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THE MEAN EULER CHARACTERISTIC AND EXCURSION PROBABILITY OF GAUSSIAN RANDOM FIELDS WITH STATIONARY INCREMENTS[1]

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Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. For any compact rectangle $T \subset \mathbb{R}^N$ and $u \in \mathbb{R}$, denote by $A_u = \{t \in T : X(t) \geq u\}$ the excursion set. Under $X(\cdot) \in C^2(\mathbb{R}^N)$ and certain regularity conditions, the mean Euler characteristic of A_u, denoted by $\mathbb{E}\{\varphi(A_u)\}$, is derived. By applying the Rice method, it is shown that, as $u \to \infty$, the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ can be approximated by $\mathbb{E}\{\varphi(A_u)\}$ such that the error is exponentially smaller than $\mathbb{E}\{\varphi(A_u)\}$. This verifies the expected Euler characteristic heuristic for a large class of Gaussian random fields with stationary increments.

1. Introduction. Let $X = \{X(t), t \in T\}$ be a real-valued Gaussian random field on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where T is the parameter set. The study of the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ is a classical but very important problem in probability theory and has many applications in statistics and related areas. Many authors have developed various methods for precise approximations of $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$. These include the double sum method [Piterbarg (1996a)], the tube method [Sun (1993)], the Euler characteristic method [Adler (2000), Taylor and Adler (2003), Taylor, Takemura and Adler (2005), Adler and Taylor (2007)] and the Rice method [Azaïs, Bardet and Wschebor (2002), Azaïs and Delmas (2002), Azaïs and Wschebor (2005, 2008, 2009)].

For a centered, unit-variance smooth Gaussian random field $X = \{X(t), t \in T\}$ parameterized on a manifold T, Adler and Taylor [(2007), Theorem 14.3.3] proved, under certain conditions on the regularity of X and topology of T, the following approximation:

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \mathbb{E}\{\varphi(A_u)\}\big(1 + o\big(e^{-\alpha u^2}\big)\big) \qquad \text{as } u \to \infty, \tag{1.1}$$

where $\varphi(A_u)$ is the Euler characteristic of excursion set $A_u = \{t \in T : X(t) \geq u\}$ and $\alpha > 0$ is a constant which relates to the curvature of the boundary of T and

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the second-order partial derivatives of X. This verifies the "Expected Euler Characteristic Heuristic" for unit-variance smooth Gaussian random fields. We refer to Takemura and Kuriki (2002), Taylor and Adler (2003) and Taylor, Takemura and Adler (2005) for similar results in special cases. It should be mentioned that Taylor, Takemura and Adler (2005) were able to provide an explicit form of α in (1.1).

The approximation (1.1) is remarkable and very accurate, since $\mathbb{E}\{\varphi(A_u)\}$ is computable and the error is exponentially smaller than this principal term. It has been applied for P-value approximation in many statistical applications to brain imaging, cosmology and environmental sciences. We refer to Adler and Taylor (2007) and its forthcoming companion Adler, Taylor and Worsley (2012) for further information. However, the above requirement of "constant variance" on the Gaussian random fields is too restrictive for many applications and excludes some important Gaussian random fields such as those with stationary increments (see Section 2 below), or more generally, Gaussian random intrinsic functions [Matheron (1973), Stein (1999, 2013)]. If the constant variance condition on X is not satisfied, then several important properties [e.g., $X(t)$ and its gradient $\nabla X(t)$ are independent for every t] are not available and the formulas for computing $\mathbb{E}\{\varphi(A_u)\}$ [cf. Theorems 12.4.1 and 12.4.2 in Adler and Taylor (2007)] cannot be applied. Little had been known on whether the approximation (1.1) still holds. The only exception is Azaïs and Wschebor [(2008), Theorem 5], where they proved (1.1) for a centered smooth Gaussian random field X whose maximum variance is attained in the interior of T.

In this paper, let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered real-valued Gaussian random field with stationary increments and $X(0) = 0$, and let $T \subset \mathbb{R}^N$ be a rectangle. Our objectives are to compute the expected Euler characteristic $\mathbb{E}\{\varphi(A_u)\}$ and to show that it can be applied to give an accurate approximation for the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$. In particular, we prove that (1.1) holds for a large class of smooth Gaussian random fields with stationary increments and $X(0) = 0$. In this generality, our main results in Sections 3 and 4 are new even for the case of $N = 1$.

The paper is organized as follows. In Section 2, we provide some preliminaries on Gaussian random fields with stationary increments and prove some basic lemmas. These are derived from the spectral representation of the random fields and will be useful for proving the main results in Sections 3 and 4.

In Section 3, we compute the mean Euler characteristic $\mathbb{E}\{\varphi(A_u)\}$ by applying the Kac–Rice metatheorem in Adler and Taylor [(2007), Theorem 11.2.1] [see also Adler and Taylor (2011), Theorem 4.1.1]. The computation of $\mathbb{E}\{\varphi(A_u)\}$ involves the conditional expectation of the determinant of the Hessian $\nabla^2 X(t)$ given $X(t)$ and $\nabla X(t)$, which is more complicated for random fields with nonconstant variance function. For Gaussian random fields with stationary increments, we are able to make use of the properties of ∇X and $\nabla^2 X$ (e.g., their stationarity) to provide an explicit formula in Theorem 3.2 for $\mathbb{E}\{\varphi(A_u)\}$, using only derivatives of up to second order of the covariance function.

Section 4 is the core part of this paper. Theorems 4.6 and 4.8 provide approximations to the excursion probability which are analogous to (1.1) for Gaussian random fields with stationary increments and $X(0) = 0$. Since these random fields do not have constant variance, it is not clear if the original method for proving Theorem 14.3.3 in Adler and Taylor (2007) is still applicable. Instead, our argument is based on the Rice method in Azaïs and Delmas (2002) [see also Adler and Taylor (2007), pages 96–99]. More specifically, we decompose the rectangle T into several faces of lower dimensions and then apply the idea of Piterbarg (1996b) and the Bonferroni inequality to derive upper and lower bounds for $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ in terms of the number of extended outward maxima [see (4.1), (4.2)] and local maxima [see (4.3)], respectively. The main idea is to show that, in both cases, the upper bound makes the major contribution for estimating $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ and the last two terms in the lower bounds in (4.2) and (4.3) are super-exponentially small. Under a mild technical condition on the variogram of X, we apply (4.3) to obtain in Theorem 4.6 an expansion of the excursion probability which is, in spirit, similar to the case of stationary Gaussian fields [cf. (14.0.3) in Adler and Taylor (2007)]. Theorem 4.8 establishes a general approximation to $\mathbb{P}\{\sup_{t\in T} X(t) \geq u\}$ in terms of $\mathbb{E}\{\varphi(A_u)\}$, which verifies the "Expected Euler Characteristic Heuristic" for smooth Gaussian random fields with stationary increments and $X(0) = 0$. For the purpose of comparison, we mention that, if $Z = \{Z(t), t \in \mathbb{R}^N\}$ is a real-valued, centered stationary Gaussian random field, then the random field X defined by $X(t) = Z(t) - Z(0)$ has stationary increments with $X(0) = 0$. Consequently, Theorems 4.6 and 4.8 provide approximations to the excursion probability $\mathbb{P}\{\sup_{t\in T} Z(t) - Z(0) \geq u\}$.

Section 5 provides further remarks on the main results and some examples where significant simplifications can be made. In Remarks 5.3 and 5.5, we show that if the variance function of the random field attains its maximum at a unique point, then one can apply the Laplace method to derive a first-order approximation for the excursion probability explicitly. Finally, the Appendix contains proofs of some auxiliary lemmas.

2. Gaussian fields with stationary increments.

2.1. *Spectral representation.* Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments. That is, for any $h \in \mathbb{R}^N$, $\{X(t+h) - X(h), t \in \mathbb{R}^N\} \overset{d}{=} \{X(t) - X(0), t \in \mathbb{R}^N\}$, where $\overset{d}{=}$ means equality in finite dimensional distributions. We assume that X has continuous covariance function $C(t,s) = \mathbb{E}\{X(t)X(s)\}$. Then it is known [cf. Yaglom (1957)] that

$$
\begin{aligned}
&C(t,s) - C(t,0) - C(0,s) + C(0,0) \\
&\quad = \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big)\big(e^{-i\langle s,\lambda\rangle} - 1\big) F(d\lambda) + \langle t, \Theta s\rangle,
\end{aligned}
\tag{2.1}
$$

where $\langle x, y\rangle$ is the ordinary inner product in $\mathbb{R}^N$, Θ is an $N \times N$ nonnegative definite matrix and F is a nonnegative symmetric measure on $\mathbb{R}^N \setminus \{0\}$ which satisfies

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1+\|\lambda\|^2} F(d\lambda) < \infty. \tag{2.2}$$

Similar to stationary random fields, the measure F and its density (if it exists) $f(\lambda)$ are called the *spectral measure* and *spectral density* of X, respectively. It is known that many probabilistic, analytic and geometric properties of $\{X(t), t \in \mathbb{R}^N\}$ can be described in terms of its spectral measure F and, on the other hand, various interesting Gaussian random fields can be constructed by choosing their spectral measures appropriately. See Xiao (2009), Xue and Xiao (2011) and the references therein for more information.

By (2.1), we see that X has the following stochastic integral representation:

$$X(t) - X(0) \overset{d}{=} \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big) W(d\lambda) + \langle \mathbf{Y}, t\rangle, \tag{2.3}$$

where $\mathbf{Y}$ is an N-dimensional Gaussian random vector and W is a complex-valued Gaussian random measure (independent of $\mathbf{Y}$) with F as its control measure. Note that in (2.3) there is no restriction on $X(0)$ other than that all joint distributions of $\{X(t), t \in \mathbb{R}^N\}$ are Gaussian.

For simplicity, we assume throughout this paper that $\mathbf{Y} = 0$. It follows from (2.1) or (2.3) that the *variogram* ν of X is given by

$$\nu(h) := \mathbb{E}\big(X(t+h) - X(t)\big)^2 = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big) F(d\lambda). \tag{2.4}$$

Mean-square directional derivatives and sample path differentiability of Gaussian random fields have been well studied. See, for example, Adler (1981), Adler and Taylor (2007), Potthoff (2010), Xue and Xiao (2011). In particular, general sufficient conditions for a Gaussian random field to have a modification whose sample functions are in $C^k(\mathbb{R}^N)$ are given by Adler and Taylor (2007). For a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with stationary increments, Xue and Xiao (2011) provided conditions for its sample path differentiability in terms of the spectral density function $f(\lambda)$. A similar argument can be applied to give the following spectral condition for the sample functions of X to be in $C^k(\mathbb{R}^N)$, whose proof is given in Cheng (2013) and is omitted here.

PROPOSITION 2.1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field with stationary increments and let k_i $(1 \le i \le N)$ be nonnegative integers. If there is a constant $\varepsilon > 0$ such that*

$$\int_{\{\|\lambda\|\ge 1\}} \prod_{i=1}^N |\lambda_i|^{2k_i+\varepsilon} F(d\lambda) < \infty, \tag{2.5}$$

then X has a modification $\widetilde{X}$ such that the partial derivative $\frac{\partial^k \widetilde{X}(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}}$ is continuous on $\mathbb{R}^N$ almost surely, where $k = \sum_{i=1}^N k_i$. Moreover, for any compact rectangle $T \subset \mathbb{R}^N$ and any $\varepsilon' \in (0, \varepsilon \wedge 1)$, there exists a constant c_1 such that

$$\mathbb{E}\left(\frac{\partial^k \widetilde{X}(t)}{\partial t_1^{k_1} \cdots \partial t_N^{k_N}} - \frac{\partial^k \widetilde{X}(s)}{\partial s_1^{k_1} \cdots \partial s_N^{k_N}}\right)^2 \le c_1 \|t - s\|^{\varepsilon'} \qquad \forall t, s \in T. \tag{2.6}$$

For notational simplicity, we will not distinguish X from its modification $\widetilde{X}$. As a consequence of Proposition 2.1, we see that, if $X = \{X(t), t \in \mathbb{R}^N\}$ has a spectral density $f(\lambda)$ which satisfies

$$f(\lambda) = O\left(\frac{1}{\|\lambda\|^{N+2k+H}}\right) \qquad \text{as } \|\lambda\| \to \infty, \tag{2.7}$$

for some integer $k \ge 1$ and $H \in (0, 1)$, then the sample functions of X are in $C^k(\mathbb{R}^N)$ a.s. Further examples of anisotropic Gaussian random fields which may have different smoothness along different directions can be found in Xue and Xiao (2011).

When $X(\cdot) \in C^2(\mathbb{R}^N)$ almost surely, we write $\frac{\partial X(t)}{\partial t_i} = X_i(t)$ and $\frac{\partial^2 X(t)}{\partial t_i \partial t_j} = X_{ij}(t)$. We will use the same notation for the partial derivatives of deterministic functions such as $\nu(\cdot)$ in Theorem 4.6.

Denote by $\nabla X(t)$ and $\nabla^2 X(t)$ the column vector $(X_1(t), \ldots, X_N(t))^T$ and the $N \times N$ matrix $(X_{ij}(t))_{i,j=1,\ldots,N}$, respectively. It follows from (2.1) that for every $t \in \mathbb{R}^N$,

$$\lambda_{ij} := \int_{\mathbb{R}^N} \lambda_i \lambda_j F(d\lambda) = \frac{\partial^2 C(t, s)}{\partial t_i \partial s_j}\bigg|_{s=t} = \mathbb{E}\{X_i(t) X_j(t)\}. \tag{2.8}$$

Let $\Lambda = (\lambda_{ij})_{i,j=1,\ldots,N}$, then (2.8) shows that $\Lambda = \mathrm{Cov}(\nabla X(t))$ for all t. In particular, the distribution of $\nabla X(t)$ is independent of t. Let

$$\lambda_{ij}(t) := \int_{\mathbb{R}^N} \lambda_i \lambda_j \cos\langle t, \lambda \rangle F(d\lambda), \qquad \Lambda(t) := (\lambda_{ij}(t))_{i,j=1,\ldots,N}.$$

Then we have

$$\begin{aligned} \lambda_{ij}(t) - \lambda_{ij} &= \int_{\mathbb{R}^N} \lambda_i \lambda_j (\cos\langle t, \lambda \rangle - 1) F(d\lambda) \\ &= \frac{\partial^2 C(t, s)}{\partial s_i \partial s_j}\bigg|_{s=t} = \mathbb{E}\{(X(t) - X(0)) X_{ij}(t)\}, \end{aligned} \tag{2.9}$$

or equivalently, $\Lambda(t) - \Lambda = \mathbb{E}\{(X(t) - X(0)) \nabla^2 X(t)\}$.

In studies of a Gaussian random field X with stationary increments in the literature, it is often assumed that $X(0) = 0$. In this case, $\Lambda(t) - \Lambda = \mathbb{E}\{X(t) \nabla^2 X(t)\}$. With little loss of generality, we will follow this convention by assuming $X(0) = 0$

in the rest of this paper. The general case can be dealt with by first applying the result of this paper to the random field $\{X(t) - X(0), t \in \mathbb{R}^N\}$ and then taking into consideration of the available information on $X(0)$ [an interesting special case is when $X(0)$ is independent of $\{X(t) - X(0), t \in \mathbb{R}^N\}$].

2.2. *Hypotheses and some important properties.* Let $T = \prod_{i=1}^N [a_i, b_i]$ be a compact rectangle in $\mathbb{R}^N$, where $a_i < b_i$ for all $1 \le i \le N$ and $0 \notin T$ (the case of $0 \in T$ will be discussed in Remark 5.1). In addition to assuming that the Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ has stationary increments and $X(0) = 0$, we will make use of the following conditions:

(H1) $X(\cdot) \in C^2(T)$ almost surely and its second derivatives satisfy the *uniform mean-square Hölder condition*: there exist constants $L > 0$ and $\eta \in (0, 1]$ such that

$$\mathbb{E}\big(X_{ij}(t) - X_{ij}(s)\big)^2 \le L\|t - s\|^{2\eta} \qquad \forall t, s \in T, i, j = 1, \dots, N. \tag{2.10}$$

(H2) For every $t \in T$, the matrix $\Lambda - \Lambda(t)$ is nondegenerate.

