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THE MAXIMUM MAXIMUM OF A MARTINGALE WITH GIVEN n MARGINALS

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We obtain bounds on the distribution of the maximum of a martingale with fixed marginals at finitely many intermediate times. The bounds are sharp and attained by a solution to n-marginal Skorokhod embedding problem in Obłój and Spoida [An iterated Azéma-Yor type embedding for finitely many marginals (2013) Preprint]. It follows that their embedding maximizes the maximum among all other embeddings. Our motivating problem is superhedging lookback options under volatility uncertainty for an investor allowed to dynamically trade the underlying asset and statically trade European call options for all possible strikes and finitely-many maturities. We derive a pathwise inequality which induces the cheapest superhedging value, which extends the two-marginals pathwise inequality of Brown, Hobson and Rogers [*Probab. Theory Related Fields* **119** (2001) 558–578]. This inequality, proved by elementary arguments, is derived by following the stochastic control approach of Galichon, Henry-Labordère and Touzi [*Ann. Appl. Probab.* **24** (2014) 312–336].

1. Introduction.

Probabilistic perspective. The problem of controlling the maximum of a continuous martingale using its terminal distribution has a long and rich history, starting with Doob's maximal inequalities. In seminal contributions, Blackwell and Dubins [7], Dubins and Gilat [21] and Azéma and Yor [3, 4] established that the distribution of the maximum $X_T^* := \sup_{t\le T} X_t$ of a martingale (X_t) is bounded above, in stochastic order, by the so-called Hardy–Littlewood transform of the distribution of X_T, and the bound is attained. This led to series of studies on the

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possible distributions of (X_T, X_T^*) including Gilat and Meilijson [24], Kertz and Rösler [31–33], Rogers [46] and Vallois [50]; see also Carraro, El Karoui and Obłój [13]. More recently, such problems appeared very naturally within the field of mathematical finance, as we explain below, which motivated further developments. The original result was generalized to the case of a nontrivial starting law in Hobson [29] and to the case of a fixed intermediate law in Brown, Hobson and Rogers [11].

In this paper, we generalize the above studies by controlling the maximum using several intermediate marginals. We consider the case when distributions of $X_{t_1}, \dots, X_{t_{n-1}}, X_{t_n}$ are given and establish an upper bound on the distribution of the maximum $X_{t_n}^*$. Our motivation comes from the mathematical finance problem of robust superhedging of a lookback option. We apply a general duality result from Possamaï et al. [45] which converts the original problem into a min-max calculus of variations problem where the Lagrange multipliers encode the intermediate marginal constraints. The multipliers have in fact an important financial interpretation as the optimal static positions in Vanilla options, which reduce the risk induced by the derivative security. Following Galichon, Henry-Labordère and Touzi [23], we apply stochastic control methods to solve the new problem explicitly. The first step of our solution recovers the extended optimal properties of the Azéma–Yor solution to the Skorokhod embedding problem (SEP) obtained by Hobson and Klimmek [27] (under slightly different conditions). The two marginal case corresponds to the work of Brown, Hobson and Rogers [11].

The stochastic control approach allows us to derive the upper bound on the distribution of $X_{t_n}^*$ in terms of the intermediate distributions $X_{t_1}, \dots, X_{t_{n-1}}, X_{t_n}$. To show that the bound is sharp, we need to construct a martingale that fits the given marginals and attains the bound. To do this we revert to the SEP methodology. First we derive the upper bound by taking expectations in a functional (pathwise) inequality, which in mathematical finance terms has the interpretation of a semi-static superhedging. This inequality is "guessed," and crucially, it is the stochastic control methodology that provides candidates for the static and the dynamic components of the optimal hedge, that is, the different terms in the pathwise inequality. Once postulated, the inequality is verified independently without any use of stochastic analysis tools. Finally, we show the optimality of the upper bound, under some technical assumptions on the marginals, by establishing that the solution to the n-marginal SEP obtained in Obłój and Spoida [41] achieves equality in our inequality. We note that the idea to derive martingale inequalities from pathwise inequalities was pivotal to the pioneering work on robust pricing and hedging of Hobson [25] and was recently underlined in Acciaio et al. [1].

Mathematical finance motivation. The problem we consider, as described above, has a clear motivation coming from mathematical finance. The classical framework underpinning much of the quantitative finance starts by postulating a stochastic universe $(\Omega, \mathbb{F}, \mathbb{P})$, which is meant to model a financial environment and capture its riskiness. What it fails to capture, however, is the uncertainty in the choice of $\mathbb{P}$, that is, the possibility that the model itself is wrong, also called *the Knightian uncertainty*; see Knight [34]. To account for model uncertainty it is natural to consider simultaneously a whole family $\{\mathbb{P}_\alpha : \alpha \in \mathcal{A}\}$ of probability measures. When all $\mathbb{P}_\alpha$ are absolutely continuous w.r.t. one reference measure $\mathbb{P}$, we speak of *drift uncertainty* or *dominated* setting. This has important implications for portfolio choice problems (see Föllmer, Schied and Weber [22]) but is not different from an incomplete market setup in terms of option pricing. However, the *nondominated* setup when $\mathbb{P}_\alpha$ may be mutually singular posed new challenges and was investigated starting with Avellaneda et al. [2] and Lyons [35], through Denis and Martini [19], to several recent works, for example, Peng [44], Soner, Touzi and Zhang [47], Dolinsky and Soner [20] and Bouchard and Nutz [9].

Naturally as one relaxes the classical setup, one has to abandon its precision: under model uncertainty we do not try to have a unique price but rather to obtain an interval of no-arbitrage prices. Its bounds are given by seller's and buyer's "safe" prices, the superreplication and the subreplication prices, which can be enforced by trading strategies that work in all considered models. These bounds can be made more efficient by enlarging the set of hedging instruments. Indeed, in the financial markets certain derivatives on the underlying we try to model are liquid and have well-defined market prices. Without one fixed model, these options can be included in traded assets without creating an arbitrage opportunity. By allowing one to trade dynamically in the underlying and statically (today) in a range of options, one hopes to have a more efficient approach with smaller intervals of possible no-arbitrage prices. This constitutes the basis of the so-called *robust approach to pricing and hedging*.

We contribute to this literature. Our aim was to derive in an explicit form the superhedging cost of a Lookback option given that the underlying asset is available for frictionless continuous-time trading, and that European options for all strikes are available for trading for a finite set of maturities. In a zero interest rate financial market, it essentially follows from the no-arbitrage condition, as observed by Breeden and Litzenberger [10], that these trading possibilities restrict the underlying asset price process into being a martingale with given marginals. Since a martingale can be written as a time changed Brownian motion, and the maximum of a continuous processes is not altered by such a time change, the one-marginal constraint version of this problem can be converted into the framework of the Skorokhod embedding problem (SEP). This observation is the starting point of the seminal paper by Hobson [25] who exploited the already known optimality result of the Azéma–Yor solution to the SEP and, importantly, provided an explicit static superhedging strategy. This methodology was subsequently used to derive robust prices and super/sub-hedging strategies for barrier options in Brown, Hobson and Rogers [12], for options on local time in Cox, Hobson and Obłój [14], for double barrier options in Cox and Obłój [15, 16] and for options on variance in Cox and Wang [17]; see Obłój [40] and Hobson [26] for more details.

The above works focused on finding explicitly robust prices and hedges for an option maturing at T and given market prices of call/put options co-maturing at T. For lookback options, an extension to the case where prices at one intermediate maturity are given can be deduced from Brown, Hobson and Rogers [11]. More recently, Hobson and Neuberger [28] treated forward starting straddle also using option prices at two maturities. Otherwise, and excluding the trivial cases when intermediate laws have no constraining effect (see, e.g., the iterated Azéma–Yor setting in Madan and Yor [36]), we are not aware of any explicit robust pricing/hedging results when prices of call options for several maturities are given. The most likely reason for this is that the SEP-based methodology pioneered in Hobson [25] starts with a good guess for the superhedge/embedding, and these become much more difficult when more marginals are involved. As explained above, our approach uses stochastic control methods to *derive* a candidate optimal superhedge strategy. On the dual side, it sees superhedging as a martingale transportation problem: maximize the expected coupling defined by the payoff so as to transport the Dirac measure along the given distributions $\mu_1, \dots, \mu_n$ by means of a continuous-time process restricted to be a martingale. This approach was simultaneously suggested by Beiglböck, Henry-Labordère and Penkner [5] in the discrete-time case, and Galichon, Henry-Labordère and Touzi [23] in continuous-time. We refer to Bonnans and Tan [8] for a numerical approximation in the context of variance options, and Tan and Touzi [49] for a general version of the optimal transportation problem under controlled dynamics.

Organization of the paper. The paper is organized as follows. Section 2 provides the precise mathematical formulation of the problem and establishes the relevant connections with martingale inequalities, the Skorohod embedding problem and the martingale optimal transport. The main results of this paper are collected in Section 3, starting from a remarkable pathwise inequality. The proofs are reported in Section 4. In particular, the pathwise inequality follows from an elementary verification. The stochastic control approach, which allowed us to derive the correct quantities for the pathwise arguments, is pursued in Section 5. Additional arguments for the one marginal case are given in Section 6. Finally, the Appendix contains some proofs of technical lemmas including some additional properties of the embedding obtained in [41].

2. Robust superhedging of Lookback options.

2.1. *Modeling the volatility uncertainty.* Let $\Omega_x := \{\omega \in C([0,T], \mathbb{R}^1) : \omega_0 = x\}$. We consider the set of paths $\Omega := \Omega_0$ as the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\le t\le T} |\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$ the filtration generated by B. Throughout the paper, X_0 is some given initial value in $\mathbb{R}$, and we denote

$$X_t := X_0 + B_t \qquad \text{for } t \in [0, T].$$

In order to model the volatility uncertainty, we introduce the set $\mathcal{P}$ of all probability measures on $(\Omega, \mathcal{F})$ such that B is a $\mathbb{P}$-martingale. The coordinate process stands for the price process of an underlying security, and the restriction to martingale measures $\mathcal{P}$ is motivated by the classical no-arbitrage results in mathematical finance and will be justified by the duality results in Theorems 3.3 and 3.5.

The quadratic variation process $\langle X\rangle = \langle B\rangle$ is universally defined and takes values in the set of all nondecreasing continuous functions with $\langle B\rangle_0 = 0$.

2.2. *Dynamic trading strategies.* For all $\mathbb{P} \in \mathcal{P}$, we denote by $\mathbb{H}^0(\mathbb{P})$ the collection of all $(\mathbb{P}, \mathbb{F})$-progressively measurable processes and

$$\mathbb{H}^2(\mathbb{P}) := \left\{ H \in \mathbb{H}^0(\mathbb{P}) : \int_0^T |H_t|^2 \, d\langle B\rangle_t < \infty, \mathbb{P}\text{-a.s.} \right\}.$$

A dynamic trading strategy is defined by a process $H \in \hat{\mathbb{H}}^2 := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathbb{H}^2(\mathbb{P})$, where H_t denotes the number of shares of the underlying asset held by the investor at each time $t \in [0, T]$. Under the self-financing condition, the portfolio value process induced by a dynamic trading strategy is

$$Y_t^H := Y_0 + \int_0^t H_s \, dB_s, \qquad t \in [0, T], \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}. \tag{2.1}$$

The stochastic integral in (2.1) is well defined and should be rather denoted $Y_t^{H^{\mathbb{P}}}$ to emphasize its dependence on $\mathbb{P}$; see, however, Nutz [38]. Nevertheless, for a large class of strategies H we may define Y_t^H pathwise. In particular, consider $H : \Omega_{X_0} \times [0, T] \to \mathbb{R}$ to be a process of finite variation which is *progressively measurable* in the sense that $H_t(\omega) = H_t(\omega')$ for any $t \in [0, T]$ and any $\omega, \omega' \in \Omega_{X_0}$ with $\omega_s = \omega'_s$, $s \le t$. We define its integral (see Dolinsky and Soner [20]) through an integration by parts formula using classical Stieltjes integration,

$$\int_0^t H_s(\omega) \, d\omega_s := H_t(\omega)\omega_t - H_0\omega_0 - \int_0^t \omega_s \, dH_s(\omega), \qquad t \in [0, T], \omega \in \Omega_{X_0}. \tag{2.2}$$

Note that this integral agrees a.s. with the Itô stochastic integral $\mathbb{P}$-a.s. for any $\mathbb{P} \in \mathcal{P}$. We will use this approach in particular in Section 2.5 and in Theorem 3.5.

2.3. *Semi-static hedging strategies.* Let n be some positive integer and $0 = t_0 < \cdots < t_n = T$ be some partition of the interval $[0, T]$. In addition to the continuous-time trading of the primitive securities, we assume that the investor can take static positions in European call or put options with all possible strikes and maturities $t_1 < \cdots < t_n$. The market price of the European call option with strike $K \in \mathbb{R}$ and maturity t_i is denoted

$$c_i(K), \qquad i = 1, \ldots, n \quad \text{and we denote} \quad c_0(K) := (X_0 - K)^+.$$

A model $\mathbb{P} \in \mathcal{P}$ is said to be calibrated to the market if $\mathbb{E}^{\mathbb{P}}[(X_{t_i} - K)^+] = c_i(K)$ for all $1 \le i \le n$ and $K \in \mathbb{R}$. For such a model, it was observed by Breeden and Litzenberger [10] that, by direct differentiation with respect to K,

$$\mathbb{P}(X_{t_i} > K) = -c_i'(K+) =: \mu_i\big((K, \infty)\big),$$

so that the marginal distributions of X_{t_i}, $i = 1, \dots, n$, are uniquely specified by the market prices and are independent of $\mathbb{P}$. Let $\mu = (\mu_1, \dots, \mu_n)$ and

$$\mathcal{P}^\mu := \{\mathbb{P} \in \mathcal{P} : X_{t_i} \sim \mu_i, 1 \le i \le n\}$$

be the set of calibrated market models. By the Strassen theorem [48], we have $\mathcal{P}^\mu \neq \varnothing$ if and only if μ_i's are nondecreasing in convex order or, equivalently,

$$\int |x|\, d\mu_i(x) < \infty, \qquad \int x\, d\mu_i(x) = X_0 \quad \text{and} \quad c_{i-1} \le c_i \qquad \text{for all } 1 \le i \le n, \tag{2.3}$$

where now $c_i(K) = \int_K^\infty (x - K)\, d\mu_i(x)$. The necessity follows from Jensen's inequality. For sufficiency, an explicit model can be constructed using techniques of Skorokhod embeddings; see Obłój [39]. Consequently, up to integrability, the t_i-maturity European derivative defined by the payoff $\lambda_i(X_{t_i})$ has an unambiguous market price

$$\mu_i(\lambda_i) := \int \lambda_i\, d\mu_i = \mathbb{E}^{\mathbb{P}}\big[\lambda_i(X_{t_i})\big] \qquad \text{for all } \mathbb{P} \in \mathcal{P}^\mu.$$

The condition $\mathcal{P}^\mu \neq \varnothing$ embodies the fact that the current observed market prices do not induce arbitrage. By this we mean that there exists a model which admits no-arbitrage (no free lunch with vanishing risk) and reprices the call options through risk neutral expectation. For this reason we sometimes refer to (2.3) as the *no-arbitrage condition*. We note, however, that the arbitrage considerations are more subtle in all generality since boundary cases may arise where $\mathcal{P}^\mu = \varnothing$ but there is no model-independent arbitrage; see Davis and Hobson [18] and Cox and Obłój [15].

REMARK 2.1. For the purpose of the present financial application, we could restrict the measures μ_i to have support in $\mathbb{R}_+$ and $\mathbb{P} \in \mathcal{P}$ to be such that $X_t \ge 0$ $\mathbb{P}$-a.s. Note, however, that this is easily achieved: it suffices to assume that $X_0 > 0$ and $c_n(K) = X_0 - K$ for $K \le 0$. Then $\mu_n((K, \infty)) = 1$, $K < 0$, and hence $\mu_n([0, \infty)) = 1$. Then for any $\mathbb{P} \in \mathcal{P}^\mu$ we have $X_t = \mathbb{E}^{\mathbb{P}}[X_T | \mathcal{F}_t] \ge 0$ $\mathbb{P}$-a.s. for $t \in [0, T]$. In particular, $\mu_i([0, \infty)) = \mathbb{P}(X_{t_i} \ge 0) = 1$.

We denote $\mathbf{t} := (t_1, \dots, t_n)$, $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\mu(\lambda) := \sum_{i=1}^n \mu_i(\lambda_i), \qquad \lambda(\omega_{\mathbf{t}}) := \sum_{i=1}^n \lambda_i(\omega_{t_i}), \tag{2.4}$$

for $\omega \in C([0,T])$. The set of Vanilla payoffs which may be used by the hedger are naturally taken in the set

$$\Lambda_n^\mu := \left\{\lambda : \int |\lambda_i| \, d\mu_i < \infty, 1 \le i \le n\right\}. \tag{2.5}$$

Thus, in addition to the dynamic hedging strategies H, the investor has access to the static hedging strategies λ, consisting of a buy-and-hold strategy in a portfolio of options. Such a pair (λ, H) is called a semi-static hedging strategy and induces the final value of the self-financing portfolio

$$\overline{Y}_T^{H,\lambda} := Y_T^H - \mu(\lambda) + \lambda(X_{\mathbf{t}}), \tag{2.6}$$

indicating that the investor has the possibility of buying at time 0 any derivative security with payoff $\lambda_i(X_{t_i})$ for the price $\mu_i(\lambda_i)$. As it will be made clear in our subsequent analysis, the functions λ_i will play the role of a Lagrange multiplier for the constraints $X_{t_i} \sim \mu_i$, $i = 1, \dots, n$.

2.4. *Robust superhedging under semi-static hedging strategies.* In this paper, we focus on the problem of robust semi-static superhedging of a lookback option defined by the payoff at the maturity T,

$\xi := \phi(X_T^*)$ where $X_t^* := \max_{s \le t} X_s$ and $\phi : \mathbb{R} \longmapsto \mathbb{R}$ is right-continuous, nondecreasing.

