REMARK 2. When $\tilde{V}_0(\varepsilon, x) = \varepsilon \tilde{V}_0(x)$, $X_t^{x,\varepsilon}$ has the following expansion:

$$\begin{aligned} X_t^{x,\varepsilon} &= x + \varepsilon \sum_{j=0}^d \tilde{V}_j(x) \int_0^t \circ dB_s^j \\ &\quad + \sum_{k=2}^m \varepsilon^k \sum_{(i_1,\dots,i_k)\in\{0,1,\dots,d\}^k} (\tilde{V}_{i_1}\cdots\tilde{V}_{i_k})(x) \int_{0<t_1<\cdots<t_k<t} \circ dB_{t_1}^{i_1} \circ \cdots \circ dB_{t_k}^{i_k} \\ &\quad + \varepsilon^{m+1} \tilde{R}_m(t, x, \varepsilon), \end{aligned}$$

where $\tilde{R}_m(t,x,\varepsilon)$ is the residual. Here, we used the notation $B_t^0 = t$. Then

$$\begin{aligned} &\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0} \\ &\quad = \sum_{(i_1,\dots,i_k)\in\{0,1,\dots,d\}^k} (\tilde{V}_{i_1}\cdots\tilde{V}_{i_k})(x) \int_{0<t_1<\cdots<t_k<t} \circ dB_{t_1}^{i_1} \circ \cdots \circ dB_{t_k}^{i_k}. \end{aligned}$$

REMARK 3. Φ_t^j is obtained by multiple Skorohod integral and each Malliavin weight is concretely calculated as follows; for $G(t,x) \in \mathcal{K}_r$ and $i =$

$1,\dots,N$,

$$
\begin{aligned}
&H_{(i)}\left(\frac{\partial}{\partial\varepsilon}X_t^{x,\varepsilon}\bigg|_{\varepsilon=0}, G(t,x)\right)\\
&\quad = G(t,x)\sum_{j=1}^{N}\sum_{k=1}^{d}[\Sigma(t)^{-1}]_{i,j}\int_0^t \big(J_t^{x,0}(J_s^{x,0})^{-1}V_k(X_s^{x,0})\big)^j\,dB_s^k \\
&\qquad - \sum_{j=1}^{N}\sum_{k=1}^{d}[\Sigma(t)^{-1}]_{i,j}\int_0^t D_{s,k}G(t,x)\big(J_t^{x,0}(J_s^{x,0})^{-1}V_k(X_s^{x,0})\big)^j\,ds
\end{aligned}
\tag{3.19}
$$

with the deterministic covariance matrix $(\Sigma_{i,j}(t))_{1\le i,j\le N}$ corresponds to (3.12), that is,

$$
\begin{aligned}
\Sigma_{i,j}(t) &= \sum_{k=1}^{d}\int_0^t D_{s,k}\frac{\partial}{\partial\varepsilon}X_t^{x,\varepsilon,i}\bigg|_{\varepsilon=0} D_{s,k}\frac{\partial}{\partial\varepsilon}X_t^{x,\varepsilon,j}\bigg|_{\varepsilon=0}\,ds \\
&= \sum_{k=1}^{d}\int_0^t \big(J_t^{x,0}(J_s^{x,0})^{-1}V_k(X_s^{x,0})\big)^i\big(J_t^{x,0}(J_s^{x,0})^{-1}V_k(X_s^{x,0})\big)^j\,ds.
\end{aligned}
\tag{3.20}
$$

Let $(P_t)_t$ be linear operators on $f\in C_b(\mathbf{R}^N;\mathbf{R})$ defined by

$$
P_t f(x) = E\big[f\big(X_t^{x,\varepsilon}\big)\big]. \tag{3.21}
$$

We remark that $(P_t)_t$ is a semigroup. Also let $(\bar{P}_t)_t$ be linear operators on $f\in C_b(\mathbf{R}^N;\mathbf{R})$ defined by

$$
\bar{P}_t f(x) = E\big[f\big(\bar{X}_t^{x,\varepsilon}\big)\big]. \tag{3.22}
$$

Next, as an approximation of P_s we introduce a linear operator $Q_{(s)}^m$ below. First, for $j\ge 1$ and $t\in(0,1]$, let $\bar{P}_{\Phi^j}(t)$ be a linear operator defined by the following expectation with Malliavin weight Φ_t^j:

$$
\bar{P}_{\Phi^j}(t)f(x) = E\big[f\big(\bar{X}_t^{x,\varepsilon}\big)\Phi_t^j\big]. \tag{3.23}
$$

Then $(Q_{(s)}^m)_{s\in(0,1]}$ is defined as linear operators:

$$
Q_{(s)}^m f(x) = \bar{P}_s f(x) + \sum_{j=1}^{m}\varepsilon^j\,\bar{P}_{\Phi^j}(s)f(x). \tag{3.24}
$$

We remark that

$$
\bar{P}_{\Phi^j}(t)f(x) = \int_{\mathbf{R}^N} f(y)E\big[\Phi_t^j|\bar{X}_t^{x,\varepsilon}=y\big]p^{\bar{X}^\varepsilon}(t,x,y)\,dy \tag{3.25}
$$

$$
= E\big[f\big(\bar{X}_t^{x,\varepsilon}\big)\mathcal{M}_{(j)}\big(t,x,\bar{X}_t^{x,\varepsilon}\big)\big], \tag{3.26}
$$

where $\mathcal{M}_{(j)}(t,x,y) = E[\Phi_t^j|\bar{X}_t^{x,\varepsilon}=y]$ and $y\mapsto p^{\bar{X}^\varepsilon}(t,x,y)$ is the density of $\bar{X}_t^{x,\varepsilon}$.

Then $Q^m_{(s)}$ can be written as follows:

$$Q^m_{(s)} f(x) = E\big[f\big(\bar{X}^{x,\varepsilon}_s\big)\mathcal{M}^m\big(s,x,\bar{X}^{x,\varepsilon}_s\big)\big], \tag{3.27}$$

where $\mathcal{M}^m(s,x,y) = 1 + \sum_{j=1}^m \varepsilon^j \mathcal{M}_{(j)}(s,x,y)$.

Then we have the following explicit representation for the Malliavin weight function $\mathcal{M}^m$.

THEOREM 2. *Under condition* [**H**], *the Malliavin weight function* $\mathcal{M}^m$ *is given by*

$$\begin{aligned}
&\mathcal{M}^m(t,x,y)\\
&\quad = 1 + \sum_{j=1}^m \varepsilon^j \sum_{k=1}^j \sum_{\beta_1+\cdots+\beta_k=j+k,\beta_l\ge 2} \sum_{\alpha^{(k)}\in\{1,\dots,N\}^k} \frac{\varepsilon^k}{k!}\\
&\qquad \times \partial^*_{\alpha_k}\circ\partial^*_{\alpha_{k-1}}\circ\cdots\circ\partial^*_{\alpha_1} E\Bigg[\prod_{l=1}^k \frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}} X^{x,\varepsilon,\alpha_l}_t\bigg|_{\varepsilon=0}\bigg|\bar{X}^{x,\varepsilon}_t = y\Bigg],
\end{aligned} \tag{3.28}$$

where ∂^* *is the divergence operator on the Gaussian space* $(\mathbf{R}^N,\nu)$, *that is*,

$$\begin{aligned}
\nu(dy) &= p^{\bar{X}^\varepsilon}(t,x,y)\,dy\\
&= \frac{1}{(2\pi\varepsilon)^{N/2}\det(\Sigma(t))^{1/2}}\\
&\qquad \times e^{-(y-X^{x,0}_t-\varepsilon\mu(t))^\top\Sigma^{-1}(t)(y-X^{x,0}_t-\varepsilon\mu(t))/(2\varepsilon^2)}\,dy,\\
\partial^*_i A(y) &= -\bigg[\frac{\partial}{\partial y_i}\log p^{\bar{X}^\varepsilon}(t,x,y)\bigg]A(y) - \frac{\partial}{\partial y_i}A(y),\\
&\qquad\qquad A\in\mathcal{S}(\mathbf{R}^N),\ 1\le i\le N.
\end{aligned} \tag{3.29}$$

Here, $\mu(t)$ *and* $\Sigma(t) = (\Sigma_{i,j}(t))_{1\le i,j\le N}$ *are defined in* (3.11) *and* (3.12), *respectively*, *that is*,

$$\mu(t) = \int_0^t J^{x,0}_t\big(J^{x,0}_u\big)^{-1}\frac{\partial}{\partial\varepsilon}V_0\big(\varepsilon,X^{x,0}_u\big)\bigg|_{\varepsilon=0}du, \tag{3.30}$$

$$\Sigma_{i,j}(t) = \sum_{k=1}^d \int_0^t \big(J^{x,0}_t\big(J^{x,0}_s\big)^{-1}V_k\big(X^{x,0}_s\big)\big)^i\big(J^{x,0}_t\big(J^{x,0}_s\big)^{-1}V_k\big(X^{x,0}_s\big)\big)^j\,ds, \tag{3.31}$$

and $\mathcal{S}(\mathbf{R}^N)$ *is the Schwartz rapidly decreasing functions on* $\mathbf{R}^N$.

PROOF. See Appendix B. □

REMARK 4. The term $\prod_{l=1}^k \frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}}X^{x,\varepsilon,\alpha_l}_t|_{\varepsilon=0}$ in each conditional expectation in (3.28) of Theorem 2 is generally expressed as a finite sum of iterated multiple Wiener–Itô integrals. Hence, we are able to explicitly compute each conditional expectation, conditioned on $\bar{X}^{x,\varepsilon}_t$ that is given by the first-order Wiener–Itô integral.

For instance, let $q_k = (q_{k,1}, \dots, q_{k,d})^\top$, $q_{k,i} \in L^2([0,t])$, $k = 1,2,3,4$, $i = 1,\dots,d$ and $h_l(\xi; v)$ be the (one dimensional) Hermite polynomial of degree l with parameter $v = \int_0^t q_1^\top(s) q_1(s)\,ds$. Then the conditional expectations of the second- and the third-order iterated multiple Wiener–Itô integrals are evaluated as the following formulas:

$$
\begin{aligned}
&E\left[\int_0^t \int_0^s q_2^\top(u)\,dB_u q_3^\top(s)\,dB_s \Big| \int_0^t q_1^\top(s)\,dB_s = \xi\right] \\
&\quad = \left(\int_0^t \int_0^s q_2^\top(u) q_1(u)\,du\, q_3^\top(s) q_1(s)\,ds\right) \frac{h_2(\xi; v)}{v^2},
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
&E\left[\int_0^t \int_0^s \int_0^u q_2^\top(r)\,dB_r q_3^\top(u)\,dB_u q_4^\top(s)\,dB_s \Big| \int_0^t q_1^\top(s)\,dB_s = \xi\right] \\
&\quad = \left(\int_0^t q_4^\top(s) q_1(s) \int_0^s q_3^\top(u) q_1(u) \int_0^u q_2^\top(r) q_1(r)\,dr\,du\,ds\right) \frac{h_3(\xi; v)}{v^3},
\end{aligned} \tag{3.33}
$$

where $h_2(\xi; v) = \xi^2 - v$ and $h_3(\xi; v) = \xi^3 - 3v\xi$.

The conditional expectations of higher order iterated multiple Wiener–Itô integrals can be evaluated in the similar manner. For the details, see Takahashi (1999) and Takahashi, Takehara and Toda (2009). In fact, we obtain the Malliavin weights appearing in the numerical examples in Section 5 as closed forms by applying the formulas.

Therefore, Theorem 1 is summarized as follows.