(H3) For every pair $(t, s) \in T^2$ with $t \ne s$, the Gaussian random vector

$$\big(X(t), \nabla X(t), X_{ij}(t), X(s), \nabla X(s), X_{ij}(s), 1 \le i \le j \le N\big)$$

is nondegenerate.

(H3′) For every $t \in T$, $(X(t), \nabla X(t), X_{ij}(t), 1 \le i \le j \le N)$ is nondegenerate.

Clearly, by Proposition 2.1, condition (H1) is satisfied if (2.7) holds for $k = 2$. Also note that (H3) implies (H3′). We shall use conditions (H1), (H2) and (H3) to prove Theorems 4.6 and 4.8 on the excursion probability. Condition (H3′) will be used for computing $\mathbb{E}\{\varphi(A_u)\}$ in Theorem 3.2.

We point out that the nondegeneracy conditions (H3) and (H3′) are standard for studying crossing problems when $N = 1$, excursion sets and excursion probabilities of smooth Gaussian random fields. In the case where $N = 1$ and X is a stationary Gaussian process, Cramér and Leadbetter [(1967), pages 203–204] showed that (H3′) is automatically satisfied if X has second-order mean square derivatives and the spectral measure of X is not purely discrete. See Exercises 3.4 and 3.5 in Azaïs and Wschebor [(2009), page 87] for similar results. Notice that (H3) and (H3′) are equivalent to saying that the corresponding covariance matrices are nondegenerate which, in turn, can be verified by establishing positive lower bounds for the conditional variances. Thus, (H3) and (H3′) are related to the properties of local nondeterminism [cf. Cuzick (1977), Xiao (2009)]. Hence, for a general Gaussian random field X with stationary increments, it is possible to provide sufficient conditions in terms of the spectral measure F for (H3) and (H3′) to hold. In order not to make this paper too lengthy, we do not give details here.

The following lemma shows that for Gaussian fields with stationary increments and $X(0) = 0$, (H2) is equivalent to $\Lambda - \Lambda(t)$ being positive definite.

LEMMA 2.2. *For every $t \in \mathbb{R}^N$, $\Lambda - \Lambda(t)$ is nonnegative definite. Hence, under* (H2), *$\Lambda - \Lambda(t)$ is positive definite for every $t \in T$.*

PROOF. Let $t \in \mathbb{R}^N$ be fixed. It follows from (2.9) that for any $(a_1, \dots, a_N) \in \mathbb{R}^N \setminus \{0\}$,

$$\sum_{i,j=1}^{N} a_i a_j \big(\lambda_{ij} - \lambda_{ij}(t)\big) = \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} a_i \lambda_i\right)^2 \big(1 - \cos\langle t, \lambda\rangle\big) F(d\lambda). \tag{2.11}$$

Since $(\sum_{i=1}^N a_i \lambda_i)^2 (1 - \cos\langle t, \lambda\rangle) \ge 0$ for all $\lambda \in \mathbb{R}^N$, (2.11) is always nonnegative, which implies that $\Lambda - \Lambda(t)$ is nonnegative definite. If (H2) is satisfied, then, for every $t \in T$, all the eigenvalues of $\Lambda - \Lambda(t)$ are positive. This completes the proof. □

It follows from (2.11) that, if the spectral measure F is *full* [i.e., not supported on any $(N-1)$-dimensional hyperplane], then (H2) holds. Hence, (H2) is in fact a mild condition for smooth Gaussian fields with stationary increments.

Lemma 2.2 and the following two lemmas indicate some significant properties of Gaussian fields with stationary increments. They will play important roles in later sections.

LEMMA 2.3. *Let $t \in \mathbb{R}^N$ be fixed. Then for all i, j, k, the random variables $X_i(t)$ and $X_{jk}(t)$ are independent. Moreover, $\mathbb{E}\{X_{ij}(t) X_{kl}(t)\}$ is symmetric in i, j, k, l.*

PROOF. By (2.1), one can verify that for $t, s \in \mathbb{R}^N$,

$$\mathbb{E}\big\{X_i(t) X_{jk}(s)\big\} = \frac{\partial^3 C(t,s)}{\partial t_i\, \partial s_j\, \partial s_k} = \int_{\mathbb{R}^N} \lambda_i \lambda_j \lambda_k \sin\langle t - s, \lambda\rangle F(d\lambda).$$

Letting $s = t$ we see that $X_i(t)$ and $X_{jk}(t)$ are independent. Similarly, we have

$$\mathbb{E}\big\{X_{ij}(t) X_{kl}(s)\big\} = \frac{\partial^4 C(t,s)}{\partial t_i\, \partial t_j\, \partial s_k\, \partial s_l} = \int_{\mathbb{R}^N} \lambda_i \lambda_j \lambda_k \lambda_l \cos\langle t - s, \lambda\rangle F(d\lambda).$$

This implies the second conclusion. □

The following lemma is a consequence of Lemma 2.3.

LEMMA 2.4. *Let $A = (a_{ij})_{1 \le i,j \le N}$ be a symmetric matrix. Then for any fixed $t \in \mathbb{R}^N$,*

$$\mathcal{S}_t(i,j,k,l) = \mathbb{E}\big\{\big(A\nabla^2 X(t) A\big)_{ij} \big(A\nabla^2 X(t) A\big)_{kl}\big\}$$

is a symmetric function of i, j, k, l.

3. The mean Euler characteristic.

3.1. *Related existing results and notation.* The rectangle $T=\prod_{i=1}^N [a_i,b_i]$ can be decomposed into several faces of lower dimensions. We use the same notation as in Adler and Taylor [(2007), page 134].

A face J of dimension k, is defined by fixing a subset $\sigma(J)\subset\{1,\dots,N\}$ of size k [if $k=0$, we have $\sigma(J)=\varnothing$ by convention] and a subset $\varepsilon(J)=\{\varepsilon_j, j\notin\sigma(J)\}\subset\{0,1\}^{N-k}$ of size $N-k$, so that

$$J=\{t=(t_1,\dots,t_N)\in T: a_j<t_j<b_j \text{ if } j\in\sigma(J),\\ t_j=(1-\varepsilon_j)a_j+\varepsilon_j b_j \text{ if } j\notin\sigma(J)\}.$$

Denote by $\partial_k T$ the collection of all k-dimensional faces in T, then the interior of T is given by $\overset{\circ}{T}=\partial_N T$ and the boundary of T is given by $\partial T=\bigcup_{k=0}^{N-1}\bigcup_{J\in\partial_k T} J$. For $J\in\partial_k T$, denote by $\nabla X_{|J}(t)$ and $\nabla^2 X_{|J}(t)$ the column vector $(X_{i_1}(t),\dots,X_{i_k}(t))^T_{i_1,\dots,i_k\in\sigma(J)}$ and the $k\times k$ matrix $(X_{mn}(t))_{m,n\in\sigma(J)}$, respectively.

If $X(\cdot)\in C^2(\mathbb{R}^N)$ and it is a Morse function a.s. [cf. Definition 9.3.1 in Adler and Taylor (2007)], then according to Corollary 9.3.5 or pages 211–212 in Adler and Taylor (2007), the Euler characteristic of the excursion set $A_u=\{t\in T: X(t)\geq u\}$ is given by

$$\varphi(A_u)=\sum_{k=0}^{N}\sum_{J\in\partial_k T}(-1)^k\sum_{i=0}^{k}(-1)^i\mu_i(J) \tag{3.1}$$

with

$$\mu_i(J):=\#\{t\in J: X(t)\geq u, \nabla X_{|J}(t)=0, \operatorname{index}(\nabla^2 X_{|J}(t))=i,\\ \varepsilon_j^* X_j(t)\geq 0 \text{ for all } j\notin\sigma(J)\}, \tag{3.2}$$

where $\varepsilon_j^*=2\varepsilon_j-1$ and the index of a matrix is defined as the number of its negative eigenvalues. We also define

$$\widetilde{\mu}_i(J):=\#\{t\in J: X(t)\geq u, \nabla X_{|J}(t)=0, \operatorname{index}(\nabla^2 X_{|J}(t))=i\}. \tag{3.3}$$

It follows from (2.4) that $\nu(t)=\operatorname{Var}(X(t))$. Let $\sigma_T^2=\sup_{t\in T}\nu(t)$ be the maximum variance. For any $t\in T$ and $J\in\partial_k T$, where $k\geq 1$, let

$$\Lambda_J=(\lambda_{ij})_{i,j\in\sigma(J)}=\operatorname{Cov}(\nabla X_{|J}(t)),\\ \Lambda_J(t)=(\lambda_{ij}(t))_{i,j\in\sigma(J)},\\ \theta_{J,t}^2=\operatorname{Var}(X(t)|\nabla X_{|J}(t)), \qquad \gamma_t^2=\operatorname{Var}(X(t)|\nabla X(t)),\\ \{J_1,\dots,J_{N-k}\}=\{1,\dots,N\}\setminus\sigma(J),\\ E(J)=\{(t_{J_1},\dots,t_{J_{N-k}})\in\mathbb{R}^{N-k}: t_j\varepsilon_j^*\geq 0, j=J_1,\dots,J_{N-k}\}. \tag{3.4}$$

Note that $\theta_{J,t}^2 \ge \gamma_t^2$ for all $t \in T$ and $\theta_{J,t}^2 = \gamma_t^2$ if $J = \partial_N T$. If $J = \{\tau\} \in \partial_0 T$ is a vertex, then $\nabla X_{|J}(t)$ is not defined and we set $\theta_{J,t}^2$ as $\nu(t)$ by convention. Moreover, if $J = \{\tau\} \in \partial_0 T$, then $E(\{\tau\})$ is a quadrant of $\mathbb{R}^N$ decided by the corresponding $\varepsilon(\{\tau\}) \in \{0,1\}^N$. In the sequel, we will write $\theta_{J,t}^2$ as θ_t^2 for simplicity of notation. This will not cause any confusion because θ_t^2 always appears together with $t \in J$.

For $t \in T$, let $C_j(t)$ be the $(1, j+1)$ entry of $(\mathrm{Cov}(X(t), \nabla X(t)))^{-1}$, that is,

$$C_j(t) = M_{1,j+1}(t)/\operatorname{detCov}\big(X(t), \nabla X(t)\big),$$

where $M_{1,j+1}(t)$ is the cofactor of the $(1, j+1)$ entry, $\mathbb{E}\{X(t)X_j(t)\}$, in the covariance matrix $\mathrm{Cov}(X(t), \nabla X(t))$. If $\{X(t), t \in \mathbb{R}^N\}$ is replaced by a Gaussian field $\{Z(t), t \in \mathbb{R}^N\}$ with constant variance, the independence of $Z(t)$ and $\nabla Z(t)$ for each t implies that $M_{1,j+1}(t)$ and hence $C_j(t)$ is zero for all $j \ge 1$.

Denote by $H_k(x)$ the Hermite polynomial of order k, that is, $H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2})$. Then it can be verified directly [cf. Adler and Taylor (2007), page 289] that

$$\int_u^\infty H_k(x) e^{-x^2/2}\, dx = H_{k-1}(u) e^{-u^2/2}, \tag{3.5}$$

where $u > 0$ and $k \ge 1$. For a matrix A, let $|A|$ denote its determinant. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$ and let $\Psi(u) = (2\pi)^{-1/2} \int_u^\infty e^{-x^2/2}\, dx$.

3.2. *Computing the mean Euler characteristic.* The following lemma is an extension of Lemma 11.7.1 in Adler and Taylor (2007). It provides a key step for computing the mean Euler characteristic in Theorem 3.2 below, and has a close connection with Theorem 4.6. It follows from (3.6) that $(-1)^k \mathbb{E}\{\sum_{i=0}^k (-1)^i \widetilde{\mu}_i(J)\}$ is always positive. This fact will be used to approximate the expected number of local maxima above level u; see Lemma 4.1.

LEMMA 3.1. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Suppose conditions* (H1), (H2) *and* (H3′) *hold. Then for each $J \in \partial_k T$ with $k \ge 1$,*

$$\begin{aligned} &\mathbb{E}\left\{\sum_{i=0}^k (-1)^i \widetilde{\mu}_i(J)\right\} \\ &\quad = \frac{(-1)^k}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\left(\frac{u}{\theta_t}\right) e^{-u^2/(2\theta_t^2)}\, dt. \end{aligned} \tag{3.6}$$

PROOF. Let $\mathcal{D}_i$ be the collection of all $k \times k$ matrices with index i. Recall the definition of $\widetilde{\mu}_i(J)$ in (3.3), and thanks to (H1) and (H3′), we can apply the Kac–Rice metatheorem [cf. Theorem 11.2.1 or Corollary 11.2.2 in Adler and Taylor

(2007)] to get that the left-hand side of (3.6) becomes

$$
\begin{aligned}
&\int_J p_{\nabla X_{|J}(t)}(0)\,dt \\
(3.7)\qquad &\quad\times \sum_{i=0}^{k}(-1)^i \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\}} \mathbb{1}_{\{X(t)\ge u\}} | \nabla X_{|J}(t)=0\}.
\end{aligned}
$$

Note that on the set $\mathcal{D}_i$, the matrix $\nabla^2 X_{|J}(t)$ has i negative eigenvalues, which implies $(-1)^i |\det \nabla^2 X_{|J}(t)| = \det \nabla^2 X_{|J}(t)$. Also, $\bigcup_{i=0}^k \{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\} = \Omega$ a.s., hence (3.7) equals

$$
\begin{aligned}
&\int_J p_{\nabla X_{|J}(t)}(0)\,dt\, \mathbb{E}\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{X(t)\ge u\}} | \nabla X_{|J}(t)=0\} \\
(3.8)\qquad &= \int_J dt \int_u^\infty dx \frac{e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} \\
&\quad\times \mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t)=x, \nabla X_{|J}(t)=0\}.
\end{aligned}
$$