The investor can trade as discussed in the previous two sections. However, we need to impose a further admissibility condition to rule out doubling strategies. We let $\mathcal{H}^\mu$ consist of all processes $H \in \hat{\mathbb{H}}^2$ whose induced portfolio value process Y^H is $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}^\mu$. The robust superhedging upper bound is then defined by

$$U_n^\mu(\xi) := \inf\{Y_0 : \exists (\lambda, H) \in \Lambda_n^\mu \times \mathcal{H}^\mu, \overline{Y}_T^{H,\lambda} \ge \xi, \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}\}. \tag{2.7}$$

Selling ξ at a price higher than $U_n^\mu(\xi)$, the hedger could set up a portfolio with a negative initial cost and a nonnegative payoff under any market scenario leading to a strong (model-independent) arbitrage opportunity. Note that, thanks to the definition of admissible trading strategies, by taking expectations in the superhedging inequality $\overline{Y}_T^{H,\lambda} \ge \xi$, and optimizing over (λ, H) and $\mathbb{P}$, we obtain the usual pricing-hedging inequality

$$U_n^\mu(\xi) \ge \sup_{\mathbb{P} \in \mathcal{P}^\mu} \mathbb{E}^{\mathbb{P}}[\xi]. \tag{2.8}$$

Theorems 3.3 and 3.5 below establish a bound on the RHS value and show that under mild technical conditions equality holds in (2.8) and exhibit both the best superhedge and the maximal $\mathbb{P}^{\max}$ on the RHS. However, before proceeding to our main results, we first discuss the connection of the superhedging problem with two important questions in applied mathematics, namely martingale inequalities in probability and the theory of optimal transport.

2.5. *Pathwise super-hedging and martingale inequalities.* In this section, we discuss briefly the connection between the robust super-hedging problem and pathwise inequalities inducing martingale inequalities, as highlighted by Acciaio et al. [1] and Beiglböck and Nutz [6]; see also Osękowski [43]. Suppose that existence holds in the super hedging problem (2.7). Then $\overline{Y}_0^{H,\lambda} = U_n^\mu(\xi)$ and $\overline{Y}_T^{H,\lambda} \geq \xi$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$, for some $\lambda \in \Lambda_n^\mu$ and $H \in \mathcal{H}^\mu$. Assume further that the dynamic hedging process H is such that its "stochastic integral" can be defined pathwise and that the super-hedging property extends to a pathwise inequality

$$\xi(\omega) \leq U_n^\mu(\xi) + \lambda(\omega_{\mathbf{t}}) - \mu(\lambda) + \int_0^T H_s \, d\omega(s) \qquad \text{for all } \omega \in \Omega_{X_0}. \tag{2.9}$$

This inequality is then sharp in the sense that the above pair (λ, H) minimizes the cost $\mu(\lambda)$ of the trading strategy among all such strategies which super-hedge ξ pathwise. In particular, $U_n^\mu(\xi)$ is the superhedging price not only in the sense of (2.7) but also in the pathwise sense. Assuming further that the stochastic integral in (2.9) defines a uniformly integrable martingale, this implies a martingale inequality

$$\begin{aligned}&\mathbb{E}[\xi(Y)] \leq U_n^{\mathcal{L}_{\mathbf{t}}^Y}(\xi) \text{ for any continuous martingale } (Y_t)_{t \leq T},\\ &\text{where } \mathcal{L}_{\mathbf{t}}^Y = (\mathcal{L}^{Y_{t_i}}, i \leq n),\end{aligned}$$

and $\mathcal{L}^{Y_{t_i}}$ is the distribution of Y_{t_i} and $Y_0 = X_0$. Moreover, by construction, this inequality is sharp: we can construct continuous martingales Y which attain equality. An example of such martingale inequality is provided in Proposition 3.2 below.

2.6. *Optimal transportation under martingale restriction.* In this short section we discuss the connection of our problem to optimal transportation theory, which will be the building block for the stochastic control approach of Section 5.

The first duality we consider is the expected quasi-sure extension of the classical dual formulation of the superhedging problem

$$U_n^\mu(\xi) = \inf_{\lambda \in \Lambda_n^\mu} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi + \lambda(X_{\mathbf{t}}) - \mu(\lambda)].$$

We show it holds, subject to technical assumptions, in Proposition 5.2 below. The second duality follows by formally inverting the inf-sup on the RHS above leading to the optimization problem

$$\sup_{\mathbb{P} \in \mathcal{P}^\mu} \mathbb{E}^{\mathbb{P}}[\xi], \tag{2.10}$$

which falls into the recently introduced class of optimal transportation problems under controlled stochastic dynamics; see Beiglböck, Henry-Labordère and Penkner [5], Galichon, Henry-Labordère and Touzi [23] and Tan and Touzi [49]. In words, the above problem consists of maximizing the expected transportation cost of the Dirac measure $\delta_{\{X_0\}}$ along the given marginals $\mu_1, \dots, \mu_n$ with transportation scheme constrained to the class of martingales. The cost of transportation in our context is defined by the path-dependent payoff $\xi(\omega)$.

The validity of the equality between the value function in (2.10) and our problem $U_n^\mu(\xi)$ was established recently by Dolinsky and Soner [20] for $n = 1$ and under strong continuity assumptions on the payoff function $\omega \longmapsto \xi(\omega)$. The corresponding duality result in the discrete time framework was obtained by Beiglböck, Henry-Labordère and Penkner [5].

Note that if we can find a trading strategy $\overline{Y}_T^{H,\lambda}$ as in (2.6) which superreplicates ξ: $\overline{Y}_T^{H,\lambda} \ge \xi$ $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}$ and a $\mathbb{P}^{\max} \in \mathcal{P}^\mu$ such that $\mathbb{E}^{\mathbb{P}^{\max}}[\xi] = Y_0$, then as in (2.8),

$$Y_0 \le \sup_{\mathbb{P} \in \mathcal{P}^\mu} \mathbb{E}^{\mathbb{P}}[\xi] \le U_n^\mu(\xi) \le Y_0,$$

and it follows that we have equalities throughout. This line of attack has been at the heart of the approach to robust pricing and hedging based on the Skorokhod embedding problem, as in Hobson [25], Brown, Hobson and Rogers [12], Cox and Obłój [15, 16] and Cox and Wang [17]. It relies crucially on the ability to make a correct guess for the cheapest superhedge $\overline{Y}_T^{H,\lambda}$. This becomes increasingly difficult when one considers information about prices at several maturities, $n > 1$. In this paper, we follow the above methodology in Section 4 to prove our main result, Theorem 3.5. Sections 5–6 then provide an alternative approach based on stochastic control methods. The latter is longer and more involved than the former and requires slight modifications for technical reasons. However, it is in fact necessary as it is the origin of the determination of the right quantities for the former, namely the pathwise inequality.

2.7. *The Skorokhod embedding problem.* We now specialize the discussion to the case of a lookback option $\xi = G(X_{\mathbf{t}}, X_T^*)$, for some payoff function G. By the Dambis–Dubins–Schwarz theorem, we may re-write problem (2.10) as a multiple stopping problem (see Proposition 3.1 of Galichon, Henry-Labordère and Touzi [23]),

$$\sup_{(\tau_1,\dots,\tau_n) \in \mathcal{T}^\mu} \mathbb{E}^{\mathbb{P}_0}\big[G\big(X_{\tau_1}, \dots, X_{\tau_n}, X_{\tau_n}^*\big)\big], \tag{2.11}$$

where the $\mathcal{T}^\mu$ is the set of ordered stopping times $\tau_1 \le \cdots \le \tau_n < \infty$ $\mathbb{P}_0$-a.s. with $X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$ for all $i = 1, \dots, n$ and $(X_{t \wedge \tau_n})$ being a uniformly integrable martingale. Elements of $\mathcal{T}^\mu$ are solutions to the iterated (multi-marginal) version of the so-called Skorokhod embedding problem (SEP); cf. Obłój [39]. Here, formulation (2.11) is directly searching for a solution to the SEP which maximizes the criterion defined by the coupling $G(x, m)$. Previous works have focused mainly on single marginal constraint ($n = 1$). The case $G(x, m) = \phi(m)$ for some nondecreasing function ϕ is solved by the so-called Azéma–Yor embedding; cf. Azéma and Yor [3, 4], Hobson [25]; see also Galichon, Henry-Labordère and Touzi [23] who recovered this result by the stochastic-control approach of Section 5. The case $G(x, m)$ was considered recently by Hobson and Klimmek [27], where the optimality of the Azéma–Yor solution of the SEP is shown to be valid under convenient conditions on the function G. This case is also solved in Section 6 of the present paper, with our approach leading to the same results as those obtained by Hobson and Klimmek [27], but under slightly different assumptions.

The case $G(x_1, \dots, x_n, m) = \phi(m)$ for some nondecreasing function ϕ is also trivially solved by $\tau^{AY}(\mu_n)$ in the following special case when the single marginal solutions are naturally ordered: $\tau^{AY}(\mu_i) \le \tau^{AY}(\mu_{i+1})$. This is called the *increasing mean residual value property* by Madan and Yor [36] who establish, in particular, a strong Markov property of the resulting time-changed process. The case of arbitrary measures which satisfy (2.3) for $n = 2$ was solved in Brown, Hobson and Rogers [11]. In this paper we consider $n \in \mathbb{N}$. For subsequent use, we recall that the Azéma–Yor embedding for μ_i is given by $\tau^{AY}(\mu_i) = \inf\{t \ge 0 : X_t \le b_i^{-1}(X_t^*)\}$, where the inverse barycentre function $b_i^{-1}(m)$ is a minimizer in

$$\min_{\zeta \le m} \frac{c_i(\zeta)}{m - \zeta}, \qquad m > X_0, \tag{2.12}$$

taken to be right-continuous in m and with $b_i^{-1}(m) = m$ for $m \ge \inf\{m : c_i(m) = 0\}$. It is easy to see that b_i^{-1} is nondecreasing. Note also that $c_i(\zeta)/(m - \zeta)$ is nonincreasing for $\zeta \le b_i^{-1}(m)$ and nondecreasing for $\zeta \ge b_i^{-1}(m)$.

3. Main results. Our main result is split into three parts. We first state a trajectorial inequality which is the building block for the solution of the robust superhedging problem. We next solve the pricing problem (2.10) in Theorem 3.3. Finally, in Theorem 3.5, we solve the superhedging problem (2.7).

3.1. *A remarkable trajectorial inequality.* The first result involves, for all $\zeta_1 \le \dots \le \zeta_n < m$, the semi-static hedging strategy

$$\lambda_i^{\zeta,m}(x) := \frac{(x - \zeta_i)^+}{m - \zeta_i} - \mathbf{1}_{\{i<n\}} \frac{(x - \zeta_{i+1})^+}{m - \zeta_{i+1}}, \qquad x \in \mathbb{R}, \tag{3.1}$$

$$H_t^{\zeta,m}(\omega) := -\frac{\mathbf{1}_{(t_{i-1},t]}(T_m(\omega)) + \mathbf{1}_{[0,t_{i-1}]}(T_m(\omega))\mathbf{1}_{\{\omega_{t_{i-1}} \ge \zeta_i\}}}{m - \zeta_i}, \qquad t \in [t_{i-1}, t_i), \tag{3.2}$$

for all $i = 1, \dots, n$, where $T_m(\omega) := \inf\{t \ge 0 : \omega_t \ge m\}$. Notice that $H^{\zeta,m}$ is a piecewise constant predictable process, so that the stochastic integral $\int H_t(\omega)\, d\omega_t$ is a well-defined, pathwise, finite sum.

PROPOSITION 3.1. *Let ω be a càdlàg path, $m > \omega_0$, and denote $\omega_t^* := \sup_{0\le s\le t}\omega_s$. Then, for all $\zeta_1 \le \cdots \le \zeta_n < m$, the following inequality holds*:

$$\mathbf{1}_{\{\omega_{t_n}^* \ge m\}} \le \sum_{i=1}^n \left\{ \lambda_i^{\zeta,m}(\omega_{t_i}) + \int_{t_{i-1}}^{t_i} H_t^{\zeta,m}(\omega)\,d\omega_t \right\}. \tag{3.3}$$

The proof is reported in Section 4.1 and is based on an elementary verification. The importance of this inequality is that it will be shown to be sharp in some precise sense, so that the solution of the robust superhedging problem is fully deduced from it. Thereore, the relevant difficulty is on how this inequality can be guessed. This issue is addressed in Section 5, where our intention is to show that the stochastic control approach is genuinely designed for this purpose.

The pathwise inequality of Proposition 3.1 is stated for the elementary lookback option defined by the payoff function $\phi = \mathbf{1}_{[m,\infty)}$. The corresponding extension to a general right-continuous, nondecreasing function ϕ follows by the obvious identity

$$\phi(\omega_{t_n}^*) = \phi(\omega_0) + \int_{(\omega_0,\infty)} \mathbf{1}_{\{\omega_{t_n}^* \ge m\}}\,d\phi(m). \tag{3.4}$$

3.2. *Financial interpretation*. We develop now a financial interpretation of the RHS of (3.3), for $\omega = X$ the price process, as a (pathwise) superhedging strategy for a simple knock-in digital barrier option with payoff $\xi = \mathbf{1}_{\{X_T^* \ge m\}}$. The semi-static hedging strategy consists of three elements: a static position in call options, a forward transaction (with the shortest available maturity) when the barrier m is hit and rebalancing thereafter at times t_i. More precisely:

(i) *Static position in calls*:

$$\lambda^{\zeta,m}(X_{\mathbf{t}}) := \sum_{i=1}^n \lambda_i^{\zeta,m}(X_{t_i}).$$

For $1 \le i < n$, we hold a portfolio long and short calls with maturity t_i and strikes ζ_i and ζ_{i+1}, respectively. This yields a "tent like" payoff which becomes negative only if the underlying exceeds level m. Note that by setting $\zeta_i = \zeta_{i+1}$ we may avoid trading the t_i-maturity calls. For maturity t_n we are only long in a call with strike ζ_n.

(ii) *Forward transaction if the barrier m is hit*:

$$\int_{t_{i-1}}^{t_i} H_t^{\zeta,m}(X)\,dX_t = \frac{m - X_{t_i}}{m-\zeta_i} \qquad \text{on } \{t_{i-1} < T_m(X) \le t_i\} = \{X_{t_{i-1}}^* < m \le X_{t_i}^*\}.$$

At the moment when the barrier m is hit,[4] say between maturities t_{i-1} and t_i, we enter into forward contracts with maturity t_i. Note that the long call position with

[4]This is well defined since we only consider continuous paths. Note, however, that even if the process was allowed to jump over the level m, the superhedging property would be preserved and only an additional profit would be realized.

maturity t_i together with the forward then superhedge the knock-in digital barrier option; cf. (4.3). This resembles the robust semi-static hedge in the one-marginal case; cf. Lemma 2.4 of Brown, Hobson and Rogers [11]. All the "tent like" payoffs up to maturity t_{i-1} are nonnegative.

(iii) *Rebalancing of portfolio to hedge calendar spreads*:

$$\int_{t_i}^{t_n} H_t^{\zeta,m}\,dX_t = \sum_{j=i}^{n} \mathbf{1}_{\{X_{t_j}\geq\zeta_{j+1}\}} \frac{X_{t_j}-X_{t_{j+1}}}{m-\zeta_{j+1}} \qquad \text{on } \{t_{i-1} < T_m(X) \leq t_i\}.$$

After the barrier m is hit between t_{i-1} and t_i, we start trading at times t_j, $j \geq i$, in such a way that a potential negative payoff of the calendar spreads $\frac{(X_{t_j}-\zeta_j)^+}{m-\zeta_j} - \frac{(X_{t_{j-1}}-\zeta_j)^+}{m-\zeta_j}$, $i < j \leq n$, is offset; cf. (4.2).

In the above, (ii) and (iii) are instances of dynamic trading which is done in a self-financing way. Their combined payoff is $\int_0^T H_s^{\zeta,m}\,dX_s$, and inequality (3.3) simply says that for any choice of $\zeta_1 \leq \cdots \leq \zeta_n < m$, the semi-static hedging strategy $(H^{\zeta,m}, \lambda^{\zeta,m})$ superreplicates ξ.

3.3. *Martingale inequalities and robust superhedging*. As a first consequence of the trajectorial inequality of Proposition 3.1, we have the following martingale inequality involving finitely-many intermediate marginals.

PROPOSITION 3.2. *Let Y be a càdlàg submartingale defined on a filtered probability space satisfying the usual conditions, and consider an arbitrary right-continuous nondecreasing function ϕ. Then, for any functions $\zeta_1(m) \leq \cdots \leq \zeta_n(m) \leq m$, we have*

$$\mathbb{E}[\phi(Y_{t_n}^*)] \leq \phi(Y_0) + \int_{(X_0,\infty)} \sum_{i=1}^{n} \biggl(\frac{\mathbb{E}[(Y_{t_i}-\zeta_i(m))^+]}{m-\zeta_i(m)} - \frac{\mathbb{E}[(Y_{t_i}-\zeta_{i+1}(m))^+]}{m-\zeta_{i+1}(m)} \mathbf{1}_{\{i<n\}} \biggr)\,d\phi(m),$$

where $Y_t^* := \sup_{u\leq t} Y_u$.