COROLLARY 1. *Assume that condition* [**H**] *holds.*

1. *There exists* $C > 0$ *such that*

$$
\|P_s f - Q^m_{(s)} f\|_\infty \le \varepsilon^{m+1} C \left(\sum_{k=1}^{m+1} s^{(m+1+k)/2} \|\nabla^k f\|_\infty\right), \tag{3.34}
$$

for any $s \in (0,1]$ *and* $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$.

2. *There exists* $C > 0$ *such that*

$$
\|P_s f - Q^m_{(s)} f\|_\infty \le \varepsilon^{m+1} C s^{(m+2)/2}, \tag{3.35}
$$

for any $s \in (0,1]$ *and Lipschitz continuous function* $f : \mathbf{R}^N \to \mathbf{R}$.

3. *There exists* $C > 0$ *such that*

$$
\|P_s f - Q^m_{(s)} f\|_\infty \le \varepsilon^{m+1} C s^{(m+1)/2}, \tag{3.36}
$$

for any $s \in (0,1]$ *and bounded Borel function* $f : \mathbf{R}^N \to \mathbf{R}$.

REMARK 5. The above results are obtained based on the integration by parts argument for $G(s,x) \in \mathcal{K}_r$ with time $s \in (0,1]$. However, we are able to show that the same results hold for $s \in (0,T]$, $T > 0$, using the properties of the elements in the space $\mathcal{K}_r^T$ defined as in Crisan, Manolarakis and Nee (2013).

Next, for $T > 0$, $\gamma > 0$, define a partition $\pi = \{(t_0, t_1, \dots, t_n) : 0 = t_0 < t_1 < \cdots < t_n = T, t_k = k^\gamma T/n^\gamma, n \in \mathbf{N}\}$ and $s_k = t_k - t_{k-1}$, $k = 1, \dots, n$. Using the asymptotic expansion operator Q^m of P, we can guess the following semigroup approximation.

$$E\big[f\big(X_T^{x,\varepsilon}\big)\big] = P_T f(x) = P_{s_n} P_{s_{n-1}} \cdots P_{s_1} f(x) \simeq Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f(x).$$

The next theorem shows our main result on the approximation error for this scheme.

THEOREM 3. *Assume that condition* [**H**] *holds. Let* $T > 0$, $\gamma > 0$ *and* $n \in \mathbf{N}$.

1. *For any* $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, *there exists* $C > 0$ *such that*

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{\gamma(m+2)/2}}, \quad 0 < \gamma < m/(m+2), \tag{3.37}$$

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}(1 + \log n), \quad \gamma = m/(m+2), \tag{3.38}$$

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}, \qquad \gamma > m/(m+2). \tag{3.39}$$

2. *For any Lipschitz continuous function* $f : \mathbf{R}^N \to \mathbf{R}$, *there exists* $C > 0$ *such that*

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{\gamma(m+2)/2}}, \quad 0 < \gamma < m/(m+2), \tag{3.40}$$

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}(1 + \log n), \quad \gamma = m/(m+2), \tag{3.41}$$

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}, \qquad \gamma > m/(m+2). \tag{3.42}$$

3. *For any bounded Borel function* $f : \mathbf{R}^N \to \mathbf{R}$, *there exists* $C > 0$ *such that*

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{\gamma(m+1)/2}}, \quad 0 < \gamma < (m-1)/(m+1), \tag{3.43}$$

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{(m-1)/2}}(1 + \log n), \quad \gamma = (m-1)/(m+1), \tag{3.44}$$

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{(m-1)/2}}, \qquad \gamma > (m-1)/(m+1). \tag{3.45}$$

PROOF. See Appendix C. □

REMARK 6. Due to the theorem above, the higher order asymptotic expansion provides the higher order weak approximation. In fact, we can mostly attain enough accuracy even when the expansion order m is low such as $m = 1, 2$. In Section 5, we confirm this fact through numerical examples.

REMARK 7. When $\gamma = 1$, that is, $s_k = T/n$ for all $k = 1, \dots, n$, we have:

1. For any $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, there exists $C > 0$ such that

$$\|P_T f - (Q^m_{(T/n)})^n f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}.$$

2. For any Lipschitz continuous function $f : \mathbf{R}^N \to \mathbf{R}$, there exists $C > 0$ such that

$$\|P_T f - (Q^m_{(T/n)})^n f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{m/2}}.$$

3. For any bounded Borel function $f : \mathbf{R}^N \to \mathbf{R}$, there exists $C > 0$ such that

$$\|P_T f - (Q^m_{(T/n)})^n f\|_\infty \le \varepsilon^{m+1} \frac{C}{n^{(m-1)/2}}.$$

4. Computation with Malliavin weights. This section illustrates computational scheme for implementation of our method.

4.1. *Backward discrete-time approximation.* For preparation, we describe a backward discrete-time approximation of our method.

For $s \in (0, 1]$ and $x, y \in \mathbf{R}^N$, define $p^m(s, x, y)$ as

$$Q^m_{(s)} f(x) = \int_{\mathbf{R}^N} f(y) p^m(s, x, y)\, dy. \tag{4.1}$$

Then $p^m(s, x, y)$ is given by using the Malliavin weight function $\mathcal{M}^m$ as follows:

$$p^m(s, x, y) = \mathcal{M}^m(s, x, y) p^{\bar{X}^\varepsilon}(s, x, y), \tag{4.2}$$

with

$$p^{\bar{X}^\varepsilon}(s, x, y) = \frac{1}{(2\pi\varepsilon^2)^{N/2} \det(\Sigma(s))^{1/2}} \times e^{-(y - \varepsilon\mu(s) - X_s^{x,0})^\top \Sigma^{-1}(s)(y - \varepsilon\mu(s) - X_s^{x,0})/(2\varepsilon^2)}, \tag{4.3}$$

where $\mu(s)$ and $\Sigma(s) = (\Sigma_{i,j}(s))_{1 \le i,j \le N}$ are defined in (3.11) and (3.12), respectively.

Then we are able to calculate $(Q^m_{(T/n)})^n f(x)$ as follows:

$$(Q^m_{(T/n)})^n f(x) \tag{4.4}$$

$$= \int_{(\mathbf{R}^N)^n} f(y_n) \prod_{i=0}^{n-1} p^m(s_i, y_i, y_{i+1})\, dy_n \cdots dy_1 \tag{4.5}$$

$$= \int_{(\mathbf{R}^N)^{n-1}} q_{n-1}(y_{n-1}) \prod_{i=0}^{n-2} p^m(s_i, y_i, y_{i+1})\, dy_{n-1} \cdots dy_1 \tag{4.6}$$

$$= \int_{(\mathbf{R}^N)^{n-2}} q_{n-2}(y_{n-2}) \prod_{i=0}^{n-3} p^m(s_i, y_i, y_{i+1})\, dy_{n-2} \cdots dy_1 \tag{4.7}$$

$$= \int_{\mathbf{R}^N} q_1(y_1) p^m(s_1, y_0, y_1)\, dy_1, \tag{4.8}$$

with $y_0 = x$.

4.2. *Example of computational scheme*. We are able to compute the expectation in the various ways such as numerical integration and Monte Carlo simulation. As an illustrative purpose and an example, this subsection briefly describes a scheme based on Monte Carlo simulation.

In computation of $(Q^m_{(T/n)})^n f(x)$ with simulation (for the case of $\gamma = 1$), we store $\bar{X}^{x,(j)}_{T/n} \equiv \bar{X}^{x,\varepsilon,(j)}_{T/n}$, which stands for the jth $(1 \le j \le M)$ independent outcome of $\bar{X}^{x,\varepsilon}$ at T/n (i.e., at $t_i + T/n$) starting from x at each grid $t_i = (iT)/n$ $(0 \le i \le n-1)$.

Then we calculate an approximate semigroup at each time grid. That is, $q_{n-1}(x)$, $q_{n-2}(x)$ are calculated as follows:

$$q_{n-1}(x) = \int_{\mathbf{R}^N} f(y) p^m(T/n, x, y)\, dy \tag{4.9}$$

$$= \int_{\mathbf{R}^N} f(y) \mathcal{M}^m(T/n, x, y) p^{\bar{X}^\varepsilon}(T/n, x, y)\, dy \tag{4.10}$$

$$\simeq \frac{1}{M} \sum_{j=1}^{M} f(\bar{X}^{x,(j)}_{T/n}) \mathcal{M}^m(T/n, x, \bar{X}^{x,(j)}_{T/n}), \tag{4.11}$$

$$q_{n-2}(x) = \int_{\mathbf{R}^N} q_{n-1}(y) p^m(T/n, x, y)\, dy \tag{4.12}$$

$$= \int_{\mathbf{R}^N} q_{n-1}(y) \mathcal{M}^m(T/n, x, y) p^{\bar{X}^\varepsilon}(T/n, x, y)\, dy \tag{4.13}$$

$$\simeq \frac{1}{M} \sum_{j=1}^{M} q_{n-1}(\bar{X}^{x,(j)}_{T/n}) \mathcal{M}^m(T/n, x, \bar{X}^{x,(j)}_{T/n}). \tag{4.14}$$

Therefore, in general,

$$q_{i-1}(x) = \int_{\mathbf{R}^N} q_i(y) p^m(T/n, x, y)\, dy \tag{4.15}$$

$$= \int_{\mathbf{R}^N} q_i(y) \mathcal{M}^m(T/n, x, y) p^{\bar{X}^\varepsilon}(T/n, x, y)\, dy \tag{4.16}$$

$$\simeq \frac{1}{M} \sum_{j=1}^{M} q_i\big(\bar{X}_{T/n}^{x,(j)}\big) \mathcal{M}^m\big(T/n, x, \bar{X}_{T/n}^{x,(j)}\big). \tag{4.17}$$

Finally, we obtain an approximation:

$$\big(Q_{(T/n)}^m\big)^n f(x) = \int_{\mathbf{R}^N} q_1(y) p^m(T/n, x, y)\, dy \tag{4.18}$$

$$= \int_{\mathbf{R}^N} q_1(y) \mathcal{M}^m(T/n, x, y) p^{\bar{X}^\varepsilon}(T/n, x, y)\, dy \tag{4.19}$$

$$\simeq \frac{1}{M} \sum_{j=1}^{M} q_1\big(\bar{X}_{T/n}^{x,(j)}\big) \mathcal{M}^m\big(T/n, x, \bar{X}_{T/n}^{x,(j)}\big). \tag{4.20}$$

We also remark that if the numerical integration method is applied, the scheme is based on equations (4.16) and (4.19).

4.3. *Comparison with Kusuoka–Lyons–Victoir* (*KLV*) *cubature method*. In this subsection, we compare our method to a related work, Kusuoka–Lyons–Victoir (KLV) cubature method on Wiener space [Kusuoka (2001, 2004), Lyons and Victoir (2004)].

As mentioned above, we defined the operator $Q_{(s)}^m$ by using the asymptotic expansion with Malliavin weights, while Kusuoka (2001, 2004) and Lyons and Victoir (2004) developed a construction method of a local approximation operator $\hat{Q}_{(s)}^m$ for P_s based on finite variation paths $\omega_1, \dots, \omega_l$ for some $l \in \mathbf{N}$ with weights $\lambda_1, \dots, \lambda_l$.

In the following, we summarize our weak approximation method and the KLV cubature scheme.