Now we turn to computing $\mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t)=x, \nabla X_{|J}(t)=0\}$ in (3.8). By Lemma 2.2, under (H2), $\Lambda - \Lambda(t)$, and hence $\Lambda_J - \Lambda_J(t)$ are positive definite for every $t\in J$. Thus, there exists a $k\times k$ positive definite matrix Q_t such that

$$
(3.9)\qquad Q_t(\Lambda_J - \Lambda_J(t))Q_t = I_k,
$$

where I_k is the $k\times k$ identity matrix. It follows from (2.9) that $\Lambda_J(t) - \Lambda_J = \mathbb{E}\{X(t)\nabla^2 X_{|J}(t)\}$. Hence,

$$
\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\} = -(Q_t(\Lambda_J - \Lambda_J(t))Q_t)_{ij} = -\delta_{ij},
$$

where δ_{ij} is the Kronecker delta function. We write

$$
(3.10)\quad \mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t) | X(t)=x, \nabla X_{|J}(t)=0\} = \mathbb{E}\{\det \Delta(t,x)\},
$$

where $\Delta(t,x) = (\Delta_{ij}(t,x))_{i,j\in\sigma(J)}$ with all elements $\Delta_{ij}(t,x)$ being Gaussian variables. To study $\Delta(t,x)$, we only need to find the mean and covariance of $\Delta_{ij}(t,x)$. Note that $\nabla X(t)$ and $\nabla^2 X(t)$ are independent by Lemma 2.3, thus

$$
\begin{aligned}
\mathbb{E}\{\Delta_{ij}(t,x)\} &= \mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij} | X(t)=x, \nabla X_{|J}(t)=0\} \\
&= (\mathbb{E}\{X(t)(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}\}, 0, \dots, 0) \\
(3.11)\qquad &\quad\times (\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(x, 0, \dots, 0)^T \\
&= (-\delta_{ij}, 0, \dots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(x, 0, \dots, 0)^T \\
&= -\frac{x}{\theta_t^2}\delta_{ij},
\end{aligned}
$$

where the last equality comes from the fact that the $(1,1)$ entry of $(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}$ is $\det\mathrm{Cov}(\nabla X_{|J}(t))/\det\mathrm{Cov}(X(t), \nabla X_{|J}(t)) = 1/\theta_t^2$. For the covariance, we have

$$\begin{aligned}
&\mathbb{E}\{(\Delta_{ij}(t,x) - \mathbb{E}\{\Delta_{ij}(t,x)\})(\Delta_{kl}(t,x) - \mathbb{E}\{\Delta_{kl}(t,x)\})\} \\
&\quad = \mathbb{E}\{(Q_t \nabla^2 X_{|J}(t) Q_t)_{ij} (Q_t \nabla^2 X_{|J}(t) Q_t)_{kl}\} \\
&\qquad - (\mathbb{E}\{X(t)(Q_t \nabla^2 X_{|J}(t) Q_t)_{ij}\}, 0, \dots, 0) \\
&\qquad \times (\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1} (\mathbb{E}\{X(t)(Q_t \nabla^2 X_{|J}(t) Q_t)_{kl}\}, 0, \dots, 0)^T \\
&\quad = \mathcal{S}_t(i,j,k,l) - (-\delta_{ij}, 0, \dots, 0)(\mathrm{Cov}(X(t), \nabla X_{|J}(t)))^{-1}(-\delta_{kl}, 0, \dots, 0)^T \\
&\quad = \mathcal{S}_t(i,j,k,l) - \frac{\delta_{ij}\delta_{kl}}{\theta_t^2},
\end{aligned}$$

where $\mathcal{S}_t$ is a symmetric function of i, j, k, l by Lemma 2.4 with A replaced by Q_t. Therefore, (3.10) becomes

$$\begin{aligned}
&\mathbb{E}\left\{\frac{1}{\theta_t^k} \det(\theta_t Q_t(\nabla^2 X_{|J}(t)) Q_t) \middle| X(t) = x, \nabla X_{|J}(t) = 0\right\} \\
&\quad = \frac{1}{\theta_t^k} \mathbb{E}\left\{\det\left(\widetilde{\Delta}(t) - \frac{x}{\theta_t} I_k\right)\right\},
\end{aligned}$$

where $\widetilde{\Delta}(t) = (\widetilde{\Delta}_{ij}(t))_{i,j\in\sigma(J)}$ and all entries $\widetilde{\Delta}_{ij}(t)$ are Gaussian variables satisfying

$$\mathbb{E}\{\widetilde{\Delta}_{ij}(t)\} = 0, \qquad \mathbb{E}\{\widetilde{\Delta}_{ij}(t)\widetilde{\Delta}_{kl}(t)\} = \theta_t^2 \mathcal{S}_t(i,j,k,l) - \delta_{ij}\delta_{kl}.$$

By Corollary 11.6.3 in Adler and Taylor (2007), (3.10) is equal to $(-1)^k \theta_t^{-k} \times H_k(x/\theta_t)$, hence

$$\begin{aligned}
&\mathbb{E}\{\det \nabla^2 X_{|J}(t) | X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = \mathbb{E}\{\det(Q_t^{-1} Q_t \nabla^2 X_{|J}(t) Q_t Q_t^{-1}) | X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = |\Lambda_J - \Lambda_J(t)| \mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) | X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = \frac{(-1)^k}{\theta_t^k} |\Lambda_J - \Lambda_J(t)| H_k\left(\frac{x}{\theta_t}\right).
\end{aligned}$$

Plugging this into (3.8) and applying (3.5), we obtain the desired result. □

The following is the main theorem of this section, which is an extension of Theorem 11.7.2 of Adler and Taylor (2007) to Gaussian random fields with stationary increments. Notice that in (3.12), for every $\{t\} \in \partial_0 T$, $\nabla X(t) \in E(\{t\})$ specifies the signs of the partial derivatives $X_j(t)$ $(j = 1, \dots, N)$ and, for $J \in \partial_k T$, the set $\{J_1, \dots, J_{N-k}\}$ is defined in (3.4).

THEOREM 3.2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Suppose conditions* (H1), (H2) *and* (H3′) *hold. Then*

$$\begin{aligned}\mathbb{E}\{\varphi(A_u)\} = \sum_{\{t\}\in\partial_0 T} \mathbb{P}\big(X(t) \geq u, \nabla X(t) \in E(\{t\})\big) + \sum_{k=1}^{N}\sum_{J\in\partial_k T} \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \\ \times \int_J dt \int_u^\infty dx \int\cdots\int_{E(J)} dy_{J_1}\cdots dy_{J_{N-k}} \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} \\ \times H_k\left(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t) y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t) y_{J_{N-k}}\right) \\ \times p_{X(t),X_{J_1}(t),\ldots,X_{J_{N-k}}(t)}\big(x, y_{J_1},\ldots,y_{J_{N-k}} | \nabla X_{|J}(t) = 0\big).\end{aligned} \tag{3.12}$$

REMARK 3.3. If $Z = \{Z(t), t \in \mathbb{R}^N\}$ is a smooth centered stationary Gaussian random field, then the mean Euler characteristic of the excursion set $\{t \in T : Z(t) \geq u\}$ is given by Theorem 11.7.2 in Adler and Taylor (2007). Applying Theorem 3.2 to $X(t) = Z(t) - Z(0)$, (3.12) computes the mean Euler characteristic of $A_u = \{t \in T : Z(t) - Z(0) \geq u\}$.

PROOF OF THEOREM 3.2. According to Corollary 11.3.2 in Adler and Taylor (2007), (H1) and (H3′) imply that X is a Morse function a.s. It follows from (3.1) that

$$\mathbb{E}\{\varphi(A_u)\} = \sum_{k=0}^{N}\sum_{J\in\partial_k T} (-1)^k \mathbb{E}\left\{\sum_{i=0}^{k} (-1)^i \mu_i(J)\right\}. \tag{3.13}$$

If $J \in \partial_0 T$, say $J = \{t\}$, it turns out that $\mathbb{E}\{\mu_0(J)\} = \mathbb{P}(X(t) \geq u, \nabla X(t) \in E(\{t\}))$. If $J \in \partial_k T$ with $k \geq 1$, we apply the Kac–Rice metatheorem in Adler and Taylor (2007) to obtain that the expectation on the right-hand side of (3.13) becomes

$$\begin{aligned}&\int_J p_{\nabla X_{|J}(t)}(0)\, dt \\ &\quad\times \sum_{i=0}^{k} (-1)^i \mathbb{E}\big\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_i\}} \mathbb{1}_{\{(X_{J_1}(t),\ldots,X_{J_{N-k}}(t))\in E(J)\}} \\ &\qquad\qquad \times \mathbb{1}_{\{X(t)\geq u\}} | \nabla X_{|J}(t) = 0\big\} \\ &= \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int\cdots\int_{E(J)} dy_{J_1}\cdots dy_{J_{N-k}} \\ &\quad\times \mathbb{E}\big\{\det \nabla^2 X_{|J}(t) | X(t) = x, X_{J_1}(t) = y_{J_1},\ldots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \\ &\qquad\qquad \nabla X_{|J}(t) = 0\big\} \\ &\quad\times p_{X(t),X_{J_1}(t),\ldots,X_{J_{N-k}}(t)}\big(x, y_{J_1},\ldots,y_{J_{N-k}} | \nabla X_{|J}(t) = 0\big).\end{aligned} \tag{3.14}$$

For fixed t, let Q_t be the positive definite matrix in (3.9). Then, similar to the proof in Lemma 3.1, we can write

$$\mathbb{E}\{\det(Q_t\nabla^2 X_{|J}(t)Q_t)|X(t)=x, X_{J_1}(t)=y_{J_1},\dots,X_{J_{N-k}}=y_{J_{N-k}}, \nabla X_{|J}(t)=0\}$$

as $\mathbb{E}\{\det\overline{\Delta}(t,x)\}$, where $\overline{\Delta}(t,x)$ is a matrix consisting of Gaussian entries $\overline{\Delta}_{ij}(t,x)$ with mean

$$\begin{aligned}
&\mathbb{E}\{(Q_t\nabla^2 X_{|J}(t)Q_t)_{ij}|X(t)=x, X_{J_1}(t)=y_{J_1},\dots,X_{J_{N-k}}=y_{J_{N-k}}, \\
&\qquad \nabla X_{|J}(t)=0\} \\
&= (-\delta_{ij},0,\dots,0)\big(\mathrm{Cov}(X(t),X_{J_1}(t),\dots,X_{J_{N-k}}(t),\nabla X_{|J}(t))\big)^{-1} \\
&\quad\times(x,y_{J_1},\dots,y_{J_{N-k}},0,\dots,0)^T \\
&= -\frac{\delta_{ij}}{\gamma_t^2}(x+\gamma_t^2 C_{J_1}(t)y_{J_1}+\cdots+\gamma_t^2 C_{J_{N-k}}(t)y_{J_{N-k}}),
\end{aligned}\tag{3.15}$$

and covariance

$$\begin{aligned}
&\mathbb{E}\{(\overline{\Delta}_{ij}(t,x)-\mathbb{E}\{\overline{\Delta}_{ij}(t,x)\})(\overline{\Delta}_{kl}(t,x)-\mathbb{E}\{\overline{\Delta}_{kl}(t,x)\})\} \\
&\quad = \mathcal{S}_t(i,j,k,l)-\frac{\delta_{ij}\delta_{kl}}{\gamma_t^2}.
\end{aligned}$$

Following the same procedure in the proof of Lemma 3.1, we obtain that the last conditional expectation in (3.14) is equal to

$$\begin{aligned}
&\frac{(-1)^k|\Lambda_J-\Lambda_J(t)|}{\gamma_t^k} \\
&\quad\times H_k\left(\frac{x}{\gamma_t}+\gamma_t C_{J_1}(t)y_{J_1}+\cdots+\gamma_t C_{J_{N-k}}(t)y_{J_{N-k}}\right).
\end{aligned}\tag{3.16}$$

Plugging this into (3.14) and (3.13) yields the desired result. □

REMARK 3.4. Usually, for a nonstationary (including constant-variance) Gaussian field X on $\mathbb{R}^N$, its mean Euler characteristic involves at least the third-order derivatives of the covariance function. For Gaussian random fields with stationary increments, as shown in Lemma 2.3, $\mathbb{E}\{X_{ij}(t)X_k(t)\}=0$ and $\mathbb{E}\{X_{ij}(t)X_{kl}(t)\}$ is symmetric in i, j, k, l. Hence, the mean Euler characteristic becomes simpler, containing only up to second-order derivatives of the covariance function. This can also be seen from the spectral representation (2.3) which implies that ∇X and $\nabla^2 X$ are stationary. In various practical applications, (3.12) can be simplified with only an exponentially smaller difference. See the discussions in Section 5.

4. Excursion probability.

4.1. *Preliminaries.* As in Section 3.1, we decompose T into its faces as $T = \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} J$. For $k \geq 1$ and any $J \in \partial_k T$, define the number of *extended outward maxima* above level u as

$$M_u^E(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \operatorname{index}(\nabla^2 X_{|J}(t)) = k,$$
$$\varepsilon_j^* X_j(t) \geq 0 \text{ for all } j \notin \sigma(J)\}.$$

In fact, $M_u^E(J)$ is the same as $\mu_k(J)$ defined in (3.2) with $i = k$. For $k = 0$ and any $\{t\} \in \partial_0 T$, let

$$M_u^E(\{t\}) := \mathbb{1}_{\{X(t) \geq u, \nabla X(t) \in E(\{t\})\}} = \mathbb{1}_{\{X(t) \geq u, \varepsilon_j^* X_j(t) \geq 0, \forall j = 1, \dots, N\}}.$$

One can show easily that, under conditions (H1) and (H3′),

$$\Big\{\sup_{t \in T} X(t) \geq u\Big\} = \bigcup_{k=0}^{N} \bigcup_{J \in \partial_k T} \{M_u^E(J) \geq 1\} \qquad \text{a.s.}$$

It follows that

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} \leq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{P}\{M_u^E(J) \geq 1\} \leq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\{M_u^E(J)\}. \tag{4.1}$$

On the other hand, by the Bonferroni inequality,

$$\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} \geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{P}\{M_u^E(J) \geq 1\} - \sum_{J \neq J'} \mathbb{P}\{M_u^E(J) \geq 1, M_u^E(J') \geq 1\}.$$

Note that [cf. Piterbarg (1996b)]

$$\mathbb{E}\{M_u^E(J)\} - \mathbb{P}\{M_u^E(J) \geq 1\} \leq \tfrac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J) - 1)\},$$

together with the obvious bound $\mathbb{P}\{M_u^E(J) \geq 1, M_u^E(J') \geq 1\} \leq \mathbb{E}\{M_u^E(J) \times M_u^E(J')\}$, we obtain the following lower bound for the excursion probability

$$\begin{aligned} \mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} \geq \sum_{k=0}^{N} \sum_{J \in \partial_k T} \Big(\mathbb{E}\{M_u^E(J)\} - \frac{1}{2}\mathbb{E}\{M_u^E(J)(M_u^E(J) - 1)\}\Big) \\ - \sum_{J \neq J'} \mathbb{E}\{M_u^E(J) M_u^E(J')\}. \end{aligned} \tag{4.2}$$

Define the number of *local maxima* above level u as

$$M_u(J) := \#\{t \in J : X(t) \geq u, \nabla X_{|J}(t) = 0, \operatorname{index}(\nabla^2 X_{|J}(t)) = k\}.$$

Then obviously $M_u(J) \ge M_u^E(J)$ and $M_u(J)$ is the same as $\widetilde{\mu}_k(J)$ defined in (3.3) with $i = k$. It follows similarly that

$$
\begin{aligned}
\sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u(J)\} &\ge \mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} \\
&\ge \sum_{k=0}^{N} \sum_{J\in\partial_k T} \Big(\mathbb{E}\{M_u(J)\} - \frac{1}{2}\mathbb{E}\{M_u(J)(M_u(J)-1)\}\Big) \\
&\quad - \sum_{J\neq J'} \mathbb{E}\{M_u(J)M_u(J')\}.
\end{aligned}
\tag{4.3}
$$

We will use (4.1) and (4.2) to estimate the excursion probability for the general case in Theorem 4.8. Inequalities in (4.3) provide another method to approximate the excursion probability in some special cases; see Theorem 4.6. The advantage of (4.3) is that the principal term induced by $\sum_{k=0}^{N}\sum_{J\in\partial_k T} \mathbb{E}\{M_u(J)\}$ is much easier to compute compared with the one induced by $\sum_{k=0}^{N}\sum_{J\in\partial_k T} \mathbb{E}\{M_u^E(J)\}$.

4.2. *Estimating the moments*: *Major terms and error terms*. The following two lemmas provide the estimations for the principal terms in approximating the excursion probability.