PROOF. Taking expectation in inequality (3.3) for any $\zeta_1 \leq \cdots \leq \zeta_n < m$, we see that

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{Y_{t_n}^*\geq m\}}] &\leq \sum_{i=1}^{n} \biggl\{ \mathbb{E}[\lambda^{\zeta,m}(Y_{t_i})] \\ &\qquad - \frac{\mathbb{E}[Y_{t_i\vee T_m} - Y_{t_{i-1}\vee T_m} + \mathbf{1}_{\{T_m\leq t_{i-1}, Y_{t_{i-1}}\geq\zeta_i\}}(Y_{t_i}-Y_{t_{i-1}})]}{m-\zeta_i} \biggr\} \\ &\leq \sum_{i=1}^{n} \mathbb{E}[\lambda^{\zeta,m}(Y_{t_i})],\end{aligned}$$

by the submartingale property of Y. Taking limits, the above extends to any $\zeta_1 \le \cdots \le \zeta_n \le m$ giving the required inequality for $\phi = \mathbf{1}_{[m,\infty)}$. Then, for a right-continuous nondecreasing function ϕ, we take expectation in (3.4), and we conclude using Fubini and the last inequality. $\square$

The particular case $\phi = \mathbf{1}_{[m,\infty)}$ provides an upper bound on $\mathbb{P}[Y^*_{t_n} \ge m]$. Note also that for some Y, the RHS may reduce to a much simpler form. In particular if Y is stopped at t_1, $Y_t = Y_{t\wedge t_1}$ for $t \ge 0$, then the RHS reduces to the one marginal case. We explore martingale inequalities of the above form and their usefulness in a short parallel note [42].

The key ingredient for the solution of the present n-marginals robust superhedging problem is to re-write the upper bound in the last martingale inequality in terms of the call prices

$$\sum_{i=1}^n \mathbb{E}^{\mathbb{P}}[\lambda^{\zeta,m}(X_{t_i})] = \sum_{i=1}^n \left(\frac{c_i(\zeta_i)}{m-\zeta_i} - \frac{c_i(\zeta_{i+1})}{m-\zeta_{i+1}}\mathbf{1}_{\{i<n\}}\right) \qquad \text{for all } \mathbb{P} \in \mathcal{P}^\mu.$$

By the arbitrariness of the parameters ζ, we are then reduced to the best upper bound

$$C(m) := \min_{\zeta_1\le\cdots\le\zeta_n\le m} \sum_{i=1}^n \left(\frac{c_i(\zeta_i)}{m-\zeta_i} - \frac{c_i(\zeta_{i+1})}{m-\zeta_{i+1}}\mathbf{1}_{\{i<n\}}\right) \qquad \text{for all } m > X_0, \tag{3.5}$$

where here, and throughout, we understand the value of the sum on the RHS of (3.5) for $\zeta_k < \zeta_{k+1} = \cdots = \zeta_n = m$ as limit of the value $\zeta_{k+1} = \cdots = \zeta_n = \zeta \to m$ which is clearly either $+\infty$ or is well defined in terms of the derivative of the call function at m.

THEOREM 3.3. *Let ϕ be a right-continuous nondecreasing function.*

(i) *Under the no-arbitrage condition* (2.3), *we have*

$$\sup_{\mathbb{P}\in\mathcal{P}^\mu} \mathbb{E}^{\mathbb{P}}[\phi(X^*_T)] \le \phi(X_0) + \int_{(X_0,\infty)} C(m)\, d\phi(m). \tag{3.6}$$

(ii) *If in addition $\mu_1, \dots, \mu_n$ satisfy Assumption ⊛ of Obłój and Spoida* [41], *then equality holds in* (3.6) *and is attained by some $\mathbb{P}^{\max} \in \mathcal{P}^\mu$.*

Part (i) of the last theorem is a direct consequence of Proposition 3.2. The proof of part (ii) is reported in Section 4.3. The upper bound (3.6) holds in all generality; in particular, both sides could be infinite.

We next focus on the existence of a semi-static superhedging strategy which induces upper bound (3.6). For the following preliminary result, we recall the inverse barycenter functions b_i introduced by (2.12).

LEMMA 3.4. *There exists a measurable minimiser $\zeta^*(m)$ for the optimization problem* (3.5) *with $\zeta_1^*(m) \geq \underline{b}^{-1}(m) := \min_{1\leq i\leq n} b_i^{-1}(m)$.*

The proof of this lemma is reported in Appendix A. Given these minimizing functions $m \longmapsto \zeta^*(m)$, we deduce from Proposition 3.1 together with (3.4) the following candidates for the optimal semi-static hedging strategies:

$$
\begin{aligned}
\hat{\lambda}_i &:= \int_{(X_0,\infty)} \lambda_i^{\zeta_i^*(m),m}\, d\phi(m), \qquad i=1,\dots,n,\\
\hat{H}_t &:= \int_{(X_0,\infty)} H_t^{\zeta^*(m),m}\, d\phi(m),
\end{aligned}
\tag{3.7}
$$

where we will impose further assumptions on ϕ under which these integrals are well defined. Note that from the definition of T_m it follows that

$$
\begin{aligned}
-\hat{H}_t = {}& \int_{(\omega^*_{t_{i-1}},\omega^*_t]} \frac{d\phi(m)}{m-\zeta_i^*(m)}\\
&+ \int_{(X_0,\omega^*_{t_{i-1}}]} \mathbf{1}_{\{\omega_{t_{i-1}}\geq \zeta_i^*(m)\}} \frac{d\phi(m)}{m-\zeta_i^*(m)}, \qquad t\in[t_{i-1},t_i).
\end{aligned}
\tag{3.8}
$$

The first term is nondecreasing in $t\in[t_{i-1},t_i)$ while the second one is constant. It follows that $\hat{H}$ is of finite variation. It is also clear that $\hat{H}$ is progressively measurable in the sense introduced in Section 2.2 and hence the stochastic integral $\int \hat{H}_t\, d\omega_t$ is well-defined pathwise for $\omega\in\Omega_{X_0}$ via (2.2).

We now show that, under mild technical assumptions, equality holds in (2.8) and that $(\hat{\lambda},\hat{H})$ is optimal and attains the minimal superhedging cost.

THEOREM 3.5. *Assume that the no-arbitrage condition* (2.3) *holds, and let $\zeta_1^*(m),\dots,\zeta_n^*(m)$ be defined by Lemma* 3.4. *Let ϕ be a right-continuous nondecreasing function such that $m\mapsto (m-\zeta_n^*(m))^{-1}$ is $d\phi$-locally integrable, $\hat{\lambda}$ and $\hat{H}$ be given by* (3.7), *and assume that*

$$
\int_{\mathbb{R}}\int_{(X_0,\infty)} \frac{(x-\zeta_i^*(m))^+}{m-\zeta_i^*(m)}\, d\phi(m)\mu_i(dx) < \infty, \qquad i=1,\dots,n,
\tag{3.9}
$$

which is equivalent to $\hat{\lambda}\in\Lambda_n^\mu$. Then $\hat{H}\in\mathcal{H}^\mu$ and:

(i) $U_n^\mu(\phi(X_T^*)) \leq \phi(X_0)+\mu(\hat{\lambda}) = \phi(X_0)+\int_{(X_0,\infty)} C(m)\,d\phi(m) < \infty$, *and*

$$
\phi(\omega_T^*) \leq \phi(X_0) + \hat{\lambda}(\omega_{\mathbf{t}}) + \int_0^T \hat{H}_t(\omega)\, d\omega_t \qquad \text{for all } \omega\in\Omega_{X_0};
\tag{3.10}
$$

(ii) *if, in addition, $(\mu_i)_{1\leq i\leq n}$ satisfy Assumption $\circledast$ of Obłój and Spoida* [41], *then $U_n^\mu(\phi(X_T^*)) = \phi(X_0)+\mu(\hat{\lambda})$, and equality holds in* (3.10) *$\mathbb{P}^{\max}$-a.s., for $\mathbb{P}^{\max}\in\mathcal{P}^\mu$ described in the proof.*

The proof is reported in Section 4.4. Note that when $\hat{\lambda} \in \Lambda_n^{\mu}$ and $\hat{H} \in \mathcal{H}^{\mu}$, claim (i) is a direct consequence of the pathwise inequality of Proposition 3.1 together with (3.4) and the assumed conditions.

REMARK 3.6. It follows from Section 4 of Obłój and Spoida [41] that if their Assumption $\circledast$ fails, then bound (3.6) is not necessarily optimal, and Theorem 3.5(ii) fails.

REMARK 3.7. In the case $\phi = \sum_{j=1}^{J} \mathbf{1}_{[m_j,\infty)}$ for some $m_j \geq X_0$ with $\zeta_j^*(m_j) < m_j$, the conditions of Theorem 3.5 are all easily satisfied; that is, both the local integrability condition above and the requirement $\hat{\lambda}_i \in \mathbb{L}^1(\mu_i)$ are immediate for $i = 1, \dots, n$.

4. Proofs of the main results.

4.1. *Proof of the pathwise inequality of Proposition* 3.1. Our objective is to prove by induction the following trajectorial inequality:

$$\begin{aligned}
\mathbf{1}_{\{\omega_{t_n}^* \geq m\}} &\leq \Upsilon_n(\omega, m, \zeta) \\
&:= \sum_{i=1}^{n} \biggl(\frac{(\omega_{t_i} - \zeta_i)^+}{m - \zeta_i} + \mathbf{1}_{\{\omega_{t_{i-1}}^* < m \leq \omega_{t_i}^*\}} \frac{(m - \omega_{t_i})}{m - \zeta_i} \biggr) \\
&\quad + \sum_{i=1}^{n-1} \biggl(\frac{(\omega_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}} + \mathbf{1}_{\{m \leq \omega_{t_i}^*, \zeta_{i+1} \leq \omega_{t_i}^*\}} \frac{(\omega_{t_{i+1}} - \omega_{t_i})}{m - \zeta_{i+1}} \biggr),
\end{aligned}$$

which is immediately seen to imply the required inequality.

For simplicity we omit the arguments (ω, m, ζ) for Υ_n below. First, in the case $n = 1$, the required inequality is the same as that of Lemma 2.1 of Brown, Hobson and Rogers [11]:

$$\begin{aligned}
\Upsilon_1 &= \frac{(\omega_{t_1} - \zeta_1)^+ + \mathbf{1}_{\{\omega_{t_0}^* < m \leq \omega_{t_1}^*\}}(m - \omega_{t_1})}{m - \zeta_1} \\
&\geq \frac{\omega_{t_1} - \zeta_1 + m - \omega_{t_1}}{m - \zeta_1} \mathbf{1}_{\{m \leq \omega_{t_1}^*\}} \qquad (4.1) \\
&\geq \mathbf{1}_{\{m \leq \omega_{t_1}^*\}}.
\end{aligned}$$

We next assume that $\Upsilon_{n-1} \geq \mathbf{1}_{\{\omega_{t_{n-1}}^* \geq m\}}$ for some $n \geq 2$, and show that $\Upsilon_n \geq \mathbf{1}_{\{\omega_{t_n}^* \geq m\}}$. We consider two cases:

Case 1: $\omega_{t_{n-1}}^* \geq m$. Then $\omega_{t_n}^* \geq m$, and it follows from the induction hypothesis that $1 = \mathbf{1}_{\{\omega_{t_n}^* \geq m\}} = \mathbf{1}_{\{\omega_{t_{n-1}}^* \geq m\}} \leq \Upsilon_{n-1}$. In order to see that $\Upsilon_{n-1} \leq \Upsilon_n$, we compute directly that, in the present case,

$$\Upsilon_n - \Upsilon_{n-1} = \frac{\omega_{t_n} - \zeta_n}{m - \zeta_n}(\mathbf{1}_{\{\omega_{t_n} \geq \zeta_n\}} - \mathbf{1}_{\{\omega_{t_{n-1}} \geq \zeta_n\}}) \geq 0. \tag{4.2}$$

Case 2: $\omega^*_{t_{n-1}} < m$. As (ω^*_t) is nondecreasing, it follows that $\omega^*_{t_i} < m$ for all $i \leq n-1$. With a direct computation we obtain

$$\Upsilon_n = \Upsilon_n^0 + \frac{(\omega_{t_n} - \zeta_n)^+}{m - \zeta_n} + \mathbf{1}_{\{m \leq \omega^*_{t_n}\}} \frac{m - \omega_{t_n}}{m - \zeta_n}$$

$$\text{where } \Upsilon_n^0 := \sum_{i=1}^{n-1} \left(\frac{(\omega_{t_i} - \zeta_i)^+}{m - \zeta_i} - \frac{(\omega_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}} \right).$$

Since $m > \omega^*_{t_i} \geq \omega_{t_i}$ for $i \leq n-1$, the functions $\zeta \longmapsto (\omega_{t_i} - \zeta)^+/(m - \zeta)$ are nonincreasing. This implies that $\Upsilon_n^0 \geq 0$ by the fact that $\zeta_i \leq \zeta_{i+1}$ for all $i \leq n$. Then

$$\begin{aligned}
\Upsilon_n &\geq \frac{(\omega_{t_n} - \zeta_n)^+ + \mathbf{1}_{\{m \leq \omega^*_{t_n}\}}(m - \omega_{t_n})}{m - \zeta_n} \\
&\geq \frac{(\omega_{t_n} - \zeta_n)^+ + m - \omega_{t_n}}{m - \zeta_n} \mathbf{1}_{\{m \leq \omega^*_{t_n}\}} \geq \frac{\omega_{t_n} - \zeta_n + m - \omega_{t_n}}{m - \zeta_n} \mathbf{1}_{\{m \leq \omega^*_{t_n}\}} \qquad (4.3) \\
&= \mathbf{1}_{\{m \leq \omega^*_{t_n}\}}.
\end{aligned}$$

□

4.2. *The iterated Azéma–Yor-type embedding of Obłój and Spoida* [41]. Before we proceed to the proof of Theorems 3.3 and 3.5, we recall the iterated Azéma–Yor-type embedding of Obłój and Spoida [41]. This embedding will allow us to identify the extremal model in the context of these theorems.

Under their Assumption ⊛, Obłój and Spoida [41] extend the Azéma–Yor embedding for $\mu_1, \ldots, \mu_n$ by introducing the stopping times based on some functions $\eta_1, \ldots, \eta_n$,

$$\tau_0 = 0 \quad \text{and} \quad \tau_i := \inf\{t \geq \tau_{i-1} : X_t \leq \eta_i(X^*_t)\}, \qquad i = 1, \ldots, n. \tag{4.4}$$

Theorem 2.6 therein asserts that for η obtained from an iterative optimization problem (these functions are called $\xi_1, \ldots, \xi_n$ in [41]), we have $X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$, and $(X_{t \wedge \tau_n})$ is uniformly integrable. Consider a time change of X.

$$Z_t := X_{\tau_i \wedge (\tau_{i-1} \vee (t - t_{i-1})/(t_i - t))} \qquad \text{for } t_{i-1} < t \leq t_i, i = 1, \ldots, n \tag{4.5}$$

with $Z_0 = X_0$, and observe that Z is a continuous, uniformly integrable martingale on $[0, t_n]$ with $Z_{t_i} = X_{\tau_i} \sim_{\mathbb{P}_0} \mu_i$. As a consequence, the distribution of Z, $\mathbb{P}^{\max} := \mathbb{P}_0 \circ (Z)^{-1}$, is an element of $\mathcal{P}^{\mu}$.

We shall argue in Appendix D that

$$\zeta^*_i(m) = \min_{j \geq i} \eta_j(m) \qquad \text{for all } i \leq n, X_0 < m < \inf\{y : c_n(y) = 0\}. \tag{4.6}$$

Then note that

$$C(m) = K_n(m) \tag{4.7}$$

for the continuous and nonincreasing function K_n defined by Obłój and Spoida [41]; see Lemma 2.14 therein.

Optimality of these iterated Azéma–Yor-type embeddings will follow from the fact that they attain a.s. equality in the pathwise inequality (3.3). We state this in a greater generality:

LEMMA 4.1 (Pathwise equality). *Let $(\eta_i)_{1\le i\le n}$ be nondecreasing, right-continuous functions, and $\zeta_i := \min_{j\ge i}\eta_j$. Let $(\tau_i)_{1\le i\le n}$ be the corresponding stopping times defined by* (4.4), *and Z the process defined by* (4.5). *Assume that $(X_{t\wedge\tau_n}: t\ge 0)$ is $\mathbb{P}_0$-uniformly integrable. Then, for any $m > Z_0$ with $\eta_n(m) < m$, Z achieves equality in* (3.3),

$$\mathbf{1}_{\{Z^*_{t_n}(\omega)\ge m\}} = \Upsilon_n\big(Z(\omega), m, \zeta(m)\big) \qquad \forall \omega \in \Omega_{X_0}. \tag{4.8}$$

PROOF. See Appendix D. □

4.3. *Proof of Theorem* 3.3(ii). Assume that (μ_i) satisfy Assumption ⊛ of Obłój and Spoida [41]. Recall that (4.6) then holds. First, suppose $\zeta_n^*(m) < m$ $d\phi(m)$-a.e. Then it follows directly from Lemma 4.1 and the proof of Proposition 3.2 that we have equality in (3.6) which is attained by $\mathbb{P}^{\max} := \mathbb{P}_0 \circ (Z)^{-1} \in \mathcal{P}^{\mu}$, as defined in Section 4.2. Finally, if $d\phi(m)$ charges the set $\{m : \zeta_n^*(m) = m\}$, the following reasoning applies. It is seen directly that Assumption ⊛ excludes that $c_n \equiv c_{n-1}$ on an open interval inside the support of μ_n. Then for every $\delta > 0$ and $m \in (X_0, \inf\{y : c_n(y) = 0\}]$ such that $\zeta_n^*(m) = m$ there exists $m' \in (m-\delta, m)$ such that $\zeta_n^*(m') < m'$. Consequently, for $\varepsilon > 0$ there exists a right-continuous nondecreasing ϕ^ε such that $0 \le \phi^\varepsilon - \phi < \varepsilon$, $\phi^\varepsilon(X_0) = \phi(X_0)$ and $\zeta_n^*(m) < m$ $d\phi^\varepsilon$-a.s. Note that clearly $\mathbb{E}^{\mathbb{P}^{\max}}[\phi(Z^*_{t_n})]$ is finite if and only if $\mathbb{E}^{\mathbb{P}^{\max}}[\phi^\varepsilon(Z^*_{t_n})]$ is. Let $\phi^{\varepsilon,-1}$ and ϕ^{-1} denote the right-continuous inverses of ϕ^ε and ϕ, respectively. Finally, recall from (4.7) above that C is nonincreasing and continuous. Hence, applying the previous case to ϕ^ε, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\max}}[\phi(Z^*_{t_n})] &= \lim_{\varepsilon\to 0}\mathbb{E}^{\mathbb{P}^{\max}}[\phi^\varepsilon(Z^*_{t_n})] \\
&= \lim_{\varepsilon\to 0}\left\{\phi(X_0) + \int_{(X_0,\infty)} C(m)\, d\phi^\varepsilon(m)\right\} \\
&= \lim_{\varepsilon\to 0}\left\{\phi(X_0) + \int_{\phi(X_0)}^{\infty} C\big(\phi^{\varepsilon,-1}(x)\big)\, dx\right\} \\
&= \phi(X_0) + \int_{(X_0,\infty)} C(m)\, d\phi(m),
\end{aligned}$$

where we used monotone convergence since $C(\phi^{\varepsilon,-1}(x)) \ge C(\phi^{-1}(x))$.