Weak approximation with asympotic expansion and Malliavin weights. Let $X_t^{x,\varepsilon}$ be a solution to the following SDE:

$$dX_t^{x,\varepsilon} = V_0\big(\varepsilon, X_t^{x,\varepsilon}\big)\, dt + \varepsilon \sum_{i=1}^{d} V_i\big(X_t^{x,\varepsilon}\big)\, dB_t^i, \qquad X_0^{x,\varepsilon} = x. \tag{4.21}$$

For a Lipschitz continuous function f, $P_t f(x) = E[f(X_t^{x,\varepsilon})]$ is approximated by $Q_{(t)}^m f(x) = E[f(\bar{X}_t^{x,\varepsilon})] + \sum_{j=1}^m \varepsilon^j E[f(\bar{X}_t^{x,\varepsilon})\Phi_t^j] = E[f(\bar{X}_t^{x,\varepsilon})\mathcal{M}^m(t, x, \bar{X}_t^{x,\varepsilon})]$ as follows:

$$\|P_t f - Q_{(t)}^m f\|_\infty = O\big(\varepsilon^{m+1} t^{(m+2)/2}\big), \qquad t \in (0, 1]. \tag{4.22}$$

Then we have the global approximation,

$$\| P_T f - (Q^m_{(T/n)})^n f \|_\infty = O(\varepsilon^{m+1} n^{-m/2}). \tag{4.23}$$

It is emphasized that we are able to evaluate Malliavin weights $\mathcal{M}^m(t, x, \bar{X}^{x,\varepsilon}_t)$ mostly as closed forms by applying computational schemes such as conditional expectation formulas in Takahashi (1999) and Takahashi, Takehara and Toda (2009). In fact, this is the case for the numerical examples in Section 5 of this paper.

KLV cubature scheme on Wiener space. Let X^x_t be a solution to the following SDE:

$$dX^x_t = V_0(X^x_t)\,dt + \sum_{i=1}^d V_i(X^x_t) \circ dB^i_t, \qquad X^x_0 = x. \tag{4.24}$$

A set of finite variation paths $\omega = (\omega_1, \dots, \omega_l)$ with $\lambda = (\lambda_1, \dots, \lambda_l)$ forms *cubature formula on Wiener space of degree m* if for any $\alpha \in \mathcal{A}_m$,

$$\begin{aligned} &E\left[\int_{0<t_1<\cdots<t_r<t} \circ dB^{\alpha_1}_{t_1} \circ \cdots \circ dB^{\alpha_r}_{t_r}\right] \\ &\qquad = \sum_{j=1}^l \lambda_j \int_{0<t_1<\cdots<t_r<t} d\omega^{\alpha_1}_{j,t_1} \cdots d\omega^{\alpha_r}_{j,t_r}. \end{aligned} \tag{4.25}$$

$\omega = (\omega_1, \dots, \omega_l)$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ are called the cubature paths and weights, respectively. Here, $\mathcal{A}_m$ is a set defined by $\mathcal{A}_m = \{(\alpha_1, \dots, \alpha_r) \in \{0, 1, \dots, d\}^r; r + \#\{j|\alpha_j = 0\} \le m, r \in \mathbf{N}\}$. For cubature paths $\omega = (\omega_1, \dots, \omega_l)$ and weights $\lambda = (\lambda_1, \dots, \lambda_l)$, consider the following ODEs:

$$\begin{aligned} &d\hat{X}^x_t(\omega_j) = V_0(\hat{X}^x_t(\omega_j))\,dt + \sum_{i=1}^d V_i(\hat{X}^x_t(\omega_j))\,d\omega^i_{j,t}, \\ &\hat{X}^x_0(\omega_j) = x, \qquad j = 1, \dots, l. \end{aligned} \tag{4.26}$$

Then, for a Lipschitz continuous function f, $P_t f(x) = E[f(X^x_t)]$ can be approximated by $\hat{Q}^m_{(t)} f(x) = \sum_{j=1}^l \lambda_j f(\hat{X}^x_t(\omega_j))$ as follows:

$$\| P_t f - \hat{Q}^m_{(t)} f \|_\infty = O(t^{(m+1)/2}), \qquad t \in (0, 1]. \tag{4.27}$$

Then it can be shown that

$$\| P_T f - (\hat{Q}^m_{(T/n)})^n f \|_\infty = O(n^{-(m-1)/2}). \tag{4.28}$$

See Kusuoka (2001, 2004) and Lyons and Victoir (2004) for the proofs. Here, we note that the Kusuoka–Lyons–Victoir's approximation is generally discussed in the case of nonuniform time grids.

Algorithm 1 Weak approximation with asympotic expansion and Malliavin weights

Define the Malliavin weight $\mathcal{M}^m(t,x,y)$.
for $i=1$ **to** n **do**
Simulate Gaussian random variable $\bar{X}^{x,(j)}_{T/n}$, $j=1,\dots,M$.
if $i=1$ **then**
$q_{n-i}(x)=\frac{1}{M}\sum_{j=1}^{M} f(\bar{X}^{x,(j)}_{T/n})\mathcal{M}^m(T/n,x,\bar{X}^{x,(j)}_{T/n})$
else
$q_{n-i}(x)=\frac{1}{M}\sum_{j=1}^{M} q_{n-i+1}(\bar{X}^{x,(j)}_{T/n})\mathcal{M}^m(T/n,x,\bar{X}^{x,(j)}_{T/n})$
end if
end for
$P_T f(x)\simeq q_0(x)$

In order to obtain a local approximation, we use Malliavin's integration by parts formula on Wiener space for a Gaussian random variable $\bar{X}^{x,\varepsilon}_t = X^{0,x}_t + \varepsilon\frac{\partial}{\partial\varepsilon}X^{x,\varepsilon}_t|_{\varepsilon=0}$. Then the local approximation $Q^m_{(t)}f(x)$ is given by multiplying the Malliavin weight function $\mathcal{M}^m(t,x,\bar{X}^{x,\varepsilon}_t)$ and $f(\bar{X}^{x,\varepsilon}_t)$. As mentioned above, we can mostly evaluate Malliavin weights $\mathcal{M}^m(t,x,\bar{X}^{x,\varepsilon}_t)$ as closed forms.

On the other hand, the Kusuoka–Lyons–Victoir's approximation based on the cubature formula on Wiener space requires to solve the ODEs (4.26) with cubature paths and weights, and then the local approximation $\hat{Q}^m_{(t)}f(x)$ is given by the weighted sum of $f(\hat{X}^x_t(\omega_j))$ with the cubature weights λ_j, $j=1,\dots,l$.

Finally, we summarize the algorithms of our weak approximation method and the KLV cubature scheme on Wiener space as Algorithms 1 and 2, respectively.

5. Numerical example. This section demonstrates the effectiveness of our method through the numerical examples for option pricing under local and stochastic volatility models.

Algorithm 2 Weak approximation: KLV cubature on Wiener space

Define the cubature paths $\omega=(\omega_1,\dots,\omega_l)$ and weights $\lambda=(\lambda_1,\dots,\lambda_l)$.
for $i=1$ **to** n **do**
Solve ODE for $\hat{X}^x_{T/n}(\omega_j)$, $j=1,\dots,l$.
if $i=1$ **then**
$\hat{q}_{n-i}(x)=\sum_{j=1}^{l}\lambda_j f(\hat{X}^x_{T/n}(\omega_j))$
else
$\hat{q}_{n-i}(x)=\sum_{j=1}^{l}\lambda_j \hat{q}_{n-i+1}(\hat{X}^x_{T/n}(\omega_j))$
end if
end for
$P_T f(x)\simeq \hat{q}_0(x)$

5.1. *Local volatility model*. The first example takes the following local volatility model:

$$(5.1)\qquad \begin{aligned} dS_t^{x,\varepsilon} &= \varepsilon\sigma\big(S_t^{x,\varepsilon}\big)\,dB_t, \\ S_0^{x,\varepsilon} &= S_0 = x. \end{aligned}$$

Then let $(\bar{S}_t^{x,\varepsilon})_{t\geq 0}$ be the solution to the following SDE:

$$(5.2)\qquad \begin{aligned} d\bar{S}_t^{x,\varepsilon} &= \varepsilon\sigma(x)\,dB_t, \\ \bar{S}_0^{x,\varepsilon} &= x. \end{aligned}$$

In this numerical example, for the payoff function $f(x) = \max\{x - K, 0\}$ or $f(x) = \max\{K - x, 0\}$ where K is a positive constant, we apply the first-order asymptotic expansion operator, that is, $m = 1$;

$$(5.3)\qquad Q_{(t)}^1 f(x) = E\big[f\big(\bar{S}_t^{x,\varepsilon}\big)\mathcal{M}^1\big(t, x, \bar{S}_t^{x,\varepsilon}\big)\big]$$

and the second-order asymptotic expansion operator, that is, $m = 2$;

$$(5.4)\qquad Q_{(t)}^2 f(x) = E\big[f\big(\bar{S}_t^{x,\varepsilon}\big)\mathcal{M}^2\big(t, x, \bar{S}_t^{x,\varepsilon}\big)\big].$$

The Malliavin weights $\mathcal{M}^1(t, x, y)$ and $\mathcal{M}^2(t, x, y)$ are given by

$$\mathcal{M}^1(t, x, y) = 1 + \varepsilon E\left[H_{(1)}\left(\frac{\partial}{\partial\varepsilon}S_t^{x,\varepsilon}\bigg|_{\varepsilon=0}, \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}S_t^{x,\varepsilon}\bigg|_{\varepsilon=0}\right)\bigg|\bar{S}_t^{x,\varepsilon} = y\right],$$

and

$$\begin{aligned} \mathcal{M}^2(t, x, y) = \mathcal{M}^1(t, x, y) &+ \varepsilon^2 E\left[H_{(1)}\left(\frac{\partial}{\partial\varepsilon}S_t^{x,\varepsilon}\bigg|_{\varepsilon=0}, \frac{1}{6}\frac{\partial^3}{\partial\varepsilon^3}S_t^{x,\varepsilon}\bigg|_{\varepsilon=0}\right)\bigg|\bar{S}_t^{x} = y\right] \\ &+ \frac{1}{2}\varepsilon^2 E\left[H_{(1,1)}\left(\frac{\partial}{\partial\varepsilon}S_t^{x,\varepsilon}\bigg|_{\varepsilon=0}, \left(\frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}S_t^{x,\varepsilon}\bigg|_{\varepsilon=0}\right)^2\right)\bigg|\bar{S}_t^{x,\varepsilon} = y\right]. \end{aligned}$$

Moreover, we remark that those Malliavin weights are obtained as closed forms.

Also, we specify the local volatility function as a log-normal scaled volatility $\varepsilon\sigma(S) = \varepsilon S_0^{1-\beta} S^\beta$ with $\beta = 0.5$. The parameters are set to be $S_0 = 100$ and $\varepsilon = 0.4$. The benchmark values are computed by Monte Carlo simulations (Benchmark MC) with 10^7 trials and 1000 time steps for the 1 year maturity case or 2000 time steps for the 10 year maturity case.

Figures 1 and 2 show the results. The vertical axis in the figures is the Error rate defined by

$$\text{Error Rate} = (\text{WeakApprox} - \text{Benchmark MC})/\text{Benchmark MC}\ (\%).$$

Here, WeakApprox is our weak approximation based on the asymptotic expansion with Malliavin weights given in previous sections. We observe that the increase in the number of the time steps improves the approximation. (See Error rate AE

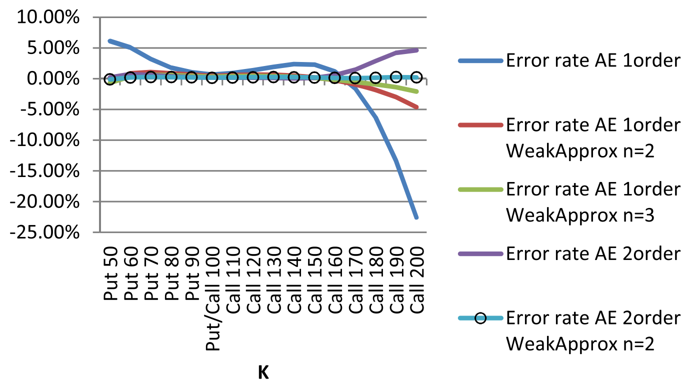

FIG. 1. $T = 1$: *Local volatility model, Error rates of the first- and second-order asymptotic expansions and their weak approximations.*

1order and Error rate AE 1order WeakApprox $n = 2, 3$ in Figure 1.) We also note that our scheme with the second-order expansion and two time steps (Error rate AE 2order WeakApprox $n = 2$) improves the base (analytical only) second-order expansion (Error rate AE 2order), and is able to provide an accurate approximation across all the strikes even for the long maturity case such as the 10-year maturity case in Figure 2.