LEMMA 4.1. *Let X be a Gaussian field as in Theorem 3.2. Then for each $J \in \partial_k T$ with $k \ge 1$, there exists some constant $\alpha > 0$ such that*

$$
\begin{aligned}
&\mathbb{E}\{M_u(J)\} \\
&\quad = \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \\
&\qquad \times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\Big(\frac{u}{\theta_t}\Big) e^{-u^2/(2\theta_t^2)}\, dt\big(1 + o(e^{-\alpha u^2})\big).
\end{aligned}
\tag{4.4}
$$

PROOF. Following the notation in the proof of Lemma 3.1, we obtain similarly that

$$
\begin{aligned}
&\mathbb{E}\{M_u(J)\} \\
&\quad = \int_J p_{\nabla X_{|J}(t)}(0)\, dt \mathbb{E}\{|\det \nabla^2 X_{|J}(t)| \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}} \mathbb{1}_{\{X(t)\ge u\}} | \nabla X_{|J}(t) = 0\} \\
&\quad = \int_J dt \int_u^\infty dx \frac{(-1)^k e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} \\
&\qquad \times \mathbb{E}\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k\}} | X(t) = x, \nabla X_{|J}(t) = 0\}.
\end{aligned}
\tag{4.5}
$$

Recall that Q_t is the $k \times k$ positive definite matrix in (3.9). We write (3.11) as

$$\mathbb{E}\{Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, \nabla X_{|J}(t) = 0\} = -\frac{x}{\theta_t^2} I_k.$$

Make change of variables

$$V(t) = Q_t \nabla^2 X_{|J}(t) Q_t + \frac{x}{\theta_t^2} I_k.$$

Then $(V(t)|X(t) = x, \nabla X_{|J}(t) = 0)$ is a Gaussian matrix whose mean is 0 and covariance is the same as that of $(Q_t \nabla^2 X_{|J}(t) Q_t | X(t) = x, \nabla X_{|J}(t) = 0)$. Write $V(t) = (V_{ij}(t))_{1 \le i,j \le k}$ and denote the density of Gaussian vectors $((V_{ij}(t))_{1 \le i \le j \le k} | X(t) = x, \nabla X_{|J}(t) = 0)$ by $h_t(v)$, where $v = (v_{ij})_{1 \le i \le j \le k} \in \mathbb{R}^{k(k+1)/2}$. Then

$$\begin{aligned}
&\mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) \mathbb{1}_{\{\nabla^2 X_{|J}(t) \in \mathcal{D}_k\}} | X(t) = x, \nabla X_{|J}(t) = 0\} \\
&\quad = \mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) \mathbb{1}_{\{Q_t \nabla^2 X_{|J}(t) Q_t \in \mathcal{D}_k\}} | X(t) = x, \nabla X_{|J}(t) = 0\} \qquad (4.6) \\
&\quad = \int_{\{v : (v_{ij}) - (x/\theta_t^2) I_k \in \mathcal{D}_k\}} \det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right) h_t(v)\, dv,
\end{aligned}$$

where (v_{ij}) is the abbreviation for the matrix $(v_{ij})_{1 \le i,j \le k}$. Since $\{\theta_t^2 : t \in T\}$ is bounded, there exists a constant $c > 0$ such that

$$(v_{ij}) - \frac{x}{\theta_t^2} I_k \in \mathcal{D}_k \qquad \forall \|(v_{ij})\| := \left(\sum_{i,j=1}^k v_{ij}^2\right)^{1/2} < \frac{x}{c}.$$

Thus, we can write (4.6) as

$$\begin{aligned}
&\int_{\mathbb{R}^{k(k+1)/2}} \det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right) h_t(v)\, dv \\
&\qquad - \int_{\{v : (v_{ij}) - (x/\theta_t^2) I_k \notin \mathcal{D}_k\}} \det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right) h_t(v)\, dv \qquad (4.7) \\
&\qquad = \mathbb{E}\{\det(Q_t \nabla^2 X_{|J}(t) Q_t) | X(t) = x, \nabla X_{|J}(t) = 0\} + Z(t,x),
\end{aligned}$$

where $Z(t,x)$ is the second integral in the first line of (4.7) and it satisfies

$$|Z(t,x)| \le \int_{\|(v_{ij})\| \ge x/c} \left|\det\left((v_{ij}) - \frac{x}{\theta_t^2} I_k\right)\right| h_t(v)\, dv.$$

Denote by $G(t)$ the covariance matrix of $((V_{ij}(t))_{1 \le i \le j \le k} | X(t) = x, \nabla X_{|J}(t) = 0)$. Then by Lemma A.1 in the Appendix, the eigenvalues of $G(t)$ and those of $(G(t))^{-1}$ are bounded for all $t \in T$. It follows that there exists some constant $\alpha' > 0$ such that $h_t(v) = o(e^{-\alpha' \|(v_{ij})\|^2})$ and hence $|Z(t,x)| = o(e^{-\alpha x^2})$ for some constant $\alpha > 0$ uniformly for all $t \in T$. Combining this with (4.5), (4.6), (4.7) and the proof of Lemma 3.1 yields (4.4). $\square$

LEMMA 4.2. *Let X be a Gaussian field as in Theorem 3.2. Then for each $J \in \partial_k T$ with $k \geq 1$, there exists some constant $\alpha > 0$ such that*

$$\begin{aligned}
&\mathbb{E}\{M_u^E(J)\} \\
&\quad = \frac{1}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} \int_J dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \\
&\qquad \times \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} H_k\biggl(\frac{x}{\gamma_t} + \gamma_t C_{J_1}(t) y_{J_1} + \cdots + \gamma_t C_{J_{N-k}}(t) y_{J_{N-k}}\biggr) \\
&\qquad \times p_{X(t), X_{J_1}(t), \dots, X_{J_{N-k}}(t)}\bigl(x, y_{J_1}, \dots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0\bigr)\bigl(1 + o\bigl(e^{-\alpha u^2}\bigr)\bigr).
\end{aligned} \tag{4.8}$$

PROOF. Similar to the proof in Theorem 3.2, we see that $\mathbb{E}\{M_u^E(J)\}$ is equal to

$$\begin{aligned}
&\int_J \frac{(-1)^k |\Lambda_J - \Lambda_J(t)|}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} dt \int_u^\infty dx \int \cdots \int_{E(J)} dy_{J_1} \cdots dy_{J_{N-k}} \\
&\quad \times \mathbb{E}\bigl\{\det\bigl(Q_t \nabla^2 X_{|J}(t) Q_t\bigr) \mathbb{1}_{\{Q_t \nabla^2 X_{|J}(t) Q_t \in \mathcal{D}_k\}} | \\
&\qquad\qquad X(t) = x, X_{J_1}(t) = y_{J_1}, \dots, X_{J_{N-k}}(t) = y_{J_{N-k}}, \nabla X_{|J}(t) = 0\bigr\} \\
&\quad \times p_{X(t), X_{J_1}(t), \dots, X_{J_{N-k}}(t)}\bigl(x, y_{J_1}, \dots, y_{J_{N-k}} | \nabla X_{|J}(t) = 0\bigr) \\
&\quad := \int_J \frac{(-1)^k |\Lambda_J - \Lambda_J(t)|}{(2\pi)^{k/2}|\Lambda_J|^{1/2}} dt \int_u^\infty dx K(t, x),
\end{aligned}$$

where Q_t is the positive definite matrix in (3.9). Then, by using a similar argument as in the proof of Lemma 4.1 to estimate $K(t, x)$, we obtain the desired result. □

We call a function $h(u)$ *super-exponentially small* [when compared with $\mathbb{P}(\sup_{t \in T} X(t) \geq u)$], if there exists a constant $\alpha > 0$ such that $h(u) = o(e^{-\alpha u^2 - u^2/(2\sigma_T^2)})$ as $u \to \infty$.

The following lemma is Lemma 4 in Piterbarg (1996b). It will be used to show that the factorial moments of $M_u(J)$ and $M_u^E(J)$ are usually super-exponentially small.

LEMMA 4.3. *Let $\{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian field satisfying* (H1) *and* (H3). *Then for any $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that for any $J \in \partial_k T$ and u large enough,*

$$\mathbb{E}\bigl\{M_u(J)\bigl(M_u(J) - 1\bigr)\bigr\} \leq e^{-u^2/(2\beta_J^2 + \varepsilon)} + e^{-u^2/(2\sigma_J^2 - \varepsilon_1)},$$

where $\sigma_J^2 = \sup_{t \in J} \operatorname{Var}(X(t))$ and $\beta_J^2 = \sup_{t \in J} \sup_{e \in \mathbb{S}^{k-1}} \operatorname{Var}(X(t) | \nabla X_{|J}(t), \nabla^2 X_{|J}(t) e)$. Here and in the sequel, $\mathbb{S}^{k-1}$ is the unit sphere in $\mathbb{R}^k$.

COROLLARY 4.4. *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be a centered Gaussian random field with stationary increments satisfying* (H1), (H2) *and* (H3). *Then for all* $J \in \partial_k T$, $\mathbb{E}\{M_u(J)(M_u(J)-1)\}$ *and* $\mathbb{E}\{M_u^E(J)(M_u^E(J)-1)\}$ *are super-exponentially small.*

PROOF. Since $M_u^E(J) \le M_u(J)$, we only need to show that $\mathbb{E}\{M_u(J) \times (M_u(J)-1)\}$ is super-exponentially small. If $k=0$, then $M_u(J)$ is either 0 or 1 and hence $\mathbb{E}\{M_u(J)(M_u(J)-1)\} = 0$. If $k \ge 1$, then, thanks to Lemma 4.3, it suffices to show that β_J^2 is strictly less than σ_T^2.

Clearly, $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e) \le \sigma_T^2$ for every $e \in \mathbb{S}^{k-1}$ and $t \in T$. On the other hand,

$$\mathrm{Var}\big(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e\big) = \sigma_T^2 \quad \Longrightarrow \quad \mathbb{E}\big\{X(t)\big(\nabla^2 X_{|J}(t)e\big)\big\} = 0. \tag{4.9}$$

Note that, by (2.9), the right-hand side of (4.9) is equivalent to $(\Lambda_J(t) - \Lambda_J)e = 0$. However, by (H2), $\Lambda_J(t) - \Lambda_J$ is negative definite, which implies $(\Lambda_J(t) - \Lambda_J)e \ne 0$ for all $e \in \mathbb{S}^{k-1}$. Thus

$$\mathrm{Var}\big(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e\big) < \sigma_T^2$$

for all $e \in \mathbb{S}^{k-1}$ and $t \in T$. This and the continuity of $\mathrm{Var}(X(t)|\nabla X_{|J}(t), \nabla^2 X_{|J}(t)e)$ in (e,t) imply $\beta_J^2 < \sigma_T^2$. $\square$

The following lemma shows that the cross terms in (4.2) and (4.3) are super-exponentially small if the two faces are not adjacent. For the case when the faces are adjacent, the proof is more technical. See the proofs in Theorems 4.6 and 4.8.

LEMMA 4.5. *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be a centered Gaussian random field with stationary increments satisfying* (H1) *and* (H3). *Let* J *and* J' *be two faces of* T *such that their distance is positive, that is,* $\inf_{t\in J, s\in J'} \|s-t\| > \delta_0$ *for some* $\delta_0 > 0$. *Then* $\mathbb{E}\{M_u(J)M_u(J')\}$ *is super-exponentially small.*

PROOF. We first consider the case when $\dim(J) = k \ge 1$ and $\dim(J') = k' \ge 1$. By the Kac–Rice metatheorem for higher moments [the proof is the same as that of Theorem 11.5.1 in Adler and Taylor (2007)],

$$\begin{aligned} &\mathbb{E}\{M_u(J)M_u(J')\} \\ &\quad = \int_J dt \int_{J'} ds \mathbb{E}\big\{\big|\det \nabla^2 X_{|J}(t)\big|\big|\det \nabla^2 X_{|J'}(s)\big| \\ &\qquad\qquad \times \mathbb{1}_{\{X(t)\ge u, X(s)\ge u\}} \\ &\qquad\qquad \times \mathbb{1}_{\{\nabla^2 X_{|J}(t)\in\mathcal{D}_k, \nabla^2 X_{|J'}(s)\in\mathcal{D}_{k'}\}}\big| \\ &\qquad\qquad X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\big\} \\ &\qquad\quad \times p_{X(t),X(s),\nabla X_{|J}(t),\nabla X_{|J'}(s)}(x,y,0,0) \end{aligned} \tag{4.10}$$

$$\le \int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy$$
$$\mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)||$$
$$\times X(t)=x, X(s)=y, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\} p_{X(t),X(s)}(x,y)$$
$$\times p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0|X(t)=x, X(s)=y).$$

Note that the following two inequalities hold: For constants a_i and b_j,

$$\prod_{i=1}^{k} |a_i| \prod_{j=1}^{k'} |b_j| \le \frac{1}{k+k'}\left(\sum_{i=1}^{k} |a_i|^{k+k'} + \sum_{j=1}^{k'} |b_j|^{k+k'}\right);$$

and for any Gaussian variable ξ and positive integer l,

$$\mathbb{E}|\xi|^l \le \mathbb{E}(|\mathbb{E}\xi| + |\xi - \mathbb{E}\xi|)^l \le 2^l(|\mathbb{E}\xi|^l + \mathbb{E}|\xi-\mathbb{E}\xi|^l) = 2^l(|\mathbb{E}\xi|^l + K_l(\mathrm{Var}(\xi))^{l/2}),$$

where the constant K_l depends only on l. It follows from these two inequalities that there exist some positive constants C_1 and N_1 such that for large x and y,

$$\begin{aligned}\sup_{t\in J, s\in J'} \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y, \nabla X_{|J}(t)=0,\\ \nabla X_{|J'}(s)=0\}\\ \le C_1 x^{N_1} y^{N_1}.\end{aligned} \tag{4.11}$$

Also, there exists a positive constant C_2 such that

$$\begin{aligned}&\sup_{t\in J, s\in J'} p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0|X(t)=x, X(s)=y)\\ &\quad\le \sup_{t\in J, s\in J'} (2\pi)^{-(k+k')/2}\\ &\qquad\times [\det\mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s)|X(t)=x, X(s)=y)]^{-1/2}\\ &\quad\le C_2.\end{aligned} \tag{4.12}$$

Let $\rho(\delta_0) = \sup_{\|s-t\|>\delta_0} \frac{|\mathbb{E}\{X(t)X(s)\}|}{\sqrt{\nu(t)\nu(s)}}$ which is strictly less than 1 due to (H3), then $\forall\varepsilon > 0$, there exists a positive constant C_3 such that for all $t \in J$, $s \in J'$ and u large enough,

$$\begin{aligned}&\int_u^\infty \int_u^\infty x^{N_1} y^{N_1} p_{X(t),X(s)}(x,y)\,dx\,dy\\ &\quad= \mathbb{E}\{[X(t)X(s)]^{N_1} \mathbb{1}_{\{X(t)\ge u, X(s)\ge u\}}\}\\ &\quad\le \mathbb{E}\{[X(t)+X(s)]^{2N_1} \mathbb{1}_{\{X(t)+X(s)\ge 2u\}}\}\\ &\quad\le C_3 \exp\left(\varepsilon u^2 - \frac{u^2}{(1+\rho(\delta_0))\sigma_T^2}\right).\end{aligned} \tag{4.13}$$

Combining (4.10) with (4.11), (4.12) and (4.13) yields that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small.

When only one of the faces, say J, is a singleton, then let $J = \{t_0\}$ and we have

$$
\begin{aligned}
&\mathbb{E}\{M_u(J)M_u(J')\} \\
(4.14)\qquad &\le \int_{J'} ds \int_u^\infty dx \int_u^\infty dy p_{X(t_0),X(s),\nabla X_{|J'}(s)}(x,y,0) \\
&\quad\times \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)||X(t_0)=x, X(s)=y, \nabla X_{|J'}(s)=0\}.
\end{aligned}
$$

Following the previous discussion yields that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small.