4.4. *Proof of Theorem* 3.5. Note that by Lemma 3.4, $\zeta_i^*(m) \geq \underline{b}^{-1}(m) \to \infty$ as $m \to \infty$, and hence the integral defining $\hat{\lambda}_i$ in (3.7) is over a bounded interval. It is therefore well defined by the assumed local integrability. The same argument, applied to representation (3.8), shows that $\hat{H}$ is well defined.

The superhedging inequality (3.10) then follows from the trajectorial inequality of Proposition 3.1 together with (3.4). If $\hat{\lambda} \in \Lambda_n^\mu$ and $\hat{H} \in \mathcal{H}^\mu$, then (3.10) instantly implies the bound $U_n^\mu(\phi(X_T^*)) \leq \phi(X_0) + \mu(\hat{\lambda})$. Finally, by Fubini, $\mu(\hat{\lambda}) = \int_{(X_0,\infty)} C(m)\, d\phi(m) < \infty$, thus establishing claim (i) of the theorem.

Note that the integrability conditions of Theorem 3.5 imply, in particular, that $\zeta_n^*(m) < m$ $d\phi(m)$-a.e. If μ satisfy Assumption $\circledast$ of Obłój and Spoida [41], then as above, it follows directly from Lemma 4.1 by integrating the pathwise equality against $d\phi$, that there is $\mathbb{P}^{\max}$-a.s. equality in (3.10). As a consequence the equality $U_n^\mu(\phi(X_T^*)) = \phi(X_0) + \mu(\hat{\lambda})$ holds. This establishes claim (ii) of the theorem.

It remains to argue the admissibility of $\hat{\lambda}$ and $\hat{H}$. Observe that for $i = n$, (3.9) is simply the required property $\int \hat{\lambda}_n \, d\mu_n < \infty$. Note also that the inner integral in (3.9) is a convex function of x so that by (2.3), we may replace μ_i with μ_{i-1} in (3.9) and the double integral remains finite. The equivalence of $\hat{\lambda} \in \Lambda_n^\mu$ and (3.9) now follows from the definition of $\hat{\lambda}$.

Finally, we show $\hat{H} \in \mathcal{H}^\mu$. It is immediate that $\hat{H} \in \hat{\mathbb{H}}^2$. It remains to prove that $\int_0^\cdot \hat{H}_s \, dX_s$ is a $\mathbb{P}$-supermartingale for any $\mathbb{P} \in \mathcal{P}^\mu$. Let us fix one such $\mathbb{P}$ and recall that X is a $\mathbb{P}$-continuous martingale. For $t \in [t_i, t_{i+1})$ we have, by (3.8),

$$\int_{t_i}^t \hat{H}_s \, dX_s = -\int_{t_i}^t \big(f_i(X_s^*) - f_i(X_{t_i}^*)\big)\, dX_s - g_i\big(X_{t_i}, X_{t_i}^*\big)(X_t - X_{t_i}), \tag{4.9}$$

where $f_i(x) := \int_{(X_0,x]} \frac{d\phi(m)}{m - \zeta_{i+1}^*(m)}$ and $g_i(x,y) = \int_{(X_0,y]} \mathbf{1}_{\{x \geq \zeta_{i+1}^*(m)\}} \frac{d\phi(m)}{m - \zeta_{i+1}^*(m)}$. In the first stochastic integral we recognize an Azéma–Yor process; see Carraro, El Karoui and Obłój [13]. We recall that $M^{F_i}(X)_t := F_i(X_t^*) - f_i(X_t^*)(X_t^* - X_t)$, where $F_i(x) = \int_{X_0}^x f_i(u)\, du = \int_{(X_0,x]} \frac{(x-m)\, d\phi(m)}{m - \zeta_{i+1}^*(m)}$, $x \geq X_0$, satisfies $M^{F_i}(X)_t = \int_0^t f_i(X_s^*)\, dX_s$ and is a local martingale.

We can extend F_i to the real line putting $F_i(x) = 0$ for $x < X_0$. This preserves convexity, and we observe that we hence have $M^{F_i}(X)_t \leq F_i(X_t)$. Also, for $t \leq t_{i+1}$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big[F_i(X_t)\big] &\leq \mathbb{E}^{\mathbb{P}}\big[F_i(X_{t_{i+1}})\big] \\
&= \int_{\mathbb{R}} \int_{X_0}^\infty \frac{(x-m)^+ \, d\phi(m)}{m - \zeta_{i+1}^*(m)} \mu_{i+1}(dx) \\
&\leq \int_{\mathbb{R}} \int_{X_0}^\infty \frac{(x - \zeta_{i+1}^*(m))^+ \, d\phi(m)}{m - \zeta_{i+1}^*(m)} \mu_{i+1}(dx) < \infty
\end{aligned}$$

by condition (3.9). It follows that $M^{F_i}(X)_t^+$ is $\mathbb{P}$-integrable.

Let (τ_k) be a localizing sequence for $M^{F_i}(X)$ and $0 \le s \le t \le t_n$. A simple monotone convergence argument shows that $\mathbb{E}^{\mathbb{P}}[F_i(X^*_{t\wedge\tau_k})|\mathcal{F}_s]$ converge to $\mathbb{E}^{\mathbb{P}}[F_i(X^*_t)|\mathcal{F}_s]$, as $k \to \infty$, and $M^{F_i}(X)_{s\wedge\tau_k}$ converge to $M^{F_i}(X)_s$ by continuity. It follows that the following limit is well defined:

$$\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}}\big[f_i(X^*_{t\wedge\tau_k})(X^*_{t\wedge\tau_k} - X_{t\wedge\tau_k})\big|\mathcal{F}_s\big] \ge \mathbb{E}^{\mathbb{P}}\big[f_i(X^*_t)(X^*_t - X_t)\big|\mathcal{F}_s\big] \qquad \text{a.s.},$$

and the inequality follows from Fatou's lemma since $f_i \ge 0$ and $X^* \ge X$. Combining these we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[M^{F_i}(X)_t|\mathcal{F}_s\big] &\ge \lim_{k\to\infty} \mathbb{E}^{\mathbb{P}}\big[F_i(X^*_{t\wedge\tau_k}) - f_i(X^*_{t\wedge\tau_k})(X^*_{t\wedge\tau_k} - X_{t\wedge\tau_k})|\mathcal{F}_s\big] \\ &= M^{F_i}(X)_s,\end{aligned}$$

where the LHS is well defined since $M^{F_i}(X)^+_t$ is $\mathbb{P}$-integrable. In particular $\mathbb{E}^{\mathbb{P}}[M^{F_i}(X)_t] \ge M^{F_i}(X)_0 = 0$, and hence $\mathbb{E}^{\mathbb{P}}[|M^{F_i}(X)_t|] < \infty$, and $M^{F_i}(X)$ is a submartingale. Finally, $-M^{F_i}(X)$ is a supermartingale.

We can compute explicitly

$$\int_{t_i}^{t} \big(f_i(X^*_s) - f_i(X^*_{t_i})\big)\, dX_s = M^{F_i}(X)_t - M^{F_i}(X)_{t_i} - f_i(X^*_{t_i})(X_t - X_{t_i}).$$

Combining with (4.9) we conclude that

$$\begin{aligned}\int_0^{t_n} \hat{H}_s\, dX_s = &-\sum_{i=0}^{n-1}\big(M^{F_i}(X)_{t_{i+1}} - M^{F_i}(X)_{t_i}\big) \\ &+ \sum_{i=0}^{n-1}\big(f_i(X^*_{t_i}) - g_i(X_{t_i}, X^*_{t_i})\big)(X_{t_{i+1}} - X_{t_i}),\end{aligned}$$

and we note that by the above, the first sum is integrable under $\mathbb{P}$. By the superhedging property (3.10), we have

$$\begin{aligned}&\sum_{i=0}^{n-1}\big(f_i(X^*_{t_i}) - g_i(X_{t_i}, X^*_{t_i})\big)(X_{t_{i+1}} - X_{t_i}) \\ &\qquad \ge \phi(X^*_{t_n}) - \hat{\lambda}(X_{\mathbf{t}}) + \sum_{i=0}^{n-1}\big(M^{F_i}(X)_{t_{i+1}} - M^{F_i}(X)_{t_i}\big),\end{aligned}$$

and the RHS is a $\mathbb{P}$-integrable r.v. by the above, the assumption that $\hat{\lambda} \in \Lambda^\mu_n$ and, as argued above, its implication that the bound in (3.6) is finite. It follows from Lemma B.1 that the simple discrete trading component of $\hat{H}$ defines a $\mathbb{P}$-martingale; that is, $(f_i(X^*_{t_i}) - g_i(X_{t_i}, X^*_{t_i}))(X_t - X_{t_i})$, $t \in [t_i, t_{i+1})$ is a $\mathbb{P}$-martingale. The above was carried under an arbitrary $\mathbb{P} \in \mathcal{P}^\mu$ and hence $\hat{H} \in \mathcal{H}^\mu$ as required. This completes the proof of the theorem.

5. The stochastic control approach. We now present the methodology which led us to identify the remarkable pathwise inequality of Proposition 3.1, and to deduce the value (3.5) as the cheapest semi-static superhedging cost.

For technical reasons and clarity of presentation, we impose the following additional conditions on the nature of the lookback payoff and the marginal constraints:

$$\begin{aligned}&\phi \in C^1 \text{ bounded nondecreasing}, \phi_{|(-\infty,X_0]} \equiv 0, \textstyle\int_{X_0}^{\infty} C(m)\,d\phi(m) < \infty,\\ &\hat{\lambda} \text{ bounded and } \zeta_i^* \text{ continuous increasing}, i = 1, \dots, n,\end{aligned} \tag{5.1}$$

where $\hat{\lambda}$ is the optimal static hedging payoff function of (3.7). The assumption that ϕ is constant on $(-\infty, X_0]$ is no loss generality since these values of ϕ are irrelevant for the payoff $\phi(X_T^*)$. Clearly adding a constant to ϕ does not change the problem, and hence we take $\phi(X_0) = 0$ for convenience.

REMARK 5.1. Given the expression of $\hat{\lambda}$ as difference of call option's payoffs, the boundedness condition imposed above may seem inappropriate. However, we observe that when the probability measures μ_i have bounded support, the values taken by $\hat{\lambda}$ outside all supports are irrelevant. Therefore, we may re-define $\hat{\lambda}$ as a bounded function.

Since the candidate optimal static hedging strategy $\hat{\lambda}$ is assumed to be bounded, the robust superhedging problem is not changed by restricting to bounded static strategies. In the present section, we even seek more simplification, and we also analyze a slightly different formulation,

$$\begin{aligned}\overline{U}_n^{\mu}(\xi) := \inf\{Y_0 : \exists(\lambda, H) \in (\mathbb{L}^{\infty})^n \times \mathcal{H}, \overline{Y}_T^{H,\lambda} \ge \xi,\\ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_S\},\end{aligned} \tag{5.2}$$

where $\mathcal{P}_S$ is the subset of $\mathcal{P}$, consisting of probability measures

$$\mathbb{P} := \mathbb{P}_0 \circ (X^{\alpha})^{-1} \qquad \text{with } X^{\alpha} := \int_0^{\cdot} \alpha_s\,dB_s \text{ for some } \alpha \in \mathbb{H}^2(\mathbb{P}_0),$$

and where $\mathcal{H}$ is the subset of dynamic trading strategies $H \in \hat{\mathbb{H}}^2$ such that the corresponding value process Y^H is a $\mathbb{P}$-supermartingale for all $\mathbb{P} \in \mathcal{P}_S$. We note that

$$\begin{aligned}\overline{U}_n^{\mu}(\xi) &\ge \inf\{Y_0 : \exists(\lambda, H) \in (\mathbb{L}^{\infty})^n \times \mathcal{H}_S^{\mu}, \overline{Y}_T^{H,\lambda} \ge \xi, \mathbb{P}\text{-a.s. } \forall \mathbb{P} \in \mathcal{P}_S\}\\ &\ge \sup_{\mathbb{P} \in \mathcal{P}_S^{\mu}} \mathbb{E}^{\mathbb{P}}[\xi],\end{aligned} \tag{5.3}$$

where $\mathcal{P}_S^{\mu} := \{\mathbb{P} \in \mathcal{P}_S : X_{t_i} \sim_{\mathbb{P}} \mu_i, i = 1, \dots, n\}$ and $\mathcal{H}_S^{\mu}$ is obtained by relaxing the supermartingale requirement in $\mathcal{H}$ to $\mathbb{P} \in \mathcal{P}_S^{\mu}$. The first inequality then follows by relaxation of conditions on (λ, H), and the second one is the usual majorization of pricing by hedging; see (2.8). We note that, given the definition of U_n^μ in (2.7) and Theorem 3.5, the middle term may appear more natural for the superhedging price. However, under some simplifying technical assumptions and for $\xi = \phi(X_T^*)$, we will show that in fact we have equalities throughout (5.3).

5.1. *Dual formulation of the robust superhedging problem.* The first step for the present approach is the observation that

$$\overline{U}_n^\mu(\xi) = \inf_{\lambda\in(\mathbb{L}^\infty)^n} \inf\{Y_0 : \exists H \in \mathcal{H}, Y_T^H \geq \xi - \lambda(X_\mathbf{t}) + \mu(\lambda), \\ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_S\}.$$

We shall continue analyzing the RHS of the last inequality, observing that for each fixed $\lambda \in (\mathbb{L}^\infty)^n$, we are reduced to the robust superhedging problem of the derivative $\xi - \lambda(X_\mathbf{t}) + \mu(\lambda)$. A dual formulation of this problem is derived in Theorem 2.1 of [23] under some uniform continuity assumptions. More recently, Neufeld and Nutz [37] relaxed the uniform continuity condition, allowing for a larger class of random variables including measurable ones. The following direct application of [45] is better suited to our context:

PROPOSITION 5.2. *For a bounded random variable ξ, we have*

$$\overline{U}_n^\mu(\xi) = \inf_{\lambda\in(\mathbb{L}^\infty)^n} \sup_{\mathbb{P}\in\mathcal{P}_S} \{\mu(\lambda) + \mathbb{E}^\mathbb{P}[\xi - \lambda(X_\mathbf{t})]\}.$$

5.2. *The one-marginal problem.* We start with an essential ingredient, namely a general one-marginal construction, which allows us to move from $(n-1)$ to n marginals.

For an inherited maximum $M_0 \geq X_0$, we introduce the process

$$M_t := M_0 \vee X_t^* \qquad \text{for } t \geq 0.$$

The process (X, M) takes values in the state space $\mathbf{\Delta} := \{(x, m) \in \mathbb{R}^2 : x \leq m\}$. Our interest in this section is on the upper bound on the price of the one-marginal $(n = 1)$ lookback option defined by the payoff

$$\xi = g(X_T, X_T^*) \qquad \text{for some } g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}. \tag{5.4}$$

ASSUMPTION A. Function $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is bounded, C^1 in m, absolutely continuous in x, and g_{xx} exists as a measure.

ASSUMPTION B. The function $x \longmapsto \frac{g_m(x,m)}{m-x}$ is nondecreasing.

For a bounded measurable function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$, we denote $g^\lambda := g - \lambda$. Similarly to Proposition 3.1 in Galichon, Henry-Labordère and Touzi [23], it follows from the Dambis–Dubins–Schwarz time change theorem that the model-free upper bound can be converted into

$$\overline{U}_1^{\mu}(\xi) = \inf_{\lambda\in\mathbb{L}^\infty} \sup_{\tau\in\mathcal{T}} \{\mu(\lambda) + J(\lambda,\tau)\} \tag{5.5}$$
$$\text{where } J(\lambda,\tau) := \mathbb{E}^{\mathbb{P}_0}\big[g^\lambda(X_\tau, X_\tau^*)\big],$$

and $\mathcal{T}$ is the collection of all stopping times τ such that

$$\{X_{t\wedge\tau}, t\ge 0\} \qquad \text{is a } \mathbb{P}_0\text{-uniformly integrable martingale.} \tag{5.6}$$

Then for every fixed multiplier $\lambda \in \mathbb{L}^\infty$, we are facing the infinite horizon optimal stopping problem

$$u^\lambda(x,m) := \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}_0}_{x,m}\big[g^\lambda(X_\tau, M_\tau)\big], \qquad (x,m)\in\mathbf{\Delta}, \tag{5.7}$$

where $\mathbb{E}^{\mathbb{P}_0}_{x,m}$ denotes the conditional expectation operator $\mathbb{E}^{\mathbb{P}_0}[\cdot|(X_0,M_0) = (x,m)]$. The dynamic programming equation corresponding to the optimal stopping problem u^λ defined in (5.7) is

$$\begin{aligned} \min\{u - g^\lambda, -u_{xx}\} &= 0 \qquad \text{for } (x,m)\in\mathbf{\Delta}, \\ u_m(m,m) &= 0 \qquad \text{for } m\in\mathbb{R}. \end{aligned} \tag{5.8}$$

It is then natural to introduce a candidate solution for the dynamic programming equation defined by a free boundary $\{x = \psi(m)\}$, for some convenient function ψ,

$$v^\psi(x,m) = g^\lambda\big(x\wedge\psi(m), m\big) + \big(x-\psi(m)\big)^+ g_x^\lambda\big(\psi(m), m\big) \tag{5.9}$$

$$= g^\lambda(x,m) - \int_{\psi(m)}^{x\vee\psi(m)} (x-\xi) g^\lambda_{xx}(d\xi, m), \qquad x\le m, \tag{5.10}$$

where g_x^λ denotes the right-derivative of g^λ with respect to x, and g^λ_{xx} is the corresponding second derivative in the sense of distributions. The existence of these derivatives is justified by the restriction of the function λ to the set $\hat{\Lambda}^\mu$ defined in (5.11) below.