5.2. *Stochastic volatility model.* The second example considers the following stochastic volatility model, which is also known as the log-normal SABR model:

$$dS_t^{(z,\sigma)} = \sigma_t^\sigma S_t^{(z,\sigma)}\,dB_t^1, \qquad S_0^{(z,\sigma)} = z, \tag{5.5}$$

$$d\sigma_t^\sigma = \nu\sigma_t^\sigma\bigl(\rho\,dB_t^1 + \sqrt{1-\rho^2}\,dB_t^2\bigr), \qquad \sigma_0^\sigma = \sigma. \tag{5.6}$$

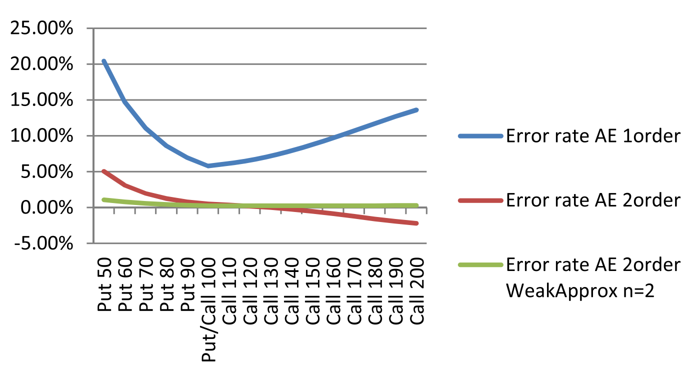

FIG. 2. $T = 10$: *Local volatility model, Error rates of the first- and second-order asymptotic expansions and the weak approximations.*

Next, let us introduce the following perturbed logarithmic SABR model:

$$(5.7)\qquad dX_{1,t}^{(x,\sigma),\varepsilon} = \varepsilon\Big[-\eta\frac{(\sigma_t^{\sigma,\varepsilon})^2}{2}\,dt + \eta\sigma_t^{\sigma,\varepsilon}\,dB_t^1\Big], \qquad X_{1,0}^{(x,\sigma),\varepsilon} = x,$$

$$(5.8)\qquad d\sigma_t^{\sigma,\varepsilon} = \varepsilon\big[\sigma_t^{\sigma,\varepsilon}\big(\rho\,dB_t^1 + \sqrt{1-\rho^2}\,dB_t^2\big)\big], \qquad \sigma_0^{\sigma,\varepsilon} = \sigma,$$

with $\varepsilon = \nu$ and $\eta = 1/\nu$. For some fixed $T > 0$ and $K > 0$, the target expectation is given by

$$\begin{aligned} &E\big[f\big(X_{1,T}^{(x,\sigma),\varepsilon}, \sigma_T^{\sigma,\varepsilon}\big)\big] \\ &\quad \equiv E\big[\hat{f}\big(X_{1,T}^{(x,\sigma),\varepsilon}\big)\big] \\ &\quad := E\big[\max\big\{e^{X_{1,T}^{(x,\sigma),\varepsilon}} - K, 0\big\}\big] \quad \text{or} \quad E\big[\max\big\{K - e^{X_{1,T}^{(x,\sigma),\varepsilon}}, 0\big\}\big]. \end{aligned}$$

Next, let $(\bar{X}_{1,t}^{(x,\sigma),\varepsilon}, \bar{\sigma}_t^{\sigma,\varepsilon})_{t\geq 0}$ be the solution to the following SDE:

$$(5.9)\qquad d\bar{X}_{1,t}^{(x,\sigma),\varepsilon} = \varepsilon\Big[-\eta\frac{\sigma^2}{2}\,dt + \eta\sigma\,dB_t^1\Big], \qquad \bar{X}_{1,0}^{(x,\sigma),\varepsilon} = x,$$

$$(5.10)\qquad d\bar{\sigma}_t^{\sigma,\varepsilon} = \varepsilon\big[\sigma\big(\rho\,dB_t^1 + \sqrt{1-\rho^2}\,dB_t^2\big)\big], \qquad \bar{\sigma}_0^{\sigma,\varepsilon} = \sigma.$$

The parameters are set to be $z = 100$, $\sigma = 0.3$, $\varepsilon = \nu = 0.1$, $\eta = 1/\nu$ and $\rho = -0.5$. The benchmark values are calculated by Monte Carlo simulations with 10^7 trials and 1000 time steps for the 1-year maturity case or 2000 times steps for the 2-year maturity case.

In this example, we use the first-order two-dimensional asymptotic expansion operator with two time steps, that is, $m = 1$ and $n = 2$. Then the calculation procedure corresponding to the one in the previous section is the following: first, set $t_0 = 0$, $t_1 = T/2$, $t_2 = T$ and $s = t_k - t_{k-1} = T/2$ $(k = 1, 2)$.

- For $(\bar{X}_{1,t_1}^{(x_1,\sigma_1),\varepsilon}, \bar{\sigma}_{1,t_1}^{\sigma_1,\varepsilon}) = (x_1, \sigma_1)$ at $t = t_1$,

$$(5.11)\qquad q_1(x_1,\sigma_1) = E\big[\hat{f}\big(\bar{X}_{1,s}^{(x_1,\sigma_1),\varepsilon}\big)\mathcal{M}^1\big(s, (x_1,\sigma_1), \big(\bar{X}_{1,s}^{(x_1,\sigma_1),\varepsilon}, \bar{\sigma}_s^{\sigma_1,\varepsilon}\big)\big)\big].$$

- At $t = t_0 = 0$,

$$(5.12)\qquad q_0(x,\sigma) = E\big[q_1\big(\bar{X}_{1,s}^{(x,\sigma),\varepsilon}, \bar{\sigma}_s^{\sigma,\varepsilon}\big)\mathcal{M}^1\big(s, (x,\sigma), \big(\bar{X}_{1,s}^{(x,\sigma),\varepsilon}, \bar{\sigma}_s^{\sigma,\varepsilon}\big)\big)\big].$$

Here, $\mathcal{M}^1(t, (x,\sigma), (x',\sigma'))$ is the two-dimensional Malliavin weight given by

$$\begin{aligned} &\mathcal{M}^1\big(t, (x,\sigma), (x',\sigma')\big) \\ &\quad = 1 + \varepsilon E\Big[H_{(1)}\Big(\Big(\frac{\partial}{\partial\varepsilon}X_{1,t}^{(x,\sigma),\varepsilon}\Big|_{\varepsilon=0}, \frac{\partial}{\partial\varepsilon}\sigma_t^{\sigma,\varepsilon}\Big|_{\varepsilon=0}\Big), \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}X_{1,t}^{(x,\sigma),\varepsilon}\Big|_{\varepsilon=0}\Big)\Big| \\ &\qquad\qquad \big(\bar{X}_{1,t}^{(x,\sigma),\varepsilon}, \bar{\sigma}_t^{\sigma,\varepsilon}\big) = (x',\sigma')\Big] \end{aligned}$$

$$+\varepsilon E\left[H_{(1)}\left(\left(\frac{\partial}{\partial\varepsilon}X_{1,t}^{(x,\sigma),\varepsilon}\Big|_{\varepsilon=0}, \frac{\partial}{\partial\varepsilon}\sigma_t^{\sigma,\varepsilon}\Big|_{\varepsilon=0}\right), \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}\sigma_t^{\sigma,\varepsilon}\Big|_{\varepsilon=0}\right)\Big| \\ (\bar{X}_{1,t}^{(x,\sigma),\varepsilon}, \bar{\sigma}_t^{\sigma,\varepsilon}) = (x', \sigma')\right].$$

Moreover, we remark that those Malliavin weights are obtained as closed forms as in the local volatility case.

Actually, at t_1 we need not implement (5.11), but just compute the first-order analytical asymptotic expansion for pricing options with the time-to-maturity $T/2$ and the initial value $(\bar{X}_{1,t_1}^{(x_1,\sigma_1),\varepsilon}, \bar{\sigma}_{t_1}^{\sigma_1,0}) = (x_1, \sigma_1)$. That is,

$$\hat{q}_1(x_1, \sigma_1) = E[\hat{f}(\bar{X}_{1,s}^{(x_1,\sigma_1),\varepsilon})\hat{\mathcal{M}}^1(s, (x_1, \sigma_1), \bar{X}_{1,s}^{(x_1,\sigma_1),\varepsilon})], \tag{5.13}$$

where $\hat{\mathcal{M}}^1(s, (x_1, \sigma_1), y) = 1 + \varepsilon\hat{\mathcal{M}}_{(1)}(s, (x_1, \sigma_1), y)$, and $\hat{\mathcal{M}}_{(1)}(s, (x_1, \sigma_1), y)$ stands for the first-order one-dimensional Malliavin weight:

$$\hat{\mathcal{M}}_{(1)}(s, (x_1, \sigma_1), y) \\ = E\left[H_{(1)}\left(\frac{\partial}{\partial\varepsilon}X_{1,s}^{(x_1,\sigma_1),\varepsilon}\Big|_{\varepsilon=0}, \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}X_{1,s}^{(x_1,\sigma_1),\varepsilon}\Big|_{\varepsilon=0}\right)\Big|\bar{X}_{1,s}^{(x_1,\sigma_1,\varepsilon)} = y\right]. \tag{5.14}$$

On the other hand, we apply a conditional expectation formula for multidimensional asymptotic expansions in Takahashi (1999) in order to evaluate the Malliavin weight $\mathcal{M}^1$ in (5.12).

Figures 3 and 4 show the results (the vertical axis in the figures is Error rate). Again, our scheme with (5.13) and (5.12) (Error rate AE 1st order WeakApprox $n = 2$) improves the base first-order expansion (Error rate AE 1st order) especially for the deep OTM calls and puts.

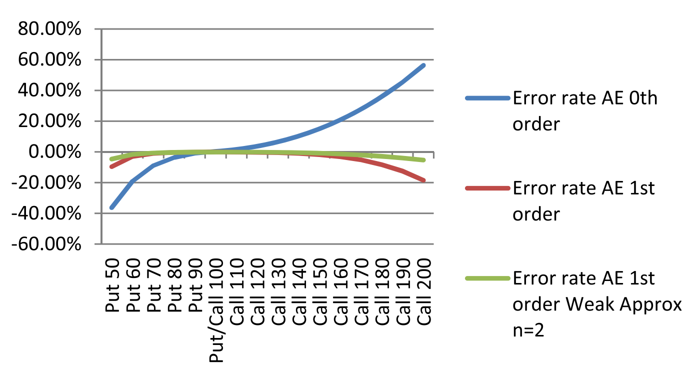

FIG. 3. $T = 1$: *Stochastic volatility model, Error rate of the first-order two-dimensional asymptotic expansion and the weak approximation.*

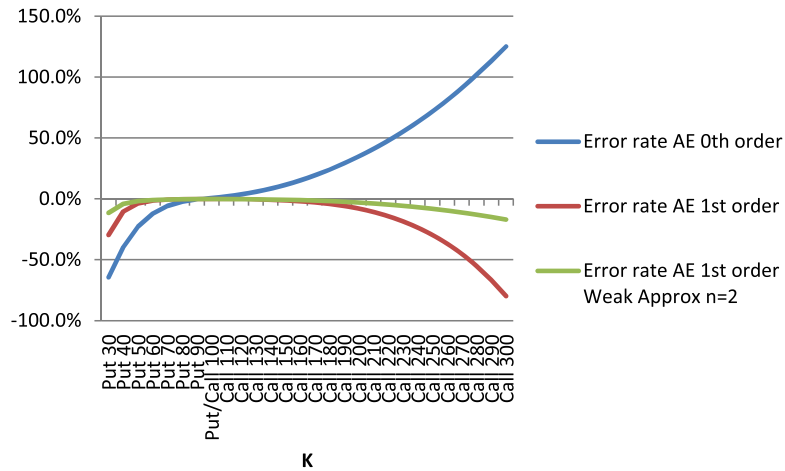

FIG. 4. $T = 2$: *Stochastic volatility model, Error rate of the first-order two-dimensional asymptotic expansion and the weak approximation.*

5.3. *Error analysis.* In this section, we investigate the validity of our approximation by comparing the theoretical and the numerical errors.