Finally, if both J and J' are singletons, then $\mathbb{E}\{M_u(J)M_u(J')\}$ becomes the joint probability of two Gaussian variables exceeding level u, and hence is trivial. □

4.3. *Main results and their proofs.* Now we are ready to prove our main results on approximating the excursion probability $\mathbb{P}\{\sup_{t\in T} X(t) \ge u\}$. Theorem 4.6 contains a mild technical condition (4.15) which specifies the way that the variogram $\nu(t)$ attains its maximum on the boundary of T. In particular, it implies that, at each point on ∂T where $\nu(t)$ achieves $\sigma_T^2 := \sup_{t\in T} \nu(t)$, $\nabla\nu(t)$ is not zero. In the case of $N = 1$ and $T = [a, b]$, if $\nu(t)$ attains its maximum σ_T^2 at the end point a or b, then (4.15) requires that $\nu(t)$ is strictly monotone in a neighborhood of that end point. Notice that, if $\nu(t)$ only attains its maximum in $\partial_N T$, the interior of T, then (4.15) is satisfied automatically. In this sense, (4.15) is more general than the corresponding condition in Theorem 5 of Azaïs and Wschebor (2008).

THEOREM 4.6. *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments such that* (H1), (H2) *and* (H3) *are fulfilled. Suppose that for any face J,*

$$
(4.15)\qquad \{t \in J : \nu(t) = \sigma_T^2, \nu_j(t) = 0 \text{ for some } j \notin \sigma(J)\} = \varnothing.
$$

Then there exists some constant $\alpha > 0$ such that

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} \\
&= \sum_{k=0}^{N} \sum_{J\in\partial_k T} \mathbb{E}\{M_u(J)\} + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big) \\
(4.16)\qquad &= \sum_{\{t\}\in\partial_0 T} \Psi\bigg(\frac{u}{\sqrt{\nu(t)}}\bigg) + \sum_{k=1}^{N} \sum_{J\in\partial_k T} \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \\
&\quad\times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\bigg(\frac{u}{\theta_t}\bigg) e^{-u^2/(2\theta_t^2)}\, dt + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big).
\end{aligned}
$$

REMARK 4.7. It should be mentioned that, for Gaussian random fields with constant variance, Taylor, Takemura and Adler (2005) provided an explicit form for the constant α in (1.1). However, in Theorems 4.6 and 4.8, we are not able to give explicit information on the value(s) of α.

PROOF OF THEOREM 4.6. Since the second equality in (4.16) follows from Lemma 4.1 directly, we only need to prove the first one. By (4.3) and Corollary 4.4, it suffices to show that the last term in (4.3) is super-exponentially small. Thanks to Lemma 4.5, we only need to consider the case when the distance of J and J' is 0, that is, $I := \bar{J} \cap \bar{J}' \neq \varnothing$. Without loss of generality, we assume

$$\begin{aligned} \sigma(J) &= \{1, \dots, m, m+1, \dots, k\}, \\ \sigma(J') &= \{1, \dots, m, k+1, \dots, k+k'-m\}, \end{aligned} \tag{4.17}$$

where $0 \le m \le k \le k' \le N$ and $k' \ge 1$. Recall that, if $k = 0$, then $\sigma(J) = \varnothing$. Under assumption (4.17), we have $J \in \partial_k T$, $J' \in \partial_{k'} T$ and $\dim(I) = m$.

Case 1: $k = 0$, that is, J is a singleton, say $J = \{t_0\}$. If $\nu(t_0) < \sigma_T^2$, then by (4.14), it is trivial to show that $\mathbb{E}\{M_u(J)M_u(J')\}$ is super-exponentially small. Now we consider the case $\nu(t_0) = \sigma_T^2$. Due to (4.15), $\mathbb{E}\{X(t_0)X_1(t_0)\} \neq 0$, and hence by continuity, there exists $\delta > 0$ such that $\mathbb{E}\{X(s)X_1(s)\} \neq 0$ for all $\|s - t_0\| \le \delta$. It follows from (4.14) that $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded from above by

$$\begin{aligned} &\int_{s \in J' : \|s - t_0\| > \delta} ds \int_u^\infty dx \int_u^\infty dy \\ &\quad \times \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| \,|\, X(t_0) = x, X(s) = y, \nabla X_{|J'}(s) = 0\} \\ &\quad \times p_{X(t_0), X(s), \nabla X_{|J'}(s)}(x, y, 0) \\ &\quad + \int_{s \in J' : \|s - t_0\| \le \delta} ds \int_u^\infty dy \\ &\qquad \times \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)| \,|\, X(s) = y, \nabla X_{|J'}(s) = 0\} p_{X(s), \nabla X_{|J'}(s)}(y, 0) \\ &\qquad := I_1 + I_2. \end{aligned}$$

Following the proof of Lemma 4.5, we can show that I_1 is super-exponentially small. Note that there exists a constant $\varepsilon_0 > 0$ such that

$$\sup_{s \in J' : \|s - t_0\| \le \delta} \mathrm{Var}(X(s) | \nabla X_{|J'}(s)) \le \sup_{s \in J' : \|s - t_0\| \le \delta} \mathrm{Var}(X(s) | X_1(s)) \le \sigma_T^2 - \varepsilon_0.$$

This implies that I_2, and hence $\mathbb{E}\{M_u(J)M_u(J')\}$ are super-exponentially small.

Case 2: $k \ge 1$. For all $t \in I$ with $\nu(t) = \sigma_T^2$, by assumption (4.15), $\mathbb{E}\{X(t)X_i(t)\} \neq 0$, $\forall i = m+1, \dots, k+k'-m$. Since I is a compact set, we see that there exist constants $\varepsilon_1, \delta_1 > 0$ such that

$$\begin{aligned} &\sup_{t \in B, s \in B'} \mathrm{Var}(X(t) | X_{m+1}(t), \dots, X_k(t), X_{k+1}(s), \dots, X_{k+k'-m}(s)) \\ &\quad \le \sigma_T^2 - \varepsilon_1, \end{aligned} \tag{4.18}$$

where $B = \{t \in J : \mathrm{dist}(t, I) \le \delta_1\}$ and $B' = \{s \in J' : \mathrm{dist}(s, I) \le \delta_1\}$. It follows from (4.10) that $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded by

$$\begin{aligned}
&\iint_{(J\times J')\setminus(B\times B')} dt\, ds \int_u^\infty dx \int_u^\infty dy p_{X(t),X(s),\nabla X_{|J}(t),\nabla X_{|J'}(s)}(x, y, 0, 0)\\
&\quad\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)||X(t)=x, X(s)=y, \nabla X_{|J}(t)=0,\\
&\qquad\qquad \nabla X_{|J'}(s)=0\}\\
&\quad+\iint_{B\times B'} dt\, ds \int_u^\infty dx p_{X(t)}\big(x|\nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\big)\\
&\qquad\qquad\times p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0)\\
&\quad\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)||X(t)=x, \nabla X_{|J}(t)=0, \nabla X_{|J'}(s)=0\}\\
&\quad:= I_3 + I_4.
\end{aligned}$$

Note that

$$\begin{aligned}
&(J\times J')\setminus(B\times B')\\
&\quad= \big((J\setminus B)\times B'\big)\cup\big(B\times(J'\setminus B')\big)\cup\big((J\setminus B)\times(J'\setminus B')\big).
\end{aligned} \tag{4.19}$$

Since each product set on the right-hand side of (4.19) consists of two sets with positive distance, following the proof of Lemma 4.5 we can verify that I_3 is super-exponentially small.

For I_4, taking into account (4.18), one has

$$\sup_{t\in B, s\in B'} \mathrm{Var}\big(X(t)|\nabla X_{|J}(t), \nabla X_{|J'}(s)\big) \le \sigma_T^2 - \varepsilon_1. \tag{4.20}$$

For any $t \in B$, $s \in B'$ with $s \ne t$, in order to estimate

$$\begin{aligned}
&p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0)\\
&\quad= (2\pi)^{-(k+k')/2}\big(\det\mathrm{Cov}(\nabla X_{|J}(t), \nabla X_{|J'}(s))\big)^{-1/2},
\end{aligned} \tag{4.21}$$

we write the determinant on the right-hand side of (4.21) as

$$\begin{aligned}
&\det\mathrm{Cov}\big(X_{m+1}(t), \dots, X_k(t), X_{k+1}(s), \dots, X_{k+k'-1}(s)|\\
&\qquad\qquad X_1(t), \dots, X_m(t), X_1(s), \dots, X_m(s)\big)\\
&\quad\times \det\mathrm{Cov}\big(X_1(t), \dots, X_m(t), X_1(s), \dots, X_m(s)\big),
\end{aligned} \tag{4.22}$$

where the first determinant in (4.22) is bounded away from zero due to (H3). By (H1), as shown in Piterbarg (1996b), applying Taylor's formula, we can write

$$\nabla X(s) = \nabla X(t) + \nabla^2 X(t)(s-t)^T + \|s-t\|^{1+\eta} Y_{t,s}, \tag{4.23}$$

where $Y_{t,s} = (Y^1_{t,s}, \dots, Y^N_{t,s})^T$ is a Gaussian vector field with bounded variance uniformly for all $t \in J$, $s \in J'$. Hence as $\|s-t\| \to 0$, the second determinant in (4.22) becomes

$$\begin{aligned}
&\operatorname{detCov}\big(X_1(t), \dots, X_m(t), X_1(t) + \langle \nabla X_1(t), s-t\rangle + \|s-t\|^{1+\eta} Y^1_{t,s}, \dots, \\
&\qquad\qquad X_m(t) + \langle \nabla X_m(t), s-t\rangle + \|s-t\|^{1+\eta} Y^m_{t,s}\big) \\
&\quad = \operatorname{detCov}\big(X_1(t), \dots, X_m(t), \langle \nabla X_1(t), s-t\rangle + \|s-t\|^{1+\eta} Y^1_{t,s}, \dots, \\
&\qquad\qquad \langle \nabla X_m(t), s-t\rangle + \|s-t\|^{1+\eta} Y^m_{t,s}\big) \\
&\quad = \|s-t\|^{2m} \operatorname{detCov}\big(X_1(t), \dots, X_m(t), \langle \nabla X_1(t), e_{t,s}\rangle, \dots, \langle \nabla X_m(t), e_{t,s}\rangle\big) \\
&\qquad \times \big(1 + o(1)\big),
\end{aligned} \tag{4.24}$$

where $e_{t,s} = (s-t)^T/\|s-t\|$ and due to (H3), the last determinant in (4.24) is bounded away from zero uniformly for all $t \in J$ and $s \in J'$. It then follows from (4.22) and (4.24) that

$$\operatorname{detCov}\big(\nabla X_{|J}(t), \nabla X_{|J'}(s)\big) \geq C_1 \|s-t\|^{2m} \tag{4.25}$$

for some constant $C_1 > 0$. Similar to (4.11), there exist constants $C_2, N_1 > 0$ such that

$$\begin{aligned}
&\sup_{t \in J, s \in J'} \mathbb{E}\big\{\big|\det \nabla^2 X_{|J}(t)\big|\big|\det \nabla^2 X_{|J'}(s)\big| \big| \\
&\qquad\qquad X(t) = x, \nabla X_{|J}(t) = 0, \nabla X_{|J'}(s) = 0\big\} \\
&\quad \leq C_2\big(1 + x^{N_1}\big).
\end{aligned} \tag{4.26}$$

Combining (4.20) with (4.21), (4.25) and (4.26), and noting that $m < k'$ implies $1/\|s-t\|^m$ is integrable on $J \times J'$, we conclude that I_4, and hence $\mathbb{E}\{M_u(J)M_u(J')\}$ are finite and super-exponentially small. $\square$

THEOREM 4.8. *Let $X = \{X(t) : t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Assume that* (H1), (H2) *and* (H3) *are fulfilled. Then there exists a constant $\alpha > 0$ such that*

$$\begin{aligned}
\mathbb{P}\Big\{\sup_{t \in T} X(t) \geq u\Big\} &= \sum_{k=0}^{N} \sum_{J \in \partial_k T} \mathbb{E}\big\{M^E_u(J)\big\} + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big) \\
&= \mathbb{E}\big\{\varphi(A_u)\big\} + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big),
\end{aligned} \tag{4.27}$$

where $\mathbb{E}\{\varphi(A_u)\}$ is the mean Euler characteristic of A_u formulated in Theorem 3.2.

The main idea for the proof of Theorem 4.8 comes from Azaïs and Delmas (2002) (especially their Theorem 4). Before showing the proof, we list the following two lemmas whose proofs are given in the Appendix.

LEMMA 4.9. *Under* (H2), *there exists a constant* $\alpha_0 > 0$ *such that*

$$\langle e, (\Lambda - \Lambda(t))e\rangle \geq \alpha_0 \qquad \forall t \in T, e \in \mathbb{S}^{N-1}.$$

LEMMA 4.10. *Let* $\{\xi_1(t) : t \in T_1\}$ *and* $\{\xi_2(t) : t \in T_2\}$ *be two centered Gaussian random fields. Let*

$$\sigma_i^2(t) = \operatorname{Var}(\xi_i(t)), \qquad \overline{\sigma}_i = \sup_{t \in T_i} \sigma_i(t), \qquad \underline{\sigma}_i = \inf_{t \in T_i} \sigma_i(t),$$

$$\rho(t,s) = \frac{\mathbb{E}\{\xi_1(t)\xi_2(s)\}}{\sigma_1(t)\sigma_2(s)}, \qquad \overline{\rho} = \sup_{t \in T_1, s \in T_2} \rho(t,s), \qquad \underline{\rho} = \inf_{t \in T_1, s \in T_2} \rho(t,s),$$

and assume $0 < \underline{\sigma}_i \leq \overline{\sigma}_i < \infty$ *for* $i = 1, 2$. *If* $0 < \underline{\rho} \leq \overline{\rho} < 1$, *then for any* $N_1, N_2 > 0$, *there exists a constant* $\alpha > 0$ *such that as* $u \to \infty$,

$$\sup_{t \in T_1, s \in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2})\mathbb{1}_{\{\xi_1(t) \geq u, \xi_2(s) < 0\}}\} = o(e^{-\alpha u^2 - u^2/(2\overline{\sigma}_1^2)}).$$

Similarly, if $-1 < \underline{\rho} \leq \overline{\rho} < 0$, *then*

$$\sup_{t \in T_1, s \in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2})\mathbb{1}_{\{\xi_1(t) \geq u, \xi_2(s) > 0\}}\} = o(e^{-\alpha u^2 - u^2/(2\overline{\sigma}_1^2)}).$$

PROOF OF THEOREM 4.8. Note that the second equality in (4.27) follows from Theorem 3.2 and Lemma 4.2, and similar to the proof in Theorem 4.6, we only need to show that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small when J and J' are neighboring. Let $I := \bar{J} \cap \bar{J'} \neq \varnothing$. We follow the assumptions in (4.17) and assume also that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, which implies $E(J) = \mathbb{R}_+^{N-k}$ and $E(J') = \mathbb{R}_+^{N-k'}$.

We first consider the case $k \geq 1$. By the Kac–Rice metatheorem, $\mathbb{E}\{M_u^E(J) \times M_u^E(J')\}$ is bounded from above by

$$\begin{aligned}
&\int_J dt \int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_{k+1} \cdots \\
&\quad \times \int_0^\infty dz_{k+k'-m} \int_0^\infty dw_{m+1} \cdots \int_0^\infty dw_k \\
&\quad \times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)|| \\
&\qquad X(t) = x, X(s) = y, \nabla X_{|J}(t) = 0, X_{k+1}(t) = z_{k+1}, \ldots, \\
&\qquad X_{k+k'-m}(t) = z_{k+k'-m}, \nabla X_{|J'}(s) = 0, \\
&\qquad\qquad X_{m+1}(s) = w_{m+1}, \ldots, X_k(s) = w_k\} \\
&\quad \times p_{t,s}(x, y, 0, z_{k+1}, \ldots, z_{k+k'-m}, 0, w_{m+1}, \ldots, w_k) \\
&\quad := \iint_{J \times J'} A(t,s)\, dt\, ds,
\end{aligned} \tag{4.28}$$

where $p_{t,s}(x, y, 0, z_{k+1}, \dots, z_{k+k'-m}, 0, w_{m+1}, \dots, w_k)$ is the density of

$$\big(X(t), X(s), \nabla X_{|J}(t), X_{k+1}(t), \dots, X_{k+k'-m}(t), \nabla X_{|J'}(s), X_{m+1}(s), \dots, X_k(s)\big)$$

evaluated at $(x, y, 0, z_{k+1}, \dots, z_{k+k'-m}, 0, w_{m+1}, \dots, w_k)$.