Here, $v^\psi(\cdot,m)$ coincides with the obstacle g^λ before the exercise boundary $\psi(m)$, and satisfies $v^\psi_{xx}(\cdot,m) = 0$ in the continuation region $(\psi(m), m]$. However, the candidate solution needs to satisfy more conditions. Namely $v^\psi(\cdot,m)$ must be above the obstacle and concave in x on $(-\infty, m]$, and it needs to satisfy the Neumann condition in (5.8).

For this reason, our strategy of proof consists of first restricting the minimization in (5.5) to those multipliers λ in the set

$$\hat{\Lambda} := \big\{\lambda\in\mathbb{L}^\infty : v^\psi \text{ concave in } x \text{ and } v^\psi \ge g^\lambda \text{ for some } \psi\in\Psi^\lambda\big\}, \tag{5.11}$$

where the set Ψ^λ is defined in (5.14) below so that our candidate solution v^ψ satisfies the Neumann condition in (5.8). Since $v(\cdot,m) = g^\lambda(\cdot,m)$ on $(-\infty,\psi(m)]$, it follows that g^λ is concave on this range, thus justifying that the second derivative g^λ_{xx} is a well-defined measure for all $\lambda \in \hat\Lambda$. Also, by Assumption A, this guarantees that λ'' is also a well-defined measure.

By formal differentiation of v^ψ, the Neumann condition reduces to the ordinary differential equation (ODE)

$$\begin{aligned} & -\psi' g^\lambda_{xx}(\psi, m) = \gamma(\psi, m) \\ & \qquad \text{where } \gamma(x,m) := (m-x)\frac{\partial}{\partial x}\left\{\frac{g_m(x,m)}{m-x}\right\} \end{aligned} \tag{5.12}$$

exists a.e. in view of Assumption B. Similarly to Galichon, Henry-Labordère and Touzi [23], we need for technical reasons to consider this ODE in the relaxed sense. We then introduce the weak formulation of the ODE (5.12),

$$\begin{aligned} & \psi(m) < m \qquad \text{for all } m \in \mathbb{R}, \\ & -\int_{\psi(E)} g^\lambda_{xx}(\cdot, \psi^{-1})(d\xi) = \int_E \gamma(\psi, \cdot)(dm) \qquad \text{for Borel subsets } E \subset \mathbb{R}, \end{aligned} \tag{5.13}$$

where ψ is chosen in its right-continuous version and is nondecreasing by the concavity of g^λ, and the nonnegativity of γ implied by Assumption B. In fact, we shall restrict to those ψ which are continuous and (strictly) increasing, so that the inverse ψ^{-1} is a well-defined continuous increasing function. This is the reason for the condition on ζ^* in (5.1), and this restriction is adopted here for the sake of technical simplicity.

We introduce the collection of all relaxed solutions of (5.12) with the additional simplifying assumption of continuity

$$\Psi^\lambda := \{\psi : \mathbb{R} \to \mathbb{R} \text{ continuous, increasing, and satisfies (5.13)}\}. \tag{5.14}$$

Notice that the ODE (5.12), which motivates the relaxation (5.13), does not characterize the free boundary ψ as it is not complemented by any boundary condition.

REMARK 5.3. For later use, we observe that (5.13) implies by direct integration that the function

$$x \longmapsto \lambda(x) - \int_{\psi(X_0)}^x \int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi), \xi)}{\xi - \psi(\xi)}\, d\xi\, dy - \int_{\psi(X_0)}^x g_x(\xi, \psi^{-1}(\xi))\, d\xi$$

is affine.

PROPOSITION 5.4. *Let Assumptions A and B hold true. Then*

$$u^\lambda \le v^\psi \qquad \textit{for any } \lambda \in \hat\Lambda \textit{ and } \psi \in \Psi^\lambda.$$

PROOF. By the definition of Ψ^λ, the function ψ that we will be manipulating has a well-defined continuous increasing inverse. We proceed in two steps:

Step 1. We first prove that v^ψ is differentiable in m on the diagonal with

$$v_m^\psi(m,m) = 0 \qquad \text{for all } m \in \mathbb{R}. \tag{5.15}$$

Indeed, since $\psi \in \Psi^\lambda$, it follows from Remark 5.3 that

$$\begin{aligned}\lambda(x) = \alpha_0 + \alpha_1 x &+ \int_{\psi(X_0)}^x \int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi),\xi)}{\xi - \psi(\xi)}\,d\xi\,dy \\ &+ \int_{\psi(X_0)}^x g_x\big(\xi, \psi^{-1}(\xi)\big)\,d\xi\end{aligned}$$

for some constants α_0, α_1. Plugging this expression into (5.9), we see that for $\psi(m) \le x \le m$,

$$\begin{aligned} v^\psi(x,m) &= g\big(\psi(m),m\big) - \bigg(\alpha_1 + \int_{X_0}^m \frac{g_m(\psi(\xi),\xi)}{\xi-\psi(\xi)}\,d\xi\bigg)\big(x - \psi(m)\big) \\ &\qquad - \bigg(\alpha_0 + \alpha_1\psi(m) + \int_{\psi(X_0)}^{\psi(m)} \int_{X_0}^{\psi^{-1}(y)} \frac{g_m(\psi(\xi),\xi)}{\xi-\psi(\xi)}\,d\xi\,dy \\ &\qquad\qquad + \int_{\psi(X_0)}^{\psi(m)} g_x\big(\xi,\psi^{-1}(\xi)\big)\,d\xi\bigg) \\ &= g\big(\psi(m),m\big) - \alpha_0 - \alpha_1 x - \int_{X_0}^m g_m\big(\psi(\xi),\xi\big)\frac{x-\psi(\xi)}{\xi-\psi(\xi)}\,d\xi \\ &\qquad - \int_{\psi(X_0)}^{\psi(m)} g_x\big(\xi,\psi^{-1}(\xi)\big)\,d\xi \\ &= g\big(\psi(X_0), X_0\big) + \int_{X_0}^m g_m\big(\psi(\xi),\xi\big)\frac{\xi - x}{\xi-\psi(\xi)}\,d\xi. \end{aligned}$$

Since g is C^1 in m by Assumption A, (5.15) follows by direct differentiation with respect to m.

Step 2. Let $\tau \in \mathcal{T}$ be arbitrary, and define the sequence of stopping times $\tau_n := \tau \wedge \inf\{t > 0 : |X_t - x| > n\}$. Since v^ψ is concave, it follows from the Itô–Tanaka formula that

$$v^\psi(x,m) \ge v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t,M_t)\,dX_t - \int_0^{\tau_n} v_m^\psi(X_t,M_t)\,dM_t.$$

Notice that $(M_t - X_t)\,dM_t = 0$. Then since $v_m^\psi(m,m) = 0$, it follows that $v_m^\psi(X_t,M_t)\,dM_t = v_m^\psi(M_t,M_t)\,dM_t = 0$, and therefore

$$v^\psi(x,m) \ge v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t,M_t)\,dX_t. \tag{5.16}$$

Taking expectations in the last inequality, we see that

$$v^\psi(x,m) \ge \mathbb{E}_{x,m}^{\mathbb{P}_0}\big[v^\psi(X_{\tau_n}, M_{\tau_n})\big] \ge \mathbb{E}_{x,m}^{\mathbb{P}_0}\big[g^\lambda(X_{\tau_n}, M_{\tau_n})\big]. \tag{5.17}$$

The required result now follows by the dominated convergence, due to the boundedness of g and λ and by the arbitrariness of $\tau \in \mathcal{T}$. $\square$

REMARK 5.5. Inequality (5.16) is the key step in order to determine the pathwise inequality of Proposition 3.1. Indeed by sending $n \to \infty$ and taking $\tau = \tau^\psi := \inf\{t : X_t \le \psi(M_t)\}$, we see that

$$\begin{aligned} v^\psi(x,m) &\ge v^\psi(X_\tau, M_\tau) - \int_0^\tau v_x^\psi(X_t, M_t)\,dX_t \\ &= g(M_\tau) - \lambda(X_\tau) - \int_0^\tau v_x^\psi(X_t, M_t)\,dX_t. \end{aligned}$$

This inequality induces the pathwise inequality once we identify the optimal $\hat\lambda$ and the corresponding free boundary $\hat\psi$. For the purpose of the pathwise inequality of Proposition 3.1, the optimal superhedging strategy is identified by using similarly the iterated values functions $(v_k)_k$ introduced in Section 5.4 below.

5.3. *Multiple-marginals penalized value function.* We now continue our general methodology and return to the multiple-marginal problem. Our aim is to prove Theorem 3.5 for the modified robust superhedging problem $\overline{U}_n^\mu(\xi)$, and derive the robust superhedging bounds for the lookback derivative security

$$\phi(X_T^*) \qquad \text{given the marginals } X_{t_i} \sim \mu_i \qquad \text{for all } i = 1, \dots, n. \tag{5.18}$$

We recall that the probability measures μ_i are defined from market prices which do not admit arbitrage; that is, (2.3) holds.

Our purpose in this section is to analyze the upper bound on the robust superhedging cost introduced in Proposition 5.2,

$$\overline{U}_n^\mu(\xi) = \inf_{\lambda \in (\mathbb{L}^\infty)^n} \{\mu(\lambda) + u^\lambda(X_0, X_0)\}, \tag{5.19}$$

where

$$u^\lambda(x,m) := \sup_{\mathbb{P} \in \mathcal{P}_S} \mathbb{E}_{x,m}^{\mathbb{P}}[\phi^\lambda(X_{\mathbf{t}}, M_{t_n})] \quad \text{and} \quad \phi^\lambda := \phi - \sum_{i=1}^n \lambda_i. \tag{5.20}$$

Our approach is to introduce the sequence of intermediate optimization problems

$$\begin{aligned} &u_n(x,m) = \phi(m) \quad \text{and} \quad u_{k-1}(x,m) = \sup_{\mathbb{P} \in \mathcal{P}_S} \mathbb{E}_{t_{k-1},x,m}^{\mathbb{P}}[u_k^\lambda(X_{t_k}, M_{t_k})], \\ &\hspace{30em} k \le n, \end{aligned} \tag{5.21}$$

where $\mathbb{E}_{t_{k-1},x,m}^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[\cdot | (X,M)_{t_{k-1}} = (x,m)]$, and

$$u_k^\lambda(x,m) := u_k(x,m) - \lambda_k(x) \qquad \text{for } (x,m) \in \mathbf{\Delta}. \tag{5.22}$$

The last iterative sequence of value functions induces $u^\lambda = u_0^\lambda$. Moreover, by the Dambis–Dubins–Schwarz theorem (see Proposition 3.1 in [23]), we may convert the stochastic control problem in (5.21) into a sequence of optimal stopping problems

$$u_{k-1}(x,m) = \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}_0}_{x,m}\big[u^\lambda_k(X_\tau, M_\tau)\big]. \tag{5.23}$$

Then, denoting by $\mathcal{S}_n := \{\tau = (\tau_1,\dots,\tau_n) \in \mathcal{T} : \tau_1 \le \cdots \le \tau_n\}$, we see that

$$\begin{aligned}\overline{U}^\mu_n(\xi) = \inf_{\lambda\in(\mathbb{L}^\infty)^n}\big\{\mu(\lambda) + u^\lambda_0(X_0,X_0)\big\}\\ \text{with } u^\lambda_0(x,m) := \sup_{\tau\in\mathcal{S}_n} \mathbb{E}^{\mathbb{P}_0}_{x,m}\big[\phi^\lambda(X_\tau, M_{\tau_n})\big].\end{aligned} \tag{5.24}$$

5.4. *Preparation for the upper bound.* The function u_{k-1} corresponds to the optimization problem considered in Section 5.2 with a payoff $g(x,m) = u_k(x,m)$ depending on the spot and the running maximum. This was our original motivation for isolating the one-marginal problem.

To solve the multiple marginals problem, we introduce the iterative sequence of candidate value functions

$$\begin{aligned} v_n(x,m) &:= \phi(m), \qquad v^\lambda_k(x,m) := v_k(x,m) - \lambda_k(x) \quad \text{and}\\ v_{k-1}(x,m) &:= v^\lambda_k\big(x\wedge\psi_k(m), m\big) + \big(x-\psi_k(m)\big)^+ \partial^+_x v^\lambda_k\big(\psi_k(m), m\big)\\ &= v^\lambda_k(x,m) - \int_{\psi_k(m)}^{x\vee\psi_k(m)} (x-\xi)\partial_{xx} v^\lambda_k(d\xi, m),\end{aligned} \tag{5.25}$$

where $\partial^+_x v^\lambda_k$ and $\partial_{xx} v^\lambda_k$ denote the right-derivative and the measure second derivative, respectively, of v^λ_k with respect to x (which are well defined for λ in the subclass $\hat{\Lambda}_n$ introduced below), and $\psi = (\psi_1,\dots,\psi_n)$ with ψ_i defined as an arbitrary solution of the ordinary differential equation

$$\begin{aligned} -\psi'_k \partial_{xx} v^\lambda_k(\psi_k, m) = \gamma_k(\psi_k, m)\\ \text{with } \gamma_k(x,m) := (m-x)\partial_x\left\{\frac{\partial_m v_k(x,m)}{m-x}\right\},\end{aligned} \tag{5.26}$$

which stays strictly below the diagonal. Notice that, in contrast to the one-marginal case, we have dropped here the dependence of v_k in ψ by simply denoting $v_k := v^\psi_k$ and $v^\lambda_k := v^{\psi,\lambda}_k$.

Similarly to the one-marginal case, we introduce the weak formulation of this ODE,

$$\begin{aligned} \psi_k(m) < m \qquad \text{for all } m \ge 0,\\ -\int_{\psi(E)} \partial_{xx} v^\lambda_k\big(\cdot, \psi^{-1}_k\big)(d\xi) = \int_E \gamma_k(\psi_k, \cdot)(dm) \qquad \text{for all } E \in \mathcal{B}(\mathbb{R}),\end{aligned} \tag{5.27}$$

and we introduce the set

$$\Psi_n^\lambda := \{\psi : \mathbb{R} \to \mathbb{R}^n \text{ with continuous increasing entries} \qquad \psi_k \text{ satisfying (5.27)}, k \le n\}. \tag{5.28}$$

We also follow the one-marginal case by restricting the minimization in (5.24) to those multipliers λ in the set

$$\hat{\Lambda}_n := \{\lambda \in (\mathbb{L}^\infty)^n : v_{k-1} \text{ concave in } x \text{ and } v_{k-1} \ge v_k^\lambda \text{ for all } k \le n\}. \tag{5.29}$$

LEMMA 5.6. *Let ϕ satisfy (5.1), $\lambda \in \hat{\Lambda}_n$ and $\psi \in \Psi_n^\lambda$. Then*:

(i) *for all $i = 1, \dots, n$, the function v_i satisfies Assumptions A and B, that is, v_i is C^1 in m, absolutely continuous in x, Lipschitz in m uniformly in x, $\partial_{xx} v_i$ exists as a measure and $x \longmapsto \partial_m v_i(x,m)/(m-x)$ is nondecreasing*;

(ii) *for all $i = 1, \dots, n$, the function $\partial_m v_i$ is concave in x*;

(iii) $u^\lambda(X_0, X_0) \le v_0(X_0, X_0)$.

PROOF. We first prove (i). First $v_n = \phi$ satisfies Assumptions A and B as it is independent of the x-variable, nondecreasing and C^1. For the remaining cases, we proceed by induction, assuming that v_i satisfies Assumptions A and B, for some $i \le n$, and we intend to show that v_{i-1} does as well. We first observe that the following condition is also satisfied by v_i:

$$v_i(x,m) = \phi(m) \quad \text{nondecreasing, or} \quad \partial_m v_i(m,m) = 0, \tag{5.30}$$

where the first alternative holds for $i = n$. v_{i-1} is clearly C^1 in m, and by using the ODE (5.26) satisfied by v_i, we directly compute that

$$\begin{aligned} &\partial_m v_{i-1}(x,m) \\ &\quad = \begin{cases} \partial_m v_i(x,m), & \text{for } x \in (-\infty, \psi_i(m)], \\ \partial_m v_i(\psi_i(m), m) \dfrac{m-x}{m-\psi_i(m)}, & \text{for } x \in [\psi_i(m), m]. \end{cases} \end{aligned} \tag{5.31}$$

Then v_{i-1} inherits the differentiability in m, and $x \longmapsto \partial_m v_{i-1}(x,m)/(m-x)$ is nondecreasing whenever $x \longmapsto \partial_m v_i(x,m)/(m-x)$ is. The remaining properties follow from the concavity of v_i.