In particular, we use the same example of our weak approximation in the local volatility model with the maturity $T = 1$ in Section 5.1. Here, we remark that this example can be regarded as the Lipschitz continuous case in Theorem 3.

Based on the results of Theorem 3, for a fixed expansion order m and a time grid parameter $\gamma = 1$, the error of the weak approximation by the mth order asymptotic expansion with discretization n approximately satisfies the following relation:

$$\begin{aligned} &\text{Error Weak Approx with Asymp Expansion } (m, n+1) \\ &\quad \simeq \{\text{Error Weak Approx with Asymp Expansion } (m, n)\} \\ &\qquad \times \bigl(n/(n+1)\bigr)^{m/2}. \end{aligned} \tag{5.15}$$

Here, "*Error Weak Approx with Asymp Expansion* (m, n)" stands for the deviation of "Weak Approximation" from "Benchmark Monte Carlo," that is the value of (WeakApprox) − (Benchmark MC).

Figure 5 checks the above relation in the case that $m = 1$ and $n = 2$. In the figure, "*Theoretical Error*: *AE* 1*order WeakApprox* $n = 3$" is calculated by equation (5.15). It is observed that the order of the theoretical error is rather similar to that of the numerical error "*Error*: *AE* 1*order WeakApprox* $n = 3$" across all the strike prices.

Next, let us check the validity of our method from another viewpoint. For a fixed partition number n and a time grid parameter $\gamma = 1$, the error of the weak approximation based on $(m + 1)$th order asymptotic expansion with discretization

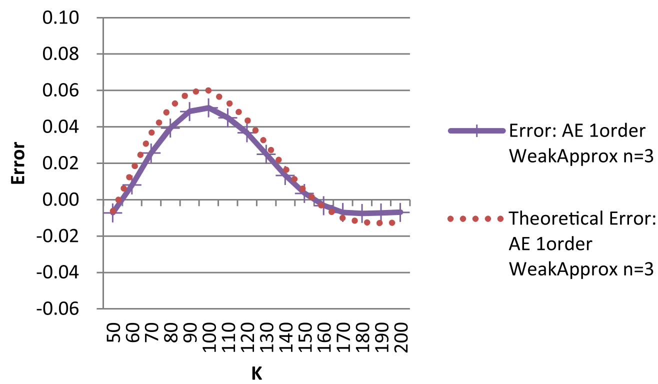

FIG. 5. *Error with respect to n of the weak approximation for fixed m.*

n labeled by "*Error Weak Approx with Asymp Expansion* $(m+1, n)$," approximately satisfies the relation:

$$\begin{aligned} &\text{Error Weak Approx with Asymp Expansion } (m+1, n) \\ &\quad \simeq \{\text{Error Weak Approx with Asymp Expansion } (m, n)\} \times \varepsilon/\sqrt{n}. \end{aligned} \tag{5.16}$$

Figure 6 examines the above relation in the case $m = 1$ and $n = 2$ with $\varepsilon = 0.4$.

"*Theoretical Error*: *AE* 2*order WeakApprox* $n = 2$" in Figure 6 is calculated by the equation (5.16). We observe that the order of the theoretical error is very close to that of the numerical error "*Error*: *AE* 2*order WeakApprox* $n = 2$" for all the strike prices.

Finally, we test the numerical errors by changing the parameter γ. Again, we fix the parameter $m = 1$. Based on the result of the Lipschitz continuous case in our main theorem (Theorem 3), the errors depend on the range of γ, that is, $\gamma < 1/3 = m/(m+2)$, $\gamma = 1/3 = m/(m+2)$, $\gamma > 1/3 = m/(m+2)$.

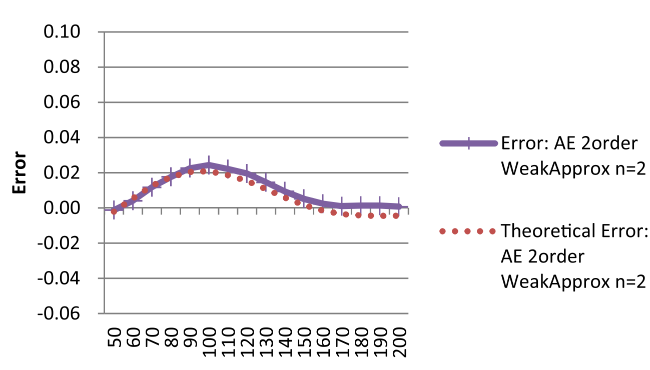

FIG. 6. *Error with respect to m of the weak approximation for fixed n.*

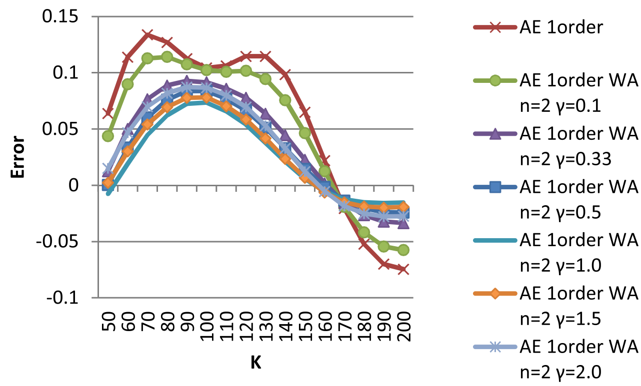

FIG. 7. *Error of the weak approximation $n = 2$ with respect to time grid parameter γ.*

In order to see the differences of the errors with the different values of γ, Figures 7 and 8 plot the errors for $\gamma = 0.1$, $\gamma = 0.33$, $\gamma = 0.5$, $\gamma = 1.0$, $\gamma = 1.5$ and $\gamma = 2.0$ with $n = 2$ and $n = 3$, respectively.

We are able to find that the errors are determined by the levels of γ and the behavior of the errors is consistent with the theoretical results in Theorem 3.

In addition, we examine which time grid parameter γ is optimal. In order to show this, we execute a simple test for the case $m = 1$ and $n = 2$. Particularly, we solve the following minimization problem:

$$\hat{\gamma} = \operatorname{argmin}\left\{ \sum_{K \in \{50,60,\dots,190,200\}} \left(\text{Error Weak Approx with Asymp Expansion } (m, n; \gamma, K)\right)^2 \right\}. \tag{5.17}$$

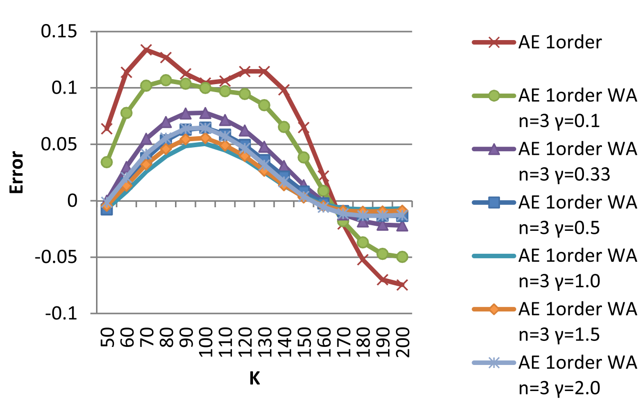

FIG. 8. *Error of the weak approximation $n = 3$ with respect to time grid parameter γ.*

We obtained a parameter $\hat{\gamma} = 1.015657$. That is, the value close to $\gamma = 1$ (the uniform time grid case) is optimal in our weak approximation with asymptotic expansion.

Therefore, we can conclude that the results of the numerical experiments of our weak approximation are consistent with the theoretical part of this paper, and confirmed the validity of our method.

6. Concluding remarks. In this paper, we have shown a new approximation method for the expectations of the functions of the solutions to SDEs by applying an asymptotic expansion with Malliavin calculus. In particular, based on Kusuoka (2001, 2003a, 2003b, 2004), we have obtained error estimates for our new weak approximation.

Moreover, we have confirmed the validity of our method through the numerical examples for option pricing under local and stochastic volatility models. The scheme is simple and we can attain enough accuracy even when the expansion order m is low such as $m = 1, 2$ with a few time steps $n = 2, 3$ as demonstrated in the previous section.

In order to obtain more accurate numerical approximation, it is natural to use many partitions n in the time scale. However, the computational cost becomes exponentially larger as the number of partitions becomes larger.

To overcome this problem, some efficient tree based (discretization) techniques can be applied. Another possible solution is to use the higher order expansion developed in Takahashi, Takehara and Toda (2012) or Violante (2012). We are convinced that the higher order expansion will improve the accuracy since the higher order mth expansion improves the error orders to $O(\varepsilon^{m+1}/n^m)$ for a Lipschitz continuous f and $O(\varepsilon^{m+1}/n^{m-1})$ for a bounded Borel f.

Further, applying our method to the higher-dimensional problems is one of the important issues. When the dimension N of the state variables becomes higher, the computational cost becomes larger. However, the multidimensional higher order expansion such as in Takahashi (1999) or Takahashi, Takehara and Toda (2012) is a tractable approach to the extension. These topics will be the main themes in our next research.

APPENDIX A: PROOF OF THEOREM 1

First, for the preparation for the proof of the theorem, we characterize the differentiations of the solution to the general perturbed SDEs $X_t^{x,\varepsilon}$ with respect to ε as elements in the space $\mathcal{K}_r$. The following lemma plays an important rule for estimating the order of the local approximation for $E[f(X_t^{x,\varepsilon})]$ in Theorem 1.

LEMMA 2.

$$\frac{1}{j!}\frac{\partial^j}{\partial\varepsilon^j}X_t^{x,\varepsilon} \in \mathcal{K}_j, \qquad j \geq 1.$$

PROOF. We prove the assertion by induction. First, the differentiation of $X_t^{x,\varepsilon}$ with respect to ε is given by

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon,l} &= \int_0^t \frac{\partial}{\partial\varepsilon} V_0^l(\varepsilon, X_s^{x,\varepsilon})\,ds + \sum_{j=1}^d \int_0^t V_j^l(X_s^{x,\varepsilon})\,dB_s^j \\
&\quad + \sum_{k=1}^N \int_0^t \partial_k V_0^l(\varepsilon, X_s^{x,\varepsilon}) \frac{\partial}{\partial\varepsilon} X_s^{x,\varepsilon,k}\,ds \\
&\quad + \varepsilon \sum_{k=1}^N \sum_{j=1}^d \int_0^t \partial_k V_j^l(X_s^{x,\varepsilon}) \frac{\partial}{\partial\varepsilon} X_s^{x,\varepsilon,k}\,dB_s^j, \qquad l = 1, \dots, N.
\end{aligned} \tag{A.1}
$$

The above SDE is linear and the order of the Kusuoka–Stroock function $\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}$ is determined by the following term:

$$
\sum_{j=1}^d \int_0^t J_t^{x,\varepsilon} (J_u^{x,\varepsilon})^{-1} V_j(X_u^{x,\varepsilon})\,dB_u^j \in \mathcal{K}_1, \tag{A.2}
$$

where $J_t^{x,\varepsilon} = \nabla_x X_t^{x,\varepsilon}$. Since this term gives the minimum order in the terms that consist of (A.1). Here, we use the properties $J_s^{x,\varepsilon}$, $(J_s^{x,\varepsilon})^{-1} \in \mathcal{K}_0$, $s \in (0, 1]$ and the boundness of V_j, $j = 1, \dots, d$. We have $\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon} \in \mathcal{K}_1$ by using the properties 2 and 3 in Lemma 1.