Let $\{e_1, e_2, \dots, e_N\}$ be the standard orthonormal basis of $\mathbb{R}^N$. For $t \in J$ and $s \in J'$, let $e_{t,s} = (s-t)^T/\|s-t\|$ and let $\alpha_i(t,s) = \langle e_i, (\Lambda - \Lambda(t))e_{t,s}\rangle$, then

$$(4.29)\qquad \big(\Lambda - \Lambda(t)\big)e_{t,s} = \sum_{i=1}^N \langle e_i, \big(\Lambda - \Lambda(t)\big)e_{t,s}\rangle e_i = \sum_{i=1}^N \alpha_i(t,s)e_i.$$

By Lemma 4.9, there exists some $\alpha_0 > 0$ such that

$$(4.30)\qquad \big\langle e_{t,s}, \big(\Lambda - \Lambda(t)\big)e_{t,s}\big\rangle \ge \alpha_0$$

for all t and s. Under the assumptions (4.17) and that all elements in $\varepsilon(J)$ and $\varepsilon(J')$ are 1, we have the following representation:

$$t = (t_1, \dots, t_m, t_{m+1}, \dots, t_k, b_{k+1}, \dots, b_{k+k'-m}, 0, \dots, 0),$$
$$s = (s_1, \dots, s_m, b_{m+1}, \dots, b_k, s_{k+1}, \dots, s_{k+k'-m}, 0, \dots, 0),$$

where $t_i \in (a_i, b_i)$ for all $i \in \sigma(J)$ and $s_j \in (a_j, b_j)$ for all $j \in \sigma(J')$. Therefore,

$$(4.31)\qquad \begin{aligned} &\langle e_i, e_{t,s}\rangle \ge 0 && \forall m+1 \le i \le k,\\ &\langle e_i, e_{t,s}\rangle \le 0 && \forall k+1 \le i \le k+k'-m,\\ &\langle e_i, e_{t,s}\rangle = 0 && \forall k+k'-m < i \le N. \end{aligned}$$

Let

$$(4.32)\qquad \begin{aligned} &D_i = \big\{(t,s) \in J \times J' : \alpha_i(t,s) \ge \beta_i\big\} \qquad \text{if } m+1 \le i \le k,\\ &D_i = \big\{(t,s) \in J \times J' : \alpha_i(t,s) \le -\beta_i\big\} \qquad \text{if } k+1 \le i \le k+k'-m,\\ &D_0 = \left\{(t,s) \in J \times J' : \sum_{i=1}^m \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \ge \beta_0\right\}, \end{aligned}$$

where $\beta_0, \beta_1, \dots, \beta_{k+k'-m}$ are positive constants such that $\beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0$. It follows from (4.31) and (4.32) that, if (t,s) does not belong to any of $D_0, D_m, \dots, D_{k+k'-m}$, then by (4.29),

$$\big\langle \big(\Lambda - \Lambda(t)\big)e_{t,s}, e_{t,s}\big\rangle = \sum_{i=1}^N \alpha_i(t,s)\langle e_i, e_{t,s}\rangle \le \beta_0 + \sum_{i=m+1}^{k+k'-m} \beta_i < \alpha_0,$$

which contradicts (4.30). Thus, $D_0 \cup \bigcup_{i=m+1}^{k+k'-m} D_i$ is a covering of $J \times J'$, by (4.28),

$$\mathbb{E}\{M_u^E(J)M_u^E(J')\} \le \iint_{D_0} A(t,s)\,dt\,ds + \sum_{i=m+1}^{k+k'-m} \iint_{D_i} A(t,s)\,dt\,ds.$$

We first show that $\iint_{D_0} A(t,s)\,dt\,ds$ is super-exponentially small. Similar to the proof of Theorem 4.6, applying (4.21), (4.25) and (4.26), we obtain

$$\begin{aligned}
&\iint_{D_0} A(t,s)\,dt\,ds \\
&\quad\le \iint_{D_0} dt\,ds \int_u^\infty dx p_{\nabla X_{|J}(t),\nabla X_{|J'}(s)}(0,0) \\
&\qquad\times p_{X(t)}\big(x|\nabla X_{|J}(t)=0,\nabla X_{|J'}(s)=0\big) \\
&\qquad\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)|| \\
&\qquad\qquad X(t)=x,\nabla X_{|J}(t)=0,\nabla X_{|J'}(s)=0\} \\
&\quad\le C_1' \iint_{D_0} dt\,ds \int_u^\infty dx\big(1+x^{N_1}\big)\|s-t\|^{-m} \\
&\qquad\times p_{X(t)}\big(x|\nabla X_{|J}(t)=0,\nabla X_{|J'}(s)=0\big),
\end{aligned}\tag{4.33}$$

for some positive constants C_1' and N_1. Due to Lemma 4.5, we only need to consider the case when $\|s-t\|$ is small. It follows from Taylor's formula (4.23) that as $\|s-t\|\to 0$,

$$\begin{aligned}
&\operatorname{Var}\big(X(t)|\nabla X_{|J}(t),\nabla X_{|J'}(s)\big) \\
&\quad\le \operatorname{Var}\big(X(t)|X_1(t),\dots,X_m(t),X_1(s),\dots,X_m(s)\big) \\
&\quad= \operatorname{Var}\big(X(t)|X_1(t),\dots,X_m(t),X_1(t)+\langle\nabla X_1(t),s-t\rangle \\
&\qquad\quad+\|s-t\|^{1+\eta}Y^1_{t,s},\dots, \\
&\qquad\qquad X_m(t)+\langle\nabla X_m(t),s-t\rangle+\|s-t\|^{1+\eta}Y^m_{t,s}\big) \\
&\quad= \operatorname{Var}\big(X(t)|X_1(t),\dots,X_m(t),\langle\nabla X_1(t),e_{t,s}\rangle+\|s-t\|^{\eta}Y^1_{t,s},\dots, \\
&\qquad\qquad\langle\nabla X_m(t),e_{t,s}\rangle+\|s-t\|^{\eta}Y^m_{t,s}\big) \\
&\quad\le \operatorname{Var}\big(X(t)|\langle\nabla X_1(t),e_{t,s}\rangle+\|s-t\|^{\eta}Y^1_{t,s},\dots,\langle\nabla X_m(t),e_{t,s}\rangle \\
&\qquad\qquad+\|s-t\|^{\eta}Y^m_{t,s}\big) \\
&\quad= \operatorname{Var}\big(X(t)|\langle\nabla X_1(t),e_{t,s}\rangle,\dots,\langle\nabla X_m(t),e_{t,s}\rangle\big)+o(1).
\end{aligned}\tag{4.34}$$

By Lemma A.1, the eigenvalues of $[\operatorname{Cov}(\langle\nabla X_1(t),e_{t,s}\rangle,\dots,\langle\nabla X_m(t),e_{t,s}\rangle)]^{-1}$ are bounded uniformly in t and s. Note that $\mathbb{E}\{X(t)\langle\nabla X_i(t),e_{t,s}\rangle\}=-\alpha_i(t,s)$. Applying these facts to the last line of (4.34), we see that there exist constants $C_4>0$ and $\varepsilon_0>0$ such that for $\|s-t\|$ sufficiently small,

$$\begin{aligned}
\operatorname{Var}\big(X(t)|\nabla X_{|J}(t),\nabla X_{|J'}(s)\big) &\le \sigma_T^2 - C_4\sum_{i=1}^m \alpha_i^2(t,s)+o(1) \\
&< \sigma_T^2-\varepsilon_0,
\end{aligned}\tag{4.35}$$

where the last inequality is due to the fact that $(t,s) \in D_0$ implies

$$\sum_{i=1}^m \alpha_i^2(t,s) \ge \sum_{i=1}^m \alpha_i^2(t,s)\big|\langle e_i, e_{t,s}\rangle\big|^2 \ge \frac{1}{m}\Bigg(\sum_{i=1}^m \alpha_i(t,s)\langle e_{t,s}, e_i\rangle\Bigg)^2 \ge \frac{\beta_0^2}{m}.$$

Plugging (4.35) into (4.33) and noting that $1/\|s-t\|^m$ is integrable on $J \times J'$, we conclude that $\iint_{D_0} A(t,s)\,dt\,ds$ is finite and super-exponentially small.

Next we show that $\iint_{D_i} A(t,s)\,dt\,ds$ is super-exponentially small for $i = m+1, \dots, k$. It follows from (4.28) that $\iint_{D_i} A(t,s)\,dt\,ds$ is bounded by

$$\begin{aligned}
&\iint_{D_i} dt\,ds \int_u^\infty dx \int_0^\infty dw_i\, p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x,0,w_i,0)\\
&\quad\times \mathbb{E}\big\{\big|\det \nabla^2 X_{|J}(t)\big|\big|\det \nabla^2 X_{|J'}(s)\big|\big|\\
&\qquad X(t)=x, \nabla X_{|J}(t)=0, X_i(s)=w_i, \nabla X_{|J'}(s)=0\big\}.
\end{aligned} \tag{4.36}$$

We can write

$$\begin{aligned}
&p_{X(t),X_i(s)}\big(x, w_i | X_i(t)=0\big)\\
&\quad= \frac{1}{2\pi\sigma_1(t)\sigma_2(t,s)(1-\rho^2(t,s))^{1/2}}\\
&\qquad\times \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)}\bigg)\bigg\},
\end{aligned}$$

where

$$\sigma_1^2(t) = \operatorname{Var}\big(X(t)|X_i(t)=0\big), \qquad \rho(t,s) = \frac{\mathbb{E}\{X(t)X_i(s)|X_i(t)=0\}}{\sigma_1(t)\sigma_2(t,s)},$$

$$\sigma_2^2(t,s) = \operatorname{Var}\big(X_i(s)|X_i(t)=0\big) = \frac{\operatorname{detCov}(X_i(s), X_i(t))}{\lambda_{ii}},$$

and $\rho^2(t,s) < 1$ due to (H3). Therefore,

$$\begin{aligned}
&p_{X(t),\nabla X_{|J}(t),X_i(s),\nabla X_{|J'}(s)}(x,0,w_i,0)\\
&\quad= p_{\nabla X_{|J'}(s),X_1(t),\dots,X_{i-1}(t),X_{i+1}(t),\dots,X_k(t)}\\
&\qquad\times \big(0|X(t)=x, X_i(s)=w_i, X_i(t)=0\big)\\
&\qquad\times p_{X(t),X_i(s)}\big(x,w_i|X_i(t)=0\big)p_{X_i(t)}(0)\\
&\quad\le C_5 \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)} + \frac{w_i^2}{\sigma_2^2(t,s)} - \frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)}\bigg)\bigg\}\\
&\qquad\times \big(\operatorname{detCov}\big(X(t), \nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)\big)\big)^{-1/2}
\end{aligned} \tag{4.37}$$

for some positive constant C_5. Also, by an argument that is similar to the proof of Theorem 4.6, we can show that there exist positive constants C_6, N_2 and N_3 such that

$$\det\mathrm{Cov}\big(\nabla X_{|J}(t), X_i(s), \nabla X_{|J'}(s)\big) \geq C_6^{-1}\|s-t\|^{2(m+1)}, \tag{4.38}$$

$$C_6^{-1}\|s-t\|^2 \leq \sigma_2^2(t,s) \leq C_6\|s-t\|^2 \tag{4.39}$$

and

$$\begin{aligned}
&\mathbb{E}\big\{\big|\det\nabla^2 X_{|J}(t)\big|\big|\det\nabla^2 X_{|J'}(s)\big| \\
&\qquad X(t)=x, \nabla X_{|J}(t)=0, X_i(s)=w_i, \nabla X_{|J'}(s)=0\big\} \\
&\quad = \mathbb{E}\big\{\big|\det\nabla^2 X_{|J}(t)\big|\big|\det\nabla^2 X_{|J'}(s)\big| \big| X(t)=x, \nabla X_{|J}(t)=0, \\
&\qquad\qquad \langle\nabla X_i(t), e_{t,s}\rangle = w_i/\|s-t\|+o(1), \nabla X_{|J'}(s)=0\big\} \\
&\quad \leq C_7\big(x^{N_2}+\big(w_i/\|s-t\|\big)^{N_3}+1\big).
\end{aligned} \tag{4.40}$$

Combining (4.36) with (4.37), (4.38) and (4.40), and making change of variable $w = w_i/\|s-t\|$, we obtain that for some positive constant C_8,

$$\begin{aligned}
&\iint_{D_i} A(t,s)\,dt\,ds \\
&\quad \leq C_8 \iint_{D_i} dt\,ds\,\|s-t\|^{-m-1}\int_u^\infty dx \int_0^\infty dw_i \\
&\qquad \times \big(x^{N_2}+\big(w_i/\|s-t\|\big)^{N_3}+1\big) \\
&\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)}+\frac{w_i^2}{\sigma_2^2(t,s)}-\frac{2\rho(t,s)xw_i}{\sigma_1(t)\sigma_2(t,s)}\bigg)\bigg\} \\
&\quad = C_8 \iint_{D_i} dt\,ds\,\|s-t\|^{-m}\int_u^\infty dx \int_0^\infty dw\big(x^{N_2}+w^{N_3}+1\big) \\
&\qquad \times \exp\bigg\{-\frac{1}{2(1-\rho^2(t,s))}\bigg(\frac{x^2}{\sigma_1^2(t)}+\frac{w^2}{\widetilde{\sigma}_2^2(t,s)}-\frac{2\rho(t,s)xw}{\sigma_1(t)\widetilde{\sigma}_2(t,s)}\bigg)\bigg\},
\end{aligned} \tag{4.41}$$

where $\widetilde{\sigma}_2(t,s) = \sigma_2(t,s)/\|s-t\|$ is bounded by (4.39). Applying Taylor's formula (4.23) to $X_i(s)$ and noting that $\mathbb{E}\{X(t)\langle\nabla X_i(t), e_{t,s}\rangle\} = -\alpha_i(t,s)$, we obtain

$$\begin{aligned}
\rho(t,s) &= \frac{1}{\sigma_1(t)\sigma_2(t,s)}\bigg(\mathbb{E}\{X(t)X_i(s)\} - \frac{1}{\lambda_{ii}}\mathbb{E}\{X(t)X_i(t)\}\mathbb{E}\{X_i(s)X_i(t)\}\bigg) \\
&= \frac{\|s-t\|}{\sigma_1(t)\sigma_2(t,s)}\bigg(-\alpha_i(t,s)+\|s-t\|^\eta\mathbb{E}\{X(t)Y_{t,s}^i\} \\
&\qquad\qquad - \frac{\|s-t\|^\eta}{\lambda_{ii}}\mathbb{E}\{X(t)X_i(t)\}\mathbb{E}\{X_i(t)Y_{t,s}^i\}\bigg).
\end{aligned}$$

By (4.39) and the fact that $(t,s) \in D_i$ implies $\alpha_i(t,s) \ge \beta_i > 0$ for $i = m+1, \dots, k$, we conclude that $\rho(t,s) \le -\delta_0$ for some $\delta_0 > 0$ uniformly for $t \in J$, $s \in J'$ with $\|s-t\|$ sufficiently small. Then applying Lemma 4.10 to (4.41) yields that $\iint_{D_i} A(t,s)\,dt\,ds$ is super-exponentially small.