We next prove (iii). By the previous step, v_i satisfies Assumptions A and B for all $i = 1, \dots, n$. Then it follows from Proposition 5.4 that $u_{n-1} \le v_{n-1}$ for all $\psi \in \Psi^{\lambda_n}$. Therefore

$$u_{n-2}(x,m) \le \sup_{\tau_{n-1} \in \mathcal{T}} \mathbb{E}_{x,m}^{\mathbb{P}}[v_{n-1}^\lambda(X_{\tau_{n-1}}, X_{\tau_{n-1}}^*)],$$

and we deduce from a second application of Proposition 5.4 that $u_{n-2} \le v_{n-2}$. The required inequality follows by a backward iteration of this argument.

We finally prove (ii). From (5.31), we see that $\partial_m v_{i-1}$ is concave in x on $(-\infty, \psi_i(m))$ and on $(\psi_i(m), m]$. It remains to verify that $\partial_m v_{i-1}$ is concave at the point $x = \psi_i(m)$. We directly calculate that

$$\partial_{xm} v_{i-1}\big(\psi_i(m)-, m\big) = \partial_{xm} v_i\big(\psi_i(m)-, m\big)$$

and

$$\partial_{xm} v_{i-1}\big(\psi_i(m)+, m\big) = \frac{-\partial_m v_i(\psi_i(m), m)}{m - \psi_i(m)}.$$

Then, by the concavity of $\partial_m v_i$ in x, together with (5.30), we have

$$\partial_m v_i\big(\psi_i(m), m\big) + \partial_{xm} v_i\big(\psi_i(m)+, m\big)\big(m - \psi_i(m)\big) \geq \partial_m v_i(m, m) \geq 0,$$

which implies that $\partial_{xm} v_{i-1}(\psi_i(m)-, m) \geq \partial_{xm} v_{i-1}(\psi_i(m)+, m)$. $\square$

LEMMA 5.7. *Let ϕ satisfy (5.1). Then, for all $\lambda \in \hat{\Lambda}_n$ and $\psi \in \Psi_n^\lambda$, we have*

$$\begin{aligned}\mu(\lambda) + u^\lambda(X_0, X_0) &\leq \mu(\lambda) + v_0(X_0, X_0)\\ &= \sum_{i=1}^n \int \big[c_i(\xi) - c_0(\xi)\mathbf{1}_{\{\xi < \psi_i(X_0)\}}\big]\lambda_i''(d\xi)\\ &\quad - \int_{\psi_i(X_0)}^\infty c_0(\xi)\partial_{xx} v_i(d\xi, X_0).\end{aligned}$$

PROOF. Denote the LHS:$= \mu(\lambda) + u^\lambda(X_0, X_0)$. Substituting the expression of the v_i's in inequality (iii) of Lemma 5.6, we see that

$$\begin{aligned}\mathrm{LHS} &\leq \sum_{i=1}^n \mu_i(\lambda_i) - \lambda_i(X_0) - \int_{\psi_i(X_0)}^\infty c_0(\xi)\partial_{xx} v_i^\lambda(d\xi, X_0)\\ &\leq \sum_{i=1}^n \int \lambda_i(\xi)(\mu_i - \delta_{X_0})(d\xi) - \int_{\psi_i(X_0)}^\infty c_0(\xi)\partial_{xx} v_i^\lambda(d\xi, X_0).\end{aligned}$$

Since $\int \xi \mu_i(d\xi) = X_0$, it follows from two integrations by parts that

$$\begin{aligned}\mathrm{LHS} \leq \sum_{i=1}^n &\int (c_i - c_0 + c_0\mathbf{1}_{[\psi_i(X_0),\infty)})(\xi)\lambda_i''(d\xi)\\ &- \int_{\psi_i(X_0)}^\infty c_0(\xi)\partial_{xx} v_i(d\xi, X_0).\end{aligned}$$

$\square$

The following result provides the necessary calculations for the terms which appear in Lemma 5.7. We denote

$$\overline{\psi}_i := \psi_i \wedge \cdots \wedge \psi_n \qquad \text{for all } i = 1, \ldots, n, \tag{5.32}$$

and we set $\overline{\psi}_{n+1}(m) := m$, $m \in \mathbb{R}$.

LEMMA 5.8. *Let ϕ satisfy (5.1), $\lambda \in \hat{\Lambda}_n$, $\psi \in \Psi_n^\lambda$ and $i \le n$. Then for any λ_i''-integrable function φ we have*

$$\int \varphi(\xi)\lambda_i''(d\xi) = \int \left(\frac{\varphi(\overline{\psi}_i(m))}{m - \overline{\psi}_i(m)} - \mathbf{1}_{\{i<n\}} \frac{\varphi(\overline{\psi}_{i+1}(m))}{m - \overline{\psi}_{i+1}(m)} \right) \mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_{i+1}\}}\, d\phi(m),$$
$$i = 1, \dots, n.$$

PROOF. See Appendix C. □

Plugging these calculations into the estimate of Lemma 5.7 provides:

LEMMA 5.9. *Let $\phi \in C^1$ be bounded nondecreasing, $\lambda \in \hat{\Lambda}_n$ and $\psi \in \Psi_n^\lambda$. Then*

$$\mu(\lambda) + u^\lambda(X_0, X_0) \le \mu(\lambda) + v_0(X_0, X_0) = \sum_{i=1}^n \int_{-\infty}^{\infty} \mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_{i+1}\}} A_i(m)\, d\phi(m),$$

where

$$A_i := \frac{c_i(\overline{\psi}_i)}{m - \overline{\psi}_i} - \mathbf{1}_{\{i<n\}} \frac{c_i(\overline{\psi}_{i+1})}{m - \overline{\psi}_{i+1}}.$$

PROOF. We proceed in two steps:

Step 1. We start by reducing the last integral in Lemma 5.7 to an integral with respect to λ_i''. Notice that $\partial_{xx} v_i(x, m) = \mathbf{1}_{\{i<n\}} \mathbf{1}_{\{x<\psi_{i+1}(m)\}} \partial_{xx} v_{i+1}^\lambda(x, m)$. Then

$$\begin{aligned}
&\int_{\psi_i(X_0)}^{\infty} c_0(\xi)\partial_{xx} v_i(d\xi, X_0) \\
&\quad = \mathbf{1}_{\{i<n\}} \int_{\psi_i(X_0)}^{\infty} \mathbf{1}_{\{\xi<\psi_{i+1}(X_0)\}} c_0(\xi) \partial_{xx} v_{i+1}^\lambda(d\xi, X_0) \\
&\quad = -\mathbf{1}_{\{i<n\}} \int_{\psi_i(X_0)}^{\infty} \mathbf{1}_{\{\xi<\psi_{i+1}(X_0)\}} c_0(\xi) \lambda_{i+1}''(d\xi) \\
&\qquad + \mathbf{1}_{\{i<n\}} \int_{\psi_i(X_0)}^{\infty} \mathbf{1}_{\{\xi<\psi_{i+1}(X_0)\}} c_0(\xi) \partial_{xx} v_{i+1}(d\xi, X_0).
\end{aligned}$$

By direct iteration, it follows from the fact that $v_n(x, m) = \phi(x)$ is independent of x that

$$\begin{aligned}
&\int_{\psi_i(X_0)}^{\infty} c_0(\xi)\partial_{xx} v_i(d\xi, X_0) \\
&\quad = -\mathbf{1}_{\{i<n\}} \sum_{j=i+1}^{n} \int_{\psi_i(X_0)}^{\infty} \mathbf{1}_{\{\xi<\psi_{i+1}\wedge\psi_j(X_0)\}} c_0(\xi) \lambda_j''(d\xi).
\end{aligned}$$

Step 2. By Lemma 5.7, we have $\mu(\lambda) + u^{\lambda}(X_0, X_0) \le A$, where, by the first step,

$$\begin{aligned}
A &:= \int \left(\sum_{i=1}^{n} (c_i - c_0 \mathbf{1}_{(-\infty, \psi_i(X_0)]}) \lambda_i'' + \sum_{i=1}^{n} \sum_{j=i+1}^{n} c_0 \mathbf{1}_{[\psi_i(X_0), \psi_{i+1} \wedge \cdots \wedge \psi_j(X_0)]} \lambda_j'' \right) \\
&= \int \left(\sum_{i=1}^{n} (c_i - c_0 \mathbf{1}_{(-\infty, \psi_i(X_0)]}) \lambda_i'' + \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_0 \mathbf{1}_{[\psi_i(X_0), \psi_{i+1} \wedge \cdots \wedge \psi_j(X_0)]} \lambda_j'' \right) \\
&= \int \left(\sum_{i=1}^{n} (c_i - c_0 \mathbf{1}_{(-\infty, \psi_i(X_0)]}) \lambda_i'' + \sum_{j=1}^{n} c_0 \mathbf{1}_{[\psi_1 \wedge \cdots \wedge \psi_i(X_0), \psi_i(X_0)]} \lambda_j'' \right) \\
&= \int \sum_{i=1}^{n} (c_i - c_0 \mathbf{1}_{(-\infty, \overline{\psi}_i(X_0))}) \lambda_i''.
\end{aligned}$$

Then it follows from Lemma 5.8 that $A = \sum_{i=1}^{n} \int \mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_{i+1}\}} A_i^0 \, d\phi$, where

$$\begin{aligned}
A_i^0 := & \frac{c_i(\psi_i) - c_0(\psi_i) \mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_i(X_0)\}}}{m - \overline{\psi}_i} \\
& - \mathbf{1}_{\{i<n\}} \frac{c_i(\overline{\psi}_{i+1}) - c_0(\overline{\psi}_{i+1}) \mathbf{1}_{\{\overline{\psi}_{i+1} < \overline{\psi}_i(X_0)\}}}{m - \overline{\psi}_{i+1}}.
\end{aligned}$$

We next observe from the increase of $\overline{\psi}_i$ that for $m \ge X_0$, we have $\overline{\psi}_i(m) \ge \overline{\psi}_i(X_0)$, implying that $\mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_i(X_0)\}} = \mathbf{1}_{(-\infty, X_0]} \mathbf{1}_{\{\overline{\psi}_i < \overline{\psi}_i(X_0)\}}$. It also follows that on $\{\overline{\psi}_i < \overline{\psi}_{i+1}\}$, we have $\mathbf{1}_{\{\overline{\psi}_{i+1} < \overline{\psi}_i(X_0)\}} = \mathbf{1}_{(-\infty, X_0]} \mathbf{1}_{\{\overline{\psi}_{i+1} < \overline{\psi}_i(X_0)\}}$. The result follows since by (5.1) we have $d\phi(m) \equiv 0$ on $(-\infty, X_0]$. $\square$

5.5. *Proof of* $\overline{U}_n^{\mu}(\xi) = \mu(\hat{\lambda})$ *under* (5.1), *Assumption* $\circledast$ *of* [41] *and Assumptions* A, B. *Step* 1. We first show that $\overline{U}_n^{\mu}(\xi) \le \mu(\hat{\lambda})$. Given the results of Lemma 5.9, we prove in this first step that the pointwise minimization in (3.5) can be achieved by some vector of Lagrange multipliers $\hat{\lambda} = (\hat{\lambda}, \dots, \hat{\lambda}) \in \hat{\Lambda}_n$, thus implying that our required upper bound satisfies

$$\overline{U}_n^{\mu}(\xi) \le \int_{X_0}^{\infty} \sum_{i=1}^{n} \left(\frac{c_i(\zeta_i^*(m))}{m - \zeta_i^*(m)} - \frac{c_i(\zeta_{i+1}^*(m), m)}{m - \zeta_{i+1}^*(m)} \mathbf{1}_{\{i<n\}} \right) d\phi(m). \tag{5.33}$$

In order to define $\hat{\lambda}$, we take the family of functions $\hat{\psi}_i$ given by the boundaries ξ_i constructed by Obłój and Spoida [41], so that

$$\hat{\psi}_1 := b_1^{-1} \quad \text{and} \quad \overline{\hat{\psi}}_i := \hat{\psi}_i \wedge \cdots \wedge \hat{\psi}_n = \zeta_i^*, \qquad 1 < i \le n.$$

Recall that b_1^{-1} is the minimizer in (2.12) and is also the right-continuous inverse of the barycentre function of μ_1; see (6.2) below. Also, under Assumption $\circledast$ of

Obłój and Spoida [41], $\hat{\psi}$ are continuous. Moreover, direct verification reveals that the functions $\hat{\lambda}_i$ solve the system of ODEs (5.26). The required result follows from our additional assumption in (5.1) that $\hat{\lambda}$ is bounded.

Step 2. Now we prove that the equality $\overline{U}_n^{\mu}(\xi) = \mu(\hat{\lambda})$ holds. In Section 4.2 we recalled the n-marginal embedding $\tau_1, \dots, \tau_n$ of Obłój and Spoida [41]. In their Theorem 2.6 they compute the law of $X^*_{\tau_n}$ as

$$\mathbb{P}\big[X^*_{\tau_n} \geq m\big] = K_n(m) = C(m); \tag{5.34}$$

see also (4.6) and (4.7).

By definition of $\overline{U}_n^{\mu}(\xi)$ in (5.2), it follows that

$$\overline{U}_n^{\mu}(\xi) \geq \mathbb{E}^{\mathbb{P}_0}\big[\phi\big(X^*_{\tau_n}\big)\big] \overset{(5.34)}{=} \int_{(X_0,\infty)} C(m)\,d\phi(m). \tag{5.35}$$

Furthermore, it follows that we have equalities throughout in (5.3). $\square$

6. The Azéma–Yor embedding solves the one-marginal problem. In this subsection, we return to the one-marginal context of Section 5.2 with the intention of revisiting the recent result of Hobson and Klimmek [27] in this setting. Our emphasis is again on the efficiency of the stochastic control approach in the present setting. Therefore, similarly to the previous section, our assumptions below will be much stronger than what one could achieve directly with the pathwise approach. The endpoints of the support of the distribution μ are denoted by

$$\ell^{\mu} := \sup\big\{x : \mu\big([x,\infty)\big) = 1\big\} \quad \text{and} \quad r^{\mu} := \inf\big\{x : \mu\big((x,\infty)\big) = 0\big\}. \tag{6.1}$$

We introduce the so-called barycentre function

$$b(x) := \frac{\int_{[x,\infty)} y\mu(dy)}{\mu([x,\infty))}\mathbf{1}_{\{x<r^{\mu}\}} + x\mathbf{1}_{\{x\geq r^{\mu}\}}, \qquad x \in \mathbb{R}. \tag{6.2}$$

The solution of Azéma and Yor [3, 4] to the *Skorokhod embedding problem* is

$$\hat{\tau} := \inf\big\{t \geq 0 : X^*_t \geq b(X_t)\big\} = \inf\big\{t \geq 0 : X_t \leq b^{-1}\big(X^*_t\big)\big\}, \tag{6.3}$$

as recalled in Section 2.7. Plugging $\hat{\psi} := b^{-1}$ into the ODE (5.13), we obtain the function

$$\hat{\lambda}(x) := \int_{\ell^{\mu}}^{x}\int_{\ell^{\mu}}^{y} g_m\big(\xi, b(\xi)\big)\frac{\mu(d\xi)}{\mu([\xi,\infty))}\,dy + \int_{\ell^{\mu}}^{x} g_x\big(\xi, b(\xi)\big)\,d\xi; \qquad x \in \big(-\infty, r^{\mu}\big), \tag{6.4}$$

whose well-posedness will be guaranteed by the following condition.

ASSUMPTION C. $\hat{\lambda}$ is well defined and bounded. Moreover, the function g has a measure second partial derivative with respect to x satisfying

$$g_{xx}(dx, m) - g_{xx}\big(dx, b(x)\big) \leq \gamma\big(x, b(x)\big)b'(dx) \qquad \text{whenever } b(x) \leq m.$$

Similarly to the previous section, we focus on the robust superhedging problem $\overline{U}_1^\mu(\xi)$, as introduced in (5.2), and re-expressed in (5.5).

THEOREM 6.1. *Let $\xi = g(X_T, X_T^*)$ for some payoff function g satisfying Assumptions A, B and C. Then the pair $(\hat{\lambda}, \hat{\tau})$ is a solution of the problem $\overline{U}_1^\mu(\xi)$ in (5.5) and*

$$\overline{U}_1^\mu(\xi) = \mu(\hat{\lambda}) + J(\hat{\lambda}, \hat{\tau}) = \mathbb{E}^{\mathbb{P}_0}\big[g\big(X_{\hat{\tau}}, X_{\hat{\tau}}^*\big)\big].$$

The remaining part of this section is dedicated to the proof of this result. Our starting point is the result of Proposition 5.4 which provides an upper bound for the value function $\overline{U}_1^\mu(\xi)$ for every choice of a multiplier $\lambda \in \hat{\Lambda}$ and a corresponding solution $\psi \in \Psi^\lambda$ of the ODE (5.13),

$$\overline{U}_1^\mu(\xi) \le \mu(\lambda) + v^\psi(X_0, X_0) \qquad \text{for all } \lambda \in \hat{\Lambda} \text{ and } \psi \in \Psi^\lambda. \tag{6.5}$$

Alternatively, for any choice of a nondecreasing function ψ with $\psi(m) < m$ for all $m \in \mathbb{R}$, we may define a corresponding multiplier function λ by (5.13), or equivalently by (5.12), in the distribution sense. Then $\psi \in \Psi^\lambda$. If in addition v^ψ is concave in x and above the corresponding obstacle g^λ, then $\lambda \in \hat{\Lambda}$, and we may conclude by Proposition 5.4 that $\overline{U}_1^\mu(\xi) \le \mu(\lambda) + v^\psi$. The next result exhibits this bound for the choice $\psi = b^{-1}$, the right-continuous inverse of the barycentre function.