For $i \geq 2$, $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} X_t^{x,\varepsilon} = (\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} X_t^{x,\varepsilon,1}, \dots, \frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} X_t^{x,\varepsilon,N})$ is recursively determined by the following:

$$
\begin{aligned}
&\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} X_t^{x,\varepsilon,n} \\
&\quad = \frac{1}{i!} \int_0^t \frac{\partial^i}{\partial\varepsilon^i} V_0^n(\varepsilon, X_u^{x,\varepsilon})\,du \\
&\qquad + \sum_{m=1}^i \sum_{i^{(k)},\alpha^{(k)}}^{(m)} \frac{1}{(i-m)!} \int_0^t \left(\prod_{l=1}^k \frac{1}{i_l!} \frac{\partial^{i_l}}{\partial\varepsilon^{i_l}} X_u^{x,\varepsilon,\alpha_l} \right) \partial_{\alpha^{(k)}} \frac{\partial^{i-m}}{\partial\varepsilon^{i-m}} V_0^n(\varepsilon, X_u^{x,\varepsilon})\,du \\
&\qquad + \sum_{i^{(k)},\alpha^{(k)}}^{(i-1)} \int_0^t \left(\prod_{l=1}^k \frac{1}{i_l!} \frac{\partial^{i_l}}{\partial\varepsilon^{i_l}} X_u^{x,\varepsilon,\alpha_l} \right) \sum_{j=1}^d \partial_{\alpha^{(k)}} V_j^n(X_u^{x,\varepsilon})\,dB_u^j \\
&\qquad + \varepsilon \sum_{i^{(\beta)},\alpha^{(k)}}^{(i)} \int_0^t \left(\prod_{l=1}^k \frac{1}{i_l!} \frac{\partial^{i_l}}{\partial\varepsilon^{i_l}} X_u^{x,\varepsilon,\alpha_l} \right) \sum_{j=1}^d \partial_{\alpha^{(k)}} V_j^n(X_u^{x,\varepsilon})\,dB_u^j, \\
&\qquad\qquad n = 1, \dots, N,
\end{aligned} \tag{A.3}
$$

where

$$\text{(A.4)}\qquad \sum_{i^{(k)},\alpha^{(k)}}^{(i)} := \sum_{k=1}^{i} \sum_{i_1+\cdots+i_k=i, i_l\ge 1} \sum_{\alpha^{(k)}\in\{1,\ldots,N\}^k} \frac{1}{k!}.$$

The above SDE is linear and the order of the Kusuoka–Stroock function $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_t^{x,\varepsilon}$ is determined inductively by the term

$$\text{(A.5)}\qquad \sum_{i^{(k)},\alpha^{(k)}}^{(i-1)} \int_0^t J_t^{x,\varepsilon}\big(J_u^{x,\varepsilon}\big)^{-1}\left(\prod_{l=1}^{k}\frac{1}{i_l!}\frac{\partial^{i_l}}{\partial\varepsilon^{i_l}}X_u^{x,\varepsilon,\alpha_l}\right)\sum_{j=1}^{d}\partial_{\alpha^{(k)}}V_j\big(X_u^{x,\varepsilon}\big)\,dB_s^j \in \mathcal{K}_i.$$

Since this term gives the minimum order in the terms that consist of (A.3). Then $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_t^{x,\varepsilon} \in \mathcal{K}_i$ by using the properties 2 and 3 in Lemma 1. □

Hereafter, we give the expansion for $E[f(X_t^{x,\varepsilon})]$ around $E[f(\bar{X}_t^{x,\varepsilon})]$. We remark that $X_t^{x,\varepsilon}$ is not uniformly nondegenerate Wiener functional in Watanabe sense because $X_t^{x,0}$ is completely degenerate as Wiener functional, that is, $X_t^{x,0}$ is the solution to ODE. Then, in order to give the expansion, we define a Wiener functional Y_t^ε given by $Y_t^\varepsilon = \varphi(X_t^{x,\varepsilon}) = \frac{X_t^{x,\varepsilon}-X_t^{x,0}}{\varepsilon}$, that is, $(Y_t^{\varepsilon,1},\ldots,Y_t^{\varepsilon,N}) = (\varphi_1(X_t^{x,\varepsilon,1}),\ldots,\varphi_N(X_t^{x,\varepsilon,N}))$, $\varphi_i(\xi) = \frac{\xi - X_t^{x,0,i}}{\varepsilon}$, $i=1,\ldots,N$. The expansion of Y_t^ε is given in the space $\mathbf{D}^\infty$, that is, for all $m\in\mathbf{N}$,

$$\text{(A.6)}\qquad \begin{aligned}&\limsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{m+1}}\left\| Y_t^\varepsilon - \left\{\frac{\partial}{\partial\varepsilon}X_t^{x,\varepsilon}\Big|_{\varepsilon=0} + \sum_{i=1}^{m}\varepsilon^i\frac{1}{(i+1)!}\frac{\partial^{i+1}}{\partial\varepsilon^{i+1}}X_t^{x,\varepsilon}\Big|_{\varepsilon=0}\right\}\right\|_{\mathbf{D}^{k,p}}\\ &\quad<\infty,\qquad \forall k\in\mathbf{N}, \forall p<\infty.\end{aligned}$$

We note that $Y_t^0 = \frac{\partial}{\partial\varepsilon}X_t^{x,\varepsilon}|_{\varepsilon=0}$ and $Y_0^0=0$. Let $\sigma^{Y_t^\varepsilon}$ be the Malliavin covariance matrix of Y_t^ε and set

$$\text{(A.7)}\qquad \tau = \inf\big\{s; \big(J_s^{x,\varepsilon}\big)^{-1}A\big(X_s^{x,\varepsilon}\big)\big(\big(J_s^{x,\varepsilon}\big)^{-1}\big)^\top \le A(x)/2\big\}.$$

Then we can see

$$\text{(A.8)}\qquad \begin{aligned}\det\big(\sigma^{Y_t^\varepsilon}\big) &\ge \det\big(J_t^{x,\varepsilon}\big)^2\det\int_0^{\min\{t,\tau\}}\big(J_s^{x,\varepsilon}\big)^{-1}A\big(X_s^{x,\varepsilon}\big)\big(\big(J_s^{x,\varepsilon}\big)^{-1}\big)^\top ds\\ &\ge \big(1/2^N\big)\det\big(J_t^{x,\varepsilon}\big)^2\det\big(A(x)\big)\min\{t,\tau\}^N,\end{aligned}$$

$$\text{(A.9)}\qquad \sup_{\varepsilon\in(0,1]}\big\|\det\big(J_t^{x,\varepsilon}\big)^{-1}\big\|_{L^p}<\infty,$$

and

$$\text{(A.10)}\qquad P(\tau<1/n)\le c_1\exp\big(-c_2 n^{c_3}\big),\qquad n\in\mathbf{N},$$

where c_i, $i = 1, 2, 3$ are positive constants [see the proofs of Theorem 3.4 of Watanabe (1987) or Theorem 10.5 of Ikeda and Watanabe (1989) for (A.8), (A.9) and (A.10)].

Therefore, under condition [**H**], we can see the nondegeneracy of the Malliavin covariance matrix of Y_t^ε

$$\sup_{\varepsilon\in(0,1]} \|\det(\sigma^{Y_t^\varepsilon})^{-1}\|_{L^p} < \infty, \qquad p < \infty. \tag{A.11}$$

Then the density $\xi \mapsto p_t^{Y^\varepsilon}(\xi)$ of Y_t^ε starting from 0 is smooth. Moreover, the Malliavin covariance matrix $\sigma^{Y_t^\varepsilon}$ is nondegenerate uniformly in ε:

$$\lim_{\varepsilon\downarrow 0}\sup \|\det(\sigma^{Y_t^\varepsilon})^{-1}\|_{L^p} = \|\det(\sigma^{Y_t^0})^{-1}\|_{L^p} < \infty, \qquad p < \infty. \tag{A.12}$$

Then we are able to give the following Taylor formulas for $\xi \mapsto p_t^{Y^\varepsilon}(\xi)$ and $E[f(Y_t^\varepsilon)]$ using the Malliavin weights:

$$\begin{aligned} p_t^{Y^\varepsilon}(\xi) &= p_t^{Y^0}(\xi) + \sum_{j=1}^m \varepsilon^j E[\Phi_t^j | Y_t^0 = \xi] p_t^{Y^0}(\xi) \\ &\quad + \varepsilon^{m+1} \int_0^1 (1-u)^m (m+1) \\ &\quad \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[H_{\alpha^{(k)}}\left(Y_t^{\varepsilon u}, \prod_{l=1}^k \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\Big|_{\eta=\varepsilon u}\right) \Big| Y_t^{\varepsilon u} = \xi \right] \\ &\quad \times p_t^{Y^{\varepsilon u}}(\xi)\, du, \end{aligned} \tag{A.13}$$

$$\begin{aligned} E[f(Y_t^\varepsilon)] &= \int_{\mathbf{R}^N} f(\xi) p_t^{Y^\varepsilon}(\xi)\, d\xi \\ &= \int_{\mathbf{R}^N} f(\xi) p_t^{Y^0}(\xi)\, d\xi + \sum_{j=1}^m \varepsilon^j \int_{\mathbf{R}^N} f(\xi) E[\Phi_t^j | Y_t^0 = \xi] p_t^{Y^0}(\xi)\, d\xi \\ &\quad + \varepsilon^{m+1} \int_0^1 (1-u)^m (m+1) \\ &\quad \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} \int_{\mathbf{R}^N} f(\xi) E\left[H_{\alpha^{(k)}}\left(Y_t^{\varepsilon u}, \prod_{l=1}^k \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\Big|_{\eta=\varepsilon u}\right) \Big| \right. \\ &\qquad\qquad \left. Y_t^{\varepsilon u} = \xi \right] p_t^{Y^{\varepsilon u}}(\xi)\, d\xi\, du \\ &= E[f(Y_t^0)] + \sum_{j=1}^m \varepsilon^j E[f(Y_t^0)\Phi_t^j] + \varepsilon^{m+1} r_m(t,x,\varepsilon). \end{aligned} \tag{A.14}$$

Here, Φ_t^j is the Malliavin weight given by

$$\Phi_t^j = \sum_{\alpha^{(k)},\beta^{(k)}}^{j} H_{\alpha^{(k)}}\left(Y_t^0, \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \varepsilon^{\beta_l}} X_t^{x,\varepsilon,\alpha_l}\bigg|_{\varepsilon=0}\right), \tag{A.15}$$

with

$$\sum_{\alpha^{(k)},\beta^{(k)}}^{j} = \sum_{k=1}^{j} \sum_{\sum_{l=1}^{k} \beta_l = j+k, \beta_l \geq 2} \sum_{\alpha^{(k)}=(\alpha_1,\dots,\alpha_k)\in\{1,\dots,N\}^k} \frac{1}{k!} \tag{A.16}$$

and $r_m(t, x, \varepsilon)$ is the residual:

1.

$$\begin{aligned} r_m(t,x,\varepsilon) &= \int_0^1 (1-u)^m (m+1) \\ &\quad \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[\partial_{\alpha^{(k)}} f\left(Y_t^{\varepsilon u}\right) \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\bigg|_{\eta=\varepsilon u}\right] du \end{aligned} \tag{A.17}$$

for $f \in C_b^\infty(\mathbf{R}^N)$,

2.

$$\begin{aligned} r_m(t,x,\varepsilon) &= \int_0^1 (1-u)^m (m+1) \\ &\quad \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[\partial_{\alpha^{(1)}} f\left(Y_t^{\varepsilon u}\right) H_{\alpha^{(k-1)}}\left(Y_t^{\varepsilon u}, \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\bigg|_{\eta=\varepsilon u}\right)\right] du \end{aligned} \tag{A.18}$$

for $f \in C_b^1(\mathbf{R}^N)$,

3.

$$\begin{aligned} r_m(t,x,\varepsilon) &= \int_0^1 (1-u)^m (m+1) \\ &\quad \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[f\left(Y_t^{\varepsilon u}\right) H_{\alpha^{(k)}}\left(Y_t^{\varepsilon u}, \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\bigg|_{\eta=\varepsilon u}\right)\right] du \end{aligned} \tag{A.19}$$

for an arbitrary bounded continuous function f.