It is similar to prove that $\iint_{D_i} A(t,s)\,dt\,ds$ is super-exponentially small for $i = k+1, \dots, k+k'-m$. In fact, in such case, $\iint_{D_i} A(t,s)\,dt\,ds$ is bounded by

$$\begin{aligned}\iint_{D_i} dt\,ds \int_u^\infty dx \int_0^\infty & dz_i\, p_{X(t),\nabla X_{|J}(t),X_i(t),\nabla X_{|J'}(s)}(x,0,z_i,0) \\ &\times \mathbb{E}\{|\det \nabla^2 X_{|J}(t)||\det \nabla^2 X_{|J'}(s)|| \\ &\qquad X(t)=x, \nabla X_{|J}(t)=0, X_i(t)=z_i, \nabla X_{|J'}(s)=0\}.\end{aligned}$$

We can follow the proof in the previous stage by exchanging the positions of $X_i(s)$ and $X_i(t)$ and replacing w_i with z_i. The details are omitted since the procedure is very similar.

If $k=0$, then $m=0$ and $\sigma(J') = \{1, \dots, k'\}$. Since J becomes a singleton, we may let $J = \{t_0\}$. By the Kac–Rice metatheorem, $\mathbb{E}\{M_u(J)M_u(J')\}$ is bounded by

$$\begin{aligned}\int_{J'} ds \int_u^\infty dx \int_u^\infty dy \int_0^\infty dz_1 \cdots \int_0^\infty dz_{k'}\, p_{t_0,s}(x,y,z_1,\dots,z_{k'},0) \\ \times \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)|| \\ X(t_0)=x, X(s)=y, X_1(t_0)=z_1, \dots, X_{k'}(t_0)=z_{k'}, \nabla X_{|J'}(s)=0\} \\ := \int_{J'} \widetilde{A}(t_0,s)\,ds,\end{aligned}$$

where $p_{t_0,s}(x,y,z_1,\dots,z_{k'},0)$ is the density of $(X(t_0), X(s), X_1(t_0), \dots, X_{k'}(t_0), \nabla X_{|J'}(s))$ evaluated at $(x,y,z_1,\dots,z_{k'},0)$. Similarly, J' could be covered by $\bigcup_{i=1}^{k'} \widetilde{D}_i$ with $\widetilde{D}_i = \{s \in J' : \alpha_i(t_0,s) \le -\widetilde{\beta}_i\}$ for some positive constants $\widetilde{\beta}_i$, $1 \le i \le k'$. On the other hand,

$$\begin{aligned}\int_{\widetilde{D}_i} \widetilde{A}(t_0,s)\,ds \le \int_{\widetilde{D}_i} ds \int_u^\infty dx \int_0^\infty dz_i\, p_{X(t_0),X_i(t_0),\nabla X_{|J'}(s)}(x,z_i,0) \\ \times \mathbb{E}\{|\det \nabla^2 X_{|J'}(s)||X(t_0)=x, X_i(t_0)=z_i, \nabla X_{|J'}(s)=0\}.\end{aligned}$$

By similar discussions, we obtain that $\mathbb{E}\{M_u^E(J)M_u^E(J')\}$ is super-exponentially small, and hence complete the proof. □

5. Further remarks and examples.

REMARK 5.1 (The case when T contains the origin). We now show that the conclusions of Theorems 4.6 and 4.8 still hold when T contains the origin. In such case, condition (H3) is actually not satisfied since $X(0)=0$ is degenerate [this is in contrast with the case when $X = \{X(t), t \in \mathbb{R}^N\}$ is assumed to have constant variance]. We will in fact prove a more general result. Let $T_0 \subset T$ be a finite union of compact rectangles such that $\sup_{t\in T_0} \nu(t) < \sigma_T^2$, then according to the Borell–TIS inequality, $\mathbb{P}\{\sup_{t\in T_0} X(t) \ge u\}$ is super-exponentially small. Let $\widehat{T} = T \setminus T_0$, then

$$
\begin{aligned}
\mathbb{P}\Big\{\sup_{t\in\widehat{T}} X(t) \ge u\Big\} &\le \mathbb{P}\Big\{\sup_{t\in T} X(t) \ge u\Big\} \\
&\le \mathbb{P}\Big\{\sup_{t\in\widehat{T}} X(t) \ge u\Big\} + \mathbb{P}\Big\{\sup_{t\in T_0} X(t) \ge u\Big\}.
\end{aligned} \tag{5.1}
$$

To estimate $\mathbb{P}\{\sup_{t\in\widehat{T}} X(t) \ge u\}$, similar to the rectangle T, we decompose $\widehat{T}$ into several faces by lower dimensions such that $\widehat{T} = \bigcup_{k=0}^N \partial_k \widehat{T} = \bigcup_{k=0}^N \bigcup_{L\in\partial_k\widehat{T}} L$. Then we can get the bounds similar to (4.3) with T replaced with $\widehat{T}$ and J replaced with L. Following the proof of Theorem 4.6 yields

$$
\mathbb{P}\Big\{\sup_{t\in\widehat{T}} X(t) \ge u\Big\} = \sum_{k=0}^N \sum_{L\in\partial_k\widehat{T}} \mathbb{E}\{M_u(L)\} + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big).
$$

Because $\sup_{t\in T_0} \nu(t) < \sigma_T^2$, we can show that terms $\mathbb{E}\{M_u(L)\}$ are super-exponentially small for all faces L such that $L \subset \partial_k \bar{T}_0$ with $0 \le k \le N-1$. The same reasoning yields that for $1 \le k \le N$, $L \in \partial_k\widehat{T}$, $J \in \partial_k T$ such that $L \subset J$, the difference between $\mathbb{E}\{M_u(L)\}$ and $\mathbb{E}\{M_u(J)\}$ is super-exponentially small. Hence, we obtain

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in\widehat{T}} X(t) \ge u\Big\} \\
&\quad = \sum_{\{t\}\in\partial_0 T} \Psi\bigg(\frac{u}{\sqrt{\nu(t)}}\bigg) + \sum_{k=1}^N \sum_{J\in\partial_k T} \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \\
&\qquad \times \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^k} H_{k-1}\bigg(\frac{u}{\theta_t}\bigg) e^{-u^2/(2\theta_t^2)}\,dt + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big).
\end{aligned} \tag{5.2}
$$

Here, by convention, if $\theta_t = 0$, we regard $e^{-u^2/(2\theta_t^2)}$ as 0. Combining (5.1) with (5.2), we conclude that Theorem 4.6 still holds when T contains the origin. The argument for Theorem 4.8 is similar.

REMARK 5.2. Based on the proofs of Theorems 4.6 and 4.8, one may expect that the approximation (1.1) holds for a much wider class of smooth Gaussian fields (not necessarily with stationary increments). Meanwhile, the argument for the parameter set could go far beyond the rectangle case. These further developments are in Cheng (2013).

REMARK 5.3 (Refinements of Theorem 4.6). Let X be a Gaussian field as in Theorem 4.6. Suppose that $\nu(t_0) = \sigma_T^2$ for some $t_0 \in J \in \partial_k T$ $(k \geq 0)$ and $\nu(t) < \sigma_T^2$ for all $t \in T \setminus \{t_0\}$.

(i) If $k = 0$, then, due to (4.15), $\sup_{t \in T\setminus\{t_0\}} \theta_t^2 \leq \sigma_T^2 - \varepsilon_0$ for some $\varepsilon_0 > 0$. This implies that $\mathbb{E}\{M_u(J')\}$ are super-exponentially small for all faces J' other than $\{t_0\}$. Therefore, there is a constant $\alpha > 0$ such that

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \Psi\left(\frac{u}{\sigma_T}\right) + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big) \qquad \text{as } u \to \infty. \tag{5.3}$$

For example, let Y be a stationary isotropic Gaussian field with covariance $\rho(t) = e^{-\|t\|^2}$ and define $X(t) = Y(t) - Y(0)$. Then X is a smooth Gaussian field with stationary increments satisfying conditions (H1)–(H3). Let $T = [0,1]^N$, then we can apply (5.3) to approximate the excursion probability of X with $t_0 = (1, \ldots, 1)$.

(ii) If $k \geq 1$, then similarly, $\mathbb{E}\{M_u(J')\}$ are super-exponentially small for all faces $J' \neq J$. It follows from Theorem 4.6 that

$$\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} = \frac{u^{k-1}}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}} \int_J \frac{|\Lambda_J - \Lambda_J(t)|}{\theta_t^{2k-1}} e^{-u^2/(2\theta_t^2)}\, dt\big(1 + o(1)\big).$$

Let $\tau(t) = \theta_t^2$, then $\forall i \in \sigma(J)$, $\tau_i(t_0) = 0$, since t_0 is a local maximum point of τ restricted on J. Assume additionally that the Hessian matrix

$$\Theta_J(t_0) := \big(\tau_{ij}(t_0)\big)_{i,j\in\sigma(J)} \tag{5.4}$$

(here $\tau_{ij} = \partial^2\tau/\partial t_i\,\partial t_j$) is negative definite, then the Hessian matrix of $1/(2\theta_t^2)$ at t_0 restricted on J,

$$\widetilde{\Theta}_J(t_0) = -\frac{1}{2\tau^2(t_0)}\big(\tau_{ij}(t_0)\big)_{i,j\in\sigma(J)} = -\frac{1}{2\sigma_T^4}\Theta_J(t_0),$$

is positive definite. Let $g(t) = |\Lambda_J - \Lambda_J(t)|/\theta_t^{2k-1}$ and $h(t) = 1/(2\theta_t^2)$, applying Lemma A.2 in the Appendix with T replaced by J yields that as $u \to \infty$,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in T} X(t) \geq u\Big\} \\
&\quad= \frac{u^{k-1}|\Lambda_J - \Lambda_J(t_0)|}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_{t_0}^{2k-1}} \frac{(2\pi)^{k/2}}{u^k|\widetilde{\Theta}_J(t_0)|^{1/2}} e^{-u^2/(2\theta_{t_0}^2)}\big(1 + o(1)\big) \qquad (5.5)\\
&\quad= \frac{2^{k/2}|\Lambda_J - \Lambda_J(t_0)|}{|\Lambda_J|^{1/2}|-\Theta_J(t_0)|^{1/2}} \Psi\left(\frac{u}{\sigma_T}\right)\big(1 + o(1)\big).
\end{aligned}$$

EXAMPLE 5.4 (The cosine field). We consider the *cosine random field* Z on $\mathbb{R}^2$ [cf. Adler and Taylor (2007), page 382]:

$$Z(t) = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} (\xi_i \cos t_i + \xi_i' \sin t_i), \qquad t = (t_1, t_2) \in \mathbb{R}^2,$$

where ξ_1, ξ_1', ξ_2, ξ_2' are independent, standard Gaussian variables. Clearly Z is a centered, unit-variance and smooth stationary Gaussian field. Moreover, Z is periodic and $Z(t) = -Z_{11}(t) - Z_{22}(t)$. Let $X(t) = Z(t) - Z(0)$ for all $t \in T \subset [0, 2\pi)^2$. Then X is a centered and smooth Gaussian field with stationary increments with $X(0) = 0$. The variogram and covariance of X are given respectively by

$$
\begin{aligned}
\nu(t) &= 2 - \cos t_1 - \cos t_2, \\
C(t,s) &= 1 + \frac{1}{2}\sum_{i=1}^{2}[\cos(t_i - s_i) - \cos t_i - \cos s_i].
\end{aligned}
\tag{5.6}
$$

Taking the partial derivatives of C gives

$$
\begin{aligned}
&\mathbb{E}\{X(t)\nabla X(t)\} = \tfrac{1}{2}(\sin t_1, \sin t_2)^T, \qquad \Lambda = \mathrm{Cov}(\nabla X(t)) = \tfrac{1}{2}I_2, \\
&\Lambda - \Lambda(t) = -\mathbb{E}\{X(t)\nabla^2 X(t)\} = \tfrac{1}{2}[I_2 - \mathrm{diag}(\cos t_1, \cos t_2)],
\end{aligned}
\tag{5.7}
$$

where I_2 is the 2×2 identity matrix and diag denotes the diagonal matrix. Therefore, X satisfies conditions (H1) and (H2) on $T \subset (0, 2\pi)^2$. Even though condition (H3) is not fully satisfied [because $X_{12}(t) \equiv 0$ and $X(t) = -2X_{11}(t) - 2X_{22}(t)$ when $t = (\pi, \pi)$], it can be shown that this does not affect the validity of Theorems 3.2, 4.6 and 4.8 for the random field $\{Z(t) - Z(0), t \in \mathbb{R}^2\}$ with $T \subset (0, 2\pi)^2$.

(i) Let $T = [0, \pi/2]^2$. Then by (5.6), $\nu(t)$ attains its maximum 2 only at the upper-right vertex $(\pi/2, \pi/2)$, where both partial derivatives of ν are positive. By Remark 5.1 (with $T_0 = [0, \varepsilon] \times [0, \pi/2] \cup [0, \pi/2] \times [0, \varepsilon]$, where $\varepsilon > 0$ is sufficiently small) and the result (i) in Remark 5.3, we obtain $\mathbb{P}\{\sup_{t\in T} X(t) \ge u\} = \Psi(u/\sqrt{2})(1 + o(e^{-\alpha u^2}))$.

(ii) Let $T = [0, 3\pi/2] \times [0, \pi/2]$. Then $\nu(t)$ attains its maximum 3 only at the boundary point $t^* = (\pi, \pi/2)$, where $\nu_2(t^*) > 0$ so that condition (4.15) is satisfied. In this case, $t^* \in J = (0, 3\pi/2) \times \{\pi/2\}$. By (5.7), we obtain $\Lambda_J = \frac{1}{2}$ and $\Lambda_J - \Lambda_J(t^*) = \frac{1}{2}(1 - \cos t_1^*) = 1$. On the other hand, for $t \in J$, by (5.7),

$$
\tau(t) = \theta_t^2 = \mathrm{Var}(X(t)|X_1(t)) = 2 - \cos t_1 - \cos t_2 - \tfrac{1}{2}\sin^2 t_1. \tag{5.8}
$$

Therefore, $\Theta_J(t^*) = \tau_{11}(t^*) = -2$. By plugging these into (5.5) with $k = 1$, we have $\mathbb{P}\{\sup_{t\in T} X(t) \ge u\} = \sqrt{2}\Psi(u/\sqrt{3})(1 + o(1))$.

(iii) Let $T = [0, 3\pi/2]^2$. Then $\nu(t)$ attains its maximum 4 only at the interior point $t^* = (\pi, \pi)$. In this case, $t^* \in J = (0, 3\pi/2)^2$. By (5.7), we obtain $\Lambda_J = \frac{1}{2}I_2$ and $\Lambda_J - \Lambda_J(t^*) = I_2$. On the other hand, for $t \in J$, by (5.7),

$$
\begin{aligned}
\tau(t) = \theta_t^2 &= \mathrm{Var}(X(t)|X_1(t), X_2(t)) \\
&= 2 - \cos t_1 - \cos t_2 - \tfrac{1}{2}\sin^2 t_1 - \tfrac{1}{2}\sin^2 t_2.
\end{aligned}
\tag{5.9}
$$

Therefore, $\Theta_J(t^*) = (\tau_{ij}(t^*))_{i,j=1,2} = -2I_2$. By plugging these into (5.5) with $k = 2$, we obtain $\mathbb{P}\{\sup_{t\in T} X(t) \ge u\} = 2\Psi(u/2)(1 + o(1))$.