PROPOSITION 6.2. *Let $\xi = g(X_T, X_T^*)$ for some payoff function g satisfying Assumptions A, B and C. Then*

$$\overline{U}_1^\mu(\xi) \le \mu(\hat{\lambda}) + J(\hat{\lambda}, \hat{\tau}) = \mathbb{E}^{\mathbb{P}_0}\big[g\big(X_{\hat{\tau}}, X_{\hat{\tau}}^*\big)\big].$$

PROOF. It is immediately checked that $\hat{\psi} := b^{-1} \in \Psi^{\hat{\lambda}}$. Moreover, by Assumption C and the subsequent discussion, we see that $\hat{\lambda} \in \hat{\Lambda}$. In view of the previous discussion, the required inequality follows from Proposition 5.4 once we prove that $v^{\hat{\psi}}$ is concave, and that $v^{\hat{\psi}} \ge g^{\hat{\lambda}}$.

(1) We first verify that $v^{\hat{\psi}}$ is concave. By direct computation using the expression of $\hat{\lambda}$ in (6.4) together with the identity

$$\frac{b'(dx)}{b(x) - x} = \frac{\mu(dx)}{\mu([x, \infty))},$$

we see that

$$g_{xx}^{\hat{\lambda}}(dx, m) = g_{xx}(dx, m) - g_{xx}\big(dx, b(x)\big) - \gamma\big(x, b(x)\big)b'(dx) \tag{6.6}$$

in the distribution sense. By Assumption C, it follows that $x \longmapsto g^{\hat{\lambda}}(x, m)$ is concave on $(-\infty, \hat{\psi}(m)]$. Since $v^{\hat{\psi}}(\cdot, m)$ is linear on $[\hat{\psi}(m), m]$ and C^1 across the boundary $\hat{\psi}$, this proves that $v^{\hat{\psi}}$ is concave.

(2) We next check that $v^{\hat{\psi}} \geq g^{\hat{\lambda}}$. Since equality holds on $(-\infty, \hat{\psi}(m)]$, we only compute for $x \in [\hat{\psi}(m), m]$ that

$$\begin{aligned}(v^{\hat{\psi}} - g^{\hat{\lambda}})(x, m) &= \int_{\hat{\psi}(m)}^{x} \big(g_x^{\hat{\lambda}}(\hat{\psi}(m), m) - g_x^{\hat{\lambda}}(\xi, m)\big)\, d\xi \\ &= -\int_{\hat{\psi}(m)}^{x} \int_{\hat{\psi}(m)}^{\xi} g_{xx}^{\hat{\lambda}}(dy, m)\, d\xi.\end{aligned}$$

By (6.6), this provides

$$\begin{aligned}&(v^{\hat{\psi}} - g^{\hat{\lambda}})(x, m) \\ &\quad = -\int_{\hat{\psi}(m)}^{x} \left(g_x(\xi, m) - g_x\big(\xi, b(\xi)\big) - \int_{\hat{\psi}(m)}^{\xi} \frac{g_m(y, b(y))}{b(y) - y} b'(dy) \right) d\xi \\ &\quad = \int_{\hat{\psi}(m)}^{x} \int_{\hat{\psi}(m)}^{\xi} \left(g_{xm}\big(\xi, b(y)\big) + \frac{g_m(y, b(y))}{b(y) - y} \right) b'(dy)\, d\xi \\ &\quad = \int_{\hat{\psi}(m)}^{x} \left(\int_{y}^{x} g_{xm}\big(\xi, b(y)\big) + \frac{g_m(y, b(y))}{b(y) - y} \right) d\xi\, b'(dy) \\ &\quad = \int_{\hat{\psi}(m)}^{x} \big(b(y) - x\big) \left(\frac{g_m(x, b(y))}{b(y) - x} - \frac{g_m(y, b(y))}{b(y) - y} \right) b'(dy) \geq 0,\end{aligned}$$

where the last inequality follows from the nondecrease of b and $x \longmapsto g_m(x, m)/(m - x)$ (Assumption B), together with the fact that $b(y) \geq x$ for $\hat{\psi}(m) \leq y \leq x \leq m$. $\square$

PROOF OF THEOREM 6.1. To complete the proof of the theorem, it remains to prove that

$$\inf_{\lambda \in \Lambda^{\mu}} \{\mu(\lambda) + u^{\lambda}(X_0, X_0)\} \geq \mathbb{E}_{X_0, X_0}^{\mathbb{P}_0}\big[g\big(X_{\hat{\tau}}, X_{\hat{\tau}}^*\big)\big].$$

To see this, we use the fact that the stopping time $\hat{\tau}$ defined in (6.3) is a solution of the Skorokhod embedding problem; that is, $X_{\hat{\tau}} \sim \mu$ and $(X_{t \wedge \hat{\tau}})_{t \geq 0}$ is a uniformly integrable martingale; see Azéma and Yor [3, 4]. Moreover $X_{\hat{\tau}}^*$ is integrable. Then, for all $\lambda \in \Lambda^{\mu}$, it follows from the definition of u^{λ} that $u^{\lambda}(X_0, X_0) \geq J(\lambda, \hat{\tau})$, and therefore

$$\begin{aligned}\mu(\lambda) + u^{\lambda}(X_0, X_0) &\geq \mu(\lambda) + \mathbb{E}_{X_0, X_0}^{\mathbb{P}_0}\big[g\big(X_{\hat{\tau}}, X_{\hat{\tau}}^*\big) - \lambda(X_{\hat{\tau}})\big] \\ &= \mathbb{E}_{X_0, X_0}^{\mathbb{P}_0}\big[g\big(X_{\hat{\tau}}, X_{\hat{\tau}}^*\big)\big].\end{aligned}$$ $\square$

We conclude this section by a formal justification that the function b^{-1} appears naturally if one searches for the best upper bound in (6.5).

Step 1. using expression (5.10) of v^ψ, we directly compute that

$$\begin{aligned}
&\mu(\lambda) + u^\lambda(X_0, X_0) \\
&\quad = \mu\big(g(\cdot, X_0)\big) + \mu\big(g^\lambda(\cdot, X_0)\big) - \int_{\psi(X_0)}^{X_0} g^\lambda_{xx}(\xi, X_0)(X_0 - \xi)\, d\xi \\
&\quad = \mu\big(g(\cdot, X_0)\big) + \int g^\lambda_{xx}(\xi, X_0)\big(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \le \psi(X_0)\}}\big)\, d\xi \\
&\quad = \mu\big(g(\cdot, X_0)\big) + \int g^\lambda_{xx}\big(\xi, \psi^{-1}(\xi)\big)\big(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \le \psi(X_0)\}}\big)\, d\xi \\
&\qquad + \int \big(g_{xx}(\xi, X_0) - g_{xx}\big(\xi, \psi^{-1}(\xi)\big)\big)\big(c(\xi) - c_0(\xi)\mathbf{1}_{\{\xi \le \psi(X_0)\}}\big)\, d\xi,
\end{aligned}$$

where the second equality follows from two integrations by parts together with the fact that $\int x\mu(dx) = X_0$; see step 1 of the proof of Lemma 3.2 of Galichon, Henry-Labordère and Touzi [23]. Then, by using ODE (5.13) satisfied by ψ to change variables in the last integral, we see that

$$\begin{aligned}
&\mu(\lambda) + u^\lambda(X_0, X_0) \\
&\quad = \mu\big(g(\cdot, X_0)\big) + \int \{-\gamma\big(\psi(m), m\big) + G\big(\psi(m), m\big)\psi'(m)\}\delta\big(\psi(m), m\big)\, dm,
\end{aligned}$$

where we denoted

$$\delta(x, m) := c(x) - c_0(x)\mathbf{1}_{\{m \le X_0\}}, \qquad c_0(x) := (X_0 - x)^+$$

and

$$G(x, m) := g_{xx}(x, X_0) - g_{xx}(x, m).$$

Step 2. The expression of $\mu(\lambda) + v^\psi$ derived in the previous step only involves the function $\psi \in \Psi^\lambda$. Forgetting about all constraints on ψ, we treat our minimization problem as a standard problem of calculus of variations. The local Euler–Lagrange equation for this problem is

$$\frac{d}{dx}(G\delta)(\psi, m) = -(\gamma\delta)_x(\psi, m) + (G\delta)_x(\psi, m)\psi'.$$

Since $(G\delta_m)(x, m) = 0$, this reduces to

$$\begin{aligned}
0 &= (G_m\delta + \gamma\delta_x + \gamma_x\delta)(\psi, m) \\
&= (m - \psi)\gamma(\psi, m)\frac{\partial}{\partial x}\left\{\frac{\delta(x, m)}{m - x}\right\}_{x=\psi}.
\end{aligned}$$

This shows formally that the solution of the minimization problem

$$\min_{\xi < m} \frac{\delta(x, m)}{m - x}$$

provides a solution to the local Euler–Lagrange equation. Finally, the solution of the above minimization problem, as recalled in Section 2.7, is known to be given by the right inverse barycenter function b^{-1}; see also the proof of Lemma 3.3 in [23].

APPENDIX A: PROOF OF LEMMA 3.4

Note that we may restrict to optimizing in (3.5) over $\underline{b}^{-1}(m) \le \zeta_1 \le \cdots \le \zeta_n \le m$. Indeed, fix some ζ as in (3.5) with $\zeta_1 = \cdots = \zeta_i < \min\{\zeta_{i+1}, \underline{b}^{-1}(m)\}$. Then all the terms featuring ζ_j for $j \le i$ reduce simply to $c_i(\zeta_i)/(m - \zeta_i)$ which is nonincreasing for $\zeta_i \le b_i^{-1}(m)$ by the discussion of (2.12). It follows that we may only decrease the value of the objective in (3.5) by setting $\zeta_1 = \cdots = \zeta_i = \min\{\zeta_{i+1}, \underline{b}^{-1}(m)\}$. In consequence the problem in (3.5) reduces to minimization of a continuous function in a compact subset of $\mathbb{R}^n$ and admits a minimizer $\zeta^*(m)$ with $\zeta_1^*(m) \ge \underline{b}^{-1}(m)$.

We finally verify that $m \longmapsto \zeta^*(m)$ can be chosen to be measurable. Indeed, for a fixed m, the set $F(m)$ of minimizers in (3.5) is closed, and for any closed $K \subset \mathbb{R}^n$, $\{m : F(m) \cap K \neq \varnothing\}$ is equal to $\{m : C(m) = C_K(m)\}$ where C_K is given as C in (3.5) but with a further requirement that $(\zeta_1, \ldots, \zeta_n) \in K$. Both C and C_K can be obtained through countable pointwise minimization of continuous functions and hence are measurable, as is $\{m : C(m) = C_K(m)\}$. Existence of a measurable selector for F now follows from Kuratowski and Ryll-Nardzewski measurable selection theorem; see, for example, Wagner [51], Theorem 4.1.

APPENDIX B: ON LOCAL MARTINGALES WITH SIMPLE INTEGRANDS

We now report a characterization of martingales defined as stochastic integrals with simple integrands, which was used in Remark 3.7 and the proof of Theorem 3.5.

Let $\{X_t, t \in [0, T]\}$ be a $(\mathbb{P}, \{\mathcal{F}_t\}_{0 \le t \le T})$-martingale, $0 = t_0 < t_1 < \cdots < t_n = T$ a partition of $[0, T]$ and H_{t_i} an $\mathcal{F}_{t_i}$-measurable r.v. for all $i = 0, \ldots, n-1$. Our interest is in the process

$$Y_t := \sum_{i=1}^{n} H_{t_{i-1}}(X_{t \wedge t_i} - X_{t \wedge t_{i-1}}), \qquad t \in [0, T].$$

Since X is a martingale, it follows that Y is a local martingale. The following result is an easy adaptation of a similar result for finite discrete-time local martingales reported in Jacod and Shiryaev [30].

LEMMA B.1. *The process Y is a martingale if and only if Y_T^- is integrable.*

PROOF. The necessary condition is obvious. We now assume Y_T^- is integrable and consider the sequence of stopping times

$$\tau_k := T \wedge \min\{t_i : 0 \le i \le n, |H_{t_i}| \ge k\}, \qquad k \in \mathbb{N}.$$

Clearly, $(\tau_k)_{k\geq 1}$ is a localizing sequence for the local martingale Y, taking values in the finite set $\{t_0, \dots, t_n\}$.

We first show that Y_t is integrable for all $t \in [0, T]$. By the Jensen inequality, we have

$$Y_t^- \leq \mathbb{E}\big[Y_T^- | \mathcal{F}_t\big] \qquad \text{on } \{\tau_k > t_{n-1}\}.$$

This shows that $\mathbb{E}[Y_t^-] < \infty$ by sending $k \to \infty$. We continue estimating

$$\begin{aligned}
\mathbb{E}\big[Y_t^+\big] &= \mathbb{E}\Big[\liminf_{k\to\infty} Y_{t\wedge\tau_k}^+\Big] \leq \liminf_{k\to\infty} \mathbb{E}\big[Y_{t\wedge\tau_k}^+\big] \\
&= \liminf_{k\to\infty} \mathbb{E}\big[Y_{t\wedge\tau_k} + Y_{t\wedge\tau_k}^-\big] \\
&= \liminf_{k\to\infty} \mathbb{E}\big[Y_{t\wedge\tau_k}^-\big] \\
&\leq \sum_{i=0}^{n} \mathbb{E}\big[Y_{t\wedge t_i}^-\big] < \infty,
\end{aligned}$$

where we used Fatou's lemma, the fact that $Y_{.\wedge\tau_k}$ is a martingale starting from the origin and the crucial property that the localizing sequence takes values in the finite set $\{t_i\}_{0\leq i\leq n}$. Hence $\mathbb{E}|Y_t| < \infty$ for all $t \in [0, T]$.

We next show that Y satisfies the martingale property. Clearly, it is sufficient to prove the martingale property on each interval $[t_{i-1}, t_i]$. For $t_{i-1} \leq s \leq t \leq t_i$, it follows from the martingale property of the stopped process $Y_{.\wedge\tau_k}$, together with the fact that the localizing sequence takes values in the finite set $\{t_i\}_{0\leq i\leq n}$, that $\mathbb{E}[Y_t|\mathcal{F}_s] = Y_s$ on $\{\tau_k > s\}$. The required result follows immediately by sending $k \to \infty$. $\square$

APPENDIX C: PROOF OF LEMMA 5.8

We start with the computation of $\gamma_i(\psi_i, \cdot)$, as defined in (5.26), in terms of ϕ and the ψ_i's.

LEMMA C.1. *For all $i < n$, we have* $\gamma_i(\psi_i(m), m) = \frac{\phi'(m)}{m-\psi_i(m)} \mathbf{1}_{\{\psi_i < \overline{\psi}_{i+1}\}}$.

PROOF. By direct differentiation of (5.25), we see that

$$\begin{aligned}
\partial_m v_{i-1}(x, m) &= \partial_m v_i\big(x \wedge \psi_i(m), m\big) \\
&\quad + \big(x - \psi_i(m)\big)^+ \big[\partial_{xx} v_i\big(\psi_i(m), m\big)\psi_i'(m) + \partial_{xm} v_i\big(\psi_i(m), m\big)\big].
\end{aligned}$$

Using the ODE satisfied by ψ_i, this provides

$$\begin{aligned}
\partial_m v_{i-1}(x, m) &= \partial_m v_i\big(x \wedge \psi_i(m), m\big) \\
&\quad - \frac{(x - \psi_i(m))^+}{m - \psi_i(m)} \partial_m v_i\big(x \wedge \psi_i(m), m\big) \\
&= \frac{m - x \vee \psi_i(m)}{m - \psi_i(m)} \partial_m v_i\big(x \wedge \psi_i(m), m\big).
\end{aligned} \tag{C.1}$$

Since $\partial_m v_i$ is concave in x, we also compute by differentiating this expression that

$$\begin{aligned}\partial_{mx} v_{i-1}(x,m) &= \mathbf{1}_{\{x<\psi_i(m)\}} \partial_{mx} v_i\big(x\wedge\psi_i(m),m\big)\\ &\quad + \mathbf{1}_{\{x>\psi_i(m)\}} \frac{-1}{m-\psi_i(m)} \partial_m v_i\big(x\wedge\psi_i(m),m\big) \qquad \text{a.e.}\end{aligned} \tag{C.2}$$

From the expression of γ_i, it follows from (C.1) and (C.2) that

$$\begin{aligned}\gamma_{i-1}(x,m) = \mathbf{1}_{\{x<\psi_i(m)\}}\gamma_i(x,m) &= \cdots = \mathbf{1}_{\{x<\overline{\psi}_i(m)\}}\gamma_n(x,m)\\ &= \mathbf{1}_{\{x<\overline{\psi}_i(m)\}}\frac{\phi'(m)}{m-x} \qquad \text{a.e.}\end{aligned}$$

□

PROOF OF LEMMA 5.8. Recall that $\psi \in \Psi_n^\lambda$ so that both ψ_i and ψ_i^{-1} are continuous and increasing. For any integrable function φ, the claim

$$\begin{aligned}&\int \varphi(\xi)\lambda_i''(d\xi)\\ &\quad = \int\Bigg(\frac{\varphi(\psi_i(m))}{m-\psi_i(m)}\mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}}\\ &\qquad\qquad - \sum_{j=i+1}^{k}\frac{\varphi(\psi_j(m))}{m-\psi_j(m)}\mathbf{1}_{\{\psi_i(m)<\psi_j(m)=\overline{\psi}_j(m)\}}\Bigg)\,d\phi(m)\\ &\qquad + \int \varphi(\xi)\big[\partial_{xx}v_k\big(\xi,\psi_i^{-1}(\xi)\big) - \partial_{xx}v_k\big(\xi,\big(\psi_{i+1}^{-1}\vee\cdots\vee\psi_k^{-1}\big)(\xi)\big)\big]\\ &\qquad\quad \times \mathbf{1}_{\{\psi_i^{-1}(\xi)>(\psi_{i+1}^{-1}\vee\cdots\vee\psi_k^{-1})(\xi)\}}\,d\xi,\end{aligned} \tag{C.3}$$

which will be proved below by induction, implies the required result for $k=n$, and uses the fact that $v_n=\phi$ is independent of x.