Then, by the transformation $X_t^{x,\varepsilon} = X_t^{x,0} + \varepsilon Y_t^\varepsilon$, the density $y \mapsto p^{X^\varepsilon}(t,x,y)$ of $X_t^{x,\varepsilon}$ is given by

$$p^{X^\varepsilon}(t,x,y) = p_t^{Y^\varepsilon}(\varphi(y)) \det\left|\frac{\partial(\varphi_1,\dots,\varphi_N)}{\partial(y_1,\dots,y_N)}\right| \tag{A.20}$$

$$= p_t^{Y^\varepsilon}((y - X_t^{x,0})/\varepsilon)\frac{1}{\varepsilon^N}. \tag{A.21}$$

Here, we note that

$$\begin{aligned} &\int_{\mathbf{R}^N} f(y) p_t^{Y^0}((y - X_t^{x,0})/\varepsilon)\frac{1}{\varepsilon^N}\,dy \\ &\quad = \int_{\mathbf{R}^N} f(y) \frac{1}{(2\pi\varepsilon^2)^{N/2}\det(\Sigma(t))^{1/2}} \\ &\qquad\quad \times e^{-(y-\varepsilon\mu(t)-X_t^{x,0})^\top \Sigma^{-1}(t)(y-\varepsilon\mu(t)-X_t^{x,0})/(2\varepsilon^2)}\,dy \\ &\quad = \int_{\mathbf{R}^N} f(y) p^{\bar{X}^\varepsilon}(t,x,y)\,dy = E[f(\bar{X}_t^{x,\varepsilon})], \end{aligned} \tag{A.22}$$

where $\mu(t)$ and $\Sigma(t)$ are the mean and the covariance matrix of Y_t^0 and $y \mapsto p^{\bar{X}^\varepsilon}(t,x,y)$ is the density of $\bar{X}_t^{x,\varepsilon}$. Also, for $G(t,x) \in \mathcal{K}_r$, we have

$$\begin{aligned} &\int_{\mathbf{R}^N} f(y) E[H_{(i)}(Y_t^0, G(t,x)) | Y_t^0 = (y - X_t^{x,0})/\varepsilon] \\ &\quad \times p_t^{Y^0}((y - X_t^{x,0})/\varepsilon)\frac{1}{\varepsilon^N}\,dy \\ &\quad = \int_{\mathbf{R}^N} f(y) E[H_{(i)}(Y_t^0, G(t,x)) | \bar{X}_t^{x,\varepsilon} = y] p^{\bar{X}^\varepsilon}(t,x,y)\,dy \\ &\quad = E[f(\bar{X}_t^{x,\varepsilon}) H_{(i)}(Y_t^0, G(t,x))], \end{aligned} \tag{A.23}$$

and

$$\begin{aligned} &\int_{\mathbf{R}^N} f(y) E[H_{(i)}(Y_t^{\varepsilon u}, G(t,x)) | Y_t^{\varepsilon u} = (y - X_t^{x,0})/\varepsilon] \\ &\quad \times p_t^{Y^{\varepsilon u}}((y - X_t^{x,0})/\varepsilon)\frac{1}{\varepsilon^N}\,dy \\ &\quad = E[f(\tilde{X}_t^{x,\varepsilon u}) H_{(i)}(Y_t^{\varepsilon u}, G(t,x))], \end{aligned} \tag{A.24}$$

with $\tilde{X}_t^{x,\varepsilon u} = X_t^{x,0} + \varepsilon Y_t^{\varepsilon u}$, $u \in [0,1]$.

Therefore, (A.14) with (A.17), (A.18) and (A.19) can be transformed into

$$E[f(X_t^{x,\varepsilon})] = E[f(\bar{X}_t^{x,\varepsilon})] + \sum_{i=1}^m \varepsilon^i E[f(\bar{X}_t^{x,\varepsilon})\Phi_t^j] + \varepsilon^{m+1} R_m(t,x,\varepsilon),$$

where:

1.

$$R_m(t,x,\varepsilon) = \int_0^1 (1-u)^m (m+1) \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[\partial_{\alpha^{(k)}} f(\tilde{X}_t^{x,\varepsilon u}) \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\bigg|_{\eta=\varepsilon u}\right] du \tag{A.25}$$

for $f \in C_b^\infty(\mathbf{R}^N)$,

2.

$$R_m(t,x,\varepsilon) = \int_0^1 (1-u)^m (m+1) \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[\partial_{\alpha^{(1)}} f(\tilde{X}_t^{x,\varepsilon u}) H_{\alpha^{(k-1)}}\left(Y_t^{\varepsilon u}, \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\bigg|_{\eta=\varepsilon u}\right)\right] du \tag{A.26}$$

for $f \in C_b^1(\mathbf{R}^N)$,

3.

$$R_m(t,x,\varepsilon) = \int_0^1 (1-u)^m (m+1) \times \sum_{\alpha^{(k)},\beta^{(k)}}^{m+1} E\left[f(\tilde{X}_t^{x,\varepsilon u}) H_{\alpha^{(k)}}\left(Y_t^{\varepsilon u}, \prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \eta^{\beta_l}} X_t^{x,\eta,\alpha_l}\bigg|_{\eta=\varepsilon u}\right)\right] du \tag{A.27}$$

for an arbitrary bounded continuous function f.

For $k \le m+1$, $\sum_{l=1}^k \beta_l = m+1+k$, $\beta_l \ge 2$, $\alpha^{(k)} = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, N\}^k$, the product of the higher derivative terms with respect to ε of $X_t^{x,\varepsilon}$ is characterized as

$$\prod_{l=1}^{k} \frac{1}{\beta_l!} \frac{\partial^{\beta_l}}{\partial \varepsilon^{\beta_l}} X_t^{x,\varepsilon,\alpha_l} \in \mathcal{K}_{m+1+k}, \tag{A.28}$$

by using Lemma 2 with Lemma 1.

For $i = 1, \dots, N$ and $G(t,x) \in \mathcal{K}_r$, we are able to see the following property for Malliavin weight as in Proposition 1:

$$H_{(i)}(Y_t^\varepsilon, G(t,x)) = \delta\left(\sum_{j=1}^N G(t,x) \gamma_{ij}^{Y_t^\varepsilon} DY_t^{\varepsilon,j}\right) \tag{A.29}$$

$$= \Bigg[G(t,x) \sum_{j=1}^{N} \sum_{k=1}^{d} \int_0^t \gamma_{ij}^{Y_t^\varepsilon} \big(J_t^{x,\varepsilon} (J_s^{x,\varepsilon})^{-1} V_k(X_s^{x,\varepsilon})\big)^j \, dB_s^k$$

$$- \sum_{j=1}^{N} \sum_{k=1}^{d} \int_0^t [D_{s,k} G(t,x)] \gamma_{ij}^{Y_t^\varepsilon} \big(J_t^{x,\varepsilon} (J_s^{x,\varepsilon})^{-1} V_k(X_s^{x,\varepsilon})\big)^j \, ds \Bigg]$$

$$\in \mathcal{K}_{r-1}.$$

Here, the first and the second terms in the second equality are characterized by

$$G(t,x) \sum_{j=1}^{N} \sum_{k=1}^{d} \int_0^t \gamma_{ij}^{Y_t^\varepsilon} \big(J_t^{x,\varepsilon} (J_s^{x,\varepsilon})^{-1} V_k(X_s^{x,\varepsilon})\big)^j \, dB_s^k \in \mathcal{K}_{r-1}, \tag{A.30}$$

$$\int_0^t [D_{s,k} G(t,x)] \gamma_{ij}^{Y_t^\varepsilon} \big(J_t^{x,\varepsilon} (J_s^{x,\varepsilon})^{-1} V_k(X_s^{x,\varepsilon})\big)^j \, ds \in \mathcal{K}_r, \tag{A.31}$$

since

$$\int_0^t \gamma_{ij}^{Y_t^\varepsilon} \big(J_t^{x,\varepsilon} (J_s^{x,\varepsilon})^{-1} V_k(X_s^{x,\varepsilon})\big)^j \, dB_s^k \in \mathcal{K}_{-2+1} = \mathcal{K}_{-1}. \tag{A.32}$$

Then, applying (A.29) with (A.28) for (A.25), (A.26) and (A.27), we obtain the following estimates according to the smoothness of f:

1.

$$\sup_{x \in \mathbf{R}^N} |R_m(t,x,\varepsilon)| \le C \Bigg(\sum_{k=1}^{m+1} t^{(m+1+k)/2} \|\nabla^k f\|_\infty \Bigg), \tag{A.33}$$

for any $f \in C_b^\infty(\mathbf{R}^N)$,

2.

$$\sup_{x \in \mathbf{R}^N} |R_m(t,x,\varepsilon)| \le C t^{(m+2)/2} \|\nabla f\|_\infty, \tag{A.34}$$

for any $f \in C_b^1$,

3.

$$\sup_{x \in \mathbf{R}^N} |R_m(t,x,\varepsilon)| \le C t^{(m+1)/2} \|f\|_\infty, \tag{A.35}$$

for an arbitrary bounded continuous function f.

Then we have the assertion.