REMARK 5.5 (Refinements of Theorem 4.8). Let X be a Gaussian field as in Theorem 4.8. Suppose $t_0 \in J \in \partial_k T$ is the only point in T such that $\nu(t_0) = \sigma_T^2$. Assume $\sigma(J) = \{1, \dots, k\}$, all elements in $\varepsilon(J)$ are 1, $\nu_{k'}(t_0) = 0$ for all $k+1 \le k' \le N$. Then by Theorem 4.8,

$$\begin{aligned} &\mathbb{P}\Big\{\sup_{t \in T} X(t) \ge u\Big\} \\ &\quad = \mathbb{E}\{M_u^E(J)\} + \sum_{k'=k+1}^{N} \sum_{J' \in \partial_{k'} T, \bar{J}' \cap \bar{J} \ne \varnothing} \mathbb{E}\{M_u^E(J')\} + o\big(e^{-\alpha u^2 - u^2/(2\sigma_T^2)}\big). \end{aligned} \tag{5.10}$$

Lemma 4.2 indicates $\mathbb{E}\{M_u^E(J)\} = (-1)^k \mathbb{E}\{\sum_{i=0}^k (-1)^i \mu_i(J)\}(1 + o(e^{-\alpha x^2}))$. Therefore,

$$\begin{aligned} \mathbb{E}\{M_u^E(J)\} &= (-1)^k \int_J p_{\nabla X_{|J}(t)}(0)\, dt \\ &\qquad \times \mathbb{E}\big\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{(X_{k+1}(t), \dots, X_N(t)) \in \mathbb{R}_+^{N-k}\}} \\ &\qquad\qquad \times \mathbb{1}_{\{X(t) \ge u\}} | \nabla X_{|J}(t) = 0\big\}\big(1 + o\big(e^{-\alpha x^2}\big)\big) \\ &= \int_u^\infty dx \int_J dt \frac{(-1)^k e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2} \theta_t} \\ &\qquad \times \mathbb{E}\big\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{(X_{k+1}(t), \dots, X_N(t)) \in \mathbb{R}_+^{N-k}\}} | \\ &\qquad\qquad X(t) = x, \nabla X_{|J}(t) = 0\big\}\big(1 + o\big(e^{-\alpha u^2}\big)\big) \\ &:= \int_u^\infty A_J(x)\, dx \big(1 + o\big(e^{-\alpha u^2}\big)\big). \end{aligned} \tag{5.11}$$

Similarly, we have

$$\begin{aligned} &\mathbb{E}\{M_u^E(J')\} \\ &\quad = \int_u^\infty dx \int_{J'} dt \frac{(-1)^{k'} e^{-x^2/(2\theta_t^2)}}{(2\pi)^{(k'+1)/2} |\Lambda_{J'}|^{1/2} \theta_t} \\ &\quad \times \mathbb{E}\big\{\det \nabla^2 X_{|J'}(t) \mathbb{1}_{\{(X_{J'_1}(t), \dots, X_{J'_{N-k'}}(t)) \in \mathbb{R}_+^{N-k'}\}} | \\ &\qquad\qquad X(t) = x, \nabla X_{|J'}(t) = 0\big\}\big(1 + o\big(e^{-\alpha u^2}\big)\big). \end{aligned}$$

(i) First, we consider the case $k \ge 1$. We use the same notation $\tau(t)$, $\Theta_J(t)$ and $\widetilde{\Theta}_J(t)$ in Remark 5.3. Let $h(t) = 1/(2\theta_t^2)$ and

$$\begin{aligned} g_x(t) = \frac{(-1)^k}{(2\pi)^{(k+1)/2} |\Lambda_J|^{1/2} \theta_t} \mathbb{E}\big\{\det \nabla^2 X_{|J}(t) \mathbb{1}_{\{(X_{k+1}(t), \dots, X_N(t)) \in \mathbb{R}_+^{N-k}\}} | \\ X(t) = x, \nabla X_{|J}(t) = 0\big\}. \end{aligned}$$

Note that $\sup_{t\in T}|g_x(t)| = o(x^{N_1})$ for some $N_1 > 0$ as $x \to \infty$, which implies that the growth of $g_x(t)$ can be dominated by the exponential decay $e^{-x^2h(t)}$, hence both Lemmas A.2 and A.3 in the Appendix are still applicable. Applying Lemma A.2 with T replaced by J and u replaced by x^2, we obtain that as $x \to \infty$,

$$A_J(x) = \frac{(2\pi)^{k/2}}{x^k(\det \widetilde{\Theta}_J(t_0))^{1/2}} g_x(t_0) e^{-x^2/(2\sigma_T^2)}\big(1 + o(1)\big). \tag{5.12}$$

On the other hand, it follows from (3.16) that

$$\begin{aligned} g_x(t) = {}& \frac{1}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\theta_t} \int \cdots \int_{\mathbb{R}_+^{N-k}} dy_{k+1}\cdots dy_N \\ & \times \frac{|\Lambda_J - \Lambda_J(t)|}{\gamma_t^k} H_k\bigg(\frac{x}{\gamma_t} + \gamma_t C_{k+1}(t) y_{k+1} + \cdots + \gamma_t C_N(t) y_N\bigg) \\ & \times p_{X_{k+1}(t),\dots,X_N(t)}\big(y_{k+1},\dots,y_N | X(t) = x, \nabla X_{|J}(t) = 0\big). \end{aligned}$$

Note that, under the assumptions on X at the beginning of Remark 5.5, $X(t_0)$ and $\nabla X(t_0)$ are independent, and $C_j(t_0) = 0$ for all $1 \le j \le N$. Therefore,

$$\begin{aligned} g_x(t_0) = {}& \frac{|\Lambda_J - \Lambda_J(t_0)|}{(2\pi)^{(k+1)/2}|\Lambda_J|^{1/2}\sigma_T^{k+1}} H_k\bigg(\frac{x}{\sigma_T}\bigg) \\ & \times \mathbb{P}\big\{\big(X_{k+1}(t_0),\dots,X_N(t_0)\big) \in \mathbb{R}_+^{N-k} | \nabla X_{|J}(t_0) = 0\big\}. \end{aligned}$$

Plugging this and (5.12) into (5.11), we obtain

$$\begin{aligned} & \mathbb{E}\{M_u^E(J)\} \\ & \quad = \frac{2^{k/2}|\Lambda_J - \Lambda_J(t_0)|}{|\Lambda_J|^{1/2}|-\Theta_J(t_0)|^{1/2}} \Psi\bigg(\frac{u}{\sigma_T}\bigg) \\ & \qquad \times \mathbb{P}\big\{\big(X_{k+1}(t_0),\dots,X_N(t_0)\big) \in \mathbb{R}_+^{N-k} | \nabla X_{|J}(t_0) = 0\big\}\big(1 + o(1)\big). \end{aligned} \tag{5.13}$$

For $J' \in \partial_{k'}T$ with $\bar{J}' \cap \bar{J} \neq \varnothing$, we apply Lemma A.3 with T replaced by J' to obtain

$$\begin{aligned} & \mathbb{E}\{M_u^E(J')\} \\ & \quad = \frac{2^{k'/2}|\Lambda_{J'} - \Lambda_{J'}(t_0)|}{|\Lambda_{J'}|^{1/2}|-\Theta_{J'}(t_0)|^{1/2}} \Psi\bigg(\frac{u}{\sigma_T}\bigg) \mathbb{P}\big\{Z_{J'}(t_0) \in \mathbb{R}_-^{k'-k}\big\} \\ & \qquad \times \mathbb{P}\big\{\big(X_{J'_1}(t_0),\dots,X_{J'_{N-k'}}(t_0)\big) \in \mathbb{R}_+^{N-k'} | \nabla X_{|J'}(t_0) = 0\big\}\big(1 + o(1)\big), \end{aligned} \tag{5.14}$$

where $Z_{J'}(t_0)$ is a centered $(k' - k)$-dimensional Gaussian vector with covariance matrix $-(\tau_{ij})_{i,j\in\sigma(J')\setminus\sigma(J)}$. Plugging (5.13) and (5.14) into (5.10), we obtain the asymptotic result.

(ii) $k=0$, say $J=\{t_0\}$. Note that $X(t_0)$ and $\nabla X(t_0)$ are independent, therefore,

$$\mathbb{E}\{M_u^E(J)\} = \Psi\left(\frac{u}{\sigma_T}\right)\mathbb{P}\{\nabla X(t_0) \in \mathbb{R}_+^N\}. \tag{5.15}$$

For $J' \in \partial_{k'}T$ with $\bar{J}' \cap \bar{J} \neq \varnothing$, then $\mathbb{E}\{M_u^E(J')\}$ is given by (5.14) with $k=0$. Plugging (5.15) and (5.14) into (5.10), we obtain the asymptotic formula for the excursion probability.

EXAMPLE 5.6 (Continued: The cosine field). We consider the Gaussian field $X=\{X(t), t \in \mathbb{R}^2\}$ defined in Example 5.4.

(i) Let $T=[0,\pi]^2$. Then $\nu(t)$ attains its maximum 4 only at the corner $t^* = (\pi,\pi)$, where $\nabla\nu(t^*)=0$ so that the condition (4.15) is not satisfied. Instead, we will use the result (ii) in Remark 5.5 with $J=\{t^*\}$ and $k=0$. Let $J'=(0,\pi)\times\{\pi\}$, $J''=\{\pi\}\times(0,\pi)$. Combining the results in Example 5.4 with (5.15) and (5.14), and noting that $\Lambda = \frac{1}{2}I_2$ implies $X_1(t)$ and $X_2(t)$ are independent for all t, we obtain

$$\mathbb{E}\{M_u^E(J)\} = \tfrac{1}{4}\Psi(u/2), \qquad \mathbb{E}\{M_u^E(\partial_2 T)\} = \tfrac{1}{2}\Psi(u/2)\big(1+o(1)\big),$$
$$\mathbb{E}\{M_u^E(J')\} = \mathbb{E}\{M_u^E(J'')\} = \tfrac{\sqrt{2}}{4}\Psi(u/2)\big(1+o(1)\big).$$

Summing these up, we have $\mathbb{P}\{\sup_{t\in T} X(t) \ge u\} = [(3+2\sqrt{2})/4]\Psi(u/2)(1+o(1))$.

(ii) Let $T=[0,3\pi/2]\times[0,\pi]$. Then $\nu(t)$ attains its maximum 4 only at the boundary point $t^*=(\pi,\pi)$, where $\nu_2(t^*)=0$. Applying the result (i) in Remark 5.5 with $J=(0,3\pi/2)\times\{\pi\}$ and $k=1$, we obtain

$$\mathbb{E}\{M_u^E(J)\} = \tfrac{\sqrt{2}}{2}\Psi(u/2), \qquad \mathbb{E}\{M_u^E(\partial_2 T)\} = \Psi(u/2)\big(1+o(1)\big),$$

which implies that $\mathbb{P}\{\sup_{t\in T} X(t) \ge u\} = [(2+\sqrt{2})/2]\Psi(u/2)(1+o(1))$.

REMARK 5.7. Note that we only provide the first-order approximation for the examples in this section. However, as shown in the theory of the approximations of integrals [see, e.g., Wong (2001)], the integrals in (4.16) and (4.27) can be expanded with more terms once the covariance function of the Gaussian field is smooth enough. Hence, for the examples above, higher-order approximation is available. Since the procedure is similar and the computation is tedious, we omit such argument here.

APPENDIX

This appendix contains proofs of Lemmas 4.9, 4.10 and some other auxiliary facts.

PROOF OF LEMMA 4.9. Let $M_{N\times N}$ be the set of all $N \times N$ matrices. Define a mapping $\phi : \mathbb{R}^N \times M_{N\times N} \to \mathbb{R}$ by $(\xi, A) \mapsto \langle \xi, A\xi\rangle$. Then ϕ is continuous. Since $\Lambda - \Lambda(t)$ is positive definite, we have $\phi(e, \Lambda - \Lambda(t)) > 0$ for all $t \in T$ and $e \in \mathbb{S}^{N-1}$. The conclusion of the lemma follows from this and the fact that $\{(e, \Lambda - \Lambda(t)) : t \in T, e \in \mathbb{S}^{N-1}\}$ is a compact subset of $\mathbb{R}^N \times M_{N\times N}$ and ϕ is continuous. □

PROOF LEMMA 4.10. We only prove the first case, since the second case follows from the first one. By elementary computation on the joint density of $\xi_1(t)$ and $\xi_2(s)$, we obtain

$$
\begin{aligned}
&\sup_{t\in T_1, s\in T_2} \mathbb{E}\{(1 + |\xi_1(t)|^{N_1} + |\xi_2(s)|^{N_2})\mathbb{1}_{\{\xi_1(t)\geq u, \xi_2(s)<0\}}\} \\
&\quad \leq \frac{1}{2\pi \underline{\sigma}_1 \underline{\sigma}_2 (1-\overline{\rho}^2)^{1/2}} \int_u^\infty \exp\left\{-\frac{x_1^2}{2\overline{\sigma}_1^2}\right\} dx_1 \\
&\qquad \times \int_{-\infty}^0 (1 + |x_1|^{N_1} + |x_2|^{N_2}) \exp\left\{-\frac{1}{2\overline{\sigma}_2^2(1-\underline{\rho}^2)}\left(x_2 - \frac{\underline{\sigma}_2 \underline{\rho} x_1}{\overline{\sigma}_1}\right)^2\right\} dx_2 \\
&\quad = o\left(\exp\left\{-\frac{u^2}{2\overline{\sigma}_1^2} - \frac{\underline{\sigma}_2^2 \underline{\rho}^2 u^2}{2\overline{\sigma}_2^2(1-\underline{\rho}^2)\overline{\sigma}_1^2} + \varepsilon u^2\right\}\right),
\end{aligned}
$$

as $u \to \infty$, for any $\varepsilon > 0$. □

A similar argument for proving Lemma 4.9 yields the following result.

LEMMA A.1. *Let $\{A(t) = (a_{ij}(t))_{1\leq i,j\leq N} : t \in T\}$ be a family of positive definite matrices such that all elements $a_{ij}(\cdot)$ are continuous. Denote by $\underline{x}$ and $\overline{x}$ the infimum and supremum of the eigenvalues of $A(t)$ over $t \in T$, respectively, then $0 < \underline{x} \leq \overline{x} < \infty$.*

The following two formulas state the results on the Laplace approximation method. Lemma A.2 can be found in many books on the approximations of integrals; here we refer to Wong (2001). Lemma A.3 can be derived by following similar arguments in the proof of the Laplace method for the case of boundary point in Wong (2001).

LEMMA A.2 (Laplace method for interior point). *Let t_0 be an interior point of T. Suppose the following conditions hold*: (i) $g(t) \in C(T)$ *and* $g(t_0) \neq 0$; (ii) $h(t) \in C^2(T)$ *and attains its unique minimum at* t_0; *and* (iii) $\nabla^2 h(t_0)$ *is positive definite. Then as* $u \to \infty$,

$$
\int_T g(t)e^{-uh(t)}\, dt = \frac{(2\pi)^{N/2}}{u^{N/2}(\det \nabla^2 h(t_0))^{1/2}} g(t_0) e^{-uh(t_0)}(1 + o(1)).
$$

LEMMA A.3 (Laplace method for boundary point). *Let* $t_0 \in J \in \partial_k T$ *with* $0 \le k \le N-1$. *Suppose that conditions* (i), (ii) *and* (iii) *in Lemma* A.2 *hold, and additionally* $\nabla h(t_0) = 0$. *Then as* $u \to \infty$,

$$\int_T g(t)e^{-uh(t)}\,dt = \frac{(2\pi)^{N/2}\mathbb{P}\{Z_J(t_0) \in (-E(J))\}}{u^{N/2}(\det \nabla^2 h(t_0))^{1/2}} g(t_0)e^{-uh(t_0)}\big(1+o(1)\big),$$

where $Z_J(t_0)$ *is a centered* $(N-k)$-*dimensional Gaussian vector with covariance matrix* $(h_{ij}(t_0))_{J_1 \le i,j \le J_{N-k}}$, $-E(J) = \{x \in \mathbb{R}^N : -x \in E(J)\}$, *and the definitions of* $J_1, \dots, J_{N-k}$ *and* $E(J)$ *are in* (3.4).

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