We start verifying (C.3) for $k=i+1$. From the expression of v_i in (5.25), we have

$$\begin{aligned}v_j = v_{j+1}^\lambda \qquad &\text{on } \{x<\psi_{j+1}(m)\} \quad \text{and}\\ \partial_{xx}v_j = 0 \qquad &\text{on } \{x>\psi_{j+1}(m)\},\end{aligned} \tag{C.4}$$

where $v_j^\lambda = v_j - \lambda_j$.

Step 1. To see that (C.3) holds true with $k=i+1$, we first decompose the integral so as to use the ODE satisfied by ψ_i,

$$\begin{aligned}\int \varphi\lambda_i'' &= -\int \varphi(\xi)\partial_{xx}v_i^\lambda\big(\xi,\psi_i^{-1}(\xi)\big)\,d\xi + \int\varphi(\xi)\partial_{xx}v_i\big(\xi,\psi_i^{-1}(\xi)\big)\,d\xi\\ &= \int\varphi\big(\psi_i(m)\big)\gamma_i\big(\psi_i(m),m\big)\,dm + \int\varphi(\xi)\partial_{xx}v_i\big(\xi,\psi_i^{-1}(\xi)\big)\,d\xi.\end{aligned}$$

Substitute the expression of $\gamma_i(\psi_i,\cdot)$ from Lemma C.1, and use (C.4) for the second integral,

$$
\begin{aligned}
\int \varphi\lambda_i'' = {} & \int \frac{\varphi(\psi_i(m))}{m-\psi_i(m)}\mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}}\,d\phi(m) \\
& + \int \varphi(\xi)\partial_{xx}v_{i+1}^{\lambda}\big(\xi,\psi_i^{-1}(\xi)\big)\mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\,d\xi \\
= {} & \int \frac{\varphi(\psi_i(m))}{m-\psi_i(m)}\mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}}\,d\phi(m) \\
& + \int \varphi(\xi)\partial_{xx}v_{i+1}^{\lambda}\big(\xi,\psi_{i+1}^{-1}(\xi)\big)\mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\,d\xi \\
& + \int \varphi(\xi)\big[\partial_{xx}v_{i+1}^{\lambda}\big(\xi,\psi_i^{-1}(\xi)\big)-\partial_{xx}v_{i+1}^{\lambda}\big(\xi,\psi_{i+1}^{-1}(\xi)\big)\big] \\
& \qquad \times \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\,d\xi.
\end{aligned}
$$

Then, by using again ODE (5.13) satisfied by ψ_{i+1} together with the expression of $\gamma_{i+1}(\psi_{i+1},\cdot)$ from Lemma C.1, we get

$$
\begin{aligned}
\int \varphi\lambda_i'' = {} & \int \frac{\varphi(\psi_i(m))}{m-\psi_i(m)}\mathbf{1}_{\{\psi_i(m)<\overline{\psi}_{i+1}(m)\}}\,d\phi(m) \\
& - \int \frac{\varphi(\psi_{i+1}(m))}{m-\psi_{i+1}(m)}\mathbf{1}_{\{\psi_i(m)<\psi_{i+1}(m)=\overline{\psi}_{i+1}(m)\}}\,d\phi(m) \\
& + \int \varphi(\xi)\big[\partial_{xx}v_{i+1}^{\lambda}\big(\xi,\psi_i^{-1}(\xi)\big)-\partial_{xx}v_{i+1}^{\lambda}\big(\xi,\psi_{i+1}^{-1}(\xi)\big)\big] \\
& \qquad \times \mathbf{1}_{\{\psi_{i+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\,d\xi,
\end{aligned}
$$

which we recognize to be the required equality (C.3) for $k=i+1$.

Step 2. We next assume that (C.3) holds for some $k<n-1$, and verify it for $k+1$. For simplicity, we denote $\psi_{i+1,j}^{-1} := \psi_{i+1}^{-1}\vee\cdots\vee\psi_j^{-1}$. By (C.4), we compute that

$$
\begin{aligned}
A := {} & \int \varphi(\xi)\big[\partial_{xx}v_k\big(\xi,\psi_i^{-1}(\xi)\big)-\partial_{xx}v_k\big(\xi,\psi_{i+1,k}^{-1}(\xi)\big)\big]\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi_{i+1,k}^{-1}(\xi)\}}\,d\xi \\
= {} & \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi_{i+1,k}^{-1}(\xi)\}} \\
& \quad \times \big[\big\{\partial_{xx}v_{k+1}\big(\xi,\psi_i^{-1}(\xi)\big)-\lambda_{k+1}''(\xi)\big\}\mathbf{1}_{\{\psi_{k+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}} \\
& \qquad - \big\{\partial_{xx}v_{k+1}\big(\xi,\psi_{i+1,k}^{-1}(\xi)\big)-\lambda_{k+1}''(\xi)\big\}\mathbf{1}_{\{\psi_{k+1}^{-1}(\xi)<\psi_{i+1,k}^{-1}(\xi)\}}\big]\,d\xi \\
= {} & \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi_{i+1,k}^{-1}(\xi)\}} \\
& \quad \times \big[\mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi)<\psi_{k+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\partial_{xx}v_{k+1}^{\lambda}\big(\xi,\psi_i^{-1}(\xi)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi)<\psi_{i+1,k}^{-1}(\xi)\}}\\
&\qquad\quad \times \{\partial_{xx} v_{k+1}(\xi,\psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}(\xi,\psi_{i+1,k}^{-1}(\xi))\}\big]\,d\xi\\
&= \int \varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi_{i+1,k}^{-1}(\xi)\}}\\
&\quad \times \big[\mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi)<\psi_{k+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\partial_{xx} v_{k+1}^{\lambda}(\xi,\psi_{k+1}^{-1}(\xi))\\
&\qquad + \mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi)<\psi_{k+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\\
&\qquad\quad \times \{\partial_{xx} v_{k+1}^{\lambda}(\xi,\psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}^{\lambda}(\xi,\psi_{k+1}^{-1}(\xi))\}\\
&\qquad + \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi)<\psi_{i+1,k}^{-1}(\xi)\}}\\
&\qquad\quad \times \{\partial_{xx} v_{k+1}(\xi,\psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}(\xi,\psi_{i+1,k}^{-1}(\xi))\}\big]\,d\xi.
\end{aligned}
$$

Putting together the two last terms, we see that

$$
\begin{aligned}
A = \int &\varphi(\xi)\mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi_{i+1,k}^{-1}(\xi)\}}\\
&\times \big[\mathbf{1}_{\{\psi_{i+1,k}^{-1}(\xi)<\psi_{k+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\partial_{xx} v_{k+1}^{\lambda}(\xi,\psi_{k+1}^{-1}(\xi))\\
&\qquad + \mathbf{1}_{\{\psi_{k+1}^{-1}(\xi)<\psi_i^{-1}(\xi)\}}\\
&\qquad\quad \times \{\partial_{xx} v_{k+1}^{\lambda}(\xi,\psi_i^{-1}(\xi)) - \partial_{xx} v_{k+1}^{\lambda}(\xi,\psi_{i+1,k+1}^{-1}(\xi))\}\big]\,d\xi.
\end{aligned}
$$

Finally, using ODE (5.26) satisfied by ψ_{k+1} in the first term, together with the expression of $\gamma_{k+1}(\psi_{k+1},\cdot)$ from Lemma C.1, we see that

$$
\begin{aligned}
A = &-\int \varphi(\psi_{k+1}(m))\frac{\varphi(\psi_{k+1}(m))}{\psi_{k+1}(m)-m}\mathbf{1}_{\{\psi_i(m)<\psi_{k+1}(m)=\overline{\psi}_{k+1}(m)\}}\,d\phi(m)\\
&+\int \varphi(\xi)\big[\partial_{xx} v_{k+2}^{\lambda}(\xi,\psi_i^{-1}(\xi)) - \partial_{xx} v_{k+2}^{\lambda}(\xi,\psi_{i+1,k+2}^{-1}(\xi))\big]\\
&\qquad \times \mathbf{1}_{\{\psi_i^{-1}(\xi)>\psi_{i+1,k+1}^{-1}(\xi)\}}\,d\xi,
\end{aligned}
$$

which is precisely the required expression in order to justify that (C.3) holds for $k+1$. $\square$

APPENDIX D: PROOFS FOR STATEMENTS IN SECTION 4.2

PROOF OF LEMMA 4.1. Fix $m > Z_0$, and write for notational convenience $\zeta_i = \zeta_i(m)$, $\eta_i = \eta_i(m)$. Let $n_1 < \cdots < n_k = n$ be such that

$$
\zeta_1 = \cdots = \zeta_{n_1} < \zeta_{n_1+1} = \cdots = \zeta_{n_2} < \cdots < \zeta_{n_{k-1}+1} = \cdots = \zeta_{n_k} = \zeta_n = \eta_n.
$$

Then by

$$Z_{t_j} \geq \zeta_j \quad \Longrightarrow \quad Z_{t_l} \geq \zeta_j \qquad \forall l \geq j$$

for all $j \leq n$, which follows directly from the definitions (4.4) and (4.5), we obtain by identifying the terms as in the proof of Proposition 3.1,

$$\begin{aligned}\Upsilon_n(Z, m, \zeta) = \sum_{j=1}^{k} \biggl(\frac{(Z_{t_{n_j}} - \zeta_i)^+}{m - \zeta_{n_j}} + \mathbf{1}_{\{Z^*_{t_{n_{j-1}}} < m \leq Z^*_{t_{n_j}}\}} \frac{m - Z_{t_{n_j}}}{m - \zeta_{n_j}} \biggr) \\ - \sum_{j=1}^{k-1} \biggl(\frac{(Z_{t_{n_j}} - \zeta_{n_{j+1}})^+}{m - \zeta_{n_{j+1}}} + \mathbf{1}_{\{m \leq Z^*_{t_{n_j}}, \zeta_{n_{j+1}} \leq Z_{t_{n_j}}\}} \frac{Z_{t_{n_{j+1}}} - Z_{t_{n_j}}}{m - \zeta_{n_{j+1}}} \biggr).\end{aligned}$$

Therefore, it is enough to prove the claim for the case

$$\zeta_1 = \eta_1 < \zeta_2 = \eta_2 < \cdots < \zeta_n = \eta_n.$$

By the same induction as in the proof of Proposition 3.1 it remains to prove that (Z, Z^*) achieves equality in (4.1), (4.2) and (4.3). As for equality in (4.1) we note that

$$Z^*_{t_1} \geq m \quad \Longrightarrow \quad Z_{t_1} \geq \zeta_1 \quad \text{and} \quad Z^*_{t_1} < m \quad \Longrightarrow \quad Z_{t_1} \leq \zeta_1.$$

Equality in (4.2) holds by

$$\begin{aligned} Z^*_{t_{n-1}} \geq m, \qquad Z_{t_{n-1}} \geq \zeta_n \quad &\Longrightarrow \quad Z_{t_n} \geq \zeta_n, \\ Z^*_{t_{n-1}} \geq m, \qquad Z_{t_{n-1}} < \zeta_n \quad &\Longrightarrow \quad Z_{t_n} < \zeta_n, \end{aligned}$$

which one verifies using the definition of the iterated Azéma–Yor-type embedding. Similarly, equality in (4.3) holds by

$$\begin{aligned} Z^*_{t_{n-1}} < m, \qquad Z^*_{t_n} \geq m \quad &\Longrightarrow \quad Z_{t_n} \geq \zeta_n, \\ Z^*_{t_{n-1}} < m, \qquad Z^*_{t_n} < m \quad &\Longrightarrow \quad Z_{t_n} \leq \zeta_n. \end{aligned}$$

The claim follows. □

PROOF OF EQUATION (4.6). Finally, we argue that (4.6) holds under Assumption ⊛ of Obłój and Spoida [41]. Let $\tilde{\zeta}_i(m) := \min_{j \geq i} \eta_j(m)$.

First fix $m \in (X_0, r^{\mu_n})$ such that $\tilde{\zeta}_n(m) < m$. Then Lemma 4.1 and Proposition 3.2 yield

$$\mathbb{P}[Z^*_{t_n} \geq m] = \mathbb{E}[\Upsilon_n(Z, m, \tilde{\zeta})] \leq C(m).$$

Next we show that under Assumption ⊛ this inequality is strict whenever $\zeta^*(m) \neq \tilde{\zeta}(m)$. This will be a contradiction to the optimality of ζ^* since, by invoking the embedding property $Z_{t_i} \sim \mu_i$, we must have $\mathbb{E}[\Upsilon_n(Z, m, \tilde{\zeta})] \geq C(m)$.

Case A. $\tilde{\zeta}_j(m) > \zeta_j^*(m)$. Assume initially that $m \in (X_0, r^{\mu_j}]$. Then on the set

$$\{Z_{t_j} > \zeta_j^*(m), Z_{t_j}^* < m\} \supseteq \{Z_{t_j}^* \in \mathcal{O}_A, Z_{t_n}^* \in \mathcal{O}_A\} =: \mathcal{Z}_A,$$

where $\mathcal{O}_A \subseteq [X_0, m)$ is a (suitable) open interval, we obtain

$$\Upsilon_n(Z, m, \zeta^*) \geq \Upsilon_j(Z, m, \zeta^*) \overset{(4.3)}{>} \mathbf{1}_{\{m \leq Z_{t_j}^*\}} = \mathbf{1}_{\{m \leq Z_{t_n}^*\}} \overset{\text{Lemma 4.1}}{=} \Upsilon_n(Z, m, \tilde{\zeta}).$$

Note that $\mathbb{E}[\Upsilon_n(Z, m, \zeta^*)] = C(m)$. It follows that if $\mathbb{P}[\mathcal{Z}_A] > 0$, then $\mathbb{E}[\Upsilon_n(Z, m, \tilde{\zeta})] < C(m)$ as required. However, this is clear by the assumption that $m \in (X_0, r^{\mu_j}]$ and elementary properties of Brownian motion.

If $m > r^{\mu_j}$, then by Lemma 4.1 we get $\mathbb{E}[\Upsilon_j(Z, m, \tilde{\zeta})] = 0$. If $\sum_{i=1}^j (\frac{c_i(\zeta_i^*)}{m-\zeta_i^*} - \frac{c_i(\zeta_{i+1}^*)}{m-\zeta_{i+1}^*}\mathbf{1}_{\{i<n\}}) > 0$, then $\mathbb{E}[\Upsilon_j(Z, m, \zeta^*)] > 0$ and hence

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{Z_{t_n}<m\}}\Upsilon_n(Z, m, \zeta^*)] &\geq \mathbb{E}[\mathbf{1}_{\{Z_{t_n}<m\}}\Upsilon_j(Z, m, \zeta^*)] > 0 \\ &= \mathbb{E}[\mathbf{1}_{\{Z_{t_n}<m\}}\Upsilon_n(Z, m, \tilde{\zeta})].\end{aligned}$$

If $\sum_{i=1}^j (\frac{c_i(\zeta_i^*)}{m-\zeta_i^*} - \frac{c_i(\zeta_{i+1}^*)}{m-\zeta_{i+1}^*}\mathbf{1}_{\{i<n\}}) = 0$, then a contradiction to Assumption $\circledast$ is obtained if $\zeta_j^*(m) < \tilde{\zeta}_j(m)$.

Case B. $\tilde{\zeta}_j(m) < \zeta_j^*(m)$. We can, without loss of generality, take $m \in (X_0, r^{\mu_j}]$. Indeed, if this is not the case, then we set $j' \geq j$ such that $\tilde{\zeta}_j(m) = \tilde{\zeta}_{j+1}(m) = \cdots = \tilde{\zeta}_{j'}(m) = \eta_{j'}(m) < \tilde{\zeta}_{j'+1}(m)$. For this j' we then have $\tilde{\zeta}_{j'}(m) < \zeta_{j'}^*(m)$ and $m \in (X_0, r^{\mu_{j'}}]$ as $\eta_{j'}(m) < m$.

Then on the set

$$\{Z_{t_j} < \zeta_j^*(m), Z_{t_j}^* \geq m\} \supseteq \{Z_{t_j}^* \in \mathcal{O}_B\} =: \mathcal{Z}_B,$$

where $\mathcal{O}_B \subseteq [m, \infty)$ is a (suitable) open interval, we obtain

$$\Upsilon_n(Z, m, \zeta^*) \geq \Upsilon_j(Z, m, \zeta^*) \overset{(4.3)}{>} 1 = \mathbf{1}_{\{m \leq Z_{t_j}^*\}} = \mathbf{1}_{\{m \leq Z_{t_n}^*\}} \overset{\text{Lemma 4.1}}{=} \Upsilon_n(Z, m, \tilde{\zeta}),$$

which yields a contradiction in a similar fashion to case A because $\mathbb{P}[\mathcal{Z}_B] > 0$.

Now consider $\tilde{\zeta}_n(m) = m$. We assume for simplicity of the argument that $\tilde{\zeta}_{n-1}(m) < \tilde{\zeta}_n(m)$. (By, e.g., use of Lemma A.1 of Obłój and Spoida [41], the general case can be reduced to this case.) Then optimality of $\zeta_n^*(m)$ yields

$$\left.\frac{c_n(z) - c_{n-1}(z)}{m - z}\right|_{z=\zeta_n^*(m)} \leq \left.\frac{c_n(z) - c_{n-1}(z)}{m - z}\right|_{z=\tilde{\zeta}_n(m)=m}.$$

From this a direct contradiction to Assumption $\circledast$ is obtained if $\zeta_n^*(m) \neq \tilde{\zeta}_n(m)$. Hence $\zeta_n^*(m) = \tilde{\zeta}_n(m)$. Then $\zeta_1^*, \ldots, \zeta_{n-1}^*$ are also optimal for the $n-1$ marginal problem (3.5). Further, the $n-1$ marginal Azéma–Yor embedding coincides with the n marginal iterated Azéma–Yor embedding until time t_{n-1}. Hence, by induction, $\zeta_i^*(m) = \tilde{\zeta}_i(m)$ for all $i < n$. $\square$

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