APPENDIX B: PROOF OF THEOREM 2

For $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, we have

$$\int_{\mathbf{R}^N} f(y) E\Big[H_{(i)}\Big(\frac{\partial}{\partial \varepsilon} X_t^{x,\varepsilon}\Big|_{\varepsilon=0}, G(t,x) \Big) \Big| \bar{X}_t^{x,\varepsilon} = y \Big] \nu(dy) \tag{B.1}$$

$$
\begin{aligned}
&= E\left[f(\bar{X}_t^{x,\varepsilon}) H_{(i)}\left(\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0}, G(t,x)\right)\right] \\
&= E\left[f(\bar{X}_t^{x,\varepsilon}) \delta\left(\sum_{j=1}^N G(t,x)\gamma_{ij}^{Y_t^0} DY_t^{0,j}\right)\right] \\
&= E\left[f(\bar{X}_t^{x,\varepsilon}) \delta\left(\sum_{j=1}^N \varepsilon G(t,x)\frac{1}{\varepsilon^2}\gamma_{ij}^{Y_t^0} \varepsilon DY_t^{0,j}\right)\right] \\
&= E\left[f(\bar{X}_t^{x,\varepsilon}) \delta\left(\sum_{j=1}^N \varepsilon G(t,x)\gamma_{ij}^{\bar{X}_t^{x,\varepsilon}} D\bar{X}_t^{x,\varepsilon,j}\right)\right] \\
&= E[f(\bar{X}_t^{x,\varepsilon}) H_{(i)}(\bar{X}_t^{x,\varepsilon}, \varepsilon G(t,x))] \\
&= E[\partial_i f(\bar{X}_t^{x,\varepsilon}) \varepsilon G(t,x)] \\
&= \int_{\mathbf{R}^N} \partial_i f(y) E[\varepsilon G(t,x)|\bar{X}_t^{x,\varepsilon} = y]\nu(dy)
\end{aligned}
$$

$$
\text{(B.2)} \qquad = \int_{\mathbf{R}^N} f(y)\partial_i^* E[\varepsilon G(t,x)|\bar{X}_t^{x,\varepsilon} = y]\nu(dy),
$$

where $\gamma^{\bar{X}_t^{x,\varepsilon}} = (\gamma_{ij}^{\bar{X}_t^{x,\varepsilon}})_{1\le i,j\le N}$ and $\gamma^{Y_t^0} = (\gamma_{ij}^{Y_t^0})_{1\le i,j\le N}$ are the inverse matrices of the Malliavin covariance matrices of $\bar{X}_t^{x,\varepsilon}$ and Y_t^0, respectively. Here, we note that $Y_t^0 = \frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}|_{\varepsilon=0}$ and $\bar{X}_t^{x,\varepsilon} = X_t^{x,0} + \varepsilon\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}|_{\varepsilon=0} = X_t^{x,0} + \varepsilon Y_t^0$. Also, we use the following relations in the above equations; for $k = 1,\dots,d$ and $j = 1,\dots,N$,

$$
\text{(B.3)} \qquad D_{s,k}\bar{X}_t^{x,\varepsilon,j} = \varepsilon D_{s,k}\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon,j}\bigg|_{\varepsilon=0} = \varepsilon D_{s,k} Y_t^{0,j}, \qquad s \le t,
$$

and, for $i, j = 1,\dots,N$,

$$
\text{(B.4)} \qquad \gamma_{ij}^{\bar{X}_t^{x,\varepsilon}} = \frac{1}{\varepsilon^2}\gamma_{ij}^{Y_t^0}.
$$

Formulas (B.1) and (B.2) hold for any Lipschitz and bounded Borel function f by using mollifier arguments. We remark that in general for any $G \in \mathbf{D}^\infty$ and nondegenerate $F \in \mathbf{D}^\infty(\mathbf{R}^N)$, the conditional expectation can be regarded as a map $\mathbf{D}^\infty \ni G \mapsto E[G|F = \cdot] \in \mathcal{S}(\mathbf{R}^N)$ by Malliavin (1997) and Malliavin and Thalmaier (2006). Therefore, for $k = 1,\dots,j \le m$, $\sum_{l=1}^k \beta_l = j + k$, $\beta_l \ge 2$, $\alpha^{(k)} = (\alpha_1,\dots,\alpha_k) \in \{1,\dots,N\}^k$, we have

$$
\text{(B.5)} \qquad
\begin{aligned}
&E\left[H_{\alpha^{(k)}}\left(\frac{\partial}{\partial\varepsilon} X_t^{x,\varepsilon}\bigg|_{\varepsilon=0}, \prod_{l=1}^k \frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}} X_t^{x,\varepsilon,\alpha_l}\bigg|_{\varepsilon=0}\right)\bigg|\bar{X}_t^{x,\varepsilon} = y\right] \\
&\quad = \varepsilon^k \partial_{\alpha_k}^* \circ \partial_{\alpha_{k-1}}^* \circ \cdots \circ \partial_{\alpha_1}^* E\left[\prod_{l=1}^k \frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}} X_t^{x,\varepsilon,\alpha_l}\bigg|_{\varepsilon=0}\bigg|\bar{X}_t^{x,\varepsilon} = y\right],
\end{aligned}
$$

and obtain the assertion.

APPENDIX C: PROOF OF THEOREM 3

We follow the similar argument as in Kusuoka (2001, 2003b, 2004) and Chapter 3 of Crisan, Manolarakis and Nee (2013).

Note first that we have the following equality:

$$\begin{aligned}
&P_T f(x) - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f(x)\\
&\quad = P_{T-t_{n-1}} P_{t_{n-1}} f(x) - Q^m_{(s_n)} P_{t_{n-1}} f(x)\\
&\qquad + Q^m_{(s_n)} P_{t_{n-1}} f(x) - Q^m_{(s_n)} Q^m_{(s_{n-1})} P_{t_{n-2}} f(x)\\
&\qquad + \cdots\\
&\qquad + Q^m_{(s_n)} \cdots Q^m_{(s_2)} P_{t_1} f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\\
&\quad = P_{T-t_{n-1}} P_{t_{n-1}} f(x) - Q^m_{(s_n)} P_{t_{n-1}} f(x)\\
&\qquad + Q^m_{(s_n)} \big(P_{s_{n-1}} P_{t_{n-2}} f(x) - Q^m_{(s_{n-1})} P_{t_{n-2}} f(x)\big)\\
&\qquad + \cdots\\
&\qquad + Q^m_{(s_n)} \cdots Q^m_{(s_2)} \big(P_{t_1} f(x) - Q^m_{(s_1)} f(x)\big).
\end{aligned}$$

Then, since Q^m is a Markov operator, we have

$$\begin{aligned}
&\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty\\
&\quad \le \big(\|P_{s_n} P_{t_{n-1}} f - Q^m_{(s_n)} P_{t_{n-1}} f\|_\infty\\
&\qquad + \|P_{s_{n-1}} P_{t_{n-2}} f - Q^m_{(s_{n-1})} P_{t_{n-2}} f\|_\infty\\
&\qquad \cdots\\
&\qquad\qquad + \|P_{t_1} f - Q^m_{(s_1)} f\|_\infty\big)\big(1 + \mathrm{O}(\varepsilon)\big)\\
&\quad = \left(\sum_{k=2}^{n} \|P_{s_k} P_{t_{k-1}} f - Q^m_{(s_k)} P_{t_{k-1}} f\|_\infty \right.\\
&\qquad\qquad \left. + \|P_{t_1} f - Q^m_{(s_1)} f\|_\infty \right)\big(1 + \mathrm{O}(\varepsilon)\big).
\end{aligned}$$

First, note that we can directly apply (3.34), (3.35) or (3.36) in Corollary 1 to obtain an estimate of $\|P_{t_1} f - Q^m_{(s_1)} f\|_\infty$ for $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, a Lipschitz continuous function or a bounded Borel function, respectively. To obtain an estimate of $\sum_{k=2}^n \|P_{s_k} P_{t_{k-1}} f - Q^m_{(s_k)} P_{t_{k-1}} f\|_\infty$, we apply the results in Corollary 1 to $P_t f$ (in stead of f) as follows:

- By (3.34) in Corollary 1, for $s, t \in (0, 1]$ and $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, there exists C such that

$$\|P_s P_t f - Q^m_{(s)} P_t f\|_\infty \le \sum_{l=1}^{m+1} s^{(m+1+l)/2} C \|\nabla^l P_t f\|_\infty \tag{C.1}$$

$$\le \sum_{l=1}^{m+1} s^{(m+1+l)/2} C \|\nabla^l f\|_\infty. \tag{C.2}$$

Hence,

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \tag{C.3}$$

$$\leq C \sum_{k=2}^{n} \sum_{l=1}^{m+1} s_k^{(m+1+l)/2} \|\nabla^l f\|_\infty \tag{C.4}$$

$$+ C \sum_{l=1}^{m+1} s_1^{(m+1+l)/2} \|\nabla^l f\|_\infty. \tag{C.5}$$

- By (3.35) in Corollary 1, for $s, t \in (0, 1]$ and $f \in C^1_b(\mathbf{R}^N; \mathbf{R})$, there exists C such that

$$\|P_s P_t f - Q^m_{(s)} P_t f\|_\infty \leq s^{(m+2)/2} C \|\nabla P_t f\|_\infty \tag{C.6}$$

$$\leq s^{(m+2)/2} C \|\nabla f\|_\infty. \tag{C.7}$$

Hence,

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \tag{C.8}$$

$$\leq C \sum_{k=2}^{n} s_k^{(m+2)/2} \|\nabla f\|_\infty \tag{C.9}$$

$$+ C s_1^{(m+2)/2} \|\nabla f\|_\infty. \tag{C.10}$$

- By (3.36) in Corollary 1, for $s, t \in (0, 1]$ and bounded Borel function f on $\mathbf{R}^N$, there exists C such that

$$\|P_s P_t f - Q^m_{(s)} P_t f\|_\infty \leq s^{(m+1)/2} C \|P_t f\|_\infty \tag{C.11}$$

$$\leq s^{(m+1)/2} C \|f\|_\infty. \tag{C.12}$$

Hence,

$$\|P_T f - Q^m_{(s_n)} Q^m_{(s_{n-1})} \cdots Q^m_{(s_1)} f\|_\infty \tag{C.13}$$

$$\leq C \sum_{k=2}^{n} s_k^{(m+1)/2} \|f\|_\infty \tag{C.14}$$

$$+ C s_1^{(m+1)/2} \|f\|_\infty. \tag{C.15}$$

Next, we obtain more explicit and compact expressions with regard to n particularly for (C.4), (C.9) and (C.14).

First, from the definition of s_k for $k \in \{2, \dots, n\}$, we have

$$s_k = \frac{\gamma T (k-1)^{\gamma-1}}{n^\gamma} \int_{k-1}^{k} \big(u/(k-1)\big)^{\gamma-1} \, du. \tag{C.16}$$

For $k \in \{2, \dots, n\}$, $(u/(k-1))^{\gamma-1} \le \max\{(k/(k-1))^{\gamma-1}, 1\} \le \max\{2^{\gamma-1}, 1\}$. Then

$$s_k^{l/2} \le \left(\frac{\gamma T (k-1)^{\gamma-1}}{n^\gamma} \max\{2^{\gamma-1}, 1\} \right)^{l/2} \tag{C.17}$$

$$\le C(1/n)^{\gamma l/2} (k-1)^{(\gamma-1)l/2}, \tag{C.18}$$

where $C = C(T, \gamma)$.

We consider the estimates for three different ranges of γ that are larger than, equal to and less than $(l-2)/l$, respectively. [$\gamma = (l-2)/l$ satisfies $(\gamma-1)l/2 = -1$.]

For $0 < \gamma < (l-2)/l$,

$$C(1/n)^{\gamma l/2} \sum_{k=2}^{n} (k-1)^{(\gamma-1)l/2} \le C(1/n)^{\gamma l/2}. \tag{C.19}$$

For $\gamma = (l-2)/l$,

$$C(1/n)^{\gamma l/2} \sum_{k=2}^{n} (k-1)^{(\gamma-1)l/2} \tag{C.20}$$

$$= C(1/n)^{(l-2)/2} \sum_{k=1}^{n} (k-1)^{-1} \tag{C.21}$$

$$\le C(1/n)^{(l-2)/2} \log n. \tag{C.22}$$

For $\gamma > (l-2)/l$,

$$C(1/n)^{\gamma l/2} \sum_{k=2}^{n} (k-1)^{(\gamma-1)l/2} \tag{C.23}$$

$$= C(1/n)^{(\gamma-1)l/2} (1/n)^{l/2} \sum_{k=2}^{n} (k-1)^{(\gamma-1)l/2} \tag{C.24}$$

$$= C(1/n)^{(l-2)/2} \sum_{k=2}^{n} \left(\frac{k-1}{n} \right)^{(\gamma-1)l/2} \frac{1}{n} \tag{C.25}$$

$$\le C(1/n)^{(l-2)/2}. \tag{C.26}$$

Then, by combining an estimate of $\|P_{t_1} f - Q_{(s_1)}^m f\|_\infty$ for $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, a Lipschitz continuous function or a bounded Borel function, we have the assertion.